SERRE'S CONJECTURE II FOR CLASSICAL GROUPS OVER IMPERFECT FIELDS

GRÉGORY BERHUY, CHRISTOPH FRINGS, AND JEAN-PIERRE TIGNOL

Introduction

In his book "Cohomologie galoisienne," Serre formulates the following conjecture:

Conjecture II: ([22, §3.1]) *For every simply connected semisimple linear algebraic group G defined over a perfect field F of cohomological dimension at most 2, the Galois cohomology set $H^1(F,G)$ is trivial.*

Every simply connected semisimple group G over a field F is isomorphic to a product of Weil transfers

$$G = \prod_{i=1}^{n} R_{K_i/F}(G_i)$$

where G_i is a simply connected absolutely simple group over a finite separable extension K_i of F, and Shapiro's lemma yields

$$H^1(F,G) = \prod_{i=1}^{n} H^1(K_i, G_i)$$

(see for instance [15, (26.8), (29.6)]). Since the cohomological dimension does not change under finite separable extensions, it suffices to consider Conjecture II for simply connected absolutely simple groups. This conjecture was proved for groups of type ${}^1A_n$ by Merkurjev–Suslin [25, Theorem 24.8] and for groups of type ${}^2A_n$, B_n, C_n, D_n (with the exception of trialitarian D_4), F_4 and G_2 by Bayer-Fluckiger–Parimala [4].

In his Bourbaki talk [23], Serre proposed a stronger version of his Conjecture II, taking into account imperfect fields. To state this stronger version, define for any prime number p and any field F the *p-separable dimension* $\mathrm{sd}_p F$ as follows (see [12, §1.1], where $\mathrm{sd}_p F$ is denoted $\dim_p^{\mathrm{sep}} F$):

- if $\operatorname{char} F \neq p$, let $\mathrm{sd}_p F = \mathrm{cd}_p F$, the p-cohomological dimension of F;
- if $\operatorname{char} F = p$, consider the p-cohomology groups H_p^r (see [14], [6]) and let $\mathrm{sd}_p F$ be the least integer $r \geq 0$ such that $H_p^{r+1}(F') = 0$ for every finite separable extension F'/F.

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Conjecture II (Strong version): ([23, §5.5]) *Let G be a simply connected absolutely simple group G defined over a field F. If $\mathrm{sd}_p F \leq 2$ for every torsion prime of the root system of G, then $H^1(F, G) = 1$.*

For the reader's convenience, we quote from [23, §2.2] the list of torsion primes:

Type	Torsion primes
A_n	2, prime divisors of $n+1$
B_n, C_n, D_n $(n \neq 4)$, G_2	2
D_4, E_6, E_7, F_4	2, 3
E_8	2, 3, 5.

Note that the hypothesis on the separable dimension of F can be translated in more elementary terms by a theorem of Gille [12, Théorème 7]: $\mathrm{sd}_p F \leq 2$ *if and only if for every finite separable extension E/F, the reduced norm map of every p-primary central simple E-algebra is surjective.* It is mostly through this characterization that the hypothesis on separable dimensions is used in this work.

The strong version of Conjecture II was proved by Serre for groups of type G_2 [23, Théorème 11], and by Gille for groups of type ${}^1A_n$ and F_4 [12, Théorème 7, Théorème 9] and for quasi-split groups of any type except E_8 [13, Théorème 4]. Our main result is the following:

Theorem. *The strong version of Conjecture II holds for groups of type ${}^2A_n$, B_n, C_n, and D_n, except perhaps for trialitarian groups D_4.*

The proof is obtained by a case-by-case analysis in Corollaries 2.6, 3.12, 4.5, and 5.5. However, the pattern is the same in all cases: we consider an isogeny $\pi\colon G \to G^{\mathrm{ad}}$ and derive from classification theorems for quadratic forms, involutions or quadratic pairs that the image of the induced map $\pi^1\colon H^1(F, G) \to H^1(F, G^{\mathrm{ad}})$ is trivial. On the other hand, a theorem of Gille [13, Théorème 6] readily shows that $\ker \pi^1 = 1$, hence $H^1(F, G) = 1$.

As this sketch of proof suggests, the essential part of our work in this paper goes to the proof of classification theorems, which roughly say that hermitian forms of various types, (generalized) quadratic forms, involutions and quadratic pairs are classified by their "classical" invariants if the separable dimension of the base field is at most 2. (The precise statements are given in Theorem 2.1 and Corollaries 2.3, 2.4, 3.9, 3.10, 4.3, 4.4, 5.2, and 5.3.) We thus follow the same approach as Bayer-Fluckiger and Parimala, whose arguments in [4] also involved classification theorems for hermitian forms. (The corresponding classification of involutions in characteristic different from 2 was derived in [16].) Our main contribution is the inclusion of the characteristic 2 case in these classification theorems. Actually, as compared to [4], we shift the emphasis from hermitian forms to involutions and quadratic pairs on central simple algebras, which allows us to give proofs valid in all characteristics. We thus recover a significant part of the results in [4], by a method which avoids Morita equivalence, and which therefore seems more transparent[1] (at least to us).

In a first section, we give the main technical tools used in the classification theorems. These tools revolve around the notion of Witt equivalence for involutions and quadratic pairs. In the next sections, we successively tackle groups of type B_n, ${}^2A_n$, C_n, and D_n. For background information on involutions and quadratic pairs,

[1]For instance, we avoid the delicate justification that the form h_0 can be chosen of the same rank as h in the proofs of Theorems 4.2.1, 4.3.1, and 4.4.1 of [4].

we refer to [15], whose notation is used throughout the paper. In particular, if K/F is a separable quadratic extension of fields (of arbitrary characteristic), and if $a \in F^\times$, we denote by $(K, a)_F$ the quaternion algebra $K \oplus Kz$ where conjugation by z restricts to the nontrivial automorphism of K/F and $z^2 = a$. If σ is an involution on a central simple algebra A, we let

$$\begin{aligned} \mathrm{Sym}(\sigma) &= \{a \in A \mid \sigma(a) = a\}, \\ \mathrm{Symd}(\sigma) &= \{a + \sigma(a) \mid a \in A\}, \\ \mathrm{Skew}(\sigma) &= \{a \in A \mid \sigma(a) = -a\}. \end{aligned}$$

Appendix A yields detailed proofs of some results on Witt kernels which do not appear in the literature in the required generality. In Appendix B, we recall the basic notions of flat cohomology for the reader's convenience.

1. Witt equivalence of involutions and quadratic pairs

1.1. **Orthogonal sums.** Let A be a central simple algebra over an arbitrary field K and let σ be an involution (of any type) on A. Suppose A contains nonzero idempotents e_1, e_2 such that $e_1 + e_2 = 1$ and $\sigma(e_1) = e_1$, $\sigma(e_2) = e_2$. The K-algebras $A_1 = e_1 A e_1$ and $A_2 = e_2 A e_2$ are central simple and Brauer-equivalent to A, and σ restricts to involutions σ_1, σ_2 on A_1 and A_2. Note that A_1, A_2 are not subalgebras of A since their unity elements e_1, e_2 are not the unity 1 of A. If $A = \mathrm{End}_D V$ for some division K-algebra D and some D-vector space V, the subspaces $V_1 = e_1(V)$ and $V_2 = e_2(V)$ satisfy $V = V_1 \oplus V_2$, and there are canonical identifications

$$A_1 = \mathrm{End}_D V_1, \qquad A_2 = \mathrm{End}_D V_2.$$

By [15, (4.2)], there is an involution θ on D and a hermitian or skew-hermitian form h on V with respect to θ such that σ is the adjoint involution ad_h of h, in the sense that

$$h\big(x, a(y)\big) = h(\sigma(a)(x), y) \qquad \text{for } x,\ y \in V \text{ and } a \in A.$$

Since $\sigma(e_1) = e_1$ and $\sigma(e_2) = e_2$, it follows that V_1 and V_2 are orthogonal. Following Dejaiffe [7], we call (A, σ) an orthogonal sum of (A_1, σ_1) and (A_2, σ_2). More precisely, we set the following definition:

Definition 1.1. A central simple K-algebra with involution (A, σ) is an *orthogonal sum* of central simple K-algebras with involution (A_1, σ_1) and (A_2, σ_2) if there are orthogonal idempotents $e_1, e_2 \in A$ such that $e_1 + e_2 = 1$ and $\sigma(e_1) = e_1$, $\sigma(e_2) = e_2$, and K-algebra isomorphisms

$$\varphi_1 \colon A_1 \xrightarrow{\sim} e_1 A e_1, \qquad \varphi_2 \colon A_2 \xrightarrow{\sim} e_2 A e_2$$

such that $\varphi_i \circ \sigma_i = \sigma \circ \varphi_i$ for $i = 1, 2$. Using φ_1 and φ_2 to identify A_1 and A_2 to subsets of A, we denote

$$(A, \sigma) = (A_1, \sigma_1) \boxplus (A_2, \sigma_2).$$

It is important to observe that an orthogonal sum is not uniquely determined up to isomorphism by its summands. Indeed, for λ_1, $\lambda_2 \in K^\times$, the involution σ' on A defined by

$$\sigma'(a) = (\lambda_1 e_1 + \lambda_2 e_2)\sigma(a)(\lambda_1^{-1} e_1 + \lambda_2^{-1} e_2)$$

is different from σ if $\lambda_1 \neq \lambda_2$, but has the same restriction as σ to A_1 and A_2.

Proposition 1.2. *If a central simple K-algebra with involution (A, σ) is an orthogonal sum*
$$(A, \sigma) = (A_1, \sigma_1) \boxplus (A_2, \sigma_2),$$
then A_1, A_2 are Brauer-equivalent to A and σ, σ_1, σ_2 have the same restriction to K,
$$\sigma|_K = \sigma_1|_K = \sigma_2|_K.$$
If $\sigma|_K = \mathrm{Id}$, then σ, σ_1, σ_2 have the same type (orthogonal or symplectic). Moreover,
$$\deg A = \deg A_1 + \deg A_2.$$

Proof. As observed before Definition 1.1, we may identify $(A, \sigma) = (\mathrm{End}_D V, \mathrm{ad}_h)$ and $(A_1, \sigma_1) = (\mathrm{End}_D V_1, \mathrm{ad}_{h_1})$, $(A_2, \sigma_2) = (\mathrm{End}_D V_2, \mathrm{ad}_{h_2})$ for some D-vector space $V = V_1 \oplus V_2$ and some hermitian or skew-hermitian form $h = h_1 \perp h_2$. The proposition follows. □

Proposition 1.3. *Let A_1, A_2 be Brauer-equivalent central simple K-algebras and let σ_1, σ_2 be involutions of the same type on A_1, A_2 respectively. If σ_1 and σ_2 are unitary, assume moreover $\sigma_1|_K = \sigma_2|_K$. Then there exists a central simple K-algebra A with involution σ such that*
$$(A, \sigma) = (A_1, \sigma_1) \boxplus (A_2, \sigma_2).$$

Proof. Let D be a central division K-algebra Brauer-equivalent to A_1 and A_2. There are D-vector spaces V_1, V_2 such that $A_1 = \mathrm{End}_D V_1$ and $A_2 = \mathrm{End}_D V_2$. By [15, (4.2)], there is an involution θ on D and hermitian or skew-hermitian forms h_1, h_2 on V_1, V_2 respectively such that $\sigma_1 = \mathrm{ad}_{h_1}$ and $\sigma_2 = \mathrm{ad}_{h_2}$. Since σ_1 and σ_2 have the same type, we may assume that h_1 and h_2 are both hermitian or both skew-hermitian. Then $(\mathrm{End}_D(V_1 \oplus V_2), \mathrm{ad}_{h_1 \perp h_2})$ is an orthogonal sum of (A_1, σ_1) and (A_2, σ_2). □

To discuss orthogonal groups in characteristic 2, we need to use also orthogonal sums of quadratic pairs. Recall from [15, (5.4)] that a quadratic pair on a central simple K-algebra A of degree n (of arbitrary characteristic) is a pair (σ, f) consisting of an involution σ of the first kind such that $\dim \mathrm{Sym}(\sigma) = \frac{1}{2}n(n+1)$, and $f\colon \mathrm{Sym}(\sigma) \to K$ is a linear map related to the reduced trace Trd_A by the following condition:
$$f\big(x + \sigma(x)\big) = \mathrm{Trd}_A(x) \qquad \text{for } x \in A.$$
If $\operatorname{char} K \neq 2$, the conditions imply that σ is an orthogonal involution and
$$f(x) = \tfrac{1}{2}\,\mathrm{Trd}_A(x) \qquad \text{for } x \in \mathrm{Sym}(\sigma).$$
The map f is thus determined by σ; it does not carry any additional structure in this case.

If $\operatorname{char} K = 2$, the involution σ is symplectic since the conditions imply that the trace of every symmetric element is zero.

Definition 1.4. A central simple K-algebra with quadratic pair (A, σ, f) is an orthogonal sum of central simple K-algebras with quadratic pair (A_1, σ_1, f_1) and (A_2, σ_2, f_2) if
$$(A, \sigma) = (A_1, \sigma_1) \boxplus (A_2, \sigma_2)$$
and, identifying A_1 and A_2 with their images in A,
$$f|_{\mathrm{Sym}(\sigma_1)} = f_1, \qquad f|_{\mathrm{Sym}(\sigma_2)} = f_2.$$

We then denote

$$(A, \sigma, f) = (A_1, \sigma_1, f_1) \boxplus (A_2, \sigma_2, f_2).$$

(See [11, p. 379] or [10].)

As for involutions, the direct sum is not uniquely determined by its summands.

Proposition 1.5. *Let A_1, A_2 be Brauer-equivalent central simple K-algebras and let (σ_1, f_1), (σ_2, f_2) be quadratic pairs on A_1 and A_2 respectively. There is a central simple K-algebra A with quadratic pair (σ, f) such that*

$$(A, \sigma, f) = (A_1, \sigma_1, f_1) \boxplus (A_2, \sigma_2, f_2).$$

Proof. We may mimic the arguments in the proof of Proposition 1.3, using generalized quadratic forms as in [11]. Alternatively, form an orthogonal sum

$$(A, \sigma) = (A_1, \sigma_1) \boxplus (A_2, \sigma_2)$$

and, identifying $A_1 = e_1Ae_1$, $A_2 = e_2Ae_2$ for some symmetric orthogonal idempotents e_1, e_2 such that $e_1 + e_2 = 1$, define

$$f(x) = f_1(e_1xe_1) + f_2(e_2xe_2) \qquad \text{for } x \in \operatorname{Sym}(\sigma).$$

□

Remark. In the notation of the proof, the map f is uniquely determined by the condition that $f|_{\operatorname{Sym}(\sigma_1)} = f_1$ and $f|_{\operatorname{Sym}(\sigma_2)} = f_2$, since every $x \in \operatorname{Sym}(\sigma)$ decomposes as

$$x = e_1xe_1 + e_2xe_2 + e_1xe_2 + e_2xe_1,$$

and $f(e_1xe_2 + e_2xe_1) = \operatorname{Trd}_A(e_1xe_2) = 0$.

1.2. **Discriminant of an orthogonal sum.** We may compare the invariants of an orthogonal sum with the invariants of the summands; see [7] and [8] for results in this direction. For our purposes, we need to consider only the discriminant algebras of unitary involutions on central simple algebras of even degree. The following result is due to Tamagawa [26, Theorem 3]. We include a proof (different from Tamagawa's) for completeness.

Proposition 1.6. *Let σ be a unitary involution on a central simple K-algebra A and let $F \subset K$ be the subfield of σ-invariant elements. Suppose (A, σ) is an orthogonal sum of central simple K-algebras with involution of even degree,*

$$(A, \sigma) = (A_1, \sigma_1) \boxplus (A_2, \sigma_2).$$

Then the Brauer classes of the discriminant algebras are related by

$$[D(A, \sigma)] = [D(A_1, \sigma_1)] + [D(A_2, \sigma_2)]$$

in the Brauer group $\operatorname{Br} F$.

Proof. Let $F(X)$ be the function field of the Weil transfer of the Severi-Brauer variety of A. By [19], the scalar extension map $\operatorname{Br} F \to \operatorname{Br} F(X)$ is injective. Therefore, it suffices to prove the claim after scalar extension to $F(X)$. We may thus assume that A (hence also A_1 and A_2) is split, hence

$$(A, \sigma) = (\operatorname{End}_K V, \operatorname{ad}_h),$$
$$(A_1, \sigma_1) = (\operatorname{End}_K V_1, \operatorname{ad}_{h_1}), \qquad (A_2, \sigma_2) = (\operatorname{End}_K V_2, \operatorname{ad}_{h_2})$$

for some K-vector space $V = V_1 \oplus V_2$ and some hermitian form $h = h_1 \perp h_2$ on V. The discriminant algebras of σ, σ_1 and σ_2 are Brauer-equivalent to quaternion F-algebras,

$$[D(A,\sigma)] = [(K, \operatorname{disc} h)_F],$$
$$[D(A_1,\sigma_1)] = [(K, \operatorname{disc} h_1)_F], \qquad [D(A_2,\sigma_2)] = [(K, \operatorname{disc} h_2)_F].$$

The lemma follows by the additivity of the quaternion symbol, since

$$\operatorname{disc}(h_1 \perp h_2) = (\operatorname{disc} h_1)(\operatorname{disc} h_2).$$

□

1.3. **Centralizers of quadratic subfields.** A major tool in our proof of the classification theorems is a reduction to centralizers of quadratic subfields. In this subsection, we consider this reduction from the viewpoint of orthogonal sums.

Let (A,σ) be a central simple K-algebra with involution (of any type) and let L/K be a separable quadratic extension with nontrivial automorphism ι. Suppose (A,σ) is an orthogonal sum

$$(A,\sigma) = (A_1,\sigma_1) \boxplus (A_2,\sigma_2)$$

and there are embeddings

$$\varepsilon_1\colon (L,\iota) \hookrightarrow (A_1,\sigma_1), \qquad \varepsilon_2\colon (L,\iota) \hookrightarrow (A_2,\sigma_2),$$

i.e. K-algebra embeddings of L in A_1, A_2 respectively such that

$$\varepsilon_1 \circ \iota = \sigma_1 \circ \varepsilon_1, \qquad \varepsilon_2 \circ \iota = \sigma_2 \circ \varepsilon_2.$$

The involutions σ_1, σ_2 then restrict to unitary involutions $\tilde\sigma_1$, $\tilde\sigma_2$ on the centralizers $\tilde A_1 = \operatorname{Cent}_{A_1} L$, $\tilde A_2 = \operatorname{Cent}_{A_2} L$ of L in A_1 and A_2.

