On isotropy of quadratic spaces in finite and infinite dimension

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Abstract

Herbert Gross asked whether there exist fields admitting anisotropic quadratic spaces of arbitrarily large finite dimensions but none of infinite dimension. This article provides examples of such fields and discusses related problems in the theory of central simple algebras and in Milnor K-theory.

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1 Introduction

Questions about isotropy are at the core of the algebraic theory of quadratic forms over fields. A natural and much studied field invariant in this context is the so-called u-invariant of a field F of characteristic different from 2. For a nonreal[1] field F it is defined to be the supremum of the dimensions of anisotropic finite-dimensional quadratic forms over F (see Section 2 for the general definition of the u-invariant). The main purpose of the present article is to give examples of fields having infinite u-invariant but not admitting any anisotropic infinite-dimensional quadratic space.

A *quadratic space* over the field F is a pair (V, q) of a vector space V over F together with a map $q : V \longrightarrow F$ such that

[1] For brevity's sake, we use the term 'real field' to denote what many authors call 'formally real field'.

- $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in F$, $x \in V$, and
- the map $b_q : V \times V \longrightarrow F$ defined by $b_q(x, y) = q(x + y) - q(x) - q(y)$ $(x, y \in V)$ is F-bilinear.[2]

If V has finite dimension n, we may identify (after fixing a basis of V) the quadratic space (V, q) with a form (a homogeneous polynomial) of degree 2 in n variables. By this identification, we will also call (V, q) a *quadratic form* over F if $\dim V < \infty$. Recall that a quadratic space (V, q) is said to be *isotropic*, if there exists $x \in V \setminus \{0\}$ such that $q(x) = 0$; otherwise, (V, q) is said to be *anisotropic*.

Assume now that the quadratic space (V, q) over F is anisotropic. For any positive integer $n \leq \dim(V)$, we may choose an n-dimensional subspace V_n of V and consider the restriction q_n of q to V_n; in this way we obtain an n-dimensional quadratic form (V_n, q_n) which is obviously again anisotropic. By this simple argument, we see that if there is an anisotropic quadratic space over F of infinite dimension, then there exist anisotropic quadratic forms over F of dimension n for all $n \in \mathbb{N}$.

While this observation is rather trivial, it motivates us to examine the converse statement. If we assume that the field F has anisotropic quadratic forms of arbitrarily large finite dimensions, does this imply the existence of some anisotropic quadratic space (V, q) over F of infinite dimension? As already mentioned in the beginning, this is generally not so.

It appears that originally this question has been formulated by Herbert Gross. He concludes the introduction to his book 'Quadratic forms in infinite-dimensional vector spaces' [12] (that appeared 1979) by the following sample of 'a number of pretty and unsolved problems' in this area, which we state in his terms (cf. [12], p. 3; here, it is assumed that the characteristic is different from 2):

1.1 Question (Gross). Is there any commutative field which admits no anisotropic $\aleph_0$-form but which has infinite u-invariant, i.e. admits, for each $n \in \mathbb{N}$, some anisotropic form in n variables?

Note that implicitly, Gross is looking for a nonreal field, i.e. a field where -1 is a sum of squares; for over a real field anisotropic quadratic spaces of infinite dimension do always exist. Indeed, one observes (or one may take it

[2]Often, one puts a factor $1/2$ in front on the right hand side of the equation, which then readily establishes the fact that quadratic and bilinear forms in characteristic not 2 are equivalent concepts. The omission has no bearing on what we will do in characteristic different from 2, but it allows us to also work in characteristic 2 where there is a difference in the concepts of quadratic and bilinear forms, see section 5.

as a definition) that the field F is real if and only if the infinite-dimensional quadratic space (V, q) given by $V = k^{(\mathbb{N})}$ and $q : V \longrightarrow k, (x_i) \longmapsto \sum x_i^2$ is anisotropic.

By restricting to those quadratic spaces that are totally indefinite (i.e. indefinite with respect to every field ordering), we obtain a meaningful analogue of the Gross Question, to which we can equally provide a solution.

Of course, one may place the Gross Question also in the context of characteristic 2; there, however, one has to distinguish between bilinear forms and quadratic forms. When considering quadratic forms, one furthermore has to distinguish the case of nonsingular forms from the case where one allows arbitrary quadratic forms. The analogue to the Gross Question for nonsingular quadratic forms in characteristic 2 can be treated in more or less the same way as in characteristic not 2, simply by invoking suitable characteristic 2 analogues of the results that we use in our proofs in the case of characteristic different from 2. Yet, if translated to bilinear forms or to arbitrary quadratic forms (possibly singular) in characteristic 2, it is not difficult to show that the Gross Question has in fact a negative answer, in other words, the 'bilinear' resp. 'general quadratic' u-invariant is infinite if and only if there exist infinite-dimensional anisotropic bilinear resp. quadratic spaces.

The paper is structured as follows. In the next section, we are going to discuss in more detail the u-invariant of a field and some related concepts. In Section 3 we will give two different constructions of nonreal fields, each giving a positive answer to the Gross Question.

All our constructions will be based on Merkurjev's method where one starts with an arbitrary field of characteristic different from 2 and then uses iterated extensions obtained by composing function fields of quadrics to produce an extension with the desired properties. Our first construction will show the following:

1.2 Theorem I. *Let F be a field of characteristic different from 2. There exists a field extension K/F with the following properties:*

(i) K has no finite extensions of odd degree.

(ii) For any binary quadratic form β over K, there is an upper bound on the dimensions of anisotropic quadratic forms over K that contain β.

(iii) For any $k \in \mathbb{N}$, there is an anisotropic k-fold Pfister form over K.

In particular, K is a perfect, nonreal field of infinite u-invariant, $I^kK \neq 0$ for all $k \in \mathbb{N}$, and any infinite-dimensional quadratic space over K is isotropic.

Here and in the sequel, I^kF stands for the kth power of IF, the fundamental ideal consisting of classes of even-dimensional forms in the Witt ring WF of F.

The proof of this theorem only uses some basic properties of Pfister forms and standard techniques from the theory of funcion fields of quadratic forms. Varying this construction and using this time products of quaternion algebras and Merkurjev's index reduction criterion (see [24] or [38], Théorème 1), we will then show the following:

1.3 Theorem II. *Let F be a field of characteristic different from 2. There exists a field extension K/F with the following properties:*

(i) *K has no finite extensions of odd degree and $I^3K = 0$.*

(ii) *For any binary quadratic form β over K, there is an upper bound on the dimensions of anisotropic quadratic forms over K that contain β.*

(iii) *For any $k \in \mathbb{N}$, there is a central division algebra over K that is decomposable into a tensor product of k quaternion algebras.*

In particular, K is a nonreal field of infinite u-invariant, and any infinite-dimensional quadratic space over K is isotropic. Furthermore, K is perfect and of cohomological dimension 2.

In Section 4, we will show two analoguous theorems for real fields.

1.4 Theorem III. *Assume that F is real. Then there exists a field extension K/F with the following properties:*

(i) *K has a unique ordering.*

(ii) *K has no finite extensions of odd degree and I^3K is torsion free.*

(iii) *For any totally indefinite quadratic form β over K, there is an upper bound on the dimensions of anisotropic quadratic forms over K that contain β.*

(iv) *For any $k \in \mathbb{N}$, there is a central division algebra over K that is decomposable into a tensor product of k quaternion algebras.*

In particular, K is real field of infinite u-invariant, and any totally indefinite quadratic space of infinite dimension over K is isotropic; moreover, the cohomological dimension of $K(\sqrt{-1})$ is 2.

While this can be seen as a counterpart to Theorem II for real fields, we can also prove an analogue of Theorem I in this situation.

1.5 Theorem IV. *Assume that F is real. Then there exists a field extension K/F with the following properties:*

(i) *K has a unique ordering.*

(ii) *K has no finite extensions of odd degree.*

(iii) *For any totally indefinite quadratic form β over K, there is an upper bound on the dimensions of anisotropic quadratic forms over K that contain β.*

(iv) *for any $k \in \mathbb{N}$, there is an element $a \in K^\times$ which is a sum of squares in K, but not a sum of k squares.*

In particular, K is a real field for which the Pythagoras number, the Hasse number, and the u-invariant are all infinite, the torsion part of $I^k K$ is nonzero for all $k \in \mathbb{N}$, and any totally indefinite quadratic space of infinite dimension over K is isotropic.

In Section 5, we will discuss the Gross Question for quadratic, nonsingular quadratic and symmetric bilinear forms in characteristic 2. As already mentioned, for nonsingular quadratic forms, we obtain similar results as in characteristic different from 2, whereas for arbitrary quadratic forms and for symmetric bilinear forms the answer turns out to be negative.

In the final Section 6, we discuss an abstract version of the Gross Question, formulated for an arbitrary monoïd together with two subsets satisfying some requirements. We give examples of such monoïds whose elements are well known objects associated to an arbitrary field, such as central simple algebras or symbols in Milnor K-theory modulo a prime p. In some of the cases that we shall discuss, the answer to (the analogue of) the Gross Question will be positive, in others it will be negative.

For all prerequisites from quadratic form theory in characteristic different from 2 needed in the sequel, we refer to the books of Lam and Scharlau (see [20], [21] and [34]). In general, we use the standard notations introduced there. However, we use a different sign convention for Pfister forms: Given $a_1, \ldots, a_r \in F^\times$, we write $\langle\langle a_1, \ldots, a_r \rangle\rangle$ for the *r-fold Pfister form* $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_r \rangle$. If φ is a quadratic form over F and $n \in \mathbb{N}$, we denote by $n \times \varphi$ the n-fold orthogonal sum $\varphi \perp \cdots \perp \varphi$. Given two quadratic forms φ and ψ over F, we write $\psi \subset \varphi$ to indicate that ψ is a *subform* of φ, in other words, that there exists another quadratic form τ over F such that $\varphi \cong \psi \perp \tau$.

A quadratic space (V, q) is said to be *nonsingular* if the radical

$$\mathrm{Rad}(V, q) = \{x \in V \mid b_q(x, y) = 0 \text{ for all } y \in V\}$$

is reduced to 0. Anisotropic quadratic spaces in characteristic different from 2 are obviously always nonsingular, but this need not be so in characteristic 2.

Unless stated otherwise, the terms 'form' or 'quadratic form' will always stand for 'nonsingular quadratic form'. A *binary form* is a 2-dimensional quadratic form.

We recall the definition of the function field $F(\varphi)$ associated to a nonsingular quadratic form φ over F in characteristic different from 2. If $\dim(\varphi) \geq 3$ or if $\dim(\varphi) = 2$ and φ is anisotropic, then $F(\varphi)$ is the function field of the projective quadric given by the equation $\varphi = 0$. We put $F(\varphi) = F$ if φ is the hyperbolic plane or if $\dim(\varphi) \leq 1$. We refer to [34], Chapter 4, §5, or [21], Chapter X, for the crucial properties of function field extensions. They will play a prominent rôle in all our constructions.

Let K/F be an arbitrary field extension. If φ is a quadratic form over F, then we denote by φ_K the quadratic form over K obtained by scalar extension from F to K. Similarly, given an F-algebra A, we write A_K for the K-algebra $A \otimes_F K$. Central simple algebras are by definition finite-dimensional. A central simple algebra without zero-divisors will be called a 'division algebra' for short. For the basics about central simple algebras and the Brauer group of a field, the reader is referred to [34], Chapter 8, or [31], Chapters 12-13.

2 The u-invariant and its relatives

In this section, all fields are assumed to be of characteristic different from 2.

The question about the existence of an anisotropic infinite-dimensional quadratic space over the field F can be rephrased within the framework of finite-dimensional quadratic form theory, as we shall see now.

