J-invariant of linear algebraic groups

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Abstract

Let G be a linear algebraic group over a field F and X be a projective homogeneous G-variety such that G splits over the function field of X. In the present paper we introduce an invariant of G called J-invariant which characterizes the motivic behaviour of X. This generalizes the respective notion invented by A. Vishik in the context of quadratic forms. As a main application we obtain a uniform proof of all known motivic decompositions of generically split projective homogeneous varieties (Severi-Brauer varieties, Pfister quadrics, maximal orthogonal Grassmannians, G_2- and F_4-varieties) as well as provide new examples (exceptional varieties of types E_6, E_7 and E_8). We also discuss relations with torsion indices, canonical dimensions and cohomological invariants of the group G.

Introduction

Let G be a simple linear algebraic group over a field F and X be a projective homogeneous G-variety. In the present paper we address the problem of computing the Grothendieck-Chow motive $\mathcal{M}(X)$ of X or, in other words, providing a direct sum decomposition of $\mathcal{M}(X)$.

When the group G is split, i.e., contains a split maximal torus, the motive of X has the simplest decomposition – it is isomorphic to a direct sum of twisted Tate motives. This was first observed by B. Köck in [Kö91]. The next step was done by V. Chernousov, S. Gille and A. Merkurjev (see

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[CGM]). They provided an algorithm of expressing the motive of any rational projective homogeneous G-variety in terms of motives of anisotropic G-varieties, i.e., those which have no rational points. Finally, P. Brosnan (see [Br05]) generalized this result to the case of an isotropic group G and possibly anisotropic X. In all these proofs one constructs a (relative) cellular filtration on X, which allows to express the motive of the total space X in terms of motives of the base. Since the latter consists of homogeneous varieties of anisotropic groups, it reduces the problem to the case of anisotropic group G.

For anisotropic groups the situation becomes more complicated: only very few partial results are known. Observe that in this case the components of a motivic decomposition are expected to have a non-geometric nature, i.e., can not be identified with (twisted) motives of some other varieties. The first examples of such decompositions were provided by M. Rost [Ro98]. He proved that the motive of a Pfister quadric decomposes as a direct sum of twisted copies of a certain non-geometric motive $\mathcal{R}$ known in the literature as Rost motive. Motives of Severi-Brauer varieties were described by N. Karpenko [Ka96]. For exceptional varieties examples of motivic decompositions were provided by J.-P. Bonnet [Bo03] (varieties of type G_2) and by S. Nikolenko, N. Semenov, K. Zainoulline [NSZ] (varieties of type F_4).

In the present paper we provide a uniform proof of all these results. Observe that in all examples the variety X has the following property: Over the generic point of X the motive $\mathcal{M}(X)$ splits as a direct sum of twisted Tate motives. Varieties which satisfy this condition will be called *generically split*. The main result of the paper says that (see Theorem 5.1)

Theorem. *Let G be a simple linear algebraic group of inner type over a field F and p be a prime integer. Let X be a generically split projective homogeneous G-variety. Then the Chow motive of X with $\mathbb{Z}/p$-coefficients is isomorphic to a direct sum*

$$\mathcal{M}(X;\mathbb{Z}/p) \simeq \bigoplus_{i\geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i}$$

of twisted copies of a certain indecomposable motive $\mathcal{R}_p(G)$, where the integers a_i are coefficients of the quotient of the respective Poincaré polynomials.

Observe that the motive $\mathcal{R}_p(G)$ depends only on G and p but not on the type of a parabolic subgroup defining X. Moreover, considered with $\mathbb{Q}$-coefficients it always splits as a direct sum of Tate motives.

Our proof is based on two different observations. The first comes from the topology of compact Lie groups. Namely, to compute Chow rings of compact Lie groups V. Kac [Kc85] invented the notion of *p-exceptional degrees* – numbers which encode the information about the Chow ring of a group modulo p. These numbers can be expressed in terms of degrees of basic polynomial invariants and, therefore, deal with the combinatorics of the Weyl group. From the other hand the results of N. Karpenko, A. Merkurjev [KM06] and K. Zainoulline [Za06] concerning *canonical p-dimensions* of algebraic groups tell us that there is a strong interrelation between p-exceptional degrees and the 'size' of the image of the restriction map $\mathrm{res}\colon \mathrm{CH}^*(X\times X) \to \mathrm{CH}^*(\overline{X}\times\overline{X})$ to the separable closure of F. To relate this image with a motivic decomposition of X we use *Rost Nilpotence Theorem*. It was first proved for quadrics by M. Rost and lately generalized to arbitrary projective homogeneous varieties by V. Chernousov, S. Gille and A. Merkurjev (see [CGM, Theorem 8.2]). Roughly speaking, it says that any decomposition which is given by idempotent cycles from this image, indeed, comes from F.