Proposition 1.7. *The embeddings ε_1, ε_2 induce an embedding*

$$\varepsilon_1 \boxplus \varepsilon_2\colon (L,\iota) \hookrightarrow (A,\sigma).$$

Moreover, letting $\tilde\sigma$ denote the restriction of σ to the centralizer $\tilde A$ of L in A,

$$(\tilde A,\tilde\sigma) = (\tilde A_1,\tilde\sigma_1) \boxplus (\tilde A_2,\tilde\sigma_2).$$

Proof. Let e_1, $e_2 \in A$ be orthogonal σ-symmetric idempotents such that $e_1+e_2 = 1$ and

$$\varphi_1\colon A_1 \xrightarrow{\sim} e_1Ae_1, \qquad \varphi_2\colon A_2 \xrightarrow{\sim} e_2Ae_2$$

be K-algebra isomorphisms under which σ_1, σ_2 correspond to the restriction of σ. For $x \in L$, define

$$(\varepsilon_1 \boxplus \varepsilon_2)(x) = \varphi_1 \circ \varepsilon_1(x) + \varphi_2 \circ \varepsilon_2(x) \in A.$$

This map is an embedding $(L,\iota) \hookrightarrow (A,\sigma)$. The idempotents e_1, e_2 centralize the image of L and φ_1, φ_2 restrict to isomorphisms

$$\tilde\varphi_1\colon \tilde A_1 \xrightarrow{\sim} e_1\tilde Ae_1, \qquad \tilde\varphi_2\colon \tilde A_2 \xrightarrow{\sim} e_2\tilde Ae_2,$$

so

$$(\tilde A,\tilde\sigma) = (\tilde A_1,\tilde\sigma_1) \boxplus (\tilde A_2,\tilde\sigma_2).$$

□

1.4. **Witt decomposition and cancellation.** Let A be a central simple algebra over an arbitrary field K, and let σ be an involution of symplectic or unitary type on A. Recall from [15, §6] that σ is called *hyperbolic* if A contains an idempotent e such that $\sigma(e) = 1 - e$; it is called *isotropic* if A contains a non-zero right ideal I such that $\sigma(I)I = 0$. To rephrase these conditions in terms of hermitian or skew-hermitian forms, choose a representation of A by endomorphisms of a vector space V over a division algebra D. The involution σ is then adjoint to a hermitian or skew-hermitian form h on V,

$$(A, \sigma) = (\operatorname{End}_D V, \operatorname{ad}_h). \tag{1}$$

Then σ is hyperbolic (resp. isotropic) if and only if h is hyperbolic (resp. isotropic), see [15, (6.2), (6.7)]. In particular, $\dim_D V$ is even in this case, hence $\operatorname{ind} A$ $(= \deg D)$ divides $\frac{1}{2}\deg A$ $(= \frac{1}{2}\deg D \dim_D V)$. Note that if σ is symplectic, we may choose a representation where h is alternating (i.e. even skew-hermitian), see [15, (4.2)]. If σ is unitary, h may be chosen hermitian with respect to a unitary involution on D. In both cases, a "Witt decomposition theorem" asserts that h decomposes into an orthogonal sum of an anisotropic and a hyperbolic form, see [21, Corollary 9.2, p. 268]. This observation yields the first part of the following proposition:

Proposition 1.8. (1) *Every central simple algebra A with symplectic or unitary involution σ has an orthogonal sum decomposition*

$$(A, \sigma) = (A_0, \sigma_0) \boxplus (A_1, \sigma_1)$$

where σ_0 is anisotropic and σ_1 is hyperbolic.

(2) *Suppose a central simple algebra with symplectic or unitary involution σ has an orthogonal sum decomposition*

$$(A, \sigma) = (A_0, \sigma_0) \boxplus (A_1, \sigma_1)$$

where σ_1 is hyperbolic. If σ is hyperbolic, then σ_0 is hyperbolic.

Proof. As observed above, the first part follows from the Witt decomposition theorem for hermitian or skew-hermitian forms. To prove the second part, choose a representation (1). The decomposition of (A, σ) yields an orthogonal decomposition of h,

$$(V, h) = (V_0, h_0) \overset{\perp}{\oplus} (V_1, h_1)$$

where h_1 is hyperbolic. If h is hyperbolic, then h_0 is hyperbolic by Witt cancellation, see [21, Corollary 9.2, p. 268]. $\square$

Remark. The second part of the proposition above may be regarded as a "Witt cancellation theorem" for involutions. Note that only hyperbolic involutions can be cancelled; Elomary has shown in [10] that general involutions do not admit cancellation.

Definition 1.9. Two central simple algebras with symplectic or unitary involutions (A, σ), (A', σ') are called *Witt-equivalent* if there are orthogonal sum decompositions

$$(A, \sigma) = (A_0, \sigma_0) \boxplus (A_1, \sigma_1), \qquad (A', \sigma') = (A'_0, \sigma'_0) \boxplus (A'_1, \sigma'_1)$$

where σ_1, σ'_1 are hyperbolic, and an isomorphism of algebras with involution

$$(A_0, \sigma_0) \simeq (A'_0, \sigma'_0).$$

From Proposition 1.8, it follows that σ is hyperbolic if (A,σ) is Witt-equivalent to a central simple algebra with hyperbolic involution.

Similar notions are defined for quadratic pairs. A quadratic pair (σ,f) on a central simple F-algebra A is called *hyperbolic* if A contains an idempotent e such that

$$f(x)=\operatorname{Trd}_A(ex) \qquad \text{for } x\in\operatorname{Sym}(\sigma), \tag{2}$$

see [15, (6.14)]. It is called *isotropic* if A contains a nonzero right ideal I such that

$$\sigma(I)I=0 \qquad \text{and} \qquad f\big(I\cap\operatorname{Sym}(\sigma)\big)=0,$$

see [15, (6.5)].

Remark 1.10. Condition (2) implies $\sigma(e)=1-e$. More precisely, for any $\ell\in A$ the following conditions are equivalent:

(a) $\operatorname{Trd}_A(\ell x)=f(x)$ for all $x\in\operatorname{Symd}(\sigma)$,
(b) $\sigma(\ell)=1-\ell$.

Indeed, for $y\in A$ we have $f(y+\sigma(y))=\operatorname{Trd}_A(y)$, hence (a) is equivalent to

$$\operatorname{Trd}_A(\ell y)+\operatorname{Trd}_A(\ell\sigma(y))=\operatorname{Trd}_A(y) \qquad \text{for all } y\in A.$$

Since $\operatorname{Trd}_A(\ell\sigma(y))=\operatorname{Trd}_A(\sigma(\ell)y)$, the last equation can be rewritten as

$$\operatorname{Trd}_A\big((\ell+\sigma(\ell))y\big)=\operatorname{Trd}_A(y) \qquad \text{for all } y\in A.$$

It is equivalent to (b) because the bilinear form $\operatorname{Trd}_A(xy)$ is regular.

The condition for isotropy or hyperbolicity of a quadratic pair can be translated in terms of generalized quadratic forms, see [11, Proposition 1.7 and Corollary 1.8]. Since hyperbolic quadratic pairs are adjoint to hyperbolic quadratic forms, it follows that $\operatorname{ind} A$ divides $\frac{1}{2}\deg A$ if A carries a hyperbolic quadratic pair. In view of the Witt decomposition and cancellation theorems for generalized quadratic forms (see [21, Corollary 9.2, p. 268]), the following result can be proved along the same lines as Proposition 1.8:

Proposition 1.11. (1) *Every central simple algebra A with quadratic pair (σ,f) has an orthogonal sum decomposition*

$$(A,\sigma,f)=(A_0,\sigma_0,f_0)\boxplus(A_1,\sigma_1,f_1)$$

where (σ_0,f_0) is anisotropic and (σ_1,f_1) is hyperbolic.

(2) *Suppose a central simple algebra with quadratic pair (σ,f) has an orthogonal sum decomposition*

$$(A,\sigma,f)=(A_0,\sigma_0,f_0)\boxplus(A_1,\sigma_1,f_1)$$

where (σ_1,f_1) is hyperbolic. If (σ,f) is hyperbolic, then (σ_0,f_0) is hyperbolic.

Definition 1.12. Two central simple F-algebras with quadratic pair (A,σ,f), (A',σ',f') are called *Witt-equivalent* if there are orthogonal sum decompositions

$$(A,\sigma,f)=(A_0,\sigma_0,f_0)\boxplus(A_1,\sigma_1,f_1), \qquad (A',\sigma',f')=(A_0',\sigma_0',f_0')\boxplus(A_1',\sigma_1',f_1')$$

where (σ_1,f_1), (σ_1',f_1') are hyperbolic, and an isomorphism of algebras with quadratic pairs

$$(A_0,\sigma_0,f_0)\simeq(A_0',\sigma_0',f_0').$$

1.5. **The Witt kernel of odd-degree and quadratic extensions.** A key tool to prove the classification theorems in Sections 3, 4, and 5 is the description of involutions and quadratic pairs which become hyperbolic under odd-degree or quadratic extensions. For odd-degree extensions, we have the following analogues of the weak version of Springer's theorem:

Theorem 1.13. *Let K be an arbitrary field. Let (A,σ) be a central simple K-algebra with involution of symplectic or unitary type, and let $F \subset K$ be the subfield of σ-invariant elements (so $F = K$ if σ is symplectic). Let E/F be an odd-degree field extension and $A_E = A \otimes_F E$, $\sigma_E = \sigma \otimes \mathrm{Id}_E$. If σ_E is hyperbolic, then σ is hyperbolic.*

Theorem 1.14. *Let (A,σ,f) be a central simple algebra with quadratic pair over an arbitrary field F, and let E/F be an odd-degree field extension. Let $A_E = A \otimes_F E$, $\sigma_E = \sigma \otimes \mathrm{Id}_E$, $f_E = f \otimes \mathrm{Id}_E$. If the quadratic pair (σ_E, f_E) is hyperbolic, then (σ, f) is hyperbolic.*

Theorem 1.13 is easily derived from the analogous statement for hermitian forms, due to Bayer-Fluckiger and Lenstra [3, Proposition 1.2]. The transfer argument used in [3] can be adapted to generalized quadratic forms to yield Theorem 1.14. For completeness, we spell out in Appendix A (see Section A.2) detailed proofs along the lines of [3, Proposition 1.2], focusing on involutions and quadratic pairs instead of hermitian forms and generalized quadratic forms.

For (separable) quadratic extensions, the key results are the following:

Theorem 1.15. *Let K be an arbitrary field, let (A,σ) be a central simple K-algebra with involution of symplectic or unitary type, and let $F \subset K$ be the subfield of σ-invariant elements (so $F = K$ if σ is symplectic). Let L/F be a separable quadratic extension with non-trivial automorphism ι and let $A_L = A \otimes_F L$, $\sigma_L = \sigma \otimes \mathrm{Id}_L$. The involution σ_L on A_L is hyperbolic if and only if there is an embedding $(L,\iota) \hookrightarrow (A,\sigma)$.*

Theorem 1.16. *Let (A,σ,f) be a central simple algebra with quadratic pair over an arbitrary field F, and let L/F be a separable quadratic extension. Let $A_L = A \otimes_F L$, $\sigma_L = \sigma \otimes \mathrm{Id}_L$ and $f_L = f \otimes \mathrm{Id}_L$. If there is an embedding $\varepsilon\colon L \hookrightarrow A$ such that*

$$\mathrm{Trd}_A(\varepsilon(\ell)x) = T_{L/F}(\ell) f(x) \qquad \text{for all } x \in \mathrm{Sym}(\sigma) \text{ and } \ell \in L, \tag{3}$$

then (σ_L, f_L) is hyperbolic. The converse holds except if A is split and (σ, f) is adjoint to a quadratic form of odd Witt index.

Remark. Note that (3) implies $\varepsilon \circ \iota = \sigma \circ \varepsilon$, since for $x = y + \sigma(y)$ it yields

$$\mathrm{Trd}_A\bigl(\varepsilon(\ell)(y + \sigma(y))\bigr) = T_{L/F}(\ell)\,\mathrm{Trd}_A(y).$$

Expanding the left side as

$$\mathrm{Trd}_A(\varepsilon(\ell)y) + \mathrm{Trd}_A(y \cdot \sigma\varepsilon(\ell)) = \mathrm{Trd}_A\bigl(y(\varepsilon(\ell) + \sigma\varepsilon(\ell))\bigr),$$

we obtain

$$\mathrm{Trd}_A(yT_{L/F}(\ell)) = \mathrm{Trd}_A\bigl(y(\varepsilon(\ell) + \sigma\varepsilon(\ell))\bigr) \qquad \text{for } y \in A \text{ and } \ell \in L.$$

Since the reduced trace bilinear form is nonsingular, it follows that $\varepsilon(\ell) + \sigma\varepsilon(\ell) = T_{L/F}(\ell)$, hence $\sigma\varepsilon(\ell) = \varepsilon\iota(\ell)$ for $\ell \in L$.

If $\operatorname{char} F \neq 2$, (3) is in fact equivalent to $\varepsilon \circ \iota = \sigma \circ \varepsilon$, as can be seen by reversing the steps in the above argument with $y = \frac{1}{2}x$.

Particular cases of Theorems 1.15 and 1.16 are known: if $\operatorname{char} F \neq 2$, Theorem 1.16 and the symplectic case of Theorem 1.15 are proved in [5, Theorem 3.3], and the unitary case of Theorem 1.15 in [17, Theorem 3.6]. If $\operatorname{char} F = 2$, the symplectic case of Theorem 1.15 and Theorem 1.16 can be found in [11, Propositions 4.1 and 4.2], except that the converse implication is proved under the additional hypothesis that σ or (σ, f) is anisotropic. Since none of the quoted references establishes Theorems 1.15 and 1.16 in the generality we require, we give a complete proof, featuring a characteristic-free approach, in Appendix A (see Section A.3).

2. Quadratic forms

Let (V, q) be a quadratic space over an arbitrary field F and let $b\colon V \times V \to F$ be the polar symmetric bilinear form, defined by

$$b(x, y) = q(x + y) - q(x) - q(y) \qquad \text{for } x,\, y \in V.$$

The radical of b is

$$\operatorname{rad} b = \{x \in V \mid b(x, y) = 0 \text{ for all } y \in V\}.$$

The quadratic form q is called *regular* if either $\operatorname{rad} b = \{0\}$ or $\operatorname{char} F = 2$ and $\dim \operatorname{rad} b = 1$. The latter case occurs only if $\dim V$ is odd, since b is alternating when $\operatorname{char} F = 2$. Henceforth, all the quadratic forms are assumed to be regular.

2.1. **Classification.**

The classical invariants of quadratic forms—the discriminant and the Clifford invariant—are recalled next.

If $\operatorname{char} F \neq 2$, we denote by $\operatorname{disc} q \in F^\times/F^{\times 2}$ the (signed) discriminant of q (see for instance [21, p. 36]). If $\operatorname{char} F = 2$, the discriminant is defined as follows: if $\dim V$ is odd (hence $\dim \operatorname{rad} b = 1$), pick a non-zero vector $x \in \operatorname{rad} b$ and let

$$\operatorname{disc} q = q(x)F^{\times 2} \in F^\times/F^{\times 2};$$

if $\dim V$ is even, $\operatorname{disc} q$ is the Arf invariant of q (see [21, p. 340]),

$$\operatorname{disc} q = \operatorname{Arf}(q) \in F/\wp(F),$$

where $\wp(F) = \{\lambda^2 - \lambda \mid \lambda \in F\}$.

The Clifford invariant $c(q)$ is defined independently of the characteristic as the Brauer class of the Clifford algebra or the even Clifford algebra,

$$c(q) = \begin{cases} [C(V, q)] \in \operatorname{Br}(F) & \text{if } \dim V \text{ is even,} \\ [C_0(V, q)] \in \operatorname{Br}(F) & \text{if } \dim V \text{ is odd.} \end{cases}$$

We can now state the classification theorem for quadratic forms:

Theorem 2.1. *If the reduced norm map of every quaternion F-algebra is surjective, then quadratic forms over F are classified up to isometry by their dimension, discriminant and Clifford invariant.*

Proof. The result was proved by Elman and Lam [9, Theorem 3.11] for fields of characteristic different from 2 and by Sah [20, Theorem 3] for quadratic forms of even dimension over fields of characteristic 2. For the rest of the proof,[2] we assume $\operatorname{char} F = 2$ and consider forms of odd dimension. We use the following notation for quadratic forms: for $a,\, b \in F$,

$$[a] = ax^2 \qquad \text{and} \qquad [a, b] = ax^2 + xy + by^2.$$

[2] We are indebted to Detlev Hoffmann for the following arguments.

Since

$$x^2 + (y^2 + yz + az^2) = (x + y)^2 + (y + az)z,$$

the change of variables $x' = x + y$, $y' = y + az$, $z' = z$ yields

$$[1] \perp [1, a] \simeq [1] \perp [0, 0]. \tag{4}$$

Let q, q' be odd-dimensional quadratic forms over F with

$$\dim q = \dim q', \qquad \operatorname{disc} q = \operatorname{disc} q', \qquad c(q) = c(q').$$

Let $a \in F^\times$ be a representative of $\operatorname{disc} q$. The restriction q_0 of q (resp. q'_0 of q') to an orthogonal complement of the radical is a regular even-dimensional quadratic form such that

$$q \simeq [a] \perp q_0, \qquad q' \simeq [a] \perp q'_0.$$

Let d (resp. d') be a representative of $\operatorname{disc}(a \cdot q_0)$ (resp. $\operatorname{disc}(a \cdot q'_0)$). It is easy to see that

$$c(q) = c(a \cdot q_0) = c([1, d] \perp a \cdot q_0), \qquad c(q') = c(a \cdot q'_0) = c([1, d'] \perp a \cdot q'_0),$$

see for instance [18, Lemma 2, Proposition 5]. By the classification theorem for even-dimensional forms it follows that

$$[1, d] \perp a \cdot q_0 \simeq [1, d'] \perp a \cdot q'_0.$$

Adding [1] to each side and using (4), we obtain

$$[1] \perp [0, 0] \perp a \cdot q_0 \simeq [1] \perp [0, 0] \perp a \cdot q'_0.$$

Since even-dimensional regular forms can be cancelled by [1, Folgerung, p. 160], it follows that

$$[1] \perp a \cdot q_0 \simeq [1] \perp a \cdot q'_0.$$

Multiplying each side by a, we obtain $q \simeq q'$. □

Remark 2.2. The reduced norm map of every quaternion F-algebra is surjective if $\operatorname{sd}_2 F \leq 2$: this is a special case of a theorem of Merkurjev–Suslin if $\operatorname{char} F \neq 2$ and of Gille if $\operatorname{char} F = 2$, see [12, Théorème 7]. Therefore, Theorem 2.1 applies in particular when $\operatorname{sd}_2 F \leq 2$.