We call a sequence of quadratic forms $(\varphi_n)_{n\in\mathbb{N}}$ over F a *chain of quadratic forms over* F if, for any $n \in \mathbb{N}$, we have $\dim(\varphi_n) = n$ and $\varphi_n \subset \varphi_{n+1}$. Given such a chain $(\varphi_n)_{n\in\mathbb{N}}$ over F, the direct limit over the quadratic spaces φ_n with the appropriate inclusions has itself a natural structure of a nonsingular quadratic space over F of dimension $\aleph_0$ (countably infinite). We denote this quadratic space over F by $\lim_{n\in\mathbb{N}}(\varphi_n)$ and observe that it is anisotropic if and only if φ_n is anisotropic for all $n \in \mathbb{N}$. Moreover, any infinite-dimensional nonsingular quadratic space over F contains a subspace isometric to the direct limit $\lim_{n\in\mathbb{N}}(\varphi_n)$ for some chain $(\varphi_n)_{n\in\mathbb{N}}$.

From these considerations we conclude:

2.1 Proposition. *There exists an anisotropic quadratic space of infinite dimension over F if and only if there exists a chain of anisotropic quadratic forms $(\varphi_n)_{n\in\mathbb{N}}$ over F.*

Recall that a form φ is *torsion* if $n \times \varphi$ is hyperbolic for some $n \geq 1$. In [9], Elman and Lam defined the *u-invariant of F* as

$$u(F) \;=\; \sup\{\dim(\varphi) \mid \varphi \text{ is an anisotropic torsion form over } F\}\,.$$

Here, 'torsion' means that the Witt class of φ is a torsion element in the Witt ring WF. It is well known that if F is nonreal, then any form over F is torsion, hence the above supremum is actually taken over all anisotropic forms over F in this case. If F is real, then Pfister's Local-Global Principle says that torsion forms are exactly those forms that have signature zero with respect to each ordering of F (i.e. that are hyperbolic over each real closure of F). In the remainder of this section, we are mainly concerned with nonreal fields.

It will be convenient to consider also the following *relative u-invariants*. Given an anisotropic quadratic form φ over F, we define

$$u(\varphi,F) \;=\; \sup\{\dim(\psi) \mid \psi \text{ anisotropic form over } F \text{ with } \varphi\subset\psi\}\,.$$

Note that, trivially, $\dim(\varphi) \leq u(\varphi,F)$. If F is nonreal, we further have that $u(\varphi,F)\leq u(F)$. Moreover, if φ_1 and φ_2 are anisotropic forms over F such that $\varphi_1\subset\varphi_2$, then $u(\varphi_1,F)\geq u(\varphi_2,F)$.

We introduce now the *derived u-invariant of F* as

$$u'(F) \;=\; \sup\{\dim(\varphi) \mid \varphi \text{ anisotropic form over } F \text{ with } u(\varphi,F)=\infty\}.$$

Whenever there exists an anisotropic form φ over F with $u(\varphi,F)=\infty$, we have $u'(F)>0$; if no such forms exist, we put $u'(F)=\sup\emptyset=0$.

2.2 Proposition. *If there exists an infinite-dimensional quadratic space over F, then $u'(F)=\infty$.*

Proof: Assume that there exists an infinite-dimensional quadratic space over F which is anisotropic. Then there is also a chain $(\varphi_n)_{n\in\mathbb{N}}$ of anisotropic forms over F. Now we have certainly $u(\varphi_n,F)=\infty$ for any $n\in\mathbb{N}$, and this implies that $u'(F)=\infty$. □

In particular, the proposition shows that $u'(F)=\infty$ if F is a real field. Certainly, one could modify the definition of u' to make this invariant more interesting for real fields, but we will not pursue this matter here.

2.3 Proposition. *Assume that F is nonreal. Then $u(F)$ is finite if and only if $u'(F) = 0$.*

Proof: Obviously one has $u(F) = u(\rho, F)$ for any 1-dimensional form ρ over F. So, if $u(F) = \infty$, then $u(\rho, F) = \infty$ for $\rho = \langle 1 \rangle$, and thus $u'(F) \geq 1$. On the other hand, if $u(F) < \infty$, then there is no anisotropic form φ over F such that $u(\varphi, F) = \infty$, and therefore $u'(F) = 0$. $\square$

By the last two statements, any nonreal field F such that $0 < u'(F) < \infty$ will provide an example which answers the Gross Question in the positive. Now, Theorem I and Theorem II each say that nonreal fields K with $u'(K) = 1$ do exist.

2.4 Lemma. *For the field $F(\!(t)\!)$ of Laurent series in the variable t over F, one has*

$$u'(F(\!(t)\!)\,) \;=\; 2\,u'(F)\,.$$

Proof: The straightforward proof is based on the well known relationship between the quadratic forms over F and over $F(\!(t)\!)$ (see [20], Chapter VI, Proposition 1.9). The details are left to the diligent reader. $\square$

Using this lemma together with the theorems mentioned in the introduction, one has the following result.

2.5 Corollary. *Let $m \in \mathbb{N}$. Then there exists a nonreal field L such that $u'(L) = 2^m$. Moreover, L can be constructed such that in addition $I^{m+3}L = 0$, or $I^rL \neq 0$ for all $r \in \mathbb{N}$, respectively.*

Proof: Theorem I or Theorem II, respectively, asserts the existence of such fields for $m = 0$. The induction step from m to $m+1$ is clear from the above lemma. $\square$

This raises the following question.

2.6 Question. Does there exist a nonreal field F with $u'(F) = \infty$ such that every infinite-dimensional quadratic space over F is isotropic?

3 Nonreal fields with infinite u-invariant

Throughout this section, all fields are assumed to be of characteristic different from 2.

We are going to give a construction, in several variants, which allows us to prove the theorems formulated in the introduction. The proof that the field obtained by this construction has infinite u-invariant will be based on known facts about the preservation of properties such as anisotropy of a fixed quadratic form, or absence of zero-divisors in a central simple algebra, under certain types of field extensions.

First, we consider a finite field extension K/F of odd degree. Springer's Theorem (see [20], Chapter VII, Theorem 2.3) says that any anisotropic quadratic form over F stays anisotropic after scalar extension from F to K. One can immediately generalise this to 'odd' algebraic extensions that are not necessarily finite.

For the following definition we allow the case where F has characteristic 2, for later reference. Note that Springer's Theorem is independent of the characteristic of F.

An algebraic extension L/F is called an *odd closure of* F if L is F-isomorphic to M^G, where M is an algebraic (resp. separable) closure of F if $\mathrm{char}(F) \neq 2$ (resp. $\mathrm{char}(F) = 2$), and G is a 2-Sylow subgroup of the Galois group of M/F. Then L itself has no odd degree extension and all finite subextensions of F inside L are of odd degree. In particular, L is perfect if $\mathrm{char}(F) \neq 2$. We call a field extension K/F an *odd extension* if it can be embedded into an odd closure of F. In this case, K/F is algebraic, thus equal to the direct limit of its finite subextensions, which are all of odd degree.

Using Springer's Theorem, we readily obtain:

3.1 Lemma. *Let K/F be an odd extension. Then any anisotropic form over F stays anisotropic over K.*

Springer's Theorem has an analogue in the theory of central simple algebras. It says that if D is a (central) division algebra over F with exponent equal to a power of 2 and if K/F is a finite field extension of odd degree, then the K-algebra $D_K = D \otimes_F K$ is also a division algebra (see [31], Section 13.4, Proposition (vi)). Therefore, we obtain in the same way as above:

3.2 Lemma. *Let K/F be an odd extension. Then any central division algebra of exponent 2 over F remains a division algebra after scalar extension to K.*

We now turn to extensions of the type $F(\varphi)/F$, where $F(\varphi)$ is the function field of a quadratic form φ over F.

3.3 Lemma. *Let π be an anisotropic Pfister form over F and φ a form over F with $\dim(\varphi) > \dim(\pi)$. Then π stays anisotropic over $F(\varphi)$.*

Proof: By the assumption on the dimensions, φ is certainly not similar to any subform of π. Therefore, by [34], Theorem 4.5.4 (ii), $\pi_{F(\varphi)}$ is not hyperbolic. Hence $\pi_{F(\varphi)}$ is anisotropic as it is a Pfister form (see [34], Lemma 2.10.4). □

3.4 Remark. The statement of the last lemma is actually a special case of a more general phenomenon. Let φ and π be anisotropic forms over F such that, for some $n \in \mathbb{N}$, one has $\dim(\pi) \leq 2^n < \dim(\varphi)$. Then π stays anisotropic over $F(\varphi)$ (see [14]). In the particular situation where π is an n-fold Pfister form, we immediately recover (3.3).

The next statement was the key in Merkurjev's construction of fields of arbitrary even u-invariant (see [24]). It is readily derived from [38], Théorème 1.

3.5 Theorem (Merkurjev). *Let D be a division algebra over F of exponent 2 and degree 2^m, where $m > 0$. Let φ be a quadratic form over F such that $\dim(\varphi) > 2m + 2$ or $\varphi \in I^3F$. Then $D_{F(\varphi)}$ is a division algebra.*

3.6 Remark. Statements analoguous to (3.1) and (3.2) hold for purely transcendental extensions. More precisely, if the field extension K/F is purely transcendental, then every anisotropic quadratic form over F stays anisotropic over K and every division algebra over F extends to a division algebra over K. We will use this fact repeatedly, especially in the case where K is the function field of an isotropic quadratic form over F. Indeed, if a quadratic form φ over F is isotropic, then $F(\varphi)/F$ is purely transcendental of transcendence degree $\dim(\varphi) - 2$ (see [34], 4.5.2 (vi)).

We are now ready for the proofs of the first two theorems formulated in the introduction.

3.7 Proof of Theorem I.

Recall that F is an arbitrary field of characteristic different from 2. We define recursively a tower of fields $(F_n)_{n\in\mathbb{N}}$, starting with $F_0 = F$. Suppose that for a certain $n \geq 1$ the field F_{n-1} has already been defined. Let $F^{\#}_{n-1}$ be an odd closure of F_{n-1} and let

$$F^{(n)}_{n-1} = F^{\#}_{n-1}(X^{(n)}_1, \ldots, X^{(n)}_n)$$

where $X^{(n)}_1, \ldots, X^{(n)}_n$ are indeterminates over $F^{\#}_{n-1}$. We define F_n as the free compositum[3] of all the function fields $F^{(n)}_{n-1}(\varphi)$ where φ is an anisotropic form

[3]See [21], p. 333, for a precise description of the notion of 'free compositum' of a family of function fields of quadratic forms.

defined over F_{n-1} such that, for some $j < n$, φ contains a binary form defined over F_j and $\dim(\varphi) = 2^j + 1$.

Let K be the direct limit of the tower of fields $(F_n)_{n\in\mathbb{N}}$. We are going to show that the field K has the following properties:

(i) K has no finite extensions of odd degree.

(ii) For any binary quadratic form β over K, there is an upper bound on the dimensions of anisotropic quadratic forms over K that contain β.

(iii) For any $k \in \mathbb{N}$, there is an anisotropic k-fold Pfister form over K.

Once these are established the remaining claims in Theorem I will follow. Indeed, (ii) implies that every infinite-dimensional quadratic space over K is isotropic and that K is nonreal, whereas (iii) implies that $u(K) = \infty$ and that $I^k K \neq 0$ for all $k \in \mathbb{N}$. Finally, since $\mathrm{char}(K) = \mathrm{char}(F) \neq 2$, it follows from (i) that K is perfect.

(i) Consider an irreducible polynomial f over K of odd degree. Then f is defined over F_n for some $n \in \mathbb{N}$. Since K contains F_{n+1} which in turn contains an odd closure of F_n, it follows that f has degree one. This shows that K is equal to its odd closure.

(ii) Consider an anisotropic binary form β over K. There is some $j \in \mathbb{N}$ such that β is defined over F_j. Let φ be a form of dimension $2^j + 1$ over K containing β. Let $n > j$ be an integer such that φ is defined over F_{n-1}. Then by construction, F_n contains $F^{(n)}_{n-1}(\varphi)$ and φ is therefore isotropic over F_n and thus over K. This shows that $u(\beta, K) \leq 2^j$. Here, j depends on the binary form β, but in any case we have that $u(\beta, K)$ is finite, proving (ii).