All this together lead to the notion of *J-invariant* $J_p(G)$ of a group G modulo p (see Definition 4.5) which generalizes the respective notion introduced by A. Vishik [Vi05] in the case of quadrics. Our main observation is that $J_p(G)$ completely characterizes the motive $\mathcal{R}_p(G)$ and, hence, the motivic decomposition of X. Observe that if the J-invariant takes its minimal possible non-trivial value $J_p(G) = (1)$, then the motive $\mathcal{R}_p(G)\otimes\mathbb{Q}$ has the following recognizable decomposition (cf. [Vo03, §5])

$$\mathcal{R}_p(G)\otimes\mathbb{Q} \simeq \bigoplus_{i=0}^{p-1} \mathbb{Q}(i\cdot \tfrac{p^{n-1}-1}{p-1})$$

where $n = 2$ or 3 (see the last section).

Apart from the notion of J-invariant we generalize some of the results of paper [CPSZ]: Using the motivic version of a result of D. Edidin and W. Graham [EG97] on cellular fibrations we provide a general formulae which expresses the motive of the total space of a cellular fibration in terms of the motives of its base (see Theorem 3.8). We also provide several criteria for the existence of liftings of motivic decompositions via the reduction map $\mathbb{Z}\to\mathbb{Z}/m$. We prove that such a lifting always exists if $m = 2$, 3, 4 or 6 (see Theorem 2.16).

The paper is organized as follows. In the first section we recall the definition of Chow motives and some properties of generically split varieties.

Rather technical section 2 is devoted to lifting of idempotents. In section 3 we discuss motives of cellular fibrations. The proof of the main result actually starts with section 4, where we introduce the notion of J-invariant and provide a motivic decomposition for the variety of complete flags. In section 5 we finish the proof and give some properties of the motive $\mathcal{R}_p(G)$. The last two sections are devoted to various applications of J-invariant and examples of motivic decompositions.

1 Motives of generically split varieties

1.1. In the present paper we will work with Chow motives of smooth projective varieties over a field F. We will use the following notation (cf. [Ma68], [CGM, §7] or [EKM, XII])

Given a smooth projective variety X over a field F we denote by $\mathcal{M}(X)$ its *Chow motive*, and by $\mathcal{M}(X)(n) = \mathcal{M}(X) \otimes \mathbb{Z}(n)$ the respective *twist* by a Tate object. A morphism of motives between $\mathcal{M}(X)(n)$ and $\mathcal{M}(Y)(m)$, where X is irreducible, is given by a class ϕ of rationally equivalent cycles of dimension $\dim X + n - m$ on $X \times Y$. Hence, the group of endomorphisms $\mathrm{End}(\mathcal{M}(X))$ coincides with the Chow group $\mathrm{CH}_{\dim X}(X \times X)$. The element ϕ is called a *correspondence* between X and Y of degree $n - m$.

Given a correspondence ϕ of degree d and $k \in \mathbb{Z}$ the composite

$$\mathrm{CH}_k(X) \xrightarrow{(\mathrm{pr}_X)^*} \mathrm{CH}_{k+\dim Y}(X \times Y) \xrightarrow{-\cap\phi} \mathrm{CH}_{k+d}(X \times Y) \xrightarrow{(\mathrm{pr}_Y)_*} \mathrm{CH}_{k+d}(Y)$$

of the pull-back $(\mathrm{pr}_X)^*$, intersection product with ϕ and the push-forward $(\mathrm{pr}_Y)_*$ is called the *realization* of ϕ and is denoted by $\phi_\star$. Given correspondences $\phi \in \mathrm{CH}_{\dim X+d}(X \times Y)$ and $\psi \in \mathrm{CH}_{\dim Y+e}(Y \times Z)$ of degrees d and e respectively the correspondence of degree $d + e$

$$(\mathrm{pr}_{X\times Z})_*((\mathrm{pr}_{Y\times Z})^* \cap (\mathrm{pr}_{X\times Y})^*) \in \mathrm{CH}_{\dim X+d+e}(X \times Z)$$

is called the *correspondence product* of ϕ and ψ and is denoted by $\psi \circ \phi$. By definition $(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star$. Given a correspondence ϕ we denote by ϕ^t its transpose.

The correspondence product endows the group $\mathrm{End}(\mathcal{M}(X))$ with the ring structure. The identity element of this ring is the class of the diagonal Δ_X.