The classification of hermitian forms over separable quadratic extensions or quaternion algebras is easily derived from Theorem 2.1 by means of a transfer argument due to Jacobson (see [21, p. 348]), as we proceed to show.

Suppose first K/F is a separable quadratic extension. Let $N(K/F) \subset F^\times$ be the image of the norm map $N_{K/F} \colon K^\times \to F^\times$. For every hermitian form h on a K-vector space V of dimension n, the discriminant $\operatorname{disc} h$ is defined by choosing a basis $(e_i)_{1 \leq i \leq n}$ of V and letting

$$\operatorname{disc} h = (-1)^{n(n-1)/2} \det\bigl(h(e_i, e_j)\bigr)_{1 \leq i,j \leq n} \cdot N(K/F) \in F^\times / N(K/F),$$

see [15, p. 114].

Corollary 2.3. *If the reduced norm map of every quaternion F-algebra is surjective (for instance if $\operatorname{sd}_2 F \leq 2$), then hermitian forms over K are classified up to isometry by their dimension and discriminant.*

Proof. For every hermitian form h on a K-vector space V, let $q_h\colon V \to F$ be the quadratic form defined by

$$q_h(x) = h(x,x) \qquad \text{for } x \in V.$$

If h and h' are hermitian forms such that

$$\dim h = \dim h', \qquad \operatorname{disc} h = \operatorname{disc} h',$$

then the formulas[3] in [21, p. 350] show that

$$\dim q_h = \dim q_{h'}, \qquad \operatorname{disc} q_h = \operatorname{disc} q_{h'}, \qquad c(q_h) = c(q_{h'}).$$

Therefore, Theorem 2.1 yields $q_h \simeq q_{h'}$, and it follows from [21, Theorem 10.1.1] that $h \simeq h'$. □

Now, let Q be a quaternion division F-algebra and let V be a finite-dimensional right Q-vector space. A hermitian form h on V (for the conjugation involution on Q) is *even* (or *trace-valued*) if $h(x,x) \in F$ for $x \in V$. If $\operatorname{char} F \neq 2$, every hermitian form is even since only F is fixed under the conjugation involution. If $\operatorname{char} F = 2$, even hermitian forms are called *alternating* or *even skew-hermitian* in [15, §4].

Corollary 2.4. *If the reduced norm map of every quaternion F-algebra is surjective (for instance if $\operatorname{sd}_2 F \leq 2$), then even hermitian forms over Q are classified up to isometry by their dimension.*

Proof. For every even hermitian form h on a Q-vector space V, let $q_h\colon V \to F$ be the quadratic form defined by

$$q_h(x) = h(x,x) \qquad \text{for } x \in V.$$

Using an orthogonal basis of h, it is easily seen that

$$\dim q_h = 4 \dim h, \qquad \operatorname{disc} q_h = \begin{cases} 1 & \text{if } \operatorname{char} F \neq 2, \\ 0 & \text{if } \operatorname{char} F = 2, \end{cases}$$

$$c(q_h) = \begin{cases} 0 & \text{if } \dim h \equiv 0 \bmod 2, \\ [Q] & \text{if } \dim h \equiv 1 \bmod 2. \end{cases}$$

Therefore, if h and h' are even hermitian forms with $\dim h = \dim h'$, then

$$\dim q_h = \dim q_{h'}, \qquad \operatorname{disc} q_h = \operatorname{disc} q_{h'}, \qquad c(q_h) = c(q_{h'}).$$

By Theorem 2.1, it follows that $q_h \simeq q_{h'}$. The same arguments as in [21, Theorem 10.1.1] then show that $h \simeq h'$. □

2.2. **Conjecture II for groups of type B_n.** The simply connected absolutely simple groups of type B_n are isomorphic to spin groups $\mathbf{Spin}(V,q)$, where (V,q) is a quadratic space of dimension $2n+1$. Let $\mathbf{O}^+(V,q)$ be the special orthogonal group of (V,q) and let $\chi\colon\ \mathbf{Spin}(V,q) \to \mathbf{O}^+(V,q)$ be the vector representation. We have an exact sequence (see for instance [15, §23])

$$(5) \qquad 1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(V,q) \xrightarrow{\chi} \mathbf{O}^+(V,q) \to 1.$$

[3]The formula for $c(q_h)$ in [21, p. 350] should be $(\delta, \operatorname{disc} h)$ instead of $(\delta, \det h)$.

It yields the following exact sequence in (flat) cohomology (see Appendix B):

$$\begin{aligned}\mathbf{O}^+(V,q)(F) \xrightarrow{\delta^0} H^1(F,\boldsymbol{\mu}_2) \to H^1(F,\mathbf{Spin}(V,q)) \xrightarrow{\chi^1} \\ \to H^1(F,\mathbf{O}^+(V,q)) \xrightarrow{\delta^1} H^2(F,\boldsymbol{\mu}_2).\end{aligned} \tag{6}$$

Recall from [15, §29.E] that the elements in $H^1(F,\mathbf{O}^+(V,q))$ can be identified with the isometry classes of quadratic forms q' on V such that $\operatorname{disc} q' = \operatorname{disc} q$. The following lemma dates back to Springer [24] for the case where $\operatorname{char} F \neq 2$. We include its short proof (in arbitrary characteristic) for the reader's convenience.

Lemma 2.5. *For $q' \in H^1(F,\mathbf{O}^+(V,q))$,*

$$\delta^1(q') = c(q') - c(q) \in {}_2\operatorname{Br}(F) = H^2(F,\boldsymbol{\mu}_2).$$

Proof. Consider the even Clifford algebra $C_0(V,q)$ and the associated group schemes $\mathbf{GL}_1(C_0(V,q))$ of invertible elements and $\mathbf{Aut}_F(C_0(V,q))$ of F-algebra automorphisms. By the Skolem-Noether theorem, there is an exact sequence

$$1 \to \mathbf{G}_{\mathrm{m}} \to \mathbf{GL}_1(C_0(V,q)) \xrightarrow{\mathrm{Int}} \mathbf{Aut}_F(C_0(V,q)) \to 1 \tag{7}$$

where the homomorphism Int maps every invertible element to the corresponding inner automorphism.

By the functorial property of Clifford algebras, every isometry in $\mathbf{O}^+(V,q)$ induces an automorphism of the even Clifford algebra $C_0(V,q)$, hence there is a commutative diagram relating (5) and (7), where the middle vertical map is the inclusion:

$$\begin{tikzcd} 1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d] & \mathbf{Spin}(V,q) \arrow[r, "\chi"] \arrow[d] & \mathbf{O}^+(V,q) \arrow[r] \arrow[d, "\rho"] & 1 \\ 1 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] & \mathbf{GL}_1(C_0(V,q)) \arrow[r, "\mathrm{Int}"] & \mathbf{Aut}(C_0(V,q)) \arrow[r] & 1. \end{tikzcd}$$

It yields the following commutative diagram, where the vertical map on the right is the inclusion ${}_2\operatorname{Br}(F) \hookrightarrow \operatorname{Br}(F)$:

$$\begin{tikzcd} H^1(F,\mathbf{O}^+(V,q)) \arrow[r, "\delta^1"] \arrow[d, "\rho^1"'] & H^2(F,\boldsymbol{\mu}_2) \arrow[d] \\ H^1\big(F,\mathbf{Aut}(C_0(V,q))\big) \arrow[r, "\partial^1"] & H^2(F,\mathbf{G}_{\mathrm{m}}). \end{tikzcd}$$

Elements in $H^1\big(F,\mathbf{Aut}_F(C_0(V,q))\big)$ can be identified with isomorphism classes of central simple F-algebras of the same degree as $C_0(V,q)$, and for $q' \in H^1(F,\mathbf{O}^+(V,q))$ the image $\rho^1(q')$ is represented by $C_0(V,q')$. Therefore,

$$\delta^1(q') = \partial^1\rho^1(q') = c(q') - c(q).$$

□

Corollary 2.6. *If $\operatorname{sd}_2 F \leq 2$, then $H^1(F,\mathbf{Spin}(V,q)) = 1$ for every quadratic space (V,q) of odd dimension.*

Proof. Consider the exact sequence (6). If $\operatorname{sd}_2 F \leq 2$, a theorem of Gille [13, Théorème 6] shows that δ^0 is surjective, hence $\ker \chi^1 = 1$. Therefore, it suffices to show that the image of χ^1 is trivial or, equivalently, that $\ker \delta^1 = 1$.

Lemma 2.5 shows that the elements in $\ker\delta^1$ are the isometry classes of quadratic forms q' on V such that $\operatorname{disc} q' = \operatorname{disc} q$ and $c(q') = c(q)$. By Theorem 2.1 and Remark 2.2 these forms are isometric to q if $\operatorname{sd}_2 F \leq 2$, hence $\ker\delta^1 = 1$. □

3. Unitary involutions

In this section, B is a central simple algebra over a field K of arbitrary characteristic, and τ is a unitary involution on B. We denote by F the subfield of τ-invariant elements in K. Our main goal is to show how the discriminant algebra $D(B,\tau)$ controls the hyperbolicity of τ when $\operatorname{sd}_2 F \leq 2$. We shall then proceed to prove classification results for hermitian forms over division algebras with unitary involution.

Lemma 3.1. *Suppose* $\deg B$ *is even.*

1. *If τ is hyperbolic, then the discriminant algebra $D(B,\tau)$ is split.*
2. *Let $M \subset B$ be a subfield of symmetric elements which is a separable extension of F, let $\tilde{B} = \operatorname{Cent}_B M$ be the centralizer of M and $\tilde{\tau} = \tau|_{\tilde{B}}$ be the restriction of τ to $\tilde{B}$. The algebra $D(B,\tau)$ is Brauer-equivalent to the norm (i.e. corestriction) of $D(\tilde{B},\tilde{\tau})$,*
$$[D(B,\tau)] = [N_{M/F}(D(\tilde{B},\tilde{\tau}))] \qquad \text{in } Br\ F.$$
3. *Let C be a central simple K-algebra and let θ be a unitary involution on C such that $\theta|_K = \tau|_K$. Then*
$$[D(B\otimes_K C, \tau\otimes\theta)] = \begin{cases} 0 & \text{if } \deg C \text{ is even,}\\ [D(B,\tau)] & \text{if } \deg C \text{ is odd.}\end{cases}$$

Proof. The first part is due to Tamagawa [26, Theorem 4]. We include a proof[4] for the reader's convenience.

As in the proof of Proposition 1.6, we may find a field $F(X)$ in which F is algebraically closed, which splits B and such that the scalar extension map $\operatorname{Br} F \to \operatorname{Br} F(X)$ is injective. Therefore, it suffices to prove the lemma in the case where B is split. Choose a representation
$$(B,\tau) = (\operatorname{End}_K V, \operatorname{ad}_h)$$
where h is a hermitian form on a K-vector space V. Then
$$[D(B,\tau)] = (K, \operatorname{disc} h)_F,$$
see [15, (10.35)]. If τ is hyperbolic, then $\operatorname{disc} h = 1$ hence $D(B,\tau)$ is split, and (1) is proved. If B contains a field M consisting of symmetric elements, then V carries a KM-vector space structure. Let $s\colon M \to F$ be a non-zero F-linear map and let $s_K\colon KM \to K$ be its K-linear extension to KM. By [15, (4.10)], we may assume h is the transfer of a hermitian form h' on the KM-vector space V with respect to the non-trivial automorphism of KM/M,
$$h = s_{K*}(h') \qquad \text{and} \qquad (\tilde{B},\tilde{\tau}) = (\operatorname{End}_{KM} V, \operatorname{ad}_{h'}).$$
Then $[D(\tilde{B},\tilde{\tau})] = (KM, \operatorname{disc} h')_M$. Since $\operatorname{disc} h = N_{M/F}(\operatorname{disc} h')$ by [21, Theorem 5.12, p. 51], the projection formula
$$N_{M/F}(KM, \operatorname{disc} h')_M = (K, N_{M/F}(\operatorname{disc} h'))_F$$

[4]Tamagawa's proof is different.

completes the proof of (2).

To prove (3), we may use the same scalar extension argument to reduce to the case where C is split as well as B. Choosing a representation

$$(C,\theta) = (\operatorname{End}_K W, \operatorname{ad}_k)$$

for some hermitian form k on a K-vector space W, we have

$$(B\otimes_K C, \tau\otimes\theta) = (\operatorname{End}_K(V\otimes_K W), \operatorname{ad}_{h\otimes k}).$$

With respect to K-bases of V and W, the matrix of $h\otimes k$ is a Kronecker product of the matrices of h and k. Therefore,

$$\operatorname{disc}(h\otimes k) = \begin{cases} 1 & \text{if } \dim_K W \text{ is even,}\\ \operatorname{disc} h & \text{if } \dim_K W \text{ is odd.}\end{cases}$$

Part (3) follows, since $[D(B\otimes_K C, \tau\otimes\theta)] = (K, \operatorname{disc}(h\otimes k))_F$. □

The main result of this section is the following:

Theorem 3.2. *Let B be a central simple K-algebra of even degree with a unitary involution τ, and let $F\subset K$ be the subfield fixed under τ. If τ is hyperbolic, then* $\operatorname{ind} B$ *divides* $\frac{1}{2}\deg B$ *and $D(B,\tau)$ is split. The converse holds if* $\operatorname{sd}_2 F\le 2$.

Proof. If τ is hyperbolic, it was already observed in Section 1.4 that $\operatorname{ind} B$ divides $\frac{1}{2}\deg B$. Moreover, $D(B,\tau)$ is split by Lemma 3.1, hence the first part is clear. In the next two subsections, we prove the second part by induction on $\operatorname{ind} B$.

For the convenience of exposition, we define the following property of a field F:

U(d): A unitary involution τ on a central simple algebra B of index d and even degree over a separable quadratic extension of F is hyperbolic if d divides $\frac{1}{2}\deg B$ and $D(B,\tau)$ is split.

We thus have to show that fields with $\operatorname{sd}_2 F\le 2$ satisfy **U(d)** for all $d\ge 1$.

3.1. **Reduction to 2-power index.** To achieve index reduction, we use the following result:

Lemma 3.3. *Let (B,τ) be as in Theorem 3.2.*

(1) *If* $\operatorname{ind} B$ *is divisible by an odd prime, then there is an odd-degree separable field extension E/F such that* $\operatorname{ind}(B\otimes_F E) < \operatorname{ind} B$.
(2) *If* $\operatorname{ind} B = 2^k$ *with $k\ge 1$, then there is an odd-degree separable field extension E/F and a separable quadratic extension L/E linearly disjoint from K such that* $\operatorname{ind}(B\otimes_F L) = 2^{k-1}$.

Proof. Let D be a central division K-algebra Brauer-equivalent to B, and let D_1, $D_2\subset D$ be central division K-algebras such that $D = D_1\otimes_K D_2$, with $\deg D_1$ a power of 2 (possibly $\deg D_1 = 1$) and $\deg D_2$ odd. Since B carries a unitary involution, a theorem of Albert–Riehm–Scharlau [15, (3.1)] shows that

$$[N_{K/F}(D)] = 0 \qquad \text{in } \operatorname{Br}(F),$$

hence $N_{K/F}(D_1)\otimes_F N_{K/F}(D_2)$ is split. Therefore, $N_{K/F}(D_1)$ and $N_{K/F}(D_2)$ are both split, since their degrees are coprime,

$$\deg N_{K/F}(D_1) = (\deg D_1)^2 \qquad \text{and} \qquad \deg N_{K/F}(D_2) = (\deg D_2)^2.$$

It follows by the Albert–Riehm–Scharlau theorem that D_1 and D_2 carry unitary involutions θ_1, θ_2 such that

$$F = K \cap \mathrm{Sym}(\theta_1) = K \cap \mathrm{Sym}(\theta_2).$$

In case (1), we have $D_2 \neq K$. The set of elements in D_2 which are separable of degree $\deg D_2$ over K is a Zariski-open subset U_2 defined by the condition that the discriminant of the reduced characteristic polynomial does not vanish. Scalar extension to an algebraic closure of F shows that the open subset $U_2 \cap \mathrm{Sym}(\theta_2)$ of $\mathrm{Sym}(\theta_2)$ is not empty. Since $D_2 \neq K$, the field F is infinite, hence the rational points in $\mathrm{Sym}(\theta_2)$ are dense. We may therefore find an element $x \in \mathrm{Sym}(\theta_2)$ which is separable of degree $\deg D_2$ over K. Since $\theta_2(x) = x$, the coefficients of the reduced characteristic polynomial of x are in F, hence $F(x)/F$ is a separable field extension of degree $\deg D_2$. Moreover, $K(x)$ is a maximal subfield of D_2, hence scalar extension to $F(x)$ splits D_2 and reduces the index of D. Part (1) of the lemma is thus proved, with $E = F(x)$.