(iii) Given positive integers n and j, we write $F_{n,j}$ for the compositum of F_n with the algebraic closure of F_j inside a fixed algebraic closure of K. Similarly, we write $F^{\#}_{n-1,j}$ and $F^{(n)}_{n-1,j}$, for the compositum of $F^{\#}_{n-1}$, $F^{(n)}_{n-1}$, respectively, with the algebraic closure of F_j.

$$
\begin{tikzcd}
 & F_{n,j} \arrow[d, dash] \\
F_n \arrow[ur, dash] \arrow[d, dash, "\textit{compositum}"'] & F^{(n)}_{n-1,j} \arrow[d, dash] \\
F^{(n)}_{n-1} \arrow[ur, dash] \arrow[d, dash, "\textit{purely transc.}"'] & F^{\#}_{n-1,j} \arrow[d, dash] \\
F^{\#}_{n-1} \arrow[ur, dash] \arrow[d, dash, "\textit{odd}"'] & F_{n-1,j} \\
F_{n-1} \arrow[ur, dash] &
\end{tikzcd}
$$

Let us assume from now on that $n > j$. Note that $F^{(n)}_{n-1,j}$ is equal to $F^{\#}_{n-1,j}(X^{(n)}_1, \ldots, X^{(n)}_n)$, which is a purely transcendental extension of $F^{\#}_{n-1,j}$. Further, $F^{\#}_{n-1,j}$ is an odd extension of $F_{n-1,j}$. Using (3.1), it follows that every anisotropic form over $F_{n-1,j}$ stays anisotropic over $F^{\#}_{n-1,j}$, hence also over $F^{(n)}_{n-1,j}$. Moreover, $F_{n,j}$ is obtained from $F^{(n)}_{n-1,j}$ as a free compositum of certain function fields $F^{(n)}_{n-1,j}(\varphi)$ where φ is a form defined over $F^{(n)}_{n-1,j}$ which is either of dimension at least $2^{j+1}+1$ or which contains a binary form defined over F_j and thus is isotropic over $F^{(n)}_{n-1,j}$.

Consider now an anisotropic m-fold Pfister form π defined over $F^{(n)}_{n-1,j}$, where $m \leq j+1$. Using (3.3) and (3.6) it follows, that π stays anisotropic over $F_{n,j}$, again by (3.6). But then π stays anisotropic over $F^{(n+1)}_{n,j}$ as well. Repeating this, we see that π stays anisotropic over all the fields $F_{n,j}$ when j is fixed and n increases.

Let now k be any positive integer. Let π denote the k-fold Pfister form $\langle\langle X^{(k)}_1, \ldots, X^{(k)}_k \rangle\rangle$. This form is defined over $F^{(k)}_{k-1}$. Since $X^{(k)}_1, \ldots, X^{(k)}_k$ are algebraically independent over F_{k-1}, hence also over its algebraic closure $F_{k-1,k-1} = F^{\#}_{k-1,k-1}$, we know that π is still anisotropic when considered as a form over the field $F^{(k)}_{k-1,k-1} = F^{\#}_{k-1,k-1}(X^{(k)}_1, \ldots, X^{(k)}_n)$. Now the above argument shows that, for any $n \geq k$, the form π is anisotropic over $F_{n,k-1}$ and, thus, over F_n. This implies that π is anisotropic over K, the direct limit of the fields F_n.

Hence we showed that for any $k \in \mathbb{N}$, there exists an anisotropic k-fold Pfister form over K.

3.8 Proof of Theorem II.

Again, we define recursively a tower of fields $(F_n)_{n\in\mathbb{N}}$, starting with $F_0 = F$. Suppose that for a certain $n \geq 1$, the field F_{n-1} is defined. As before, let $F^{\#}_{n-1}$ denote an odd closure of F_{n-1}. This time we define

$$F^{(n)}_{n-1} = F^{\#}_{n-1}(\, X^{(n)}_1, Y^{(n)}_1, \ldots, X^{(n)}_n, Y^{(n)}_n)$$

where $X^{(n)}_1, Y^{(n)}_1, \ldots, X^{(n)}_n, Y^{(n)}_n$ are indeterminates over $F^{\#}_{n-1}$. Let F_n denote the free compositum of the function fields $F^{(n)}_{n-1}(\varphi)$ where φ is an anisotropic form over F_{n-1} such that

- φ is a 3-fold Pfister form, or
- $\dim(\varphi) = 2j + 3$ for some $j < n$ and φ contains a binary form defined over F_j.

Let K be the direct limit of the tower of fields $(F_n)_{n\in\mathbb{N}}$. We want to show that K has the following properties:

(i) K has no finite extensions of odd degree and $I^3K = 0$.

(ii) For any binary quadratic form β over K, there is an upper bound on the dimensions of anisotropic quadratic forms over K which contain β.

(iii) For any $k \in \mathbb{N}$, there is a central division algebra over K that is decomposable into a tensor product of k quaternion algebras.

Note that (iii) implies that $u(K) = \infty$ (see [24] or [28], Lemma 1.1(d)), while (ii) excludes the possibility that there is an infinite-dimensional anisotropic quadratic space over K. As before, the field K is perfect by (i) and nonreal by (iii). Furthermore, (i) and (iii) together imply that the cohomological dimension of K is exactly 2 (see [24]).

(i) As in the proof of Theorem I, we see that K has no finite extensions of odd degree.

Let π be an arbitrary 3-fold Pfister form over K. It is defined as a 3-fold Pfister form over F_{n-1} for some $n \geq 1$. By the construction of the field F_n, π becomes isotropic over F_n and thus over K. Hence, every 3-fold Pfister form over K is isotropic and therefore hyperbolic. Since I^3K is additively generated by the 3-fold Pfister forms over K (see [34], p. 156), we conclude that $I^3K = 0$.

(ii) Let β be an anisotropic binary form over K. There is an integer $j \in \mathbb{N}$ such that β is defined over F_j. Let φ be any form of dimension $2j + 3$

over K containing β. There is some integer $n > j$ such that φ is defined over F_{n-1}. Since $F^{\#}_{n-1}(\varphi)$ is part of the compositum F_n, φ becomes isotropic over F_n and thus over K. Therefore $u(\beta, K) \leq 2j + 2$, establishing (ii).

(iii) For positive integers n and j, we denote by $F_{n,j}$, $F^{\#}_{n-1,j}$, $F^{(n)}_{n-1,j}$ the composita of the fields F_n, $F^{\#}_{n-1}$, $F^{(n)}_{n-1}$, respectively, with the algebraic closure of F_j inside a fixed algebraic closure of K.

Assume from now on that $n > j$. Similarly as in the proof of Theorem I, we have that $F^{(n)}_{n-1,j}$ is equal to $F^{\#}_{n-1,j}(X^{(n)}_1, Y^{(n)}_1, \ldots, X^{(n)}_n, Y^{(n)}_n)$, a purely transcendental extension of $F^{\#}_{n-1,j}$, which in turn is an odd extension of $F_{n-1,j}$. Using (3.2) and (3.6), it follows that every division algebra of exponent 2 over $F_{n-1,j}$ remains a division algebra after scalar extension to $F^{(n)}_{n-1,j}$.

Moreover, $F_{n,j}$ is obtained from $F^{(n)}_{n-1,j}$ as a free compositum of certain function fields $F^{(n)}_{n-1,j}(\varphi)$ where φ is a form defined over $F^{(n)}_{n-1,j}$ which is either a 3-fold Pfister form, or which has dimension at least $2j+3$, or which contains a binary form defined over F_j and thus is isotropic over $F^{(n)}_{n-1,j}$. Hence, by Merkurjev's Criterion (3.5) and by (3.6), any division algebra over $F^{(n)}_{n-1,j}$ of exponent 2 and of degree at most 2^j remains a division algebra after scalar extension to the field $F_{n,j}$.

Consider now a central simple algebra D of exponent 2 and degree 2^j over $F^{(j)}_{j-1}$ for some $j \in \mathbb{N}$. Assume that for some $n > j$, the algebra D will stay a division algebra after extending scalars to $F^{(n)}_{n-1,j}$. Combining the observations above, we see that D also remains a division algebra when we extend scalars to $F_{n,j}$, or even to $F^{(n+1)}_{n,j}$. Repeating this argument shows that D will stay a division algebra after scalar extension to $F^{(N)}_{N-1,j}$ for any $N \geq n$.

Let now k be a positive integer and let D denote the tensor product of quaternion algebras $(X^{(k)}_1, Y^{(k)}_1) \otimes \cdots \otimes (X^{(k)}_k, Y^{(k)}_k)$ over the field $F^{(k)}_{k-1}$. This is a central simple algebra over $F^{(k)}_{k-1}$ of degree 2^k and of exponent 2. Since $X^{(k)}_1, Y^{(k)}_1, \ldots, X^{(k)}_k, Y^{(k)}_k$ are algebraically independent over the field F_{k-1}, hence also over its algebraic closure $F_{k-1,k-1} = F^{\#}_{k-1,k-1}$, it follows that $D_{F^{(k)}_{k-1,k-1}}$ is a division algebra over the field $F^{(k)}_{k-1,k-1}$. Now the argument above applies, showing that $D_{F_{n,k-1}}$ is a division algebra over $F_{n,k-1}$ for any $n \geq k$. But then D_{F_n} is a division algebra for any $n \geq k$, implying that the tensor product of k quaternion algebras D_K is a division algebra over K.

3.9 Remark. At first glance, it may seem that the fields K constructed in the proofs of the theorems are horrendously big. However, a closer inspection of the proofs reveals that if the field F we start with is infinite, the field K obtained by the construction will have the same cardinality as F. For example, if we start with $F = \mathbb{Q}$, then the field K we end up with is countable and thus can be embedded into $\mathbb{C}$.

4 Real fields and totally indefinite spaces

In our answer to the Gross Question, we had to construct a field F which in particular has the property that all infinite-dimensional quadratic spaces over F are isotropic. A real such field cannot exist as mentioned previously. In fact, for a quadratic space φ (of finite or infinite dimension) over a real field F to be isotropic, a necessary condition is that φ be *totally indefinite*, i.e. indefinite with respect to each ordering. To get a meaningful analogue to the Gross Question in the case of real fields, it is therefore reasonable to restrict our attention to quadratic spaces that are totally indefinite. We start this section with the definition of this notion and some general observations before proving the 'real' analogues to the constructions that answer the Gross Question.

We consider an ordering P on F and denote by $<_P$ the corresponding order relation on F. A quadratic space (V, q) over F is said to be *indefinite at* P, if there exist elements $v_1, v_2 \in V$ such that $q(v_1) <_P 0 <_P q(v_2)$. If (V, q) is indefinite at every ordering of F, then we say that (V, q) is *totally indefinite*. Note that this definition of (total) indefiniteness extends the common one for quadratic forms.

The *Hasse number* $\tilde{u}$ *of* F is defined by

$$\tilde{u}(F) \;=\; \sup \{\dim(\varphi) \mid \varphi \text{ anisotropic, totally indefinite form over } F\}\,.$$

Since any nontrivial torsion form is obviously totally indefinite, one has $u(F) \leq \tilde{u}(F)$. On the other hand, there are examples of real fields F where $u(F) < \infty$ while $\tilde{u}(F) = \infty$. For a survey on the possible pairs of values $(u(F), \tilde{u}(F))$, we refer to [15].

In view of Theorem IV, we recall that the *Pythagoras number* $p(F)$ of F is the least integer $m \geq 1$ such that every sum of squares is a sum of m squares in F if such an m exists, otherwise $p(F) = \infty$. It is well known and not difficult to see that if $p(F) = \infty$, then also $u(F) = \tilde{u}(F) = \infty$, and if $u(F) > 0$ then $p(F) \leq u(F)$.