Finally, by $\mathcal{M}(X;\Lambda)$, where Λ is a commutative ring, we denote an object of the category of motives with Λ-coefficients obtained by taking correspondences ϕ with Λ-coefficients, i.e., replacing $\mathrm{CH}(X \times Y)$ by $\mathrm{CH}(X \times Y) \otimes_{\mathbb{Z}} \Lambda$.

1.2 Definition. Let L/F be a field extension. We say L is a *splitting field* of a smooth projective variety X or, equivalently, a variety X *splits* over L if the motive $\mathcal{M}(X)$ splits over L as a finite direct sum of twisted Tate motives, i.e.,

$$\mathcal{M}(X)_L \simeq \bigoplus_n \mathbb{Z}_L(n)$$

We say a variety X is *generically split* if it splits over the field of rational functions $F(X)$.

1.3 Example. A variety X is called *generically cellular* if it is cellular over its generic point, i.e., $X_{F(X)} = X \times_F F(X)$ has a proper descending filtration by closed subvarieties X_i such that each complement $X_i \setminus X_{i+1}$ is a disjoint union of affine spaces defined over $F(X)$. According to [EKM, Corollary 67.2] if X is generically cellular, then X is generically split.

1.4 Example. Let G be a semisimple linear algebraic group over a field F and X be a projective homogeneous G-variety. Let L/F be a field extension. If the group G splits over L, i.e., $G_L = G \times_F L$ contains a split maximal torus defined over L, then X_L is cellular and, hence, L is a splitting field of X. In particular, if G splits over the function field of X, then X is generically split. Many examples of such varieties can be provided investigating Tits indices of G (see Example 3.7).

1.5. Assume X has a splitting field L. We will write $\mathrm{CH}(\overline{X};\Lambda)$ for $\mathrm{CH}(X_L;\Lambda)$ and $\overline{\mathrm{CH}}(X;\Lambda)$ for the image of the restriction map $\mathrm{CH}(X;\Lambda) \to \mathrm{CH}(\overline{X};\Lambda)$ (cf. [KM06, 1.2]). Similarly, by $\mathcal{M}(\overline{X})$ we denote the motive of X considered over L. If M is a direct summand of $\mathcal{M}(X)(n)$, by $\overline{M}$ we denote the motive M_L. Elements of $\overline{\mathrm{CH}}(X)$ will be called *rational* cycles on X_L with respect to the field extension L/F. If L' is another splitting field of X, then there is a chain of canonical isomorphisms $\mathrm{CH}(X_L) \simeq \mathrm{CH}(X_{LL'}) \simeq \mathrm{CH}(X_{L'})$, where LL' is the composite of L and L'. Hence, the groups $\mathrm{CH}(\overline{X})$ and $\overline{\mathrm{CH}}(X)$ do not depend on the choice of L.

There is the Künneth decomposition $\mathrm{End}(\mathcal{M}(\overline{X})) = \mathrm{CH}(\overline{X} \times \overline{X}) = \mathrm{CH}(\overline{X}) \otimes \mathrm{CH}(\overline{X})$ and Poincaré duality. The latter means that given a basis of $\mathrm{CH}(\overline{X})$ there is a dual one with respect to the pairing $(\alpha,\beta) \mapsto \deg(\alpha\cdot\beta)$, where deg is the degree map. In view of the Künneth decomposition the correspondence product of cycles in on $\mathrm{CH}(\overline{X}\times\overline{X})$ is given by the formula $(\alpha_1\times\beta_1)\circ(\alpha_2\times\beta_2) = \deg(\alpha_1\beta_2)(\alpha_2\times\beta_1)$, the realization by $(\alpha\times\beta)_\star(\gamma) = \deg(\alpha\gamma)\beta$ and the transpose by $(\alpha\times\beta)^t = \beta\times\alpha$. Since

$\mathrm{CH}_*(\overline{X})$ is a free graded $\mathbb{Z}$-module, we may define its Poincaré polynomial as $P(\mathrm{CH}_*(\overline{X}),t)=\sum_{i\geq 0}\mathrm{rk}_{\mathbb{Z}}\,\mathrm{CH}_i(\overline{X})\cdot t^i$.

Sometimes we will use contravariant notation CH^* for Chow groups, where $\mathrm{CH}^k(X)=\mathrm{CH}_{\dim X-k}(X)$ for irreducible X.

1.6 Lemma. *Let X and Y be two smooth projective varieties such that $F(Y)$ is a splitting field of X and Y has a splitting field. Consider the projection in the Künneth decomposition*

$$\mathrm{pr}_0\colon\ \mathrm{CH}^r(\overline{X}\times\overline{Y})=\bigoplus_{i=0}^{r}\mathrm{CH}^{r-i}(\overline{X})\otimes\mathrm{CH}^i(\overline{Y})\to\mathrm{CH}^r(\overline{X}).$$

Then for any $\rho\in\mathrm{CH}^r(\overline{X})$ we have $\mathrm{pr}_0^{-1}(\rho)\cap\overline{\mathrm{CH}}^r(X\times Y)\neq\emptyset$.