In case (2), we have $D_2 = K$ and $D = D_1 \neq K$. Arguing as above, we may find $x \in \mathrm{Sym}(\theta_1)$ such that $F(x)/F$ is a separable field extension of degree $\deg D$. Let R be a Galois closure of $F(x)/F$ and let $E \subset R$ be the subfield fixed under a 2-Sylow subgroup of $\mathrm{Gal}(R/F)$. Then E/F is a separable field extension of odd degree, and R/E is a Galois extension whose Galois group is a 2-group. It follows that $E(x)/E$ is a separable 2-extension, hence we may find a subfield $L \subset E(x)$ such that L/E is a separable quadratic extension. Since the dimension of $K(x)/F$ is a power of 2, the extensions $K(x)/F$ and E/F are linearly disjoint. Moreover,

$$K(x) \otimes_F E = K \otimes_F F(x) \otimes_F E = K \otimes_F E(x),$$

so K/F is linearly disjoint from $E(x)/F$ hence also from L/F. As

$$L \subset F(x) \otimes_F E \subset D \otimes_F E,$$

we have $\mathrm{ind}(D \otimes_F L) = \frac{1}{2}\,\mathrm{ind}(D \otimes_F E)$. The conditions are thus fulfilled, since $\mathrm{ind}(D \otimes_F E) = \mathrm{ind}\, D = \mathrm{ind}\, B$ as the degree of E/F is prime to the degree of D. □

Corollary 3.4. *Let $d \geq 1$ be an integer. Suppose every separable odd-degree extension of a field F satisfies $\mathbf{U}(k)$ for every $k < d$. If d has an odd prime factor, then $\mathbf{U}(d)$ holds for F.*

Proof. Let τ be a unitary involution on a central simple K-algebra B of index d and even degree over a separable quadratic extension K of F, such that $\tau|_F = \mathrm{Id}_F$. Suppose d divides $\frac{1}{2}\deg B$ and $D(B,\tau)$ is split. By Lemma 3.3, there is a separable odd-degree extension E/F such that $\mathrm{ind}(B \otimes_F E) < d$. Since $\mathbf{U}(\mathrm{ind}(B \otimes_F E))$ holds for E, it follows that $\tau \otimes \mathrm{Id}_E$ is a hyperbolic involution on $B \otimes_F E$. By Theorem 1.13, τ is hyperbolic. □

3.2. **The 2-power index case.** In this subsection, we complete the proof of Theorem 3.2. The key tool is the following:

Lemma 3.5. *Let (B,τ) be as in Theorem 3.2. Suppose $\deg B \equiv 0 \bmod 4$, $\mathrm{ind}\, B$ divides $\frac{1}{2}\deg B$, and let $\deg B = 4n$. Assume that for every quaternion F-algebra Q split by K there exists a central simple K-algebra B_0 of even degree with unitary involution τ_0 such that*

$$[B_0] = [B] \quad \text{in } \mathrm{Br}\, K \qquad \text{and} \qquad [D(B_0,\tau_0)] = [Q] \quad \text{in } \mathrm{Br}\, F.$$

Assume moreover that there is a separable quadratic extension L/F, linearly disjoint from K, with non-trivial automorphism ι, such that

$$(L,\iota)\hookrightarrow(B,\tau)$$

and that the centralizer $\tilde{B}=\operatorname{Cent}_B L$ satisfies $n[\tilde{B}]=0$ in $\operatorname{Br}KL$. If $D(B,\tau)$ is split, then there exists a central simple K-algebra with unitary involution (B',τ') Witt-equivalent to (B,τ), of degree divisible by 4, endowed with an embedding $(L,\iota)\hookrightarrow(B',\tau')$, satisfying the following conditions:

1. *for the centralizer $\widetilde{B'}=\operatorname{Cent}_{B'}L$ and $\widetilde{\tau'}=\tau'|_{B'}$, the discriminant algebra $D(\widetilde{B'},\widetilde{\tau'})$ is split;*
2. *$\operatorname{ind}B'$ divides $\frac{1}{2}\deg B'$.*

Proof. The centralizer $\tilde{B}$ has center KL and degree $\frac{1}{2}\deg B=2n$. Let $M\subset KL$ be the subfield fixed under $\tilde{\tau}$. By [15, (10.30)] we have

$$[D(\tilde{B},\tilde{\tau})\otimes_M KL]=n[\tilde{B}]\qquad\text{in }\operatorname{Br}KL,$$

hence the hypotheses imply that $D(\tilde{B},\tilde{\tau})$ is split by KL. It is therefore Brauer-equivalent to a quaternion M-algebra Q_0 containing K. By Lemma 3.1(2), we have

$$[N_{M/F}(D(\tilde{B},\tilde{\tau}))]=[D(B,\tau)]\qquad\text{in }\operatorname{Br}F.$$

Since we assume that $D(B,\tau)$ is split, it follows by the Albert–Riehm–Scharlau theorem [15, (3.1)] that Q_0 has unitary involutions fixing F. By [15, (4.14)], we may find such a unitary involution which restricts to the non-trivial automorphism of K/F. A theorem of Albert [15, (2.22)] then yields a quaternion F-algebra Q containing K such that

$$Q_0\simeq Q\otimes_F M.$$

By hypothesis, we may find a central simple K-algebra B_0 of even degree with a unitary involution τ_0 such that B_0 is Brauer-equivalent to B and $D(B_0,\tau_0)$ to Q. Let J be the (unique) symplectic involution on $\operatorname{End}_F L$ and set

$$B_1=B_0\otimes_F\operatorname{End}_F L,\qquad \tau_1=\tau_0\otimes J.$$

Clearly, B_1 is Brauer-equivalent to B, and τ_1 is hyperbolic since J is hyperbolic. Through the regular representation $L\hookrightarrow\operatorname{End}_F L$, we may embed

$$(L,\iota)\hookrightarrow(\operatorname{End}_F L,J)\hookrightarrow(B_1,\tau_1).$$

Let $\widetilde{B_1}=\operatorname{Cent}_{B_1}L$ and $\widetilde{\tau_1}=\tau_1|_{\widetilde{B_1}}$. We have

$$\widetilde{B_1}=B_0\otimes_F L=B_0\otimes_F M\qquad\text{and}\qquad\widetilde{\tau_1}=\tau_0\otimes\iota=\tau_0\otimes\operatorname{Id}_M,$$

hence

$$[D(\widetilde{B_1},\widetilde{\tau_1})]=[D(B_0,\tau_0)\otimes_F M]=[Q\otimes_F M]=[D(\tilde{B},\tilde{\tau})].\tag{8}$$

Since B and B_1 are Brauer-equivalent and $\tau|_K=\tau_1|_K$, we may consider an orthogonal sum (B',τ') of (B,τ) and (B_1,τ_1), see Proposition 1.3. It is a central simple K-algebra with involution which is Witt-equivalent to (B,τ), since τ_1 is hyperbolic. The sum of the embeddings of (L,ι) in (B,τ) and (B_1,τ_1) is an embedding

$$(L,\iota)\hookrightarrow(B',\tau')$$

such that, using the ˜ notation for the centralizer,

$$(\widetilde{B'},\widetilde{\tau'})=(\tilde{B},\tilde{\tau})\boxplus(\widetilde{B_1},\widetilde{\tau_1}),$$

by Proposition 1.7. It follows from Proposition 1.6 and (8) that

$$[D(\widetilde{B'},\widetilde{\tau'})] = [D(\tilde{B},\tilde{\tau})] + [D(\widetilde{B_1},\widetilde{\tau_1})] = 0.$$

To complete the proof, we compute the degree of B',

$$\deg B' = \deg B + \deg B_1 = 4n + 2\deg B_0.$$

Since $\operatorname{ind} B' = \operatorname{ind} B = \operatorname{ind} B_0$ divides $\deg B_0$ and $2n$, it also divides $\frac{1}{2}\deg B'$. □

To enable us to use Lemma 3.5, we need examples where the hypothesis on the existence of B_0 holds.

Lemma 3.6. *Let (B,τ) be as in Theorem 3.2. Suppose $\operatorname{ind} B$ is a power of 2. If $\operatorname{sd}_2 F \le 2$, then for every quaternion F-algebra Q split by K there exists a central simple K-algebra B_0 of even degree and a unitary involution τ_0 on B_0 such that $\tau_0|_F = \operatorname{Id}_F$ and*

$$[B_0] = [B] \quad \text{in } \operatorname{Br} K, \qquad [D(B_0,\tau_0)] = [Q] \quad \text{in } \operatorname{Br} F.$$

Proof. Let $Q = (K,y)_F$ for some $y \in F^\times$. By a theorem of Gille [12, Théorème 7], the condition $\operatorname{sd}_2 F \le 2$ implies that the reduced norm map Nrd_B is surjective. Let $x \in B^\times$ be such that $\operatorname{Nrd}_B(x) = y$. A theorem of Yanchevskiĭ [28, Theorem 1] then shows that x lies in the subgroup of $B^\times$ generated by the invertible elements in $\operatorname{Sym}(\tau)$. Therefore, we may find invertible elements $x_1, \ldots, x_n \in \operatorname{Sym}(\tau)$ such that

$$\operatorname{Nrd}_B(x_1 \ldots x_n) = y.$$

Adjoining some $x_i = 1$ if necessary, we may assume n is even. Let $B_0 = M_n(B) = M_n(F) \otimes_F B$ and let $\theta = t \otimes \tau$, where t is the transpose involution on $M_n(F)$. Clearly, B_0 is Brauer-equivalent to B. Consider the following diagonal matrices in B_0:

$$\Delta_0 = \operatorname{diag}(x_1,\ldots,x_n), \qquad \Delta_1 = \operatorname{diag}(1,-1,\ldots,1,-1)$$

and the unitary involutions τ_1, τ_0 defined by

$$\tau_1(x) = \Delta_1\theta(x)\Delta_1^{-1}, \qquad \tau_0(x) = \Delta_0\tau_1(x)\Delta_0^{-1} \qquad \text{for } x \in B_0.$$

The involution τ_1 is hyperbolic, hence $D(B_0,\tau_1)$ is split, by Lemma 3.1(1). On the other hand, [15, (10.36)] yields

$$[D(B_0,\tau_0)] = [D(B_0,\tau_1)] + (K, \operatorname{Nrd}_{B_0}(\Delta_0))_F,$$

hence $[D(B_0,\tau_0)] = [Q]$. □

Corollary 3.7. *Fields F with $\operatorname{sd}_2 F \le 2$ satisfy $\mathbf{U}(2^k)$ for all $k \ge 0$.*

Proof. We argue by induction on k. The case $k = 0$ follows from Corollary 2.3. Let B be a central simple algebra with $\operatorname{ind} B = 2^k \ge 2$ over a separable quadratic extension K of a field F with $\operatorname{sd}_2 F \le 2$, and let τ be a unitary involution on B such that $\tau|_F = \operatorname{Id}_F$. Assume moreover $\deg B \equiv 0 \bmod 2^{k+1}$ and $D(B,\tau)$ is split. We have to show τ is hyperbolic.

By Lemma 3.3(2), we may find a separable odd-degree extension E of F and a separable quadratic extension L of E such that

$$\operatorname{ind}(B \otimes_F L) = 2^{k-1}.$$

By the induction hypothesis, $\tau \otimes \mathrm{Id}_L$ is hyperbolic. Let ι denote the non-trivial automorphism of L/E and let $B_E = B \otimes_F E$, $\tau_E = \tau \otimes \mathrm{Id}_E$. By Theorem 1.15, there is an embedding of E-algebras with involution

$$(L, \iota) \hookrightarrow (B_E, \tau_E).$$

The centralizer $\widetilde{B_E} = \mathrm{Cent}_{B_E} L$ is Brauer-equivalent to

$$B_E \otimes_E L = B \otimes_F L.$$

Therefore, $\operatorname{ind} \widetilde{B_E} = 2^{k-1}$ and

$$(\tfrac{1}{4} \deg B)[\widetilde{B_E}] = 0 \qquad \text{in } \operatorname{Br} KE.$$

Since $D(B_E, \tau_E) = D(B, \tau) \otimes_F E$ is split and $\operatorname{sd}_2 E \leq 2$, we may apply Lemmas 3.5 and 3.6 to find a central simple KE-algebra with unitary involution (B', τ') Witt-equivalent to (B_E, τ_E) and an embedding $(L, \iota) \hookrightarrow (B', \tau')$ such that $\operatorname{ind} B'$ divides $\frac{1}{2} \deg B'$ and $D(\widetilde{B'}, \widetilde{\tau'})$ is split. The algebra $\widetilde{B'}$ is Brauer-equivalent to $B' \otimes_E L$ hence also to $B \otimes_F L$ since B' is Brauer-equivalent to B_E. Therefore, $\operatorname{ind} \widetilde{B'} = 2^{k-1}$. Since $\operatorname{sd}_2 L \leq 2$, the induction hypothesis shows that $\widetilde{\tau'}$ is hyperbolic. We may therefore find an idempotent $e \in \widetilde{B'}$ such that $\widetilde{\tau'}(e) = 1 - e$. Since $\widetilde{B'} \subset B'$ and $\widetilde{\tau'} = \tau'|_{\widetilde{B'}}$, this idempotent lies in B' and satisfies $\tau'(e) = 1 - e$, hence τ' is hyperbolic. By Witt cancellation (Proposition 1.8), it follows that τ_E is hyperbolic, since (B', τ') is Witt-equivalent to (B_E, τ_E). Now, E/F is an odd-degree extension, so Theorem 1.13 shows that τ is hyperbolic. □

Conclusion of the proof of Theorem 3.2. To establish that fields with $\operatorname{sd}_2 F \leq 2$ satisfy $\mathbf{U}(d)$ for all $d \geq 1$, it now suffices to use induction and Corollaries 3.4 and 3.7: if d is a power of 2 the result follows from Corollary 3.7; if d is not a power of 2 then it has an odd prime factor, hence induction and Corollary 3.4 prove that $\mathbf{U}(d)$ holds for F. □

3.3. **Classification of hermitian forms.** In this subsection, we use Theorem 3.2 to obtain a classification result for hermitian forms over a division algebra with unitary involution.

Let D be a central division algebra over an arbitrary field K. Suppose D carries a unitary involution θ, and let $F \subset K$ be the subfield of θ-invariant elements in K. Let also $\iota = \theta|_K$ be the non-trivial automorphism of K/F. The discriminant $\operatorname{disc} h$ of a hermitian form h on a D-vector space V with respect to θ is defined as in the case where $D = K$ (see Section 2): considering a D-basis $(e_i)_{1 \leq i \leq n}$ of V, set $m = n \deg D = \deg \operatorname{End}_D V$ and

$$\operatorname{disc} h = (-1)^{m(m-1)/2} \operatorname{Nrd}_{M_n(D)}\big(h(e_i, e_j)\big)_{1 \leq i,j \leq n} \cdot N(K/F) \in F^\times / N(K/F).$$

Even though this element lies in the factor group $F^\times/N(K/F)$, the quaternion F-algebra $(K, \operatorname{disc} h)_F$ is well-defined.

Lemma 3.8. *If m is even,*

$$[D(\operatorname{End}_D V, \operatorname{ad}_h)] = \begin{cases} [(K, \operatorname{disc} h)_F] & \text{if } \dim_D V \text{ is even,} \\ [(K, (-1)^{m/2} \operatorname{disc} h)_F] + [D(D, \theta)] & \text{if } \dim_D V \text{ is odd.} \end{cases}$$

Proof. Using the basis $(e_i)_{1\leq i\leq n}$ of V, we may identify

$$\operatorname{End}_D V = M_n(D) = M_n(F)\otimes_F D.$$

Under this identification, the involution ad_h is given by

$$\operatorname{ad}_h(x) = \Delta^{-1}\cdot (t\otimes\theta)(x)\cdot\Delta \qquad \text{for } x\in M_n(D),$$

where t is the transpose involution on $M_n(F)$ and

$$\Delta = \big(h(e_i,e_j)\big)_{1\leq i,j\leq n}\in M_n(D).$$

By [15, (10.36)], it follows that

$$[D(\operatorname{End}_D V,\operatorname{ad}_h)] = [(K,\operatorname{Nrd}\Delta)_F] + [D(M_n(D),t\otimes\theta)].$$

The last term on the right side is computed via Lemma 3.1(3):

$$[D(M_n(D),t\otimes\theta)] = \begin{cases} [D(M_n(K),t\otimes\iota)] & \text{if } \deg D \text{ is odd and } n \text{ is even,}\\ 0 & \text{if } \deg D \text{ and } n \text{ are even,}\\ [D(D,\theta)] & \text{if } \deg D \text{ is even and } n \text{ is odd.}\end{cases}$$

To complete the proof, observe that

$$(M_n(K),t\otimes\iota)\simeq(\operatorname{End}_K K^n,\operatorname{ad}_{\langle 1,\dots,1\rangle}),$$

hence if n is even

$$[D(M_n(K),t\otimes\iota)] = [(K,(-1)^{n/2})_F].$$

□

Corollary 3.9. *Suppose* $\operatorname{sd}_2 F\leq 2$. *Let h, h' be hermitian forms on a D-vector space V with respect to θ. The forms h, h' are isometric if and only if* $\operatorname{disc} h = \operatorname{disc} h'$.

Proof. Since Witt cancellation holds for hermitian forms with respect to unitary involutions by [21, Corollary 7.9.2], it suffices to prove that $h\perp -h'$ is hyperbolic if and only if $\operatorname{disc} h = \operatorname{disc} h'$.

A computation yields

$$\operatorname{disc}(h\perp -h') = \operatorname{disc} h\operatorname{disc} h',$$

hence Lemma 3.8 shows that $\operatorname{disc} h = \operatorname{disc} h'$ if and only if

$$[D(\operatorname{End}_D(V\oplus V),\operatorname{ad}_{h\perp -h'})] = 0.$$

By Theorem 3.2, this equation holds if and only if $\operatorname{ad}_{h\perp -h'}$ is hyperbolic, i.e. $h\perp -h'$ is hyperbolic. □

3.4. **Classification of unitary involutions.**

Corollary 3.10. *Let B be a central simple algebra over an arbitrary field K, and let τ, τ' be unitary involutions on B such that $\tau|_K = \tau'|_K$. Let $F\subset K$ be the fixed subfield of $\tau|_K$ and $\tau'|_K$. Suppose* $\operatorname{sd}_2 F\leq 2$. *If* $\deg B$ *is odd, then τ and τ' are conjugate. If* $\deg B$ *is even, then τ and τ' are conjugate if and only if* $D(B,\tau)\simeq D(B,\tau')$.

Proof. Choose a representation $B = \operatorname{End}_D V$ for some vector space V over a central division K-algebra D, and a unitary involution θ on D fixing F. By [15, (4.2)], there are hermitian forms h, h' on V with respect to θ such that

$$\tau = \operatorname{ad}_h, \qquad \tau' = \operatorname{ad}_{h'} .$$

If $\deg B$ is even, Lemma 3.8 shows that $D(B,\tau) \simeq D(B,\tau')$ if and only if $\operatorname{disc} h = \operatorname{disc} h'$. This condition is equivalent to $h \simeq h'$ by Corollary 3.9. Therefore, it implies that τ and τ' are conjugate.