The following observation is useful when dealing with infinite-dimensional totally indefinite quadratic spaces.

4.1 Proposition. *Every totally indefinite quadratic space over F contains a finite-dimensional, nonsingular, totally indefinite quadratic subspace.*

Proof: Let (V,q) be a totally indefinite quadratic space over F. We may assume (V,q) nonsingular. If (V,q) is isotropic then it contains a hyperbolic plane which yields the desired subspace. Hence, we may assume that (V,q) is anisotropic. In particular, any subspace of (V,q) is nonsingular. After scaling we may furthermore assume that there exists a vector $v_0 \in V$ with $q(v_0) = 1$. Since (V,q) is totally indefinite, for each ordering P there exists a vector $v_P \in V$ such that $q(v_P) <_P 0$.

Recall that the set of all orderings of F, denoted by X_F, is a compact topological space that has as a subbasis the clopen sets

$$H(a) \;=\; \{\, P \in X_F \mid a \in P\}$$

(see [32], Theorem 6.5). We put $a_P = q(v_P)$ for every $P \in X_F$. The above choice of the family $(v_P)_{P\in X_F} \subset F$ implies that $X_F = \bigcup_{P\in X_F} H(-a_P)$. The compactness of X_F thus yields that there are finitely many orderings $P_1, \ldots, P_n \in X_F$ such that

$$X_F \;=\; H(-a_{P_1}) \cup \cdots \cup H(-a_{P_n}).$$

We put $v_i = v_{P_i}$ for $1 \le i \le n$. By the last equality, for each ordering P of F we have $q(v_i) <_P 0$ for at least one $i \in \{1, \ldots, n\}$.

Let W be the subspace of V generated by the vectors $v_0, v_1, \ldots, v_n$. Then it follows that (W,q) is an anisotropic, finite-dimensional, totally indefinite subspace of (V,q). □

Recall that any ordering P of F can be extended to the odd closure of F as well as to any purely transcendental extension of F. From [10], Theorem 3.5, Remark 3.6, we cite the following simple criterion for when an ordering can be extended to the function field of a given quadratic form.

4.2 Lemma. *Let P be an ordering of F and let $\{\varphi_i\}$ be any family of quadratic forms over F of dimension at least 2. Then P can be extended to the free compositum of the $F(\varphi_i)$ if and only if each φ_i is indefinite at P.*

We are now going to modify the constructions presented in the last section and prove the remaining two theorems formulated in the introduction.

4.3 Proof of Theorem III.

This time, we construct a tower of fields with orderings $(F_n, P_n)_{n\in\mathbb{N}}$, where the ordering P_{n+1} on F_{n+1} extends the ordering P_n on F_n for all n. Let $F_0 = F$ and let P_0 be any ordering of this field. Suppose now that the pair (F_{n-1}, P_{n-1}) has been defined for a certain $n \ge 1$. Let $F^{\#}_{n-1}$ denote an odd closure of F_{n-1} and let $P^{\#}_{n-1}$ be any ordering on $F^{\#}_{n-1}$ extending P_{n-1}. Let

$$F^{(n)}_{n-1} \;=\; F^{\#}_{n-1}(X^{(n)}_1, Y^{(n)}_1, \ldots, X^{(n)}_n, Y^{(n)}_n)$$

where $X_1^{(n)}, Y_1^{(n)}, \ldots, X_n^{(n)}, Y_n^{(n)}$ are indeterminates over $F_{n-1}^{\#}$. Let $P_{n-1}^{(n)}$ be any ordering on $F_{n-1}^{(n)}$ extending $P_{n-1}^{\#}$. Let now F_n be the free compositum of the function fields $F_{n-1}^{(n)}(\varphi)$ where φ is an anisotropic form over F_{n-1} such that

- φ is a 3-fold Pfister form and indefinite at P_{n-1}, or
- $\dim(\varphi) = 2j+3$ for some $j < n$, and φ contains a binary form defined over F_j and indefinite at P_j.

Note that considered as forms over $F_{n-1}^{(n)}$ and by the construction of our orderings, all the above forms are in fact totally indefinite at $P_{n-1}^{(n)}$. By (4.2), the ordering $P_{n-1}^{(n)}$ extends to an ordering P_n on F_n. In particular, F_n is a real field.

Note that, for any 2-fold Pfister form ρ over F_{n-1} and any $a \in F_{n-1}$, at least one of the 3-fold Pfister forms $\rho \otimes \langle\langle a \rangle\rangle$ and $\rho \otimes \langle\langle -a \rangle\rangle$ is indefinite at P_{n-1} and thus becomes hyperbolic over F_n by the construction of this field.

Let K be the direct limit of the tower of fields $(F_n)_{n \in \mathbb{N}}$. We will show that K has the following properties:

(i) K has a unique ordering.

(ii) K has no finite extensions of odd degree and I^3K is torsion free.

(iii) For any totally indefinite quadratic form β over K, there is an upper bound on the dimensions of anisotropic quadratic forms over K that contain β.

(iv) For any $k \in \mathbb{N}$, there is a central division algebra over K that is decomposable into a tensor product of k quaternion algebras.

Once these properties of K are established, the remaining claims in Theorem III are immediate consequences:

- K is a real field and by (iii) and (4.1), every infinite-dimensional anisotropic quadratic space over K is definite with respect to the unique ordering.
- (i) implies that K is *SAP* (see, e.g., [32], § 9, for the definition of and some facts about *SAP*), I^3K is torsion free, and (iv) implies that the symbol length $\lambda(K)$ of K is infinite. (Recall that the symbol length $\lambda(K)$ is the smallest $m \in \mathbb{N}$ such that each central simple algebra of exponent 2 over K is Brauer equivalent to a tensor product of at most m quaternion algebras provided such an integer exists, otherwise $\lambda(K) = \infty$.) It follows from [15], Theorem 1.5, that $u(K) = \infty$.

- (i) and (ii) imply that the cohomological dimension of $K(\sqrt{-1})$ is at most 2. That it is exactly 2 then follows from (iv).

We now proceed to the proof of (i)–(iv).

(i) Since all the fields F_n ($n \in \mathbb{N}$) are real, the same holds for K. It follows from what we observed during the construction above that, for any $a \in K^\times$, one of the forms $\langle\langle -1,-1,a\rangle\rangle$ and $\langle\langle -1,-1,-a\rangle\rangle$ is hyperbolic over K, which means that either a or $-a$ is a sum of (four) squares in K. This shows that K is uniquely ordered. It is clear that the unique ordering on K is the direct limit of the orderings P_n.

(ii) There is no change — compared to the previous constructions — in the argument that K has no finite extensions of odd degree.

The torsion subgroup of I^3K is generated by those 3-fold Pfister forms over K that are torsion. Indeed, this is a general fact (see [2], Corollary 2.7) which, however, could be proven very easily in our particular situation where K is uniquely ordered.

Let π be any torsion 3-fold Pfister form over K. Then π is defined as a 3-fold Pfister form over F_{n-1} for some $n \geq 1$. Since the unique ordering on K extends the ordering P_{n-1} on F_{n-1}, it follows that π (considered as 3-fold Pfister form over F_{n-1}) is indefinite at P_{n-1}. The construction of F_n then yields that π becomes isotropic and hence hyperbolic over F_n. Therefore, π is hyperbolic over K. This shows that I^3K is torsion free.

(iii) Since K has a unique ordering, every (totally) indefinite form over K contains an indefinite binary subform. Hence, (iii) needs only to be proven for binary indefinite forms β. The proof goes along the same lines as that of (ii) in Theorem II.

(iv) This part is identical to the corresponding part (iii) in the proof of Theorem II.

4.4 Proof of Theorem IV.

Again, we define a tower of ordered fields $(F_n, P_n)_{n\in\mathbb{N}}$ where the ordering P_{n+1} on F_{n+1} extends the ordering P_n on F_n for all n.

Let F_0 be the given real field F and P_0 any ordering of this field. Suppose that for a certain $n \geq 1$ the pair (F_{n-1}, P_{n-1}) is already defined. Let $F_{n-1}^{\#}$ be an odd closure of F_{n-1} and let $F_{n-1}^{(n)}$ be the rational function field $F_{n-1}^{\#}(X^{(n)})$. As before, P_{n-1} extends to some ordering $P_{n-1}^{\#}$ of $F_{n-1}^{\#}$ which in turn extends to an ordering $P_{n-1}^{(n)}$ on $F_{n-1}^{(n)} = F_{n-1}^{\#}(X^{(n)})$ at which $X^{(n)}$ is positive.

We define F_n to be the free compositum of all function fields $F_{n-1}^{(n)}(\varphi)$ where φ is an anisotropic form defined over F_{n-1} such that, for some $j < n$, we have $\dim(\varphi) = 2^j + 1$ and φ contains an binary form which is defined over F_j and indefinite at P_j. By (4.2), $P_{n-1}^{(n)}$ extends to an ordering P_n of F_n.

Let K be the direct limit of the tower $(F_n)_{n\in\mathbb{N}}$. We are going to establish the following properties:

(i) K has a unique ordering which is given by $P = \bigcup_{n\in\mathbb{N}} P_n$.

(ii) K has no finite extensions of odd degree.

(iii) For any totally indefinite quadratic form β over K, there is an upper bound on the dimensions of anisotropic quadratic forms over K which contain β.

(iv) for any $k \in \mathbb{N}$, there is an element $a \in K^\times$ which is a sum of squares in K, but not a sum of k squares.

Note that (iv) implies that the Pythagoras number of K is infinite, which in turn forces the Hasse number and the u-invariant of K to be infinite as well. As before, (iii) implies that every infinite-dimensional anisotropic quadratic space over K is definite with respect to the unique ordering of K.

(i) Since each F_n is real, so is the direct limit K. Consider an arbitrary element $a \in K^\times$. Then $a \in F_n$ for some $n \in \mathbb{N}$. Now either $\langle 1, -a\rangle$ or $\langle 1, a\rangle$ is indefinite at P_n. Therefore, by construction, either $2^n \times \langle 1\rangle \perp \langle -a\rangle$ or $2^n \times \langle 1\rangle \perp \langle a\rangle$ becomes isotropic over the field F_{n+1}. Hence, a or $-a$ is a sum of (in fact 2^n) squares in K. This shows that K is uniquely ordered. To show that P is this unique ordering, it therefore suffices to show that P consists exactly of all nonzero sums of squares.

Any sum of squares $s \in K^\times$ is already a sum of squares in F_n for some n and hence in P_n. Thus, $s \in P$.

Conversely, any $s \in P$ is in P_n for some n, which by the above reasoning implies that s is a sum of (in fact 2^n) squares in K.

(ii) K is equal to its odd closure, by the same arguments as before.

(iii) The argument here is the same as for (iii) in the last proof.

(iv) We denote by $F_{n-1,j}$, $F_{n-1,j}^{\#}$, and $F_{n-1,j}^{(n)}$, the composita of F_{n-1}, $F_{n-1}^{\#}$, and $F_{n-1}^{(n)}$, respectively, with the real closure of F_j at the ordering P_j. Assume now that $n > j$. Then we observe as before that every anisotropic quadratic form defined over $F_{n-1,j}$ stays anisotropic over $F_{n-1,j}^{(n)}$. Note that $F_{n,j}$ is obtained from $F_{n-1,j}^{(n)}$ as a compositum of function fields $F_{n-1,j}^{(n)}(\varphi)$ where φ is a form defined over $F_{n-1,j}^{(n)}$ which either is of dimension at least $2^{j+1}+1$, or which contains a binary form defined over F_j and indefinite at P_j and which is therefore isotropic over $F_{n-1,j}^{(n)}$. As in part (iii) of the proof of

Theorem I, we conclude that if π is an anisotropic m-fold Pfister form over $F^{(n)}_{n-1,j}$ with $m \leq j+1$, then π stays anisotropic over $F_{n,j}$.