Proof. Let L be a common splitting field of X and Y. Lemma follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^r(X\times_F Y) \arrow[r, "\mathrm{res}_{L/F}"] \arrow[d] & \mathrm{CH}^r(X_L\times_L Y_L) \arrow[d] \arrow[dr, "\mathrm{pr}_0"] & \\
\mathrm{CH}^r(X_{F(Y)}) \arrow[r, "\simeq"] & \mathrm{CH}^r((X_L)_{L(Y_L)}) \arrow[r, "\simeq"] & \mathrm{CH}^r(X_L)
\end{tikzcd}$$

where the left square is obtained by taking the generic fiber of the base change morphism $X_L\to X$; the vertical arrows are taken from the localization sequence for Chow groups and, hence, are surjective; and the bottom horizontal maps are isomorphisms since L is a splitting field. $\square$

We will extensively use the following version of Rost Nilpotence Theorem.

1.7 Lemma. *Let X be a smooth projective variety such that it splits over any field K over which it has a rational point. Then for any α in the kernel of the natural map $\mathrm{End}(\mathcal{M}(X))\to\mathrm{End}(\mathcal{M}(\overline{X}))$ we have $\alpha^{\circ(\dim X+1)}=0$.*

Proof. See [EKM, Theorem 68.1]. $\square$

2 Lifting of idempotents

2.1 Definition. Given a $\mathbb{Z}$-graded ring A^* and two idempotents ϕ_1, $\phi_2\in A^0$ we say ϕ_1 and ϕ_2 are orthogonal if $\phi_1\phi_2=\phi_2\phi_1=0$. We say an element θ_{12} provides an isomorphism of degree d between idempotents ϕ_1 and ϕ_2 if $\theta_{12}\in\phi_2A^{-d}\phi_1$ and there exists $\theta_{21}\in\phi_1A^d\phi_2$ such that $\theta_{12}\theta_{21}=\phi_2$ and $\theta_{21}\theta_{12}=\phi_1$.

2.2 Example. Let Λ be a commutative ring. Set $A^* = \mathrm{End}^*(\mathcal{M}(X;\Lambda))$, where

$$\mathrm{End}^k(\mathcal{M}(X;\Lambda)) = \mathrm{CH}^k(X \times X;\Lambda), \quad k \in \mathbb{Z}$$

and the multiplication is given by the correspondence product. By definition $\mathrm{End}^0(\mathcal{M}(X;\Lambda))$ is the ring of endomorphisms of the motive $\mathcal{M}(X;\Lambda)$. Note that a direct summand of $\mathcal{M}(X;\Lambda)$ can be identified with a pair (X,ϕ), where ϕ is an idempotent, i.e., $\phi \circ \phi = \phi$ (see [EKM, ch. XII]). Then an isomorphism θ_{12} of degree d between ϕ_1 and ϕ_2 can be identified with an isomorphism between the motives (X,ϕ_1) and $(X,\phi_2)(d)$.

2.3 Definition. Let $f\colon A^* \to B^*$ be a homomorphism of $\mathbb{Z}$-graded rings. We say that f is *decomposition preserving* if given a family $\phi_i \in B^0$ of pair-wise orthogonal idempotents such that $\sum_i \phi_i = 1_B$, there exists a family of pair-wise orthogonal idempotents $\varphi_i \in A^0$ such that $\sum_i \varphi_i = 1_A$ and each $f(\varphi_i)$ is isomorphic to ϕ_i by means of an isomorphism of degree 0. We say f is *strictly decomposition preserving* if, moreover, one can choose φ_i such that $f(\varphi_i) = \phi_i$.

We say f is *isomorphism preserving* if for any idempotents φ_1 and φ_2 in A^0 and any isomorphism θ_{12} of degree d between idempotents $f(\varphi_1)$ and $f(\varphi_2)$ in B^0 there exists an isomorphism ϑ_{12} of degree d between φ_1 and φ_2. We say f is *strictly isomorphism preserving* if, moreover, one can choose ϑ_{12} such that $f(\vartheta_{12}) = \theta_{12}$.

2.4. By definition we have the following properties of *(strictly) decomposition and isomorphism preserving* morphisms:

(i) Let $f\colon A^* \to B^*$ and $g\colon B^* \to C^*$ be homomorphisms such that $g \circ f$ is *decomposition* (resp. *isomorphism*) *preserving* and g is *isomorphism preserving*. Then f is *decomposition* (resp. *isomorphism*) *preserving*.