If $\deg B$ is odd, let δ, $\delta' \in F^\times$ be such that

$$\operatorname{disc} h = \delta N(K/F), \qquad \operatorname{disc} h' = \delta' N(K/F).$$

Then $\operatorname{disc}(\delta h) = \operatorname{disc}(\delta' h') = 1$, hence $\delta h \simeq \delta' h'$ by Corollary 3.9. On the other hand, $\operatorname{ad}_{\delta h} = \operatorname{ad}_h = \tau$ and $\operatorname{ad}_{\delta' h'} = \operatorname{ad}_{h'} = \tau'$, so τ and τ' are conjugate. □

3.5. **Conjecture II for groups of outer type A_n.** Every simply connected absolutely simple group of type ${}^2A_n$ over a field F is isomorphic to $\mathbf{SU}(B,\tau)$ for some central simple algebra B of degree $n+1$ over a quadratic extension K of F and some unitary involution τ on B fixing F. Denote by $\boldsymbol{\mu}$ the center of $\mathbf{SU}(B,\tau)$. There is an exact sequence

$$1 \to \boldsymbol{\mu} \to \mathbf{SU}(B,\tau) \xrightarrow{\pi} \mathbf{PGU}(B,\tau) \to 1$$

inducing the following exact sequence in (flat) cohomology:

$$\mathbf{PGU}(B,\tau)(F) \xrightarrow{\delta^0} H^1(F,\boldsymbol{\mu}) \to H^1(F,\mathbf{SU}(B,\tau)) \xrightarrow{\pi^1} H^1(F,\mathbf{PGU}(B,\tau)). \tag{9}$$

Recall from [15, §29.D] that $H^1(F,\mathbf{PGU}(B,\tau))$ can be identified with the set of isomorphism classes of triples (B',τ',φ) consisting of a central simple algebra B' of degree $n+1$ over a quadratic extension K' of F, a unitary involution τ' on B' fixing F and an F-algebra isomorphism $\varphi\colon K' \to K$.

Lemma 3.11. *If n is even, the image of π^1 is the set of isomorphism classes of triples $(B,\tau',\operatorname{Id}_K)$ where τ' is a unitary involution on B fixing F. If n is odd, the image of π^1 is the set of isomorphism classes of triples $(B,\tau',\operatorname{Id}_K)$ where τ' is a unitary involution such that $D(B,\tau') \simeq D(B,\tau)$.*

Proof. By [15, §29.D], the set $H^1(F,\mathbf{SU}(B,\tau))$ is in one-to-one correspondence with the set of equivalence classes

$$\{(s,z) \in \operatorname{Sym}(\tau) \times K^\times \mid \operatorname{Nrd}_B(s) = N_{K/F}(z)\}/{\approx}$$

where $(s,z) \approx (s',z')$ if and only if there exists $b \in B^\times$ such that

$$s' = bs\tau(b) \qquad \text{and} \qquad z' = \operatorname{Nrd}_B(b)z.$$

The image under π^1 of the class of (s,z) is represented by the triple $(B,\tau',\operatorname{Id}_K)$ where $\tau' = \operatorname{Int}(s)\circ\tau$. If n is odd, [15, (10.36)] yields

$$[D(B,\tau')] = [D(B,\tau)] + [(K,\operatorname{Nrd}_B(s))_F] \qquad \text{in } \operatorname{Br}(F). \tag{10}$$

Since $\operatorname{Nrd}_B(s) = N_{K/F}(z)$, the last term vanishes and therefore $D(B,\tau') \simeq D(B,\tau)$. The image of π^1 is therefore contained in the set described in the statement of the lemma.

To prove the reverse inclusion, let τ' be an arbitrary unitary involution on B fixing F. By [15, (2.18)], there exists a unit $s \in \operatorname{Sym}(\tau)$ such that $\tau' = \operatorname{Int}(s)\circ\tau$. If n is odd and $D(B,\tau') \simeq D(B,\tau)$, (10) shows that the quaternion algebra

$(K, \mathrm{Nrd}_B(s))_F$ splits, hence there exists $z \in K^\times$ such that $\mathrm{Nrd}_B(s) = N_{K/F}(z)$. The isomorphism class of $(B, \tau', \mathrm{Id}_K)$ is the image under π^1 of the class of (s, z).

If n is even, we have

$$\mathrm{Nrd}_B(\mathrm{Nrd}_B(s)s) = \mathrm{Nrd}_B(s)^{n+2} = N_{K/F}(\mathrm{Nrd}_B(s)^{(n/2)+1}).$$

The isomorphism class of $(B, \tau', \mathrm{Id}_K)$ is the image under π' of the class of the pair $(\mathrm{Nrd}_B(s)s, \mathrm{Nrd}_B(s)^{(n/2)+1})$. □

Corollary 3.12. *If $\mathrm{sd}_p F \leq 2$ for $p = 2$ and for every prime factor of $n+1$, then $H^1(F, \mathbf{SU}(B, \tau)) = 1$.*

Proof. A theorem of Gille [13, Théorème 6] shows that the map δ^0 in (9) is surjective, hence $\ker \pi^1 = 1$. On the other hand, Corollary 3.10 and Lemma 3.11 show that $\mathrm{im}\, \pi^1 = 1$. □

4. Symplectic involutions

4.1. **Classification.** The main result of this section is the following:

Theorem 4.1. *Let A be a central simple algebra of even degree over an arbitrary field F, and let σ be a symplectic involution on A. If σ is hyperbolic, then $\mathrm{ind}\, A$ divides $\frac{1}{2}\deg A$. The converse holds if $\mathrm{sd}_2 F \leq 2$.*

The fact that $\mathrm{ind}\, A$ divides $\frac{1}{2}\deg A$ if σ is hyperbolic was already observed in Section 1.4. The proof of the converse follows the same general pattern as Theorem 3.2, which is the corresponding result for unitary involutions: we use induction on $\mathrm{ind}\, A$ to restrict to the centralizer of a quadratic extension, and thus reduce to the unitary case. The main technical tool is the following lemma:

Lemma 4.2. *Let (A, σ) be a central simple F-algebra with symplectic involution. Assume $\deg A \equiv 0 \bmod 8$ and A contains a separable quadratic extension L of F such that $\sigma|_L$ is the non-trivial automorphism ι of L/F, i.e.*

$$(L, \iota) \hookrightarrow (A, \sigma).$$

If the reduced norm map $\mathrm{Nrd}_A \colon A^\times \to F^\times$ is surjective, there exists a central simple F-algebra with symplectic involution (A', σ') and an embedding $(L, \iota) \hookrightarrow (A', \sigma')$ with the following properties:

1. *$\deg A' \equiv 0 \bmod 4$;*
2. *(A', σ') is Witt-equivalent to (A, σ);*
3. *for $\widetilde{A'} = \mathrm{Cent}_{A'} L$ and $\widetilde{\sigma'} = \sigma'|_{\widetilde{A'}}$, the discriminant algebra $D(\widetilde{A'}, \widetilde{\sigma'})$ is split.*

Moreover, if $\mathrm{ind}\, A$ divides $\frac{1}{2}\deg A$, then $\mathrm{ind}\, A'$ divides $\frac{1}{2}\deg A'$.

Proof. Let $\tilde{A} = \mathrm{Cent}_A L$ and $\tilde{\sigma} = \sigma|_{\tilde{A}}$. We have $\deg \tilde{A} = \frac{1}{2}\deg A$, so $\deg \tilde{A}$ is even. We may thus consider the discriminant algebra $D(\tilde{A}, \tilde{\sigma})$. By [15, (10.30)],

$$[D(\tilde{A}, \tilde{\sigma}) \otimes_F L] = (\tfrac{1}{2}\deg \tilde{A})[\tilde{A}],$$

hence the hypothesis that $\deg A \equiv 0 \bmod 8$ implies that $D(\tilde{A}, \tilde{\sigma})$ is split by L. Therefore, we may find $y \in F^\times$ such that

$$[D(\tilde{A}, \tilde{\sigma})] = (L, y)_F. \tag{11}$$

Let D be a central division F-algebra which is Brauer-equivalent to A and let ρ be an orthogonal involution on D. Since $\mathrm{Nrd}(A^\times) = \mathrm{Nrd}(D^\times) = F^\times$, there exists

$x \in D^\times$ such that $\mathrm{Nrd}_D(x) = y$. The following dimension count argument due to Dieudonné [15, p. 266] shows that x is the product of two symmetric elements: since $\dim \mathrm{Sym}(\rho) > \frac{1}{2}\dim D$,

$$\mathrm{Sym}(\rho) \cap (x\,\mathrm{Sym}(\rho)) \neq \{0\},$$

hence we may find non-zero elements $s_1,\ s_2 \in \mathrm{Sym}(\rho)$ such that $s_1 = xs_2$, i.e. $x = s_1 s_2^{-1}$.

Let $\Delta = \mathrm{diag}(s_1, -s_2^{-1}) \in M_2(D) = M_2(F) \otimes_F D$. Define an orthogonal involution σ_1 on $M_2(D)$ by

$$\sigma_1(x) = \Delta \cdot (t \otimes \rho)(x) \cdot \Delta^{-1} \qquad \text{for } x \in M_2(D),$$

where t is the transpose involution on $M_2(F)$. By [15, (7.3)],

$$\mathrm{disc}\,\sigma_1 = \mathrm{Nrd}_{M_2(D)}(\Delta) F^{\times 2} = \mathrm{Nrd}_D(s_1 s_2^{-1}) F^{\times 2} = yF^{\times 2}. \tag{12}$$

Let J be the (unique) symplectic involution on $\mathrm{End}_F L \simeq M_2(F)$. Let also $A_0 = M_2(D) \otimes_F \mathrm{End}_F L$ and $\sigma_0 = \sigma_1 \otimes J$, a symplectic involution on A_0 which is hyperbolic since J is hyperbolic. The regular representation $L \to \mathrm{End}_F L$ is an embedding

$$(L, \iota) \hookrightarrow (\mathrm{End}_F L, J) \hookrightarrow (A_0, \sigma_0).$$

Using the $\tilde{}$ notation for the centralizer, we have

$$\widetilde{A_0} = M_2(D) \otimes_F L \qquad \text{and} \qquad \widetilde{\sigma_0} = \sigma_1 \otimes \iota.$$

Therefore, [15, (10.33)] yields

$$[D(\widetilde{A_0}, \widetilde{\sigma_0})] = (L, \mathrm{disc}\,\sigma_1)_F + (\tfrac{1}{2}\deg \widetilde{A_0})[D].$$

Since $\deg A_0 = 4 \deg D$, the last term on the right side vanishes. By (11) and (12), it follows that

$$[D(\widetilde{A_0}, \widetilde{\sigma_0})] = [D(\tilde{A}, \tilde{\sigma})]. \tag{13}$$

Since A_0 and A are Brauer-equivalent and σ_0, σ have the same type, we may consider an orthogonal sum

$$(A', \sigma') = (A, \sigma) \boxplus (A_0, \sigma_0),$$

see Proposition 1.3. This sum is Witt-equivalent to (A, σ) since σ_0 is hyperbolic, and

$$\deg A' = \deg A + \deg A_0 \equiv 0 \bmod 4. \tag{14}$$

The direct sum of the embeddings of L in A and A_0 is an embedding

$$(L, \iota) \hookrightarrow (A', \sigma')$$

such that, by Proposition 1.7,

$$(\widetilde{A'}, \widetilde{\sigma'}) = (\tilde{A}, \tilde{\sigma}) \boxplus (\widetilde{A_0}, \widetilde{\sigma_0}).$$

By Proposition 1.6 and (13),

$$[D(\widetilde{A'}, \widetilde{\sigma'})] = [D(\tilde{A}, \tilde{\sigma})] + [D(\widetilde{A_0}, \widetilde{\sigma_0})] = 0.$$

Since $\mathrm{ind}\,A = \mathrm{ind}\,A'$ and $\deg A_0 = 4 \deg D = 4\,\mathrm{ind}\,A$, it follows from (14) that $\mathrm{ind}\,A'$ divides $\frac{1}{2}\deg A'$ if $\mathrm{ind}\,A$ divides $\frac{1}{2}\deg A$. $\square$

Proof of Theorem 4.1. We argue by induction on $\operatorname{ind} A$, considering separately the cases where $\operatorname{ind} A = 1$ or 2. If $\operatorname{ind} A = 1$ the result is clear (without any hypothesis on $\operatorname{sd}_2 F$) since symplectic involutions on split algebras are hyperbolic. If $\operatorname{ind} A = 2$, then A is Brauer-equivalent to a quaternion division F-algebra D, and we may choose a representation

$$(A, \sigma) = (\operatorname{End}_D V, \operatorname{ad}_h)$$

where h is an even hermitian form on the even-dimensional D-vector space V with respect to the conjugation involution on D. By Corollary 2.4, h (hence also σ) is hyperbolic.

Now, suppose $\operatorname{ind} A = 2^k$ with $k \geq 2$. Since $\operatorname{ind} A$ divides $\frac{1}{2}\deg A$, it follows that $\deg A \equiv 0 \bmod 8$. We show that σ is hyperbolic by induction on k.

The same arguments as in Lemma 3.3(2) (or [4, Lemma 3.3.3]) yield a separable odd-degree extension E/F and a separable quadratic extension L/E such that $\operatorname{ind}(A \otimes_F L) = 2^{k-1}$. By the induction hypothesis, the involution $\sigma_L = \sigma \otimes \operatorname{Id}_L$ on $A_L = A \otimes_F L$ is hyperbolic. Therefore, Theorem 1.15 yields an embedding of E-algebras with involution

$$(L, \iota) \hookrightarrow (A_E, \sigma_E).$$

Since $\operatorname{sd}_2 F \leq 2$, a theorem of Gille [12, Théorème 7] shows that the reduced norm map on A_E is surjective. We may therefore apply Lemma 4.2 to find a central simple E-algebra with symplectic involution (A', σ') which is Witt-equivalent to (A_E, σ_E) and such that $D(\widetilde{A'}, \widetilde{\sigma'})$ is split. Moreover, $\operatorname{ind} A' = \operatorname{ind} A_E = 2^k$ divides $\frac{1}{2}\deg A'$, so $\deg A' \equiv 0 \bmod 2^{k+1}$ and $\deg \widetilde{A'} \equiv 0 \bmod 2^k$. On the other hand, since $\widetilde{A'}$ is Brauer-equivalent to $A' \otimes_F L$, we have $\operatorname{ind} \widetilde{A'} = 2^{k-1}$, hence $\operatorname{ind} \widetilde{A'}$ divides $\frac{1}{2}\deg \widetilde{A'}$. By Theorem 3.2, it follows that $\widetilde{\sigma'}$ is hyperbolic, hence there is an idempotent $e \in \widetilde{A'}$ such that $\widetilde{\sigma'}(e) = 1 - e$. This idempotent also lies in A' and satisfies $\sigma'(e) = 1 - e$, hence σ' is hyperbolic. As (A_E, σ_E) is Witt-equivalent to (A', σ'), the involution σ_E also is hyperbolic. Theorem 1.13 then shows that σ is hyperbolic, since the degree of E/F is odd. □

We next apply Theorem 4.1 to obtain a classification of skew-hermitian forms and symplectic involutions.

Let D be a central division algebra over an arbitrary field F, and let θ be an orthogonal involution on D. Recall from [15, (4.2)] that the adjoint involutions of alternating skew-hermitian forms with respect to θ are symplectic.

Corollary 4.3. *Suppose $\operatorname{sd}_2 F \leq 2$ and let V be a D-vector space. Any two alternating skew-hermitian forms h, h' on V with respect to θ are isometric.*

Proof. Since Witt cancellation holds for alternating skew-hermitian forms by [21, Corollary 7.9.2], it suffices to prove that $h \perp -h'$ is hyperbolic or, equivalently, that $\operatorname{ad}_{h \perp -h'}$ is hyperbolic. This follows from Theorem 4.1 since $\operatorname{ind} \operatorname{End}_D(V \oplus V) = \deg D$ divides $\frac{1}{2}\deg \operatorname{End}_D(V \oplus V) = \dim_D V \deg D$. □

Corollary 4.4. *Suppose $\operatorname{sd}_2 F \leq 2$. Any two symplectic involutions σ, σ' on a central simple F-algebra A are conjugate.*

Proof. Choose a representation $A = \operatorname{End}_D V$ for some vector space V over a central division F-algebra D. Let also θ be an orthogonal involution on D. Then $\sigma = \operatorname{ad}_h$ and $\sigma' = \operatorname{ad}_{h'}$ for some alternating skew-hermitian forms h, h' on V by [15, (4.2)]. Corollary 4.3 shows that $h \simeq h'$, hence σ and σ' are conjugate. □

4.2. **Conjecture II for groups of type C_n.** Every simply connected absolutely simple group of type C_n over a field F is isomorphic to the symplectic group $\mathbf{Sp}(A,\sigma)$ for some central simple F-algebra A of degree $2n$ and some symplectic involution σ on A. There is an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Sp}(A,\sigma) \xrightarrow{\pi} \mathbf{PGSp}(A,\sigma) \to 1$$

and an induced exact sequence in (flat) cohomology:

$$\mathbf{PGSp}(A,\sigma)(F) \xrightarrow{\delta^0} H^1(F,\boldsymbol{\mu}_2) \to H^1(F,\mathbf{Sp}(A,\sigma)) \xrightarrow{\pi^1} H^1(F,\mathbf{PGSp}(A,\sigma)).$$

Corollary 4.5. *If* $\operatorname{sd}_2 F \leq 2$*, then* $H^1(F,\mathbf{Sp}(A,\sigma)) = 1$*.*

Proof. By a theorem of Gille [13, Théorème 6], the map δ^0 is surjective, hence $\ker \pi^1 = 1$. To complete the proof, we show $\operatorname{im}\pi^1 = 1$.

Note that π factors through the inclusion

$$\mathbf{Sp}(A,\sigma) \hookrightarrow \mathbf{GSp}(A,\sigma).$$

By [15, (29.23)], the set $H^1(F,\mathbf{GSp}(A,\sigma))$ can be identified with the set of conjugacy classes of symplectic involutions on A. Corollary 4.4 then yields

$$H^1(F,\mathbf{GSp}(A,\sigma)) = 1,$$

hence $\operatorname{im}\pi^1 = 1$. $\square$

5. Quadratic pairs

5.1. **Classification.** For the next theorem, recall from [15, (8.10)] that the discriminant of a quadratic pair (σ,f) on a central simple F-algebra A determines the center of the Clifford algebra $C(A,\sigma,f)$. In particular, if the discriminant is trivial, then $C(A,\sigma,f)$ decomposes as a direct product of two central simple F-algebras,

$$C(A,\sigma,f) \simeq C^+ \times C^-.$$

Theorem 5.1. *Let F be an arbitrary field and let (σ,f) be a quadratic pair on a central simple F-algebra of even degree. If (σ,f) is hyperbolic, then* $\operatorname{ind} A$ *divides* $\frac{1}{2}\deg A$*,* $\operatorname{disc}(\sigma,f)$ *is trivial and one (at least) of the components $C^{\pm}$ of $C(A,\sigma,f)$ is split. The converse holds if* $\operatorname{sd}_2 F \leq 2$*.*

Proof. If (σ,f) is hyperbolic, then $\operatorname{ind} A$ divides $\frac{1}{2}\deg A$ as was observed in Section 1.4. Moreover, $\operatorname{disc}(\sigma,f)$ is trivial and one of the components of $C(A,\sigma,f)$ is split by [15, (8.31)].