Let now $k \in \mathbb{N}$. Then the $(k+1)$-fold Pfister form $2^k \times \langle\langle X^{(k)} \rangle\rangle$ is defined over $F^{(k)}_{k-1}$ and is still anisotropic over $F^{(k)}_{k-1,k-1}$. It follows now from the above arguments that this form stays anisotropic over $F_{n,k-1}$, for all $n > k$. In particular, $2^k \times \langle\langle X^{(k)} \rangle\rangle$ is anisotropic over all fields F_n for $n \geq k$, thus also over K. This shows that the element $X^{(k)}$ is not a sum of 2^k squares in K. On the other hand, by the construction we have $X^{(k)} \in P$, so that $X^{(k)}$ is a sum of squares in K, by (i).

5 Fields of characteristic 2

Throughout this section, all fields considered will be of characteristic 2. To translate the Gross Question into this setting, we have to take into account the different types of objects for which analogous problems might be formulated: quadratic, nonsingular quadratic, and symmetric bilinear spaces. We maintain the convention to use the term 'form(s)' for finite-dimensional spaces. For nonsingular quadratic forms we shall obtain analogues to Theorems I and II stated in the introduction, thus obtaining a positive answer to (the corresponding formulation of) the Gross Question in this case, too. On the other hand, for arbitrary quadratic forms as well as for symmetric bilinear forms, the corresponding answer turns out to be negative. In fact, this is relatively easy to prove, so we treat these types of forms first.

We refer the reader to [3], [30] or [16] for further details on notation, terminology and basic results concerning quadratic and bilinear forms in characteristic 2.

Let (V, q) be a quadratic space over a field F of characteristic 2, and let $b_q : V \times V \to F$ be the associated bilinear form, given by $b_q(x, y) = q(x+y) + q(x) + q(y)$. Recall that the *radical of* (q, V) is the F-subspace

$$V^\perp = \mathrm{Rad}(q, V) = \{x \in V \mid b_q(x, y) = 0 \text{ for all } y \in V\}.$$

The quadratic space (V, q) is said to be

- *nonsingular* if $V^\perp = 0$;
- *singular* if $V^\perp \neq 0$;
- *totally singular* if $V^\perp = V$.

If we write $V = V_0 \oplus V^\perp$ and we put $q_0 = q|_{V_0}$ and $q_{ts} = q|_{V^\perp}$, then $q \cong q_0 \perp q_{ts}$ with q_0 nonsingular and q_{ts} totally singular. If we also have $q \cong \varphi_0 \perp \varphi_{ts}$

with φ_0 nonsingular and φ_{ts} totally singular, then $q_{ts} \cong \varphi_{ts}$ (any isometry maps radicals bijectively to radicals), but q_0 and φ_0 might not be isometric. Note that (V, q) is totally singular if and only if $q(x+y) = q(x) + q(y)$ for all $x, y \in V$.

For $a, b \in F$, the 2-dimensional quadratic form $aX^2 + XY + bY^2$ is nonsingular, and we will denote it by $[a, b]$. The *hyperbolic plane* is then the form $\mathbb{H} = [0, 0] = XY$. For $a_1, \ldots, a_s \in F$, the s-dimensional quadratic form $\sum_{i=1}^{s} a_i X_i^2$ is totally singular, and it will be denoted by $\langle a_1, \ldots, a_s \rangle$.

Let now q be a quadratic form over F and let $n = \dim(q)$. Then there exist $r, s \in \mathbb{N}$ with $2r + s = n$ and $a_1, b_1, \ldots, a_r, b_r \in F$ and $c_1, \ldots, c_s \in F$ such that

$$q \cong [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, \ldots, c_s \rangle \,,$$

and we clearly have $q_{ts} \cong \langle c_1, \ldots, c_s \rangle$. In particular, nonsingular quadratic forms are always of even dimension.

There are two versions of the u-invariant in characteristic 2, referring to the different types of quadratic forms, denoted by u and $\widehat{u}$, respectively. They are defined as follows:

$$\begin{aligned} u(F) &= \sup\{\dim(q) \mid q \text{ anisotropic nonsingular quadratic form over } F\} \\ \widehat{u}(F) &= \sup\{\dim(q) \mid q \text{ anisotropic quadratic form over } F\} \end{aligned}$$

Clearly, we have $u(F) \leq \widehat{u}(F)$, and $u(F)$ is always even if finite.

One can define corresponding u-invariants also for the classes of anisotropic symmetric bilinear forms, and of anisotropic totally singular quadratic forms, respectively, but (5.3) below will show that both suprema thus obtained just coincide with $[F : F^2]$, the *degree of inseparability* of F.

We will now concentrate for a moment on totally singular quadratic spaces. These are, in fact, very easy to treat.

For a field F of characteristic 2 we fix an algebraic closure $\overline{F}$ and put $\sqrt{F} = \{x \in \overline{F} \mid x^2 \in F\}$. Note that $\sqrt{F}/F$ is a purely inseparable algebraic field extension of degree $[F : F^2]$. Hence the squaring map $\mathrm{sq} : x \mapsto x^2$ yields a quadratic map $\mathrm{sq}_F : \sqrt{F} \to F$ over F, and the quadratic space $(\sqrt{F}, \mathrm{sq}_F)$ is of dimension $[F : F^2]$.

5.1 Proposition. *Let F be a field of characteristic 2. The quadratic space $(\sqrt{F}, \mathrm{sq}_F)$ is anisotropic and totally singular. Any anisotropic totally singular quadratic space over F is isometric to a subspace of $(\sqrt{F}, \mathrm{sq}_F)$.*

Proof: The first part is obvious. Consider now a totally singular quadratic space (V, q) over F and assume that it is anisotropic. We define

$$\rho : \; V \longrightarrow \sqrt{F} \,, \; v \longmapsto \sqrt{q(v)} \,.$$

Since q is totally singular, ρ is a homomorphism of F-vector spaces and we have $sq_F \circ \rho = q$. Since furthermore q is anisotropic, ρ is injective and thus (V, q) is isometric to the subspace $(\rho(V), \mathrm{sq}_F|_{\rho(V)})$ of $(\sqrt{F}, \mathrm{sq_F})$. $\square$

We will now briefly look at symmetric bilinear spaces (V, b) over a field F of characteristic 2. A symmetric bilinear space (V, b) is said to be *isotropic* if there exists $x \in V \setminus \{0\}$ such that $b(x, x) = 0$, *anisotropic* otherwise. In other words, (V, b) is anisotropic if and only if (V, q_b) is so, where $q_b : V \to F$ is the induced quadratic map defined by $q_b(x) = b(x, x)$.

5.2 Lemma. *Let F be a field of characteristic 2 and V an F-vector space. There exists an anisotropic symmetric bilinear map $b : V \times V \to F$ if and only if there exists an anisotropic totally singular quadratic map $q : V \to F$.*

Proof: By definition, a symmetric bilinear map $b : V \times V \to F$ is anisotropic if and only if the associated totally singular quadratic map $q_b : V \to F$ is so. Now, given an anisotropic totally singular quadratic map $q : V \to F$, it is not difficult to construct a symmetric bilinear map $b : V \times V \to F$ such that $q = q_b$. In fact, picking some F-basis $(e_i)_{i \in I}$ of V, we can define b by $b(e_i, e_j) = \delta_{ij} q(e_i)$ for $i, j \in I$. All this implies the claim. $\square$

5.3 Corollary. *For any field F of characteristic 2, we have*

$$\begin{aligned} [F : F^2] &= \sup \{\dim(q) \mid q \textit{ anisotr. tot. singular quadratic form over } F\} \\ &= \sup \{\dim(b) \mid b \textit{ anisotr. symmetric bilinear form over } F\} \end{aligned}$$

Moreover, if $[F : F^2] = \infty$, then there exist anisotropic totally singular spaces and anisotropic symmetric bilinear spaces of infinite dimension over F.

Proof: If $[F : F^2] < \infty$, then the claim is obvious from the previous results in this section. If $[F : F^2] = \infty$ then the same results yield the existence of infinite-dimensional F-spaces (e.g. $\sqrt{F}$) carrying anisotropic totally singular quadratic forms and anisotropic symmetric bilinear forms. By restricting to subspaces of dimension n for arbitrary $n \in \mathbb{N}$, we obtain that the corresponding suprema for the anisotropic dimensions of totally singular forms and symmetric bilinear forms are infinite. $\square$

We next consider general quadratic forms in characteristic 2 and the corresponding $\widehat{u}$-invariant. The first part of the following statement is [22], Corollary 1.

5.4 Proposition. *Let F be a field of characteristic 2. Then*

$$[F : F^2] \leq \widehat{u}(F) \leq 2[F : F^2] \,.$$

Furthermore, $\widehat{u}(F) = \infty$ if and only if there exists a totally singular anisotropic quadratic space of infinite dimension over F.

Proof: The first inequality is obvious from the last corollary. To prove the second inequality, we may assume that $[F : F^2] < \infty$. Let q be an anisotropic quadratic form over F. We may write

$$q = [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, \ldots, c_s \rangle$$

with $a_1, b_1, \ldots, a_r, b_r, c_1, \ldots, c_s \in F$. Since this form is anisotropic, the totally singular subform $\langle a_1, \ldots, a_r, c_1, \ldots, c_s \rangle$ is anisotropic as well, whence $r + s \leq [F : F^2]$ and thus $\dim(q) \leq 2\,[F : F^2]$. Therefore $\widehat{u}(F) \leq 2\,[F : F^2]$. The last part of the statement now also follows from the last corollary. □

So far we have shown in this section that the Gross Question (1.1) has actually a negative answer when it is reformulated for general quadratic forms, for totally singular quadratic forms, or for symmetric bilinear forms over a field of characteristic 2.

Let us now return to the case of nonsingular quadratic forms and spaces. To motivate the Gross Question (1.1), we first shall show that the existence of an infinite-dimensional anisotropic nonsingular quadratic space implies the existence of such spaces in every finite even dimension. Again, for quadratic forms φ and ψ over F we write $\varphi \subset \psi$ if there exists a quadratic form τ such that $\psi \cong \varphi \perp \tau$. It is clear that if any two of the quadratic forms φ, ψ, τ are nonsingular, then so is the third.

We call a sequence of nonsingular quadratic forms $(\varphi_n)_{n\in\mathbb{N}}$ over F a *chain of nonsingular quadratic forms over* F if, for any $n \in \mathbb{N}$, we have $\dim(\varphi_n) = 2n$ and $\varphi_n \subset \varphi_{n+1}$. Note that we need even dimension for nonsingularity. Given such a chain $(\varphi_n)_{n\in\mathbb{N}}$ over F, the direct limit over the quadratic spaces φ_n with the appropriate inclusions is again a nonsingular quadratic space over F of countably infinite dimension. We denote this quadratic space over F by $\lim_{n\in\mathbb{N}}(\varphi_n)$ and observe that it is anisotropic if and only if φ_n is anisotropic for all $n \in \mathbb{N}$.

5.5 Lemma. *Any infinite-dimensional nonsingular quadratic space over F contains a subspace isometric to the direct limit $\lim_{n\in\mathbb{N}}(\varphi_n)$ for some chain $(\varphi_n)_{n\in\mathbb{N}}$ of nonsingular quadratic forms.*

Proof: Let (V, q) be nonsingular with $\dim(V) = \infty$ and let $b = b_q$.

(i) Let $x \in V \setminus \{0\}$. The nonsingularity implies the existence of $y \in V$ such that $b(x, y) \neq 0$. Clearly, x and y are linearly independent as $b(x, x) = 0$. Let $U_1 \subset V$ be the subspace spanned by x and y. Let $\varphi_1 = q|_{U_1}$. One readily sees that φ_1 is nonsingular.