(ii) Assume we are given a commutative diagram with $\ker f' \subset \operatorname{im} i$

$$\begin{tikzcd} A^* \arrow[r, two heads, "f"] \arrow[d, hook, "i"'] & B^* \arrow[d, hook, "i'"] \\ A'^* \arrow[r, two heads, "f'"] & B'^*. \end{tikzcd}$$

If f' is *strictly decomposition* (resp. *strictly isomorphism*) *preserving*, then so is f.

2.5 Proposition. *Let $f\colon A^* \to B^*$ be a surjective homomorphism such that the kernel of the restriction of f to A^0 consists of nilpotent elements. Then f is strictly decomposition (cf. [EKM, Proposition 95.1]) and strictly isomorphism preserving.*

Proof. The fact that f is *strictly decomposition preserving* follows from [AF92, Proposition 27.4]. The fact that f is *strictly isomorphism preserving* follows from Lemma 2.6 below. $\square$

2.6 Lemma. *Let A, B be two rings, A^0, B^0 be their subrings, $f^0\colon A^0 \to B^0$ be a ring homomorphism, $f\colon A \to B$ be a map of sets satisfying the following conditions:*

- *$f(\alpha)f(\beta)$ equals either $f(\alpha\beta)$ or 0 for all $\alpha, \beta \in A$;*
- *$f^0(\alpha)$ equals $f(\alpha)$ if $f(\alpha) \in B^0$ or 0 otherwise;*
- $\ker f^0$ *consists of nilpotent elements.*

Let φ_1 and φ_2 be two idempotents in A^0, ψ_{12} and ψ_{21} be elements in A such that $\psi_{12}A^0\psi_{21} \subset A^0$, $\psi_{21}A^0\psi_{12} \subset A^0$, $f(\psi_{21})f(\psi_{12}) = f(\varphi_1)$, $f(\psi_{12})f(\psi_{21}) = f(\varphi_2)$. Then there exist elements $\vartheta_{12} \in \varphi_2 A^0\psi_{12}A^0\varphi_1$ and $\vartheta_{21} \in \varphi_1 A^0\psi_{21}A^0\varphi_2$ such that $\vartheta_{21}\vartheta_{12} = \varphi_1$, $\vartheta_{12}\vartheta_{21} = \varphi_2$, $f(\vartheta_{12}) = f(\varphi_2)f(\psi_{12}) = f(\psi_{12})f(\varphi_1)$, $f(\vartheta_{21}) = f(\varphi_1)f(\psi_{21}) = f(\psi_{21})f(\varphi_2)$.

Proof. Since $\ker f^0$ consists of nilpotents, f^0 sends non-zero idempotents in A^0 to non-zero idempotents in B^0; in particular, $f(\varphi_1) = f^0(\varphi_1) \neq 0$, $f(\varphi_2) = f^0(\varphi_2) \neq 0$. Observe that

$$f(\psi_{12})f(\varphi_1) = f(\psi_{12})f(\psi_{21})f(\psi_{12}) = f(\varphi_2)f(\psi_{12})$$

and, similarly, $f(\psi_{21})f(\varphi_2) = f(\varphi_1)f(\psi_{21})$. Changing ψ_{12} to $\varphi_2\psi_{12}\varphi_1$ and ψ_{21} to $\varphi_1\psi_{21}\varphi_2$ we may assume that $\psi_{12} \in \varphi_2 A\varphi_1$ and $\psi_{21} \in \varphi_1 A\varphi_2$. We have

$$f^0(\varphi_2) = f(\varphi_2) = f(\psi_{12})f(\psi_{21}) = f(\psi_{12}\psi_{21}) = f^0(\psi_{12}\psi_{21});$$

therefore $\alpha = \psi_{12}\psi_{21} - \varphi_2 \in A^0$ is nilpotent, say $\alpha^n = 0$. Note that $\varphi_2\alpha = \alpha = \alpha\varphi_2$. Set $\alpha^\vee = \varphi_2 - \alpha + \ldots + (-1)^{n-1}\alpha^{n-1} \in A^0$; then $\alpha\alpha^\vee = \varphi_2 - \alpha^\vee$, $\varphi_2\alpha^\vee = \alpha^\vee = \alpha^\vee\varphi_2$ and $f(\varphi_2) = f^0(\varphi_2) = f^0(\alpha^\vee) = f(\alpha^\vee)$. Therefore setting $\vartheta_{21} = \psi_{21}\alpha^\vee$ we have $\vartheta_{21} \in \varphi_1 A\varphi_2$, $\psi_{12}\vartheta_{21} = \varphi_2$ and $f(\vartheta_{21}) = f(\psi_{21})$.