To prove the converse, we argue by induction on $\operatorname{ind} A$. If $\operatorname{ind} A = 1$, then A is split and the theorem follows from Theorem 2.1. For the rest of the proof, assume $\operatorname{ind} A = 2^k$ with $k \geq 1$, hence $\deg A$ is divisible by 2^{k+1}. Arguing as in Lemma 3.3(2) (or by [4, Lemma 3.3.3]), we may find a separable odd-degree extension E/F and a separable quadratic extension L/E such that $\operatorname{ind} A_L = 2^{k-1}$. Note that $\operatorname{ind} A_E = 2^k$ since the degree of E/F is odd, hence A_E is not split. By the induction hypothesis, the quadratic pair (σ_L, f_L) is hyperbolic. Therefore, Theorem 1.16 yields an embedding $\varepsilon\colon L \hookrightarrow A_E$ such that

$$\operatorname{Trd}_{A_E}(\varepsilon(\ell)x) = T_{L/E}(\ell) f_E(x) \qquad \text{for all } x \in \operatorname{Sym}(\sigma_E) \text{ and } \ell \in L.$$

Let $\tilde{A} = \operatorname{Cent}_{A_E} L$ be the centralizer of $\varepsilon(L)$ and let $\tilde{\sigma} = \sigma|_{\tilde{A}}$, a unitary involution on $\tilde{A}$. Since $k \geq 1$ and 2^{k+1} divides $\deg A$, we have $\deg A \equiv 0 \bmod 4$. It follows from [11, Lemma 4.5][5] that, substituting for (A, σ, f) a Witt-equivalent algebra with quadratic pair if necessary, we may assume that $D(\tilde{A}, \tilde{\sigma})$ is split. Since $\operatorname{ind} \tilde{A} = \operatorname{ind} A_L = 2^{k-1}$ divides $\frac{1}{2} \deg \tilde{A} = \frac{1}{4} \deg A$ and $\operatorname{sd}_2 E \leq 2$, Theorem 3.2 shows that $\tilde{\sigma}$ is hyperbolic. Therefore (σ_E, f_E) is hyperbolic by [11, Lemma 4.3], hence (σ, f) is hyperbolic by Theorem 1.14. □

Following the pattern of Sections 3 and 4, we now derive from Theorem 5.1 the classification of generalized quadratic forms.

Let D be a central division algebra over an arbitrary field F, and let V be a D-vector space. Assume $\deg D \dim_D V$ is even, and consider the factor group

$$B_D = \operatorname{Br}(F)/\{0, [D]\}.$$

For generalized quadratic forms q, q' on V with $\operatorname{disc} q = \operatorname{disc} q'$, the quadratic pair (σ, f) adjoint to $q \perp -q'$ has trivial discriminant, hence

$$C(\operatorname{End}_D(V \oplus V), \sigma, f) = C^+ \times C^-$$

for some central simple F-algebras C^+, C^-. Since $\deg D \dim_D V$ is even, it follows that $\deg \operatorname{End}_D(V \oplus V) \equiv 0 \bmod 4$, hence by [15, (9.12)]

$$2[C] = 2[C'] = 0 \qquad \text{and} \qquad [C^+] + [C^-] = [D] \quad \text{in } \operatorname{Br}(F).$$

Therefore, $[C^+]$ and $[C^-]$ have the same image in B_D, and we may set

$$c(q, q') = \text{image of } [C^+] \text{ or } [C^-] \text{ in } B_D.$$

This relative invariant of q and q' was first defined by Bartels through Galois cohomology under the assumption that $\operatorname{char} F \neq 2$, see [2, Section 7]. The definition above comes from [11, Section 2].

Corollary 5.2. *Suppose* $\operatorname{sd}_2 F \leq 2$. *The forms q, q' are isometric if and only if* $\operatorname{disc} q = \operatorname{disc} q'$ *and* $c(q, q') = 0$.

Proof. The forms q, q' are isometric if and only if the quadratic pair (σ, f) adjoint to $q \perp -q'$ is hyperbolic. By Theorem 5.1, this condition holds if and only if one (at least) of the components $C^{\pm}$ is split, i.e. $c(q, q') = 0$. □

Corollary 5.3. *Suppose* $\operatorname{sd}_2 F \leq 2$. *Quadratic pairs (σ, f), (σ', f') on a central simple F-algebra A of even degree are conjugate if and only if $C(A, \sigma, f) \simeq C(A, \sigma', f')$ as F-algebras.*

Proof. The "only if" part is clear. We only sketch the arguments for the "if" part and refer to [16, Proposition 6] and [11, Theorem 5.3] for details. Choose a representation $A = \operatorname{End}_D V$ and quadratic forms q, q' on V whose adjoint quadratic pairs are (σ, f) and (σ', f'). For any $\lambda \in F^{\times}$, let $(\sigma_\lambda, f_\lambda)$ be the quadratic pair on $M_2(A) = \operatorname{End}_D(V \oplus V)$ adjoint to $q \perp \langle -\lambda \rangle q'$. Since $C(A, \sigma, f) \simeq C(A, \sigma', f')$, we have $\operatorname{disc} q = \operatorname{disc} q'$, hence there are central simple F-algebras C_λ^+, C_λ^- such that

$$C(M_2(A), \sigma_\lambda, f_\lambda) \simeq C_\lambda^+ \times C_\lambda^-.$$

The hypothesis $C(A, \sigma, f) \simeq C(A, \sigma', f')$ further implies that the center Z of $C(A, \sigma, f)$ splits one of C_λ^+, C_λ^-. If it splits C_λ^+, then there exists $\mu \in F^{\times}$ such that

$$[C_\lambda^+] = [(Z, \mu)_F].$$

[5]The arguments in §4 of [11] hold without change in characteristic different from 2.

Then $C^+_{\lambda\mu}$ splits, hence $c(q,\langle\lambda\mu\rangle q')=0$ and it follows from Corollary 5.2 that $q\simeq\langle\lambda\mu\rangle q'$. Since (σ',f') is also adjoint to $\langle\lambda\mu\rangle q'$, we conclude that (σ,f) and (σ',f') are conjugate. □

5.2. **Conjecture II for groups of type D_n.** The simply connected absolutely simple groups of type D_n (trialitarian D_4 excluded) over a field F are isomorphic to spin groups $\mathbf{Spin}(A,\sigma,f)$ where A is a central simple F-algebra of degree $2n$ and (σ,f) is a quadratic pair on A. Let $\boldsymbol{\mu}$ be the center of $\mathbf{Spin}(A,\sigma,f)$. There is an exact sequence (see [15, §31.A])

$$1\to\boldsymbol{\mu}\to\mathbf{Spin}(A,\sigma,f)\xrightarrow{\chi}\mathbf{PGO}^+(A,\sigma,f)\to 1 \tag{15}$$

and an induced exact sequence in (flat) cohomology

$$\begin{aligned}\mathbf{PGO}^+(A,\sigma,f)(F)&\xrightarrow{\delta^0}H^1(F,\boldsymbol{\mu})\to\\ &\to H^1(F,\mathbf{Spin}(A,\sigma,f))\xrightarrow{\chi^1}H^1(F,\mathbf{PGO}^+(A,\sigma,f)).\end{aligned} \tag{16}$$

Let Z be the center of the Clifford algebra $C(A,\sigma,f)$. Recall from [15, §29.F] that the elements in $H^1(F,\mathbf{PGO}^+(A,\sigma,f))$ can be identified with the F-isomorphism classes of 4-tuples (A',σ',f',φ) consisting of a central simple F-algebra A' of degree $2n$, a quadratic pair (σ',f') on A' and an isomorphism $\varphi\colon Z'\to Z$ from the center Z' of $C(A',\sigma',f')$ to Z.

Lemma 5.4. *If (A',σ',f',φ) represents an element in the image of χ^1, then $A'\simeq A$ and φ extends to an F-algebra isomorphism $C(A',\sigma',f')\simeq C(A,\sigma,f)$.*

Proof. As in the proof of Lemma 2.5, we relate the exact sequence (15) to the exact sequence derived from the Skolem-Noether theorem, using the homomorphism ρ arising from the functorial property of Clifford algebras:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu} & \longrightarrow & \mathbf{Spin}(A,\sigma,f) & \xrightarrow{\chi} & \mathbf{PGO}^+(A,\sigma,f) & \longrightarrow & 1\\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\rho} & & \\ 1 & \longrightarrow & \mathbf{GL}_1(Z) & \longrightarrow & \mathbf{GL}_1(C(A,\sigma,f)) & \xrightarrow{\mathrm{Int}} & \mathbf{Aut}_Z(C(A,\sigma,f)) & \longrightarrow & 1.\end{array}$$

There is an induced commutative diagram

$$\begin{array}{ccc} H^1(F,\mathbf{PGO}^+(A,\sigma,f)) & \xrightarrow{\delta^1} & H^2(F,\boldsymbol{\mu})\\ {\scriptstyle\rho^1}\downarrow & & \downarrow\\ H^1\big(F,\mathbf{Aut}_Z(C(A,\sigma,f))\big) & \xrightarrow{\partial^1} & H^2(F,\mathbf{GL}_1(Z)).\end{array}$$

If (A',σ',f',φ) represents an element in $\operatorname{im}\chi^1=\ker\delta^1$, then

$$\partial^1\circ\rho^1(A',\sigma',f',\varphi)=1. \tag{17}$$

To compute the left side, recall from [15, (29.13)] that $H^1\big(F,\mathbf{Aut}_Z(C(A,\sigma,f))\big)$ can be identified with the set of F-isomorphism classes of pairs (C,ψ) consisting of an F-algebra C and an F-algebra embedding $\psi\colon Z\hookrightarrow C$ which become isomorphic over the separable closure of F to the pair $(C(A,\sigma,f),i)$ where $i\colon Z\hookrightarrow C(A,\sigma,f)$ is the inclusion. Under this identification, $\rho^1(A',\sigma',f',\varphi)$ is represented by the pair $(C(A',\sigma',f'),\varphi^{-1})$. On the other hand, $\mathbf{GL}_1(Z)$ can be viewed as the Weil transfer of the multiplicative group over Z, and we have by Shapiro's lemma

$$H^2(F,\mathbf{GL}_1(Z))=H^2(F,R_{Z/F}(\mathbf{G}_{\mathrm{m},Z}))=H^2(Z,\mathbf{G}_{\mathrm{m},Z})=\mathrm{Br}(Z).$$

Under these identifications,

$$\partial \circ \rho^1(A', \sigma', f', \varphi) = [C(A', \sigma', f') \otimes_{Z'} Z] - [C(A, \sigma, f)] \qquad \text{in } \mathrm{Br}(Z),$$

where the tensor product is taken with respect to φ. Therefore, (17) yields

$$C(A', \sigma', f') \otimes_{Z'} Z \simeq C(A, \sigma, f) \qquad \text{as } Z\text{-algebras},$$

hence φ extends to an F-algebra isomorphism $C(A', \sigma', f') \simeq C(A, \sigma, f)$.

To prove $A' \simeq A$, observe that the map χ^1 factors through the canonical map

$$H^1(F, \mathbf{GO}^+(A, \sigma, f)) \to H^1(F, \mathbf{PGO}^+(A, \sigma, f)).$$

From the description of the set on the left in [15, §29.F], it follows that the image consists of F-isomorphism classes of 4-tuples $(A', \sigma', f', \varphi)$ where $A' = A$. □

Corollary 5.5. *If* $\mathrm{sd}_2 F \leq 2$*, then* $H^1(F, \mathbf{Spin}(A, \sigma, f)) = 1$.

Proof. Consider the exact sequence (16). By a theorem of Gille [13, Théorème 6] the map δ^0 is surjective, hence $\ker \chi^1 = 1$. On the other hand, if $(A', \sigma', f', \varphi)$ represents an element in $\mathrm{im}\, \chi^1$, then by Lemma 5.4 and Corollary 5.3

$$(A', \sigma', f', \varphi) \simeq (A, \sigma, f, \mathrm{Id}_Z).$$

Therefore $\mathrm{im}\, \chi^1 = 1$ and the proof is complete. □

Appendix A. The Witt kernel of odd-degree and separable quadratic extensions

In this appendix, we give the proofs of the results stated in Section 1.5.

A.1. **Metabolic involutions.** Besides hyperbolic involutions, we need in Section A.2 the weaker notion of metabolic involution, which we define below after the following easy observation on idempotents in central simple algebras:

Lemma A.1. *Let A be a central simple algebra of even degree over a field F and let e, $e' \in A$ be two idempotents. Any two of the following conditions imply the third one:*

(a) $ee' = 0$,
(b) $(1 - e')(1 - e) = 0$,
(c) $\dim_F eA + \dim_F e'A = \dim_F A$.

In particular, if $\dim_F eA = \dim_F e'A = \frac{1}{2} \dim_F A$*, then* $ee' = 0$ *if and only if* $(1 - e')(1 - e) = 0$.

Proof. Condition (a) is equivalent to $e' = (1 - e)e'$, hence also to $e'A \subset (1 - e)A$. Likewise, condition (b) is equivalent to the reverse inclusion $(1 - e)A \subset e'A$. Since

$$A = eA \oplus (1 - e)A,$$

we have $\dim_F (1 - e)A = \dim_F A - \dim_F eA$, hence condition (c) is equivalent to

$$\dim_F e'A = \dim_F (1 - e)A.$$

The lemma is now clear. □

Definition A.2. Let A be a central simple algebra of even degree over an arbitrary field F. An involution μ (of any type) on A is called *metabolic* if A contains an idempotent e such that $\mu(e)e = 0$ and

$$\dim_F eA = \tfrac{1}{2}\dim_F A. \tag{18}$$

Since

$$\dim \mu(e)A = \dim \mu(Ae) = \dim Ae = \dim eA, \tag{19}$$

it follows by Lemma A.1 that

$$(1-e)(1-\mu(e)) = 0. \tag{20}$$

Note that if the idempotent e satisfies $\mu(e)e = 0$ and (20), then Lemma A.1 and (19) show that it also satisfies (18). Therefore, we may substitute (20) for (18) in the requirements for e.

If A is represented as in (1) as the algebra of endomorphisms of some hermitian or skew-hermitian space (V,h), the involution $\mu = \mathrm{ad}_h$ is metabolic if and only if V contains a totally isotropic subspace U with $\dim_D U = \frac{1}{2}\dim_D V$ (or, equivalently, $U^\perp = U$).

Proposition A.3. *A symplectic or unitary involution is metabolic if and only if it is hyperbolic.*

Proof. If μ is symplectic or unitary, we may find $x \in A$ such that $\mu(x) = 1-x$, see [15, (2.6), (2.17)]. If $e \in A$ is an idempotent such that $\mu(e)e = 0$ and $\dim eA = \frac{1}{2}\dim A$, let

$$e' = e - ex\mu(e).$$

Computation shows that e' is an idempotent. Moreover,

$$\mu(e') = \mu(e) - e\mu(x)\mu(e) = \mu(e) - e\mu(e) + ex\mu(e).$$

In view of (20), the right side is $1-e'$, hence μ is hyperbolic.

Conversely, any idempotent $e \in A$ such that $\mu(e) = 1-e$ clearly satisfies $\mu(e)e = 0$ and $(1-e)(1-\mu(e)) = 0$. □

Corollary A.4. *Let K be an arbitrary field. Let (A,σ) be a central simple K-algebra with involution of symplectic or unitary type, and let $F \subset K$ be the subfield of σ-invariant elements (so $F = K$ if σ is symplectic). If B is a central simple F-algebra with a metabolic orthogonal involution μ, then the involution $\sigma\otimes\mu$ on $A\otimes_F B$ is hyperbolic.*

Proof. To simplify notation, let $\tau = \sigma\otimes\mu$. Let $e \in B$ be an idempotent such that $\mu(e)e = 0$ and $(1-e)(1-\mu(e)) = 0$, and let $e' = 1\otimes e \in A\otimes B$. This is an idempotent such that $\tau(e')e' = 0$ and $(1-e')(1-\tau(e')) = 0$, hence τ is metabolic. Moreover, since μ is orthogonal, τ has the same type as σ, hence the corollary follows from Proposition A.3. □

The next proposition gives a corresponding result for quadratic pairs. Recall from [15, (5.18)] that if (σ, f) is a quadratic pair on a central simple F-algebra A and μ is an orthogonal involution on a central simple F-algebra B, a quadratic pair $(\sigma\otimes\mu, f_*)$ is defined on $A\otimes_F B$ by the condition

$$f_*(x\otimes y) = f(x)\operatorname{Trd}_B(y) \qquad \text{for } x \in \operatorname{Sym}(\sigma) \text{ and } y \in \operatorname{Sym}(\mu).$$

Proposition A.5. *The quadratic pair $(\sigma\otimes\mu, f_*)$ is hyperbolic if μ is metabolic.*

Proof. Let $e \in B$ be an idempotent such that $\mu(e)e = 0$ and $(1-e)(1-\mu(e)) = 0$, and let $\ell \in A$ be such that
$$\operatorname{Trd}_A(\ell x) = f(x) \qquad \text{for } x \in \operatorname{Sym}(\sigma).$$
Let $e' = 1 \otimes e - \ell \otimes e\mu(e)$. To prove the proposition, we show that
$$f_*(z) = \operatorname{Trd}_{A\otimes B}(e'z) \qquad \text{for } z \in \operatorname{Sym}(\sigma\otimes\mu). \tag{21}$$
By Remark 1.10, we have $\sigma(\ell) = 1-\ell$, hence the same computation as in the proof of Proposition A.3 yields
$$(\sigma\otimes\mu)(e') = 1 - e'.$$
By Remark 1.10, it follows that (21) holds for $z \in \operatorname{Symd}(\sigma\otimes\mu)$. Since
$$\operatorname{Sym}(\sigma\otimes\mu) = \operatorname{Symd}(\sigma\otimes\mu) + \operatorname{Sym}(\sigma)\otimes\operatorname{Sym}(\mu)$$
by [15, (5.17)], it suffices to prove (21) for $z \in \operatorname{Sym}(\sigma)\otimes\operatorname{Sym}(\mu)$.