(ii) Suppose $U \subset V$ is a $2m$-dimensional subspace with $\varphi = q|_U$ nonsingular. Let $V = U \oplus W$ and let $(w_i)_{i\in I}$ be a basis of W. Let $(x_1, y_1, \ldots, x_m, y_m)$ be a basis of U such that, with respect to this basis, $\varphi = [a_1, b_1] \perp \cdots \perp [a_m, b_m]$ holds for suitable $a_1, b_1, \ldots, a_m, b_m \in F$.

For each $i \in I$, let

$$z_i = w_i + \sum_{j=1}^{m} \big(b(w_i, x_j) y_j + b(w_i, y_j) x_j \big) \ .$$

A straightforward check shows that $b(z_i, x_\ell) = b(z_i, y_\ell) = 0$ for $1 \leq \ell \leq m$. Let Z be the subspace of V generated by $(z_i)_{i \in I}$. Then $V = U \oplus Z$ and $Z \subset U^\perp = \{v \in V \mid b(v, U) = 0\}$. Since $U \cap U^\perp = 0$ it follows that $Z = U^\perp$ and $V = U \oplus U^\perp$. If $\varphi^\perp = q|_{U^\perp}$, we thus have $q \cong \varphi \perp \varphi^\perp$ with φ nonsingular of dimension $2m$ and $\varphi^\perp$ nonsingular.

Using (i) and (ii), the lemma follows immediately by induction. □

As a direct consequence, we obtain the following:

5.6 Proposition. *There exists an anisotropic nonsingular quadratic space of infinite dimension over F if and only if there exists a chain of anisotropic nonsingular quadratic forms $(\varphi_n)_{n \in \mathbb{N}}$ over F.*

Before we state the analogues of Theorems I and II in characteristic 2, we have to recall a few more definitions and facts.

Let WF denote the Witt ring of nonsingular bilinear forms over F, and $W_q F$ the Witt group of nonsingular quadratic forms, which is in fact a WF-module. The fundamental ideal of classes of even-dimensional bilinear forms in WF will be denoted by IF, and its n^{th} power by $I^n F$. We put $I_q^n F = I^{n-1} F \cdot W_q F$. Then $I_q^n F$ is the submodule of $W_q F$ generated (as a group) by the *n-fold quadratic Pfister forms*

$$\langle\langle a_1, \cdots, a_n]] \ = \ \langle 1, a_1 \rangle_b \otimes \cdots \otimes \langle 1, a_{n-1} \rangle_b \otimes [1, a_n] \,,$$

with $a_1, \ldots, a_{n-1} \in F^\times$ and $a_n \in F$; here, we denote a diagonal bilinear form with $c_1, \ldots, c_m$ in the diagonal by $\langle c_1, \ldots, c_m \rangle_b$.

Quadratic Pfister forms in characteristic 2 have properties quite analogous to those in characteristic different from 2. For example they are either anisotropic or hyperbolic (i.e. isometric to an orthogonal sum of hyperbolic planes).

Function fields of nonsingular quadratic forms are defined as in characteristic different from 2, again with the convention that $F(\mathbb{H}) = F$. If q is a nonsingular quadratic form of dimension $2m > 0$, then $F(q)/F$ can be realized as a purely transcendental extension of F of transcendence degree $2m-2$ followed by a separable quadratic extension, and $F(q)/F$ is purely transcendental if and only if q is isotropic.

Lemma (3.1) is still true in characteristic 2, i.e. an anisotropic quadratic form (possibly singular) stays anisotropic over any odd extension of F and equally over any purely transcendental extension of F.

Also, (3.3) stays true in characteristic 2 for nonsingular forms. More precisely, if π is an anisotropic n-fold quadratic Pfister form and q is any nonsingular form of dimension $> 2^n$, then $\pi_{F(q)}$ is anisotropic. This follows simply by invoking the characteristic 2 analogues of the facts referred to in the proof of (3.3), or in (3.4). See, e.g. [16], Theorem 4.2(i), 4.4, for the precise formulation in characteristic 2 of these facts.

We can now state the characteristic 2 version of Theorem I.

5.7 Theorem I(2). *Let F be a field with* $\mathrm{char}(F) = 2$. *There exists a field extension K/F with the following properties:*

(i) K has no finite extensions of odd degree.

(ii) For any binary nonsingular quadratic form β over K, there is an upper bound on the dimensions of anisotropic nonsingular quadratic forms over K that contain β.

(iii) For any $k \in \mathbb{N}$, there is an anisotropic k-fold quadratic Pfister form over K.

In particular, K has infinite u-invariant, $I_q^k K \neq 0$ for all $k \in \mathbb{N}$, and any infinite-dimensional nonsingular quadratic space over K is isotropic.

Note that we cannot possibly expect K to be perfect. Indeed, $u(F) = \infty$ implies $\widehat{u}(F) = \infty$ and thus $[K : K^2] = \infty$ by (5.4).

Using the above mentioned facts on nonsingular forms, quadratic Pfister forms and function fields of nonsingular forms, the proof of Theorem I now easily adapts to become a proof of Theorem I(2). Indeed, it simply suffices to add the adjective 'nonsingular' whenever a quadratic form is mentioned in the proof and to replace 'Pfister form' by 'quadratic Pfister form' (with the appropriate notation). Also, expressions of type $2^j + 1$ referring to the dimension of a form must be replaced by $2^j + 2$ as nonsingularity requires even dimension. We leave the details to the reader.

To treat the characteristic 2 version of Theorem II, we need a few more facts about quaternion algebras and their tensor products over fields of characteristic 2.

A *quaternion algebra* $(a, b]_F$, with $a \in F^\times$ and $b \in F$, is a 4-dimensional central simple F-algebra generated by two elements x, y subject to the relations $x^2 = a$, $y^2 + y = b$, $xy = (y + 1)x$.

We now list some relevant facts that allow us to carry over the proofs from characteristic different from 2 to characteristic 2.

5.8 Proposition. *Let $a_1 \dots, a_n \in F^\times$ and $b_1, \dots, b_n \in F$ be such that $A = (a_1, b_1]_F \otimes \cdots \otimes (a_n, b_n]_F$ is a division algebra. Then the following hold:*

(i) The nonsingular $(2n+2)$-dimensional quadratic form

$$\varphi = [1, b_1 + \cdots + b_n] \perp a_1[1, b_1] \perp \cdots \perp a_n[1, b_n]$$

is anisotropic.

(ii) For any field extension K/F of one of the following types, the K-algebra $A_K = A \otimes_F K$ is a division algebra and φ_K is anisotropic:

- *K/F is an odd extension;*
- *$K = F(q)$ where q is a nonsingular quadratic form q such that $\dim q \geq 2n + 4$ or $q \in I_q^3 F$;*
- *K/F is purely transcendental.*

Proof: (i) This is [23], Proposition 6.

(ii) By Part (i) it suffices to prove that A_K is a division algebra. For a purely transcendental extension K/F this is obvious, and for an odd extension it is also clear as the index of A is a 2-power; for $K = F(q)$, this follows from [23], Theorems 3 and 4. □

5.9 Corollary. *Suppose that for every $n \in \mathbb{N}$ there exist $a_1, \dots, a_n \in F^\times$ and $b_1, \dots, b_n \in F$ such that $(a_1, b_1]_F \otimes \cdots \otimes (a_n, b_n]_F$ is a division algebra. Then $u(F) = \infty$.*

The characteristic 2 version of Theorem II now reads as follows.

5.10 Theorem II(2). *Let F be a field with $\mathrm{char}(F) = 2$. There exists a field extension K/F with the following properties:*

(i) K has no finite extensions of odd degree and $I_q^3 K = 0$.

(ii) For any binary nonsingular quadratic form β over K, there is an upper bound on the dimensions of anisotropic nonsingular quadratic forms over K that contain β.

(iii) For any $k \in \mathbb{N}$, there is a central division algebra over K that is decomposable into a tensor product of k quaternion algebras.

In particular, K has infinite u-invariant and every infinite-dimensional nonsingular quadratic space over K is isotropic.

Using (5.8) and (5.9), it is now straightforward to obtain a proof of Theorem II(2) by applying the appropriate changes to the proof of Theorem II, in a similar fashion as was done in the case of Theorem I(2). This time, it is expressions of type $2j+3$ in the proof of Theorem II which must be replaced by $2j+4$ because of the nonsingularity of the forms considered. Again, we leave the details to the reader.

5.11 Remark. In Theorem II (where $\mathrm{char}(K) \neq 2$), the facts that K has no odd degree extensions and that $I^3K = 0$ but $I^2K \neq 0$ together imply that K has cohomological dimension $\mathrm{cd}(K) = 2$.

In Theorem II(2) (where $\mathrm{char}(K) = 2$) we have again that K has no odd degree extension. This implies in particular that any finite separable extension L/K also has this property, and therefore $H^1(L, \mu_p) = L^\times/L^{\times p}$ vanishes for every finite separable extension L/K and every odd prime p. This implies that $\mathrm{cd}_p(K) = 0$ for the cohomological p-dimension of K for any odd prime p (see [37], II.1.2, II.2.3).

On the other hand, $\mathrm{cd}_2(F) \leq 1$ for any field F of characteristic 2 (see [37], II.2.2). In our case, there exist anisotropic nonsingular forms of dimension at least 2 over K, thus there certainly are separable quadratic extensions over K. This readily implies that $\mathrm{cd}_2(K) = 1$ and therefore $\mathrm{cd}(K) = \sup\{\mathrm{cd}_p(K) \,|\, p \text{ prime}\} = 1$.

However, rather than considering $\mathrm{cd}_2(F)$ for a field F with $\mathrm{char}(F) = 2$, it is perhaps more meaningful to ask for the *separable* 2*-dimension* $\dim_2^{\mathrm{sep}}(F)$ as defined by P. Gille [11]:

$$\dim_2^{\mathrm{sep}}(F) = \sup\{r \geq 0 \,|\, H_2^r(E) \neq 0 \text{ for some finite separable ext. } E/F\}\,,$$

where the $H_2^n(F)$ ($n \geq 0$) are Kato's cohomology groups for a field F with $\mathrm{char}(F) = 2$ (see, e.g., [19]).

In the situation of Theorem II(2), we have a field K of characteristic 2 with no odd degree extension and $I_q^3K = 0$. By Kato's proof of the Milnor conjecture in characteristic 2 in [19], we have $H_2^3(K) = 0$. Furthermore, by Galois theory, if L/K is a finite separable extension then $[L : K]$ is a 2-power and L/K can be obtained as a tower of separable quadratic extensions. But for any field F of characteristic 2 and any separable quadratic extension E/F, we have that $H_2^n(F) = 0$ implies $H_2^n(E) = 0$ (see, e.g., [4], 6.6). All this together implies that $H_2^3(L) = 0$ for every finite separable extension L of K, therefore $\dim_2^{\mathrm{sep}}(K) = 2$ (note that $I_q^2K \neq 0$).

6 Analogues of the Gross Question

Let $(\mathcal{M}, *, \varepsilon)$ be a monoïd (associative semi-group) with neutral element ε. Let $\mathcal{A}$ and $\mathcal{S}$ be nonempty subsets of $\mathcal{M}$ with $\varepsilon \notin \mathcal{S} \subset \mathcal{A} \subset \mathcal{M}$. Denoting by $\langle S \rangle$ the submonoïd of M generated by $\mathcal{S}$, we furthermore assume that for any $a, b \in \langle S \rangle$, if $a * b \in \mathcal{A}$ then $a, b \in \mathcal{A}$.