Now $\vartheta_{21}\psi_{12}$ is an idempotent. We have

$$f^0(\varphi_1) = f(\varphi_1) = f(\vartheta_{21})f(\psi_{12}) = f(\vartheta_{21}\psi_{12}) = f^0(\vartheta_{21}\psi_{12});$$

therefore $\beta = \vartheta_{21}\psi_{12} - \varphi_1 \in A^0$ is nilpotent. Note that $\beta\varphi_1 = \beta = \varphi_1\beta$. Now $\varphi_1 + \beta = (\varphi_1 + \beta)^2 = \varphi_1 + 2\beta + \beta^2$ and therefore $\beta(1+\beta) = 0$. But $1+\beta$ is invertible and hence we have $\beta = 0$. It means that $\vartheta_{21}\psi_{12} = \varphi_1$ and we can set $\vartheta_{12} = \psi_{12}$. $\square$

2.7 Corollary. *The map* $\mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/p^n)) \to \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/p))$ *is strictly decomposition (cf. [EKM, Corollary 95.3]) and strictly isomorphism preserving.*

Proof. Apply Proposition 2.5 to the case $A^* = \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/p^n))$, $B^* = \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/p))$ and the reduction map $f\colon A^* \to B^*$. $\square$

2.8 Lemma. *Let $m = m_1m_2$ be a product of two coprime integers. Then the map* $\mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m)) \to \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m_1)) \times \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m_2))$ *is an isomorphism.*

Proof. Apply Chinese Remainder Theorem. $\square$

2.9. From now on assume that X is a smooth projective variety which has a splitting field such that the kernel of the restriction map

$$\mathrm{res}_E\colon\ \mathrm{End}^*(\mathcal{M}(X_E;\Lambda)) \to \mathrm{End}^*(\mathcal{M}(\overline{X};\Lambda))$$

consists of nilpotent elements for all extensions E/F and all rings of coefficients Λ. By Lemma 1.7 the latter holds if X splits over any field extension over which it has a rational point. We denote by $\overline{\mathrm{End}^*}(\mathcal{M}(X;\Lambda))$ the image of res_F.

2.10 Corollary. *The map* $\mathrm{End}^*(\mathcal{M}(X_E;\Lambda)) \to \overline{\mathrm{End}^*}(\mathcal{M}(X_E;\Lambda))$ *is strictly decomposition and strictly isomorphism preserving for any field extension E/F.*

Proof. Apply Proposition 2.5 to the homomorphism $\mathrm{res}_E\colon A^* \to B^*$ between the graded rings $A^* = \mathrm{End}^*(\mathcal{M}(X_E;\Lambda))$ and $B^* = \overline{\mathrm{End}^*}(\mathcal{M}(X_E;\Lambda))$. $\square$

2.11 Definition. We say that a field extension E/F is *rank preserving* with respect to X if the restriction map $\mathrm{res}_{E/F}\colon\ \mathrm{CH}(X) \to \mathrm{CH}(X_E)$ becomes an isomorphism after tensoring with $\mathbb{Q}$.

2.12 Lemma. *Assume X has a splitting field. Then for any rank preserving finite field extension E/F we have $[E:F]\cdot\overline{\mathrm{CH}}(X_E)\subset\overline{\mathrm{CH}}(X)$.*

Proof. Let L be a splitting field containing E. Let γ be any element in $\overline{\mathrm{CH}}(X_E)$. By definition there exists $\alpha\in\mathrm{CH}(X_E)$ such that $\gamma=\mathrm{res}_{L/E}(\alpha)$. Since $\mathrm{res}_{E/F}\otimes\mathbb{Q}$ is an isomorphism, there exists an element $\beta\in\mathrm{CH}(X)$ and a non-zero integer n such that $\mathrm{res}_{E/F}(\beta)=n\alpha$. By projection formula

$$n\cdot\mathrm{cores}_{E/F}(\alpha)=\mathrm{cores}_{E/F}(\mathrm{res}_{E/F}(\beta))=[E:F]\cdot\beta.$$

Applying $\mathrm{res}_{L/E}$ to both sides we obtain $n(\mathrm{res}_{L/E}(\mathrm{cores}_{E/F}(\alpha)))=n[E:F]\cdot\gamma$. Therefore, $\mathrm{res}_{L/E}(\mathrm{cores}_{E/F}(\alpha))=[E:F]\cdot\alpha$. □