For $x \in \operatorname{Sym}(\sigma)$ and $y \in \operatorname{Sym}(\mu)$, we have
$$\operatorname{Trd}_{A\otimes B}(e'\cdot x\otimes y) = \operatorname{Trd}_A(x)\operatorname{Trd}_B(ey) - \operatorname{Trd}_A(\ell x)\operatorname{Trd}_B(e\mu(e)y).$$
Since $\operatorname{Trd}_A(x) = 2f(x) = 2\operatorname{Trd}_A(\ell x)$, the right side is
$$f(x)\big(2\operatorname{Trd}_B(ey) - \operatorname{Trd}_B(e\mu(e)y)\big). \tag{22}$$
Now, use $\operatorname{Trd}_B(ey) = \operatorname{Trd}_B(\mu(ey)) = \operatorname{Trd}_B(\mu(e)y)$ to rewrite (22) in the form
$$f(x)\operatorname{Trd}_B((e+\mu(e)-e\mu(e))y).$$
Since $(1-e)(1-\mu(e)) = 0$ we have $e+\mu(e)-e\mu(e) = 1$, hence finally
$$\operatorname{Trd}_{A\otimes B}(e'\cdot x\otimes y) = f(x)\operatorname{Trd}_B(y),$$
proving (21) for $z = x\otimes y$. □

A.2. **The Witt kernel of an odd-degree extension.** We now turn to the proof of Theorems 1.13 and 1.14, and use the notation of these theorems. Arguing by induction on the number of generators, we may restrict to the case where E/F is a simple extension. Let $E = F(u)$, $\dim_F E = n$, and define a linear map $s\colon E \to F$ by
$$s(\alpha_0 + \alpha_1 u + \cdots + \alpha_{n-1}u^{n-1}) = \alpha_0 \qquad \text{for } \alpha_0,\,\ldots,\,\alpha_{n-1} \in F.$$
Consider the symmetric bilinear form $b\colon E\times E \to F$ defined by
$$b(x,y) = s(xy) \qquad \text{for } x,\, y \in E.$$
This form is regular, so we may consider the adjoint orthogonal involution ad_b on $\operatorname{End}_F E$. The regular representation $E \to \operatorname{End}_F E$ is an embedding
$$(E, \operatorname{Id}_E) \hookrightarrow (\operatorname{End}_F E, \operatorname{ad}_b).$$
Composing s with the embedding $F \hookrightarrow E$, we may view s as an idempotent in $\operatorname{End}_F E$. For $x,\, y \in E$,
$$b(x, s(y)) = s(x)s(y) = b(s(x), y),$$
hence $\operatorname{ad}_b(s) = s$. Let $E_0 = \ker s$ and $s_0 = 1 - s$. Let also b_0 be the restriction of b to E_0.

Lemma A.6. *The involution* ad_{b_0} *is metabolic.*

Proof. The span of $u,\ldots,u^{(n-1)/2}$ is a totally isotropic subspace of E_0 of dimension $\frac{1}{2}\dim E_0$. Any projection on this subspace is an idempotent e such that $\operatorname{ad}_{b_0}(e)e = 0$ and $\dim_F(e\operatorname{End}_F E_0) = \frac{1}{2}\dim_F \operatorname{End}_F E$. □

Proposition A.7. *Let (A, σ) be a central simple algebra with involution as in Theorem 1.13. The algebra with involution $(A \otimes_F \operatorname{End}_F E, \sigma \otimes \operatorname{ad}_b)$ is Witt-equivalent to (A, σ). Similarly, every central simple algebra with quadratic pair (A, σ, f) as in Theorem 1.14 is Witt-equivalent to $(A \otimes_F \operatorname{End}_F E, \sigma \otimes \operatorname{ad}_b, f_*)$, where f_* is defined as in Proposition A.5.*

Proof. For every symmetric idempotent $e \in \operatorname{End}_F E$, we may identify

$$e(\operatorname{End}_F E)e = \operatorname{End}_F \operatorname{im} e,$$

and the restriction of ad_b to this algebra is the adjoint involution with respect to the restriction of b to $\operatorname{im} e$. In particular, for $e = s$ and $e = s_0$,

$$s(\operatorname{End}_F E)s = \operatorname{End}_F F = F \qquad \text{and} \qquad s_0(\operatorname{End}_F E)s_0 = \operatorname{End}_F E_0,$$

and the restriction of ad_b to these algebras is Id_F, resp. ad_{b_0}. Tensoring with A, we obtain

$$(1 \otimes s)(A \otimes_F \operatorname{End}_F E)(1 \otimes s) = A$$

and

$$(1 \otimes s_0)(A \otimes_F \operatorname{End}_F E)(1 \otimes s_0) = A \otimes_F \operatorname{End}_F E_0,$$

hence an orthogonal sum decomposition

$$(A \otimes_F \operatorname{End}_F E, \sigma \otimes \operatorname{ad}_b) = (A, \sigma) \boxplus (A \otimes_F \operatorname{End}_F E_0, \sigma \otimes \operatorname{ad}_{b_0}).$$

The first part follows, since the last summand is hyperbolic by Lemma A.6 and Corollary A.4.

If (σ, f) is a quadratic pair on a central simple F-algebra A, we obtain likewise an orthogonal sum decomposition

$$(A \otimes_F \operatorname{End}_F E, \sigma \otimes \operatorname{ad}_b, f_*) = (A, \sigma, f) \boxplus (A \otimes_F \operatorname{End}_F E_0, \sigma \otimes \operatorname{ad}_{b_0}, f_*),$$

and the second part follows by Lemma A.6 and Proposition A.5. □

Proof of Theorem 1.13. The regular representation $E \to \operatorname{End}_F E$ yields an embedding

$$(E, \operatorname{Id}_E) \hookrightarrow (\operatorname{End}_F E, \operatorname{ad}_b).$$

Tensoring with (A, σ), we obtain

$$(A_E, \sigma_E) \hookrightarrow (A \otimes_F \operatorname{End}_F E, \sigma \otimes \operatorname{ad}_b).$$

If σ_E is hyperbolic, we may find an idempotent $e \in A_E$ such that $\sigma_E(e) = 1 - e$. Viewing this idempotent in $A \otimes \operatorname{End}_F E$, we also have

$$(\sigma \otimes \operatorname{ad}_b)(e) = 1 - e,$$

hence $\sigma \otimes \operatorname{ad}_b$ is hyperbolic. It follows that σ is hyperbolic since (A, σ) and $(A \otimes_F \operatorname{End}_F E, \sigma \otimes \operatorname{ad}_b)$ are Witt-equivalent by Proposition A.7. □

Proof of Theorem 1.14. The proof follows the same pattern as the proof of Theorem 1.13. The regular representation of E yields an embedding

$$(A_E, \sigma_E) \hookrightarrow (A \otimes_F \operatorname{End}_F E, \sigma \otimes \operatorname{ad}_b).$$

The main point is to show that $(\sigma \otimes \operatorname{ad}_b, f_*)$ is hyperbolic if (σ_E, f_E) is hyperbolic. The conclusion then follows from Proposition A.7.

Let $e \in A_E$ be an idempotent such that

$$f_E(x) = \operatorname{Trd}_{A_E}(ex) \qquad \text{for all } x \in \operatorname{Sym}(\sigma_E). \tag{23}$$

To complete the proof, we show that

$$f_*(z) = \operatorname{Trd}_{A\otimes \operatorname{End} E}(ez) \qquad \text{for all } z \in \operatorname{Sym}(\sigma\otimes \operatorname{ad}_b). \tag{24}$$

Since $\sigma_E(e) = 1-e$ by Remark 1.10, we have $(\sigma\otimes\operatorname{ad}_b)(e) = 1-e$. Therefore, as in the proof of Proposition A.5, it suffices to prove (24) for $z\in\operatorname{Sym}(\sigma)\otimes\operatorname{Sym}(\operatorname{ad}_b)$. Thus, we aim to prove

$$f(x)\operatorname{Tr}(y) = \operatorname{Trd}_{A\otimes\operatorname{End} E}(e\cdot x\otimes y) \qquad \text{for } x\in\operatorname{Sym}(\sigma) \text{ and } y \in \operatorname{End}_F E. \tag{25}$$

Let $\rho_v \in \operatorname{End} E$ denote the image of $v\in E$ under the regular representation, i.e. $\rho_v(w) = vw$ for v, $w\in E$. Using the basis $1, u, \ldots, u^{n-1}$ of E, it is easily seen that $(\rho_{u^i}\circ s\circ\rho_{u^j})_{0\le i,j\le n-1}$ is a basis of $\operatorname{End} E$, hence $\operatorname{End} E$ is spanned by elements of the form $\rho_v\circ s\circ\rho_w$ with v, $w\in E$. Moreover, computation shows

$$\operatorname{Tr}(\rho_v\circ s) = s(v) \qquad \text{for } v\in E.$$

Claim. *For $a\in A_E$, $\operatorname{Trd}_{A\otimes\operatorname{End} E}(a\cdot 1\otimes s) = s\big(\operatorname{Trd}_{A_E}(a)\big)$.*

Since both sides are linear in a, it suffices to prove the claim for $a = a_0\otimes v$ with $a_0\in A$ and $v\in E$. Then

$$\operatorname{Trd}(a\cdot 1\otimes s) = \operatorname{Trd}_A(a_0)\operatorname{Tr}(\rho_v\circ s) = \operatorname{Trd}_A(a_0)s(v) = s\big(\operatorname{Trd}_{A_E}(a)\big),$$

proving the claim.

We may now prove (25). Since both sides are linear in y, it suffices to prove it when $y = \rho_v\circ s\circ\rho_w$ for some v, $w\in E$. We may then decompose

$$x\otimes y = (x\otimes\rho_v)(1\otimes s)(1\otimes\rho_w).$$

Since $e\in A_E\subset A\otimes\operatorname{End} E$ commutes with $1\otimes\rho_w$, we have

$$\operatorname{Trd}(e\cdot x\otimes y) = \operatorname{Trd}\big(e\cdot(x\otimes\rho_w\circ\rho_v)\cdot(1\otimes s)\big).$$

Using the claim proved above and (23), we may simplify the right side to

$$s\big(\operatorname{Trd}_{A_E}(e\cdot x\otimes vw)\big) = s\big(f(x)vw\big) = f(x)s(vw).$$

Equation (25) follows, since

$$\operatorname{Tr}(y) = \operatorname{Tr}(\rho_v\circ s\circ\rho_w) = \operatorname{Tr}(\rho_w\circ\rho_v\circ s) = s(vw).$$

□

A.3. **The Witt kernel of a separable quadratic extension.** We now turn to the proof of Theorems 1.15 and 1.16. The following lemma already yields the "if" part of these results:

Lemma A.8. *Let A be an F-algebra, and let σ be an F-linear involution on A. Let also L/F be a separable quadratic field extension with non-trivial automorphism ι. If there is an embedding $\varepsilon\colon (L,\iota)\hookrightarrow(A,\sigma)$, then $A\otimes_F L$ contains an idempotent e such that $\sigma_L(e) = 1-e$. If A is central simple and (σ,f) is a quadratic pair on A, and if there is an embedding $\varepsilon\colon L\hookrightarrow A$ such that (3) holds, then $A\otimes_F L$ contains an idempotent e such that $f_L(x) = \operatorname{Trd}_{A_L}(ex)$ for all $x\in\operatorname{Sym}(\sigma_L)$.*

Proof. Let $\ell\in L\setminus F$ and

$$s = (1\otimes\ell - \ell\otimes 1)(1\otimes(\ell-\iota(\ell))^{-1})\in L\otimes_F L.$$

The element s is the separability idempotent of L. It satisfies

$$(\iota\otimes\operatorname{Id}_L)(s) = 1-s = (\operatorname{Id}_L\otimes\iota)(s),$$

and therefore it is mapped under $\varepsilon\otimes \mathrm{Id}_L\colon L\otimes_F L\hookrightarrow A\otimes_F L$ to an idempotent e such that $\sigma_L(e)=1-e$.

Now, assume (σ,f) is a quadratic pair on the central simple F-algebra A and ε satisfies (3). For $x\in\mathrm{Sym}(\sigma)$,

$$\begin{aligned}\mathrm{Trd}_{A_L}(e(x\otimes 1)) &= \mathrm{Trd}_A(x)\frac{\ell}{\ell-\iota(\ell)}-\mathrm{Trd}_A(\varepsilon(\ell)x)\frac{1}{\ell-\iota(\ell)}\\ &= \mathrm{Trd}_A(x)\frac{\ell}{\ell-\iota(\ell)}-T_{L/F}(\ell)f(x)\frac{1}{\ell-\iota(\ell)}.\end{aligned}$$

Since $T_{L/F}(\ell)=\ell+\iota(\ell)$ and $\mathrm{Trd}_A(x)=2f(x)$, the right side is $f(x)$. By linearity, it follows that $\mathrm{Trd}_{A_L}(ex)=f_L(x)$ for all $x\in\mathrm{Sym}(\sigma_L)$. $\square$

We now consider the "only if" part of Theorem 1.15 in the anisotropic case.

Lemma A.9. *Using the same notation as in Theorem 1.15, suppose (A,σ) is anisotropic. If σ_L is hyperbolic, then there is an embedding $\varepsilon\colon (L,\iota)\hookrightarrow(A,\sigma)$.*

Proof. Let $e\in A_L$ be an idempotent such that $\sigma_L(e)=1-e$. As observed in the proof of Proposition A.3, this condition implies

$$\dim_L(1-e)A_L=\frac{1}{2}\dim_L A_L,\qquad \text{hence}\quad \dim_F(1-e)A_L=\dim_F A. \tag{26}$$

It also implies $\sigma(x)x=0$ for $x\in(1-e)A_L$, hence

$$(A\otimes 1)\cap(1-e)A_L=\{0\}$$

since σ on A is anisotropic. Moreover, we have

$$\dim_F(A\otimes 1)+\dim_F(1-e)A_L=2\dim_F A=\dim_F A_L,$$

hence

$$A_L=(A\otimes 1)\oplus(1-e)A_L.$$

Therefore, for $x\in A_L$ there is a unique $a\in A$ such that $x-(a\otimes 1)\in(1-e)A_L$, i.e.

$$e(x-(a\otimes 1))=0.$$

We may then define a map $\varepsilon\colon L\to A$ as follows: for $\ell\in L$, $\varepsilon(\ell)\in A$ is the unique element such that

$$e(1\otimes\ell-\varepsilon(\ell)\otimes 1)=0.$$

The map ε clearly is F-linear and injective, and $\varepsilon(1)=1$. Moreover, for ℓ, $\ell'\in L$ we have

$$1\otimes\ell'\ell-\varepsilon(\ell)\varepsilon(\ell')\otimes 1=(1\otimes\ell'-\varepsilon(\ell')\otimes 1)(1\otimes\ell)+(1\otimes\ell-\varepsilon(\ell)\otimes 1)(\varepsilon(\ell')\otimes 1),$$

so

$$e(1\otimes\ell\ell'-\varepsilon(\ell)\varepsilon(\ell')\otimes 1)=0,$$

and therefore

$$\varepsilon(\ell\ell')=\varepsilon(\ell)\varepsilon(\ell').$$

This proves that ε is an F-algebra embedding $L\hookrightarrow A$. To show that $\varepsilon\circ\iota=\sigma\circ\varepsilon$, consider $\ell\in L\setminus F$ and

$$e'=(1\otimes\ell-\varepsilon(\ell)\otimes 1)(1\otimes(\ell-\iota(\ell))^{-1})\in A_L. \tag{27}$$

This is an idempotent (it is the image under $\varepsilon\otimes\mathrm{Id}_L$ of the separability idempotent of L), and it satisfies

$$(\mathrm{Id}_A\otimes\iota)(e')=1-e'.$$

Since $\dim_L(\mathrm{Id}_A \otimes \iota)(e')A_L = \dim_L e'A_L$ and $A_L = e'A_L \oplus (1-e')A_L$,

$$\dim_L e'A_L = \tfrac{1}{2}\dim_L A_L, \qquad \text{and therefore} \quad \dim_F e'A_L = \dim_F A. \tag{28}$$

By definition of $\varepsilon(\ell)$, we have $ee' = 0$, hence by (26) and (28) Lemma A.1 yields

$$(1-e')(1-e) = 0.$$

Applying σ_L, we obtain

$$e(1-\sigma_L(e')) = 0. \tag{29}$$

On the other hand,

$$1-e' = (\mathrm{Id}_A \otimes \iota)(e') = (1\otimes\iota(\ell) - \varepsilon(\ell)\otimes 1)(1\otimes(\iota(\ell)-\ell)^{-1}),$$

hence (29) yields $\varepsilon\circ\iota(\ell) = \sigma\circ\varepsilon(\ell)$. □

The corresponding result for quadratic pairs is the following:

Lemma A.10. *Using the same notation as in Theorem 1.16, suppose (σ, f) is anisotropic. If (σ_L, f_L) is hyperbolic, then there is an embedding $\varepsilon\colon L \hookrightarrow A$ satisfying (3).*

Proof. Let $e \in A_L$ be an idempotent such that $f_L(x) = \mathrm{Trd}_{A_L}(ex)$ for $x \in \mathrm{Sym}(\sigma_L)$. By Remark 1.10,

$$e + \sigma_L(e) = 1.$$

The ideal $(1-e)A_L$ is isotropic, hence the same arguments as in the proof of Lemma A.9 show that

$$A_L = (A\otimes 1) \oplus (1-e)A_L$$

and yield an F-algebra embedding $\varepsilon\colon L \hookrightarrow A$ defined by

$$e(1\otimes\ell - \varepsilon(\ell)\otimes 1) = 0 \qquad \text{for } \ell \in L.$$

As in the proof of Lemma A.9, we have

$$\sigma\circ\varepsilon = \varepsilon\circ\iota.$$

Condition (3) holds for $\ell \in F$ since then

$$T_{L/F}(\ell)f(x) = 2\ell f(x) = \ell\,\mathrm{Trd}_A(x) \qquad \text{for } x \in \mathrm{Sym}(\sigma).$$