We now define a *U-invariant* for this triple $(\mathcal{M}, \mathcal{A}, \mathcal{S})$ by

$$U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}) \;=\; \sup\{m \in \mathbb{N} \mid \exists s_1, \ldots, s_m \in \mathcal{S} \text{ with } s_1 * \cdots * s_m \in \mathcal{A}\}\,.$$

These definitions have of course been motivated by our investigations of quadratic forms. More precisely, let F be a field with $\mathrm{char}(F) \neq 2$. Then we take $\mathcal{M}$ to be the set of nonsingular quadratic forms (up to isometry) over F, the operation $*$ the orthogonal sum, ε the trivial (0-dimensional) quadratic form, $\mathcal{A}$ the set of anisotropic forms over F, and $\mathcal{S}$ the set of 1-dimensional (nonzero) quadratic forms over F. In this setting, $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S})$ is nothing else but $u(F)$.

The Gross Question has now an obvious reformulation in this more abstract setting.

6.1 Question. Suppose that $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}) = \infty$. Does there exist a sequence $(s_n)_{n \in \mathbb{N}} \subset \mathcal{S}$ such that $s_1 * \cdots * s_n$ belongs to $\mathcal{A}$ for every $n \in \mathbb{N}$?

We proved that this does not always hold for anisotropy of quadratic forms over a field F. We will now pass from quadratic forms to other types of algebraic objects defined over a field that also naturally give rise to a triple $(\mathcal{M}, \mathcal{A}, \mathcal{S})$, and we will sketch answers to the above question in these new contexts.

Symbol algebras

Let F be a field and $n \geq 2$ be an integer. We assume that $\mathrm{char}(F)$ does not divide n, and that F contains a primitive n^{th} root of unity ζ which we fix. An F-algebra generated by two elements x, y subject to the relations $x^n = a$, $y^n = b$, $xy = \zeta yx$, where $a, b \in F^\times$ is denoted by $(a, b)_n$ and called an *n-symbol algebra* over F. Note that $(a, b)_n$ is a central simple F-algebra of degree n. For $n = 2$, we recover the case of quaternion algebras. For basic properties of such symbol algebras, we refer to [7], §11 (there, such algebras are called 'power norm residue algebras'). In the sequel, we will concentrate on the case where $n = p$ is a prime number.

With F as above, let $\mathcal{M}$ be the set of isomorphism classes of central simple algebras over F. The tensor product $\otimes$, taken over F, endows $\mathcal{M}$

with a monoïd structure, where the neutral element is given by the class of F. Let $\mathcal{A} \subset \mathcal{M}$ be the subset of (finite dimensional) central division algebras over F. Further, let $\mathcal{S}_p \subset \mathcal{A}$ be the subset given by the non-split p-symbol algebras over F.

The Gross Question in this context now becomes the following:

6.2 Question. Suppose that $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}_p) = \infty$, i.e. suppose that to every $n \in \mathbb{N}$ there exist p-symbol algebras $Q_1, \ldots, Q_n$ such that $\bigotimes_{i=1}^n Q_i$ is a division algebra. Does there exist a sequence $(A_i)_{i\in\mathbb{N}}$ of p-symbol algebras A_i over F such that $\bigotimes_{i=1}^n A_i$ is a division algebra for all $n \in \mathbb{N}$?

Let us first consider the case $p = 2$. If we take $F = K$ to be the field constructed in the proof of Theorem II, then we have in fact shown there that $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}_2) = \infty$, while for any sequence $(A_n)_{n\in\mathbb{N}}$ of quaternion algebras over K, the product $A_1 \otimes \cdots \otimes A_n$ fails to be a division algebra for $n \in \mathbb{N}$ sufficiently large. Actually, these two facts do not only follow from the way in which K was constructed, but already from the properties (i)–(iii). We omit the details.

So for $p = 2$, the answer to (6.2) is negative in general. We will sketch in the sequel that there are counterexamples for arbitrary primes p. Our construction is to some extent similar to the one in the proof of Theorem II, but function fields of quadratic forms will now have to be replaced by function fields of generic partial splitting varieties, also called generalized Severi-Brauer (or Brauer-Severi) varieties, and the special case in (3.5) of Merkurjev's index reduction results for function fields of quadratic forms will have to be replaced by an appropriate version concerning index reduction for function fields of generic partial splitting varieties.

Such generic partial splitting varieties have been studied systematically perhaps for the first time by Heuser [13], and then later by Schofield and Van den Bergh [35], [36], and Blanchet [5]. Blanchet derives in particular an index reduction formula for central simple algebras over function fields of generic partial splitting varieties. This formula has been simplified by Wadsworth [39], and it is the latter formula which we will use. The reader interested in the most general results on index reduction of central simple algebras over function fields of varieties is referred to the two papers by Merkurjev, Panin and Wadsworth [25], [26].

Let A be a central simple algebra over F of degree n, and let s be a divisor of n. To A we can now associate a generalized Severi-Brauer variety $X = SB(A, n, s)$ such that for any field exension L/F, the L-points $X(L)$ are the sn-dimensional right ideals in $A_L = A \otimes_F L$. In the case where L is a splitting field, so that $A \otimes_F L \cong \mathrm{End}_L(V)$ for an n-dimensional L-vector space

V, then $X(L)$ is isomorphic to the Grassmannian $Gr(V,s)$ of s-dimensional subspaces of V.

The function field $F(X)$ has the property that $\mathrm{ind}(A_{F(X)})$ divides s, and it is *generic* for that property in the following sense: If L is any field extension such that $\mathrm{ind}(A_L)$ divides s, then there exists an F-place $F(X) \longrightarrow L \cup \{\infty\}$ (see [13]). More precisely, we have the following (see [5], Proposition 3):

6.3 Lemma. *Let A, n, s, $X = SB(A,n,s)$ be as above and let L/F be a field extension. Then the following statements are equivalent:*

(i) *X has an L-rational point.*

(ii) $\mathrm{ind}(A_L)$ *divides s.*

(iii) *The free compositum $L \cdot F(X)$ is a purely transcendental extension of L.*

We now have the following index reduction formula for function fields of generic partial splitting varieties, see [39], Theorem 2:

6.4 Theorem. *Let A, n, s, $X = SB(A,n,s)$ be as above, let $K = F(X)$ and let D be a central simple algebra over F. Then*

$$\mathrm{ind}(D_K) = \gcd \left\{ \frac{s}{\gcd(i,s)} \,\mathrm{ind}(D \otimes_F A^{-i}) \;\middle|\; 1 \le i \le n \right\} .$$

6.5 Corollary. *Let p be a prime, let D be a central division algebra of index p^r over F, and let A be a central simple algebra of degree p^m over F $(m \ge 1)$ and of exponent dividing p. Let $X = SB(A, p^m, p^{m-1})$. If A is not a division algebra, or if $m > r$, then $D_{F(X)}$ is a division algebra.*

Proof: If A is not a division algebra, then $\mathrm{ind}(A)$ divides p^{m-1} and $F(X)/F$ is purely transcendental by (6.3). This clearly implies that D will stay a division algebra over $F(X)$.

Now assume that $m > r$. We apply the above index reduction formula with $n = p^m$ and $s = p^{m-1}$. Let $i \in \{1, \cdots, n\}$.

If $p \,|\, i$, then A^{-i} is split, because $\exp A$ divides p, and it follows immediately that $\frac{s}{\gcd(i,s)} \mathrm{ind}(D \otimes_F A^{-i})$ is divisible by $\mathrm{ind}(D \otimes_F A^{-i}) = \mathrm{ind}\, D$. Furthermore, for $i = p^m$ we have $\frac{s}{\gcd(i,s)} \mathrm{ind}(D \otimes_F A^{-i}) = \mathrm{ind}\, D$.

If $p \not| \, i$ then $\gcd(i,s) = 1$. Therefore,

$$\frac{s}{\gcd(i,s)} \mathrm{ind}(D \otimes_F A^{-i}) = p^{m-1} \cdot \mathrm{ind}(D \otimes_F A^{-i}) \,,$$

and this number is divisible by p^{m-1} and thus by $\mathrm{ind}(D) = p^r \le p^{m-1}$.

We conclude that

$$\mathrm{ind}(D_{F(X)}) = \gcd \left\{ \frac{s}{\gcd(i,s)} \mathrm{ind}(D \otimes_F A^{-i}) \;\middle|\; 1 \le i \le n \right\} = \mathrm{ind}(D) \ ,$$

in other words, D stays a division algebra over $F(X)$. □

6.6 Theorem. *Let p be a prime and let F be a field with $\mathrm{char}(F) \neq p$. Then there exists a field extension K/F containing a primitive p^{th} root of unity ζ such that the following holds:*

(i) For any $a_1 \in K^\times$, there exists an $n \in \mathbb{N}$ such that for any choice of $a_2, \dots, a_n, b_1, \dots b_n \in K^\times$, the product $\bigotimes_{i=1}^n (a_i, b_i)_p$ is not a division algebra.

(ii) For every $n \in \mathbb{N}$ there exist p-symbol algebras $A_1, \cdots, A_n$ over K such that $\bigotimes_{i=1}^n A_i$ is a division algebra.

Proof: Let $F_0 = F(\zeta)$ where ζ is a primitive p^{th} root of unity in an algebraic closure of F. Let $n \ge 1$ and suppose we have have constructed F_{n-1}. Let now

$$F_{n-1}^{(n)} = F_{n-1}(\, X_1^{(n)}, Y_1^{(n)}, \dots, X_n^{(n)}, Y_n^{(n)})$$

where $X_1^{(n)}, Y_1^{(n)}, \dots, X_n^{(n)}, Y_n^{(n)}$ are indeterminates over F_{n-1}. Let F_n denote the free compositum of function fields $F_{n-1}^{(n)}(SB(A, p^{j+1}, p^j))$ for all central simple algebras A over F_{n-1} of type $A \cong (a_0, b_0)_p \otimes (a_1, b_1)_p \otimes \cdots \otimes (a_j, b_j)_p$ with $j < n$ and $a_0 \in F_j^\times$ and $a_1, \dots, a_n, b_0, \dots, b_n \in F_{n-1}^{(n)\times}$.

Finally, we define $K = \bigcup_{i=0}^\infty F_n$ and claim that K has the desired properties.

(i) Let $a_1 \in K^\times$. Then there exists $j \in \mathbb{N}$ such that $a = a_0 \in F_j$. Let $a_1, \dots, a_j, b_0, \dots, b_j \in K^\times$ and consider $B = \bigotimes_{i=0}^j (a_i, b_i)_p$. It suffices to show that B is not a division algebra over K. Now there exists $n > j$ such that $a_1, \dots, a_j, b_0, \dots b_j \in F_{n-1}$, so B is defined over F_{n-1}, and since $F_{n-1}^{(n)}(SB(B, p^{j+1}, p^j))$ is part of the compositum F_n, we have that $\mathrm{ind}(B_{F_n})$ divides p^j, which implies that B is not a division algebra over F_n and thus also not over K.

(ii) For $n \ge 1$, consider over F_n the algebra

$$C_n = (X_1^{(n)}, Y_1^{(n)})_p \otimes \cdots \otimes (X_n^{(n)}, Y_n^{(n)})_p \ .$$

It is well known that C_n is a division algebra over F_n (see, e.g., [25], Corollary 5.2). Part (ii) now follows if we can show that C_n will stay a division algebra over K. This can be achieved by mimicking the argument in part (iii) of the proof of Theorem II, this time by invoking (6.3) and (6.5). We omit the details. □

Symbols in Milnor K-theory

Recall the definition of the Milnor K-groups K_nF of a field F (see [27]). By definition, $K_0F = \mathbb{Z}$, and K_1F is the multiplicative group $F^\times$, written additively with the elements denoted by $\{a\}$, $a \in F^\times$: $\{ab\} = \{a\} + \{b\}$. For $n \geq 2$, K_nF is then defined to be the quotient of the tensor product $K_1F^{\otimes n}$ by the subgroup generated by all $\{a_1\}\otimes\cdots\otimes\{a_n\}$ satisfying $a_i+a_{i+1} = 1$ for some i. The image of an element $\{a_1\}\otimes\cdots\otimes\{a_n\}$ in the quotient group K_nF is denoted by $\{a_1, \cdots, a_n\}$ and called a *symbol*. We then define the Milnor K-ring as the $\mathbb{Z}$-algebra $K_*F = \bigcup_{n=0}^{\infty} K_nF$ with multiplication defined on symbols in the obvious way: $\{a_1, \dots, a_n\} \cdot \{b_1, \dots, b_m\} = \{a_1, \dots, a_n, b_1, \dots, b_m\}$.