2.13 Corollary. *Assume X has a splitting field, E/F is a field extension of degree coprime with m, which is rank preserving with respect to $X\times X$. Then the map $\mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m))\to\mathrm{End}^*(\mathcal{M}(X_E;\mathbb{Z}/m))$ is decomposition and isomorphism preserving.*

Proof. By Lemma 2.12 we have $\overline{\mathrm{End}^*}(\mathcal{M}(X_E;\mathbb{Z}/m))=\overline{\mathrm{End}^*}(\mathcal{M}(X;\mathbb{Z}/m))$. Now apply Corollary 2.10 and 2.4.(i) with $A^*=\mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m))$, $B^*=\mathrm{End}^*(\mathcal{M}(X_E;\mathbb{Z}/m))$ and $C^*=\overline{\mathrm{End}^*}(\mathcal{M}(X_E;\mathbb{Z}/m))$. □

2.14 Lemma. *The map $\mathrm{SL}_l(\mathbb{Z})\to\mathrm{SL}_l(\mathbb{Z}/m)$ induced by the reduction modulo m is surjective.*

Proof. Since $\mathbb{Z}/m$ is a semi-local ring, the group $\mathrm{SL}_l(\mathbb{Z}/m)$ is generated by elementary matrices (see [HOM, Theorem 4.3.9]). □

Given a free graded $\mathbb{Z}$-module V^* set $\mathrm{End}^{-d}(V^*)$, $d\in\mathbb{Z}$, to be the group of endomorphisms of V^* decreasing the degree by d.

2.15 Proposition. *(cf. [EKM, §96]) Consider a free graded $\mathbb{Z}$-module V^* of finite rank and the reduction map $f\colon\mathrm{End}^*(V^*)\to\mathrm{End}^*(V^*\otimes_{\mathbb{Z}}\mathbb{Z}/m)$. Assume that the graded components of the respective $\mathrm{im}\,\phi_i$ (see Definition 2.3) are free $\mathbb{Z}/m$-modules. Then f is strictly decomposition preserving. Moreover, if $(\mathbb{Z}/m)^\times=\{\pm1\}$, then f is strictly isomorphism preserving.*

Proof. We are given a decomposition $V^k\otimes_{\mathbb{Z}}\mathbb{Z}/m=\oplus_i W_i^k$, where W_i^k is the k-graded component of $\mathrm{im}\,\phi_i$. Present V^k as a direct sum $V^k=\bigoplus_i V_i^k$ of free $\mathbb{Z}$-modules such that $\mathrm{rk}_{\mathbb{Z}}V_i^k=\mathrm{rk}_{\mathbb{Z}/m}W_i^k$. Fix a $\mathbb{Z}$-basis $\{v_{ij}^k\}_j$ of V_i^k. For each W_i^k choose a basis $\{w_{ij}^k\}_j$ such that the linear transformation D^k of

$V^k \otimes_{\mathbb{Z}} \mathbb{Z}/m$ sending each $v_{ij}^k \otimes 1$ to w_{ij}^k has determinant 1. By Lemma 2.14 there is a lifting $\tilde{D}^k$ of D^k over $\mathbb{Z}$. So we obtain $V^k = \bigoplus_i \tilde{W}_i^k$, where $\tilde{W}_i^k = \tilde{D}^k(V_i^k)$ satisfies $\tilde{W}_i^k \otimes_{\mathbb{Z}} \mathbb{Z}/m = W_i^k$. Define φ_i on each V^k to be the projection onto $\tilde{W}_i^k$.

Now let φ_1, φ_2 be two idempotents in $\mathrm{End}^*(V^*)$. Denote by V_i^k the k-graded component of $\mathrm{im}\, \varphi_i$. An isomorphism θ_{12} between $\varphi_1 \otimes 1$ and $\varphi_2 \otimes 1$ of degree d can be identified with a family of isomorphisms $\theta_{12}^k \colon V_1^k \otimes \mathbb{Z}/m \to V_2^{k-d} \otimes \mathbb{Z}/m$. In the case $(\mathbb{Z}/m)^\times = \{\pm 1\}$ all these isomorphisms are given by matrices with determinants $\{\pm 1\}$ and, hence, can be lifted to isomorphisms $\vartheta_{12}^k \colon V_1^k \to V_2^{k-d}$ by Lemma 2.14. $\square$