Now, let $\ell \in L \setminus F$ and consider the idempotent e' defined in (27). To simplify notation, denote $\overline{} = \mathrm{Id}_A \otimes \iota$. Since $\sigma\circ\varepsilon = \varepsilon\circ\iota$, we have

$$\sigma_L(e') = 1 - e' = \overline{e'}. \tag{30}$$

For $x \in \mathrm{Sym}(\sigma)$,

$$\mathrm{Trd}_{A_L}(e(x\otimes 1)) = f(x) \in F,$$

hence

$$\mathrm{Trd}_{A_L}(e(x\otimes 1)) = \mathrm{Trd}_{A_L}(\overline{e}(x\otimes 1)).$$

By linearity, it follows that

$$\mathrm{Trd}_{A_L}((e-\overline{e})x) = 0 \qquad \text{for } x \in \mathrm{Sym}(\sigma_L).$$

In particular,

$$\mathrm{Trd}_{A_L}((e-\overline{e})\sigma_L(e')xe') = 0 \qquad \text{for } x \in \mathrm{Sym}(\sigma_L). \tag{31}$$

Applying $\overline{}$ to $ee' = 0$, and taking (30) into account, we obtain

$$\overline{e}\sigma_L(e') = 0.$$

On the other hand, it also follows from $ee' = 0$ and (30) that

$$e\sigma_L(e') = e.$$

Therefore, (31) yields

$$\mathrm{Trd}_{A_L}(exe') = 0 \qquad \text{for } x \in \mathrm{Sym}(\sigma_L).$$

Substituting for e' the right side of (27), we obtain

$$\mathrm{Trd}_{A_L}((\varepsilon(\ell)\otimes 1)ex) = \mathrm{Trd}_{A_L}(ex)\ell = f_L(x)\ell \qquad \text{for } x \in \mathrm{Sym}(\sigma_L). \tag{32}$$

Similarly, $ee' = 0$ yields $\mathrm{Trd}_{A_L}(ee'x) = 0$, hence

$$\mathrm{Trd}_{A_L}(e(\varepsilon(\ell)\otimes 1)x) = \mathrm{Trd}_{A_L}(ex)\ell = f_L(x)\ell \qquad \text{for } x \in \mathrm{Sym}(\sigma_L).$$

Substituting $\iota(\ell)$ for ℓ in the last equation, and using $\sigma\circ\varepsilon = \varepsilon\circ\iota$, we also have for $x \in \mathrm{Sym}(\sigma_L)$

$$\mathrm{Trd}_{A_L}((\varepsilon(\ell)\otimes 1)\sigma_L(e)x) = \mathrm{Trd}_{A_L}(xe(\varepsilon\circ\iota(\ell)\otimes 1)) = f_L(x)\iota(\ell). \tag{33}$$

Adding (32) and (33), and using $\sigma_L(e) + e = 1$, we finally obtain

$$\mathrm{Trd}_{A_L}((\varepsilon(\ell)\otimes 1)x) = T_{L/F}(\ell)f_L(x) \qquad \text{for } x \in \mathrm{Sym}(\sigma_L),$$

proving (3). □

Lemma A.11. *Using the same notation as in Theorem 1.15, suppose σ is hyperbolic. Then there is an embedding $\varepsilon\colon (L,\iota) \hookrightarrow (A,\sigma)$.*

Proof. Since σ is hyperbolic, we have $A \simeq M_2(A') \simeq M_2(F)\otimes_F A'$ for some central simple K-algebra A' which has involutions of the same kind as σ. Let σ' be an involution on A', of orthogonal type if σ is symplectic, of unitary type if σ is unitary, with $\sigma'|_K = \sigma|_K$, and let J be the (unique) symplectic involution on $M_2(F)$,

$$J(x) = \mathrm{Tr}(x) - x \qquad \text{for } x \in M_2(F).$$

Since J is hyperbolic, the involution $J\otimes\sigma'$ on $M_2(F)\otimes_F A'$ is hyperbolic. As hyperbolic involutions of a given type on a central simple algebra are all conjugate (since hyperbolic hermitian or skew-hermitian forms of a given dimension are isometric), it follows that

$$(A,\sigma) \simeq (M_2(F)\otimes_F A', J\otimes\sigma').$$

Now, let $\ell \in L\setminus F$ and define

$$b(x,y) = \frac{\iota(x)y - x\iota(y)}{\ell - \iota(\ell)} \qquad \text{for } x,\, y \in L.$$

The map $b\colon L\times L \to F$ is a nonsingular alternating bilinear form, hence its adjoint involution ad_b on $\mathrm{End}_F L$ is symplectic, and

$$(\mathrm{End}_F L, \mathrm{ad}_b) \simeq (M_2(F), J). \tag{34}$$

Moreover, for $u,\, x,\, y \in L$,

$$b(ux,y) = b(x,\iota(u)y),$$

hence the regular representation $L \to \mathrm{End}_F L$ is an embedding

$$(L,\iota) \hookrightarrow (\mathrm{End}_F L, \mathrm{ad}_b). \tag{35}$$

The lemma follows by composing the maps

$$(L,\iota) \hookrightarrow (\mathrm{End}_F L, \mathrm{ad}_b) \simeq (M_2(F), J) \hookrightarrow (M_2(F)\otimes_F A', J\otimes\sigma') \simeq (A,\sigma).$$

□

Lemma A.12. *Using the same notation as in Theorem 1.16, suppose* (σ, f) *is hyperbolic and* $\deg A \equiv 0 \bmod 4$. *Then there is an embedding* $\varepsilon\colon L \hookrightarrow A$ *satisfying* (3).

Proof. Since (σ, f) is hyperbolic, we have $A \simeq M_2(A') \simeq M_2(F) \otimes_F A'$ for some central simple F-algebra A' with involutions of the first kind. The degree of A' is even since $\deg A$ is divisible by 4, hence A' carries an involution σ' of symplectic type. Let J be the (unique) symplectic involution on $M_2(F)$. We may consider the quadratic pair $(J \otimes \sigma', f_\otimes)$ determined by the condition

$$f_\otimes(x_1 \otimes x_2) = 0 \qquad \text{for } x_1 \in \operatorname{Skew}(J) \text{ and } x_2 \in \operatorname{Skew}(\sigma'),$$

see [15, (5.20)].

Claim: The quadratic pair $(J \otimes \sigma', f_\otimes)$ is hyperbolic.

Consider $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes 1 \in M_2(F) \otimes A'$. To prove the claim, we show that

$$f_\otimes(x) = \operatorname{Trd}(ex) \qquad \text{for } x \in \operatorname{Sym}(J \otimes \sigma'). \tag{36}$$

By [15, (5.17)],

$$\operatorname{Sym}(J \otimes \sigma') = \operatorname{Symd}(J \otimes \sigma') + \operatorname{Sym}(J) \otimes \operatorname{Sym}(\sigma'), \tag{37}$$

hence it suffices to prove (36) for $x \in \operatorname{Symd}(J \otimes \sigma')$ and for $x \in \operatorname{Sym}(J) \otimes \operatorname{Sym}(\sigma')$. Remark 1.10 shows that (36) holds for $x \in \operatorname{Symd}(J \otimes \sigma')$.

Suppose next $x = x_1 \otimes x_2$ with $x_1 \in \operatorname{Sym}(J)$ and $x_2 \in \operatorname{Sym}(\sigma')$. Then

$$\operatorname{Trd}(ex) = \operatorname{Tr}\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) x_1\big) \operatorname{Trd}_{A'}(x_2).$$

If $\operatorname{char} F = 2$, then $\operatorname{Trd}_{A'}(x_2) = 0$ since σ' is symplectic. On the other hand, $\operatorname{Sym}(J) = \operatorname{Skew}(J)$ and $\operatorname{Sym}(\sigma') = \operatorname{Skew}(\sigma')$, so $f_\otimes(x) = 0$ by definition.

If $\operatorname{char} F \neq 2$, then $\operatorname{Sym}(J) = F$, hence

$$\operatorname{Trd}(ex) = x_1 \operatorname{Trd}_{A'}(x_2).$$

On the other hand,

$$f_\otimes(x) = \tfrac{1}{2} \operatorname{Trd}(x) = x_1 \operatorname{Trd}_{A'}(x_2),$$

so the claim is proved.

Since all the hyperbolic quadratic pairs on a given central simple algebra are conjugate (because hyperbolic quadratic forms of a given dimension are isometric), we may assume henceforth $(A, \sigma, f) = (M_2(F) \otimes_F A', J \otimes \sigma', f_\otimes)$.

To complete the proof, we show that the embedding $\varepsilon\colon L \hookrightarrow A$ which factors through the regular representation $\rho\colon L \to \operatorname{End}_F L$ satisfies (3). Again, by (37), it suffices to prove (3) for $x \in \operatorname{Symd}(J \otimes \sigma')$ and for $x \in \operatorname{Sym}(J) \otimes \operatorname{Sym}(\sigma')$.

If $x \in \operatorname{Symd}(J \otimes \sigma')$, let $x = y + (J \otimes \sigma')(y)$. For $\ell \in L$,

$$\operatorname{Trd}_A(\varepsilon(\ell)x) = \operatorname{Trd}_A(\varepsilon(\ell)y) + \operatorname{Trd}_A(\sigma \circ \varepsilon(\ell)y).$$

Since $\varepsilon(\ell) + \sigma \circ \varepsilon(\ell) = \rho(\ell) + J \circ \rho(\ell) = T_{L/F}(\ell)$, the right side is

$$T_{L/F}(\ell) \operatorname{Trd}_A(y) = T_{L/F} f_\otimes(x),$$

and it follows that (3) holds for $x \in \operatorname{Symd}(J \otimes \sigma')$.

If $x = x_1 \otimes x_2$ with $x_1 \in \operatorname{Sym}(J)$ and $x_2 \in \operatorname{Sym}(\sigma')$, then

$$\operatorname{Trd}_A(\varepsilon(\ell)x) = \operatorname{Tr}(\rho(\ell)x_1) \operatorname{Trd}_{A'}(x_2).$$

If $\operatorname{char} F \neq 2$, then $\operatorname{Sym}(J) = F$, hence the right side is $x_1 T_{L/F}(\ell) \operatorname{Trd}_{A'}(x_2)$. On the other hand,

$$f_\otimes(x) = \tfrac{1}{2} \operatorname{Trd}_A(x) = x_1 \operatorname{Trd}_{A'}(x_2),$$

so (3) holds for $x \in \mathrm{Sym}(J) \otimes \mathrm{Sym}(\sigma')$ if $\operatorname{char} F \neq 2$.

If $\operatorname{char} F = 2$, then $\mathrm{Trd}_{A'}(x_2) = 0$ since σ' is symplectic. On the other hand, $\mathrm{Sym}(J) = \mathrm{Skew}(J)$ and $\mathrm{Sym}(\sigma') = \mathrm{Skew}(\sigma')$ so $f_\otimes(x) = 0$ by definition. Therefore, (3) holds for $x \in \mathrm{Sym}(J) \otimes \mathrm{Sym}(\sigma')$ if $\operatorname{char} F = 2$. □

Proof of Theorem 1.15. If there is an embedding $(L, \iota) \hookrightarrow (A, \sigma)$, then Lemma A.8 shows that σ_L is hyperbolic. For the converse, use a Witt decomposition

$$(A, \sigma) = (A_0, \sigma_0) \boxplus (A_1, \sigma_1)$$

where σ_0 is anisotropic and σ_1 is hyperbolic, see Proposition 1.8. If σ_L is hyperbolic, Witt cancellation implies that σ_{0L} is hyperbolic. By Lemma A.9, it follows that there is an embedding $\varepsilon_0\colon (L, \iota) \hookrightarrow (A_0, \sigma_0)$. On the other hand, Lemma A.11 yields an embedding $\varepsilon_1\colon (L, \iota) \hookrightarrow (A_1, \sigma_1)$. The direct sum of these embeddings is an embedding

$$\varepsilon_0 \boxplus \varepsilon_1\colon (L, \iota) \hookrightarrow (A_0, \sigma_0) \boxplus (A_1, \sigma_1) = (A, \sigma).$$

□

The proof of Theorem 1.16 follows the same lines:

Proof of Theorem 1.16. The "if" part readily follows from Lemma A.8. For the converse, use a Witt decomposition as in Proposition 1.11,

$$(A, \sigma, f) = (A_0, \sigma_0, f_0) \boxplus (A_1, \sigma_1, f_1)$$

where (σ_0, f_0) is anisotropic and (σ_1, f_1) is hyperbolic. Then $\operatorname{ind} A = \operatorname{ind} A_1$ divides $\frac{1}{2} \deg A_1$. If $\deg A_1 \equiv 2 \bmod 4$, then $\operatorname{ind} A$ is odd, hence the algebra is split and (σ, f) is adjoint to a quadratic form with odd Witt index. This special case is excluded for the converse, so we assume $\deg A_1$ is divisible by 4. Lemma A.12 then yields an embedding $\varepsilon_1\colon L \hookrightarrow A_1$ for which (3) holds. If (σ_L, f_L) is hyperbolic, then Witt cancellation implies (σ_{0L}, f_{0L}) is hyperbolic, hence Lemma A.10 yields an embedding $\varepsilon_0\colon L \hookrightarrow A_0$ for which (3) holds. To complete the proof, we show that the direct sum $\varepsilon = \varepsilon_0 \boxplus \varepsilon_1\colon L \hookrightarrow A$ also satisfies (3).

Let e_0, $e_1 \in A$ be the symmetric idempotents such that $A_i = e_i A e_i$ for $i = 0$, 1. Since $e_0 + e_1 = 1$, we have

$$x = e_0 x e_0 + e_0 x e_1 + e_1 x e_0 + e_1 x e_1 \qquad \text{for } x \in A.$$

The reduced trace of the two middle terms vanishes since $e_0 e_1 = e_1 e_0 = 0$ implies

$$\mathrm{Trd}_A(e_0 x e_1) = \mathrm{Trd}_A(e_1 e_0 x) = 0 \qquad \text{and} \qquad \mathrm{Trd}_A(e_1 x e_0) = \mathrm{Trd}_A(e_0 e_1 x) = 0.$$

Therefore,

$$\mathrm{Trd}_A(x) = \mathrm{Trd}_{A_0}(e_0 x e_0) + \mathrm{Trd}_{A_1}(e_1 x e_1).$$

For $\ell \in L$, we have $\varepsilon(\ell) = \varepsilon_0(\ell) e_0 + \varepsilon_1(\ell) e_1$, hence

$$\mathrm{Trd}_A(\varepsilon(\ell) x) = \mathrm{Trd}_{A_0}(\varepsilon_0(\ell) e_0 x e_0) + \mathrm{Trd}_{A_1}(\varepsilon_1(\ell) e_1 x e_1) \qquad \text{for } x \in \mathrm{Sym}(\sigma).$$

Since ε_0 and ε_1 satisfy (3), it follows that

$$\mathrm{Trd}_A(\varepsilon(\ell) x) = T_{L/F}(\ell)\big(f_0(e_0 x e_0) + f_1(e_1 x e_1)\big).$$

Since, by definition of the orthogonal sum,

$$f(x) = f_0(e_0 x e_0) + f_1(e_1 x e_1),$$

the proof is complete. □

Appendix B. Flat cohomology

Let G be an affine algebraic group scheme (not necessarily smooth) defined over a field F and let $\overline{F}$ be a fixed algebraic closure of F. For any integers $n \geq 1$ and i with $n \geq i \geq 0$, define

$$\varepsilon^i\colon \otimes_F^n \overline{F} \to \otimes_F^{n+1}\overline{F}, \qquad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_n,$$

and let

$$d^i\colon G(\otimes_F^n\overline{F}) \to G(\otimes_F^{n+1}\overline{F})$$

be the induced map.

An element $g \in G(\overline{F} \otimes \overline{F})$ is a 1-*cocycle* if

$$d^1 g = (d^0 g)(d^2 g) \text{ in } G(\overline{F} \otimes \overline{F} \otimes \overline{F}).$$

Two cocycles g, $g' \in G(\overline{F} \otimes \overline{F})$ are *cohomologous* if there exists $h \in G(\overline{F} \otimes \overline{F})$ such that

$$g' = (d^0 h) g (d^1 h)^{-1}.$$

We thus obtain an equivalence relation, and we denote by $H^1(F, G)$ the quotient set. This a pointed set, the distinguished element being the class of the neutral element. If G is abelian, this set has a natural group structure, and one can define higher cohomology groups. We recall only the definition of $H^2(F, G)$. A 2-*cocycle* is an element $g \in G(\overline{F} \otimes \overline{F} \otimes \overline{F})$ satisfying

$$(d^0 g)(d^1 g)^{-1}(d^2 g)(d^3 g)^{-1} = 1.$$

A 2-*coboundary* is a 2-cocycle $g \in G(\overline{F} \otimes \overline{F} \otimes \overline{F})$ of the form $(d^0 h)(d^1 h)^{-1}(d^2 h)$ for some $h \in G(\overline{F} \otimes \overline{F})$. We define $H^2(F, G)$ to be the quotient group of the group of 2-cocycles by the group of 2-coboundaries.

If $1 \to N \to G \to H \to 1$ is an exact sequence of algebraic groups, we have an exact sequence in flat cohomology

$$1 \to N(F) \to G(F) \to H(F) \to H^1(F, N) \to H^1(F, G) \to H^1(F, H),$$

see [27, §18.1]. If N, G, H are abelian, all the morphisms are group morphisms. If N is a central subgroup of G, the exact sequence extends to

$$1 \to N(F) \to G(F) \to H(F) \to H^1(F, N) \to H^1(F, G) \to H^1(F, H) \to H^2(F, N).$$

If G is smooth, the flat cohomology sets can be identified with the Galois cohomology sets

$$H^1(F, G) = H^1\big(\mathrm{Gal}(F_s/F), G(F_s)\big)$$

where F_s is a separable closure of F, see [27, §18.5]. Moreover, connecting maps induced by exact sequences agree under this isomorphism. In view of these properties, the standard identifications valid in Galois cohomology under assumptions on the characteristic of F extend to arbitrary fields; for example $H^1(F, \mu_n) \simeq F^\times/F^{\times n}$ and $H^2(F, \mu_n) \simeq {}_n\mathrm{Br}(F)$.

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School of Mathematics, University of Southampton, Highfield, SO17 1BJ, United Kingdom

E-mail address: G.W.Berhuy@soton.ac.uk

URL: http://www.maths.soton.ac.uk/staff/Berhuy

Institut de Mathématique Pure et Appliquée, Université catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

E-mail address: tignol@math.ucl.ac.be

URL: http://www.math.ucl.ac.be/membres/tignol