We are interested in K_nF/p, the Milnor K-groups modulo p for some prime p. The image of a symbol $\{a_1, \cdots, a_n\}$ in K_nF/p will again be called a symbol and denoted in the same way.

For $p = 2$, these groups are linked to quadratic form theory through the Milnor conjecture (now a theorem due to Orlov, Vishik, and Voevodsky [29]) which asserts that if $\mathrm{char}(F) \neq 2$ then $K_nF/2$ is isomorphic to $I^nF/I^{n+1}F$, via the isomorphism mapping $\{a_1, \dots, a_n\}$ to the class of $\langle\langle a_1, \dots, a_n\rangle\rangle$ modulo $I^{n+1}F$.

We now consider the abstract version of the Gross Question (6.1) in the following setting, where we assume $F^\times \neq F^{\times p}$ because otherwise $K_nF/p = 0$ for all $n \geq 1$. Let $\mathcal{M} = K_*F/p$, $\mathcal{S} = \{\{a\} \,|\, a \in F^\times \setminus F^{\times p}\}$ (this is nonempty by assumption), $\mathcal{A} = \{\{a_1, \cdots, a_n\} \neq 0 \,|\, n \in \mathbb{N},\ a_i \in F^\times\}$. It is obvious that for $n \geq 1$ we have $K_nF/p \neq 0$ if and only if there exist $a_1, \cdots, a_n \in F^\times$ with $\{a_1, \cdots, a_n\} \neq 0$. In this setting, Question (6.1) becomes

6.7 Question. Suppose that $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}) = \infty$, i.e. $K_nF/p \neq 0$ for all $n \in \mathbb{N}$. Does there exist a sequence $(a_n)_{n\in\mathbb{N}} \subset F^\times$ such that $\{a_1, \cdots, a_n\} \neq 0$ for every $n \in \mathbb{N}$?

Let us first consider the case where $\mathrm{char}(F) = p$. Then the answer to the above question is positive by the following:

6.8 Proposition. *Let F be a field of charateristic $p > 0$. Then the following are equivalent:*

(i) $[F : F^p] = \infty$.

(ii) $K_nF/p \neq 0$ *for all* $n \in \mathbb{N}$.

(iii) There exists a sequence $(a_n)_{n\in\mathbb{N}} \subset F^\times$ *such that* $\{a_1, \cdots, a_n\} \neq 0$ *for every* $n \in \mathbb{N}$.

For $p = 2$, the above statements are further equivalent to any of the following:

(iv) $\widehat{u}(F) = \infty$.

(v) $\sup\{\dim(b) \mid b$ *anisotropic symmetric bilinear form over* $F\} = \infty$.

(vi) There exists an infinite-dimensional anisotropic quadratic space over F.

(vii) There exists an infinite-dimensional anisotropic symmetric bilinear space over F.

Proof: Recall that a subset $T \subset F$ is called *p-independent* if, for any finite subset $\{a_1, \cdots, a_n\} \subset T$, one has $[F^p(a_1, \cdots, a_n) : F^p] = p^n$, and that $T \subset F$ is called a p-basis of F if T is a minimal generating set of the extension F/F^p, i.e. $F = F^p(T)$ and T is p-independent.

The key observation here is the fact that for $a_1, \cdots, a_n \in F^\times$ we have that $\{a_1, \cdots, a_n\} \neq 0$ if and only if $a_1, \cdots, a_n$ are p-independent, in other words $[F^p(a_1, \cdots, a_n) : F^p] = p^n$. This is an immediate consequence of the Bloch-Kato-Gabber Theorem (see [6], Theorem 2.1, or Appendix 2 by Fesenko in [17]). The equivalence of the first three statements is now immediate and we leave the details to the reader.

For $p = 2$ it readily follows from (5.3) and (5.4) that (i) is equivalent to any of the statements (iv) to (vii). $\square$

Let us now turn to the case $\mathrm{char}(F) \neq p$. For $p = 2$, the answer to the above question will be negative in general, i.e. there are fields such that $K_nF/2 \neq 0$ for all $n \in \mathbb{N}$, but for any sequence $(a_n)_{n\in\mathbb{N}} \subset F^\times$ one has $\{a_1, \cdots, a_m\} = 0$ for sufficiently large m.

Indeed, any field as constructed in Theorem I will do. To see this, it suffices to note that the map from P_nF, the set of isometry classes of n-fold Pfister forms over F, into $K_nF/2$ defined by $\langle\langle a_1, \cdots, a_n\rangle\rangle \mapsto \{a_1, \cdots, a_n\}$ is well-defined and injective (mapping a hyperbolic Pfister form to zero), see [8], Main Theorem 3.2 (here, we do not need the full thrust of the Milnor Conjecture). We leave the details to the reader.

Now if $p \neq 2$ (and $\mathrm{char}(F) \neq p$), we believe (but have not checked) that in general the answer to the above question should be negative as well. To construct counterexamples, it seems reasonable to try a similar approach as in our other constructions using a tower of iterated function fields. Candidates for these functions fields will naturally be function fields of (generic) splitting varieties of symbols in Milnor K-theory modulo p. The norm varieties as constructed by Rost (see [33], also [18]) provide examples for such splitting varieties.

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References

[1] *J.K. Arason*, Cohomologische Invarianten Quadratischer Formen. J. Algebra **36** (1975), 448–491.

[2] *J.K. Arason, R. Elman,* Powers of the fundamental ideal in the Witt ring. J. Algebra **239** (2001), 150–160.

[3] *R. Baeza*, Quadratic forms over semilocal rings. Lect. Notes Math. vol. **655**, Springer, Berlin–Heidelberg–New York 1978.

[4] *R. Aravire, R. Baeza*, Milnor's K-theory and quadratic forms over fields of characteristic two. Comm. Algebra **20** (1992), 1087–1107.

[5] *A. Blanchet*, Function fields of generalized Brauer-Severi varieties. Comm. Algebra **19** (1991), 97–118.

[6] *S. Bloch, K. Kato*, p-adic étale cohomology. Inst. Hautes Études Sci. Publ. Math. **63** (1986), 107–152.

[7] *P.K. Draxl*, Skew Fields. London Mathematical Society Lecture Notes series, vol. **81**, Cambridge University Press, Cambridge 1983.

[8] *R. Elman, T.Y. Lam*, Pfister forms and K-theory of fields. J. Algebra **23** (1972), 181–213.

[9] *R. Elman, T.Y. Lam*, Quadratic forms and the u-invariant I. Math. Z. **131** (1973), 283–304.

[10] *R. Elman, T.Y. Lam, A.R. Wadsworth*, Orderings under field extensions. J. reine angew. Math. **306** (1970), 7–27.

[11] *P. Gille*, Invariants cohomologiques de Rost en caractéristique positive. K-Theory **21** (2000), 57–100.

[12] *H. Gross*, Quadratic forms in infinite-dimensional vector spaces. Birkhäuser, Boston/Basel/Stuttgart, 1979.

[13] *A. Heuser* Partielle Zerfällung einfacher Algebren. PhD thesis, Universität Regensburg, 1976.

[14] *D.W. Hoffmann*, Isotropy of quadratic forms over the function field of a quadric. Math. Z. **220** (1995), 461–476.

[15] *D.W. Hoffmann*, Dimensions of Anisotropic Indefinite Quadratic Forms, I. Documenta Math., Extra Volume, Quadratic Forms LSU (2001), 183–200.

[16] *D.W. Hoffmann, A. Laghribi*, Quadratic forms and Pfister neighbors in characteristic 2. Transactions Amer. Math. Soc. **356** (2004), 4019–4053.

[17] *O. Izhboldin*, p-primary part of the Milnor K-groups and Galois cohomologies of fields of characteristic p. With an appendix by *Masato Kurihara, Ivan Fesenko*. Invitation to higher local fields (Münster, 1999), 19–41. Geom. Topol. Monogr. **3**, Geom. Topol. Publ., Coventry, 2000.

[18] *S. Joukhovitski, A. Suslin*, Norm varieties. J. Pure Appl. Algebra **206** (2006), 245–276.

[19] *K. Kato*, Symmetric bilinear forms, quadratic forms and Milnor K-theory in characteristic two. Invent. math. **66** (1982), 493–510.

[20] *T.Y. Lam*, The algebraic theory of quadratic forms. W. A. Benjamin, Inc., Reading, MA, 1973.

[21] *T.Y. Lam*, Introduction to Quadratic Forms over Fields. Graduate Studies in Mathematics **67**, Amer. Math. Soc., Providence, RI, 2005.

[22] *P. Mammone, R. Moresi, A.R. Wadsworth*, u-invariants of fields of characteristic 2. Math. Z. **208** (1991), 335–347.

[23] *P. Mammone, J.-P. Tignol, A.R. Wadsworth*, Fields of characteristic 2 with prescribed u-invariants. Math. Ann. **290** (1991), 109–128.

[24] *A.S. Merkurjev*, Simple algebras and quadratic forms. Izv. Akad. Nauk SSSR Ser. Mat. **55** (1991), 218–224; English translation in: Math. USSR-Izv. **38** (1992), 215–221

[25] *A.S. Merkurjev, I.A. Panin, A.R. Wadsworth*, Index reduction formulas for twisted flag varieties. I. K-theory **10** (1996), 517–596.

[26] *A.S. Merkurjev, I.A. Panin, A.R. Wadsworth*, Index reduction formulas for twisted flag varieties. II. K-theory **14** (1998), 101–196.

[27] *J. Milnor*, Algebraic K-theory and quadratic forms. Invent. Math. **9** (1970), 318–344.

[28] *J. Mináč, A. Wadsworth*, The u-invariant for algebraic extensions, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), AMS, Providence, RI, 1995, 333–358.

[29] *D. Orlov, A. Vishik, V. Voevodsky*, An exact sequence for Milnor's K-theory with applications to quadratic forms. K-theory Preprint Archives, `http://www.math.uiuc.edu/K-theory/0454/` (2000).

[30] *A. Pfister*, Quadratic forms with applications to algebraic geometry and topology. London Mathematical Society Lecture Notes series, vol. **217**, Cambridge University Press, Cambridge, 1995.

[31] *R.S. Pierce*, Associative algebras. Graduate Texts in Mathematics **88**, Springer-Verlag, New York, 1982.

[32] *A. Prestel*, Lectures on formally real fields. Lecture Notes in Mathematics **1093**, Springer-Verlag, Berlin, 1984.

[33] *M. Rost*, Norm varieties and algebraic cobordism. Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 77–85, Higher Ed. Press, Beijing, 2002. (Errata: Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 649, Higher Ed. Press, Beijing, 2002.)

[34] *W. Scharlau*, Quadratic and Hermitian forms. Grundlehren 270, Springer-Verlag, Berlin, 1985.

[35] *A. Schofield, M. van den Berg*, The index of a Brauer class on a Brauer-Severi variety. Trans. Amer. Math. Soc. **333** (1992), 729–739.

[36] *A. Schofield, M. van den Berg*, Division algebra coproducts of index n. Trans. Amer. Math. Soc. **341** (1994), 505–517.

[37] *J.-P. Serre*, Galois Cohomology. Springer-Verlag, Berlin Heidelberg, 1997.

[38] *J.-P. Tignol*, Réduction de l'indice d'une algèbre simple centrale sur le corps des fonctions d'une quadrique. Bull. Soc. Math. Belg. Sér. A **42** (1990), 735–745.

[39] *A.R. Wadsworth*, The index reduction formula for generic partial splitting varieties. Comm. Algebra **21** (1993), 1063–1070.

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