Now we are ready to formulate and prove the main result of this section

2.16 Theorem. *Assume X has a splitting field of degree m which is rank preserving with respect to $X \times X$. Then the map*

$$\mathrm{End}^*(\mathcal{M}(X)) \to \mathrm{End}^*(\mathcal{M}(X; \mathbb{Z}/m))$$

preserves decompositions with the property that $\mathrm{im}\,\mathrm{res}(\phi_i)$ *are free $\mathbb{Z}/m$-modules, where*

$$\mathrm{res} \colon \mathrm{End}^*(\mathcal{M}(X; \mathbb{Z}/m)) \to \mathrm{End}^*(\mathcal{M}(\overline{X}; \mathbb{Z}/m))$$

is the restriction. If additionally $(\mathbb{Z}/m)^\times = \{\pm 1\}$ then this map is isomorphism preserving.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{End}^*(\mathcal{M}(X)) \arrow[r, two heads, "f"] \arrow[d] & \mathrm{End}^*(\mathcal{M}(X; \mathbb{Z}/m)) \arrow[d] \\
\overline{\mathrm{End}^*}(\mathcal{M}(X)) \arrow[r, two heads, "\bar{f}"] \arrow[d, hook, "i"'] & \overline{\mathrm{End}^*}(\mathcal{M}(X; \mathbb{Z}/m)) \arrow[d, hook] \\
\mathrm{End}^*(\mathcal{M}(\overline{X})) \arrow[r, two heads, "f'"] & \mathrm{End}^*(\mathcal{M}(\overline{X}; \mathbb{Z}/m)).
\end{tikzcd}$$

Note that using Poincaré duality (see 1.5) we can identify $\mathrm{End}^{-d}(\mathcal{M}(\overline{X}))$ with the group of endomorphisms of $\mathrm{CH}^*(\overline{X})$ which decrease the grading by d. Applying Proposition 2.15 to the case $V^* = \mathrm{CH}^*(\overline{X})$ we obtain that the map f' is *strictly decomposition preserving*. Moreover, if $(\mathbb{Z}/m)^\times = \{\pm 1\}$ then f' is *strictly isomorphism preserving*. By Lemma 2.12 $\ker f' \subset \mathrm{im}\, i$ and, therefore, applying 2.4.(ii) we obtain that $\bar{f}$ is *strictly decomposition preserving* and, moreover, $\bar{f}$ is *strictly isomorphism preserving* if $(\mathbb{Z}/m)^\times = \{\pm 1\}$. Now by Corollary 2.10 the vertical arrows of the top square are *strictly decomposition* and *strictly isomorphism preserving.* It remains to apply 2.4.(i). $\square$

3 Motives of fibered spaces

3.1 Definition. Let X be a smooth projective variety over a field F. We say a smooth projective morphism $f\colon Y \to X$ is a *cellular fibration* if it is a locally trivial fibration whose fiber $\mathcal{F}$ is cellular, i.e., has a decomposition into affine cells (see [EKM, §67]).

3.2 Lemma. *Let $f\colon Y \to X$ be a cellular fibration. Then $\mathcal{M}(Y)$ is isomorphic to $\mathcal{M}(X) \otimes \mathcal{M}(\mathcal{F})$.*

Proof. We follow the proof of [EG97, Proposition 1]. Define the morphism

$$\varphi\colon \bigoplus_{i\in\mathcal{I}} \mathcal{M}(X)(\operatorname{codim} B_i) \to \mathcal{M}(Y)$$

to be the direct sum $\varphi = \bigoplus_{i\in\mathcal{I}} \varphi_i$, where each φ_i is given by the cycle $[\mathrm{pr}_Y^*(B_i)\cdot\Gamma_f] \in \mathrm{CH}(X\times Y)$ produced from the graph cycle Γ_f and the chosen (non-canonical) basis $\{B_i\}_{i\in\mathcal{I}}$ of $\mathrm{CH}(Y)$ over $\mathrm{CH}(X)$. The realization of φ coincides exactly with an isomorphism of abelian groups $\mathrm{CH}(X)\otimes\mathrm{CH}(\mathcal{F}) \to \mathrm{CH}(Y)$ constructed in [EG97, Proposition 1]. Then, by Manin's identity principle (see [Ma68, §3]) φ is an isomorphism. $\square$

3.3 Lemma. *Let G be a linear algebraic group over a field F, X be a projective homogeneous G-variety and Y be a G-variety. Let $f\colon Y \to X$ be a G-equivariant projective morphism. Assume that the fiber of f over $F(X)$ is isomorphic to $\mathcal{F}_{F(X)}$ for some variety $\mathcal{F}$ over F. Then f is a locally trivial fibration with the fiber $\mathcal{F}$.*

Proof. By the assumptions, we have $Y\times_X \mathrm{Spec}\, F(X) \simeq (\mathcal{F}\times X)\times_X \mathrm{Spec}\, F(X)$ as schemes over $F(X)$. Since $F(X)$ is a direct limit of $\mathcal{O}(U)$ taken over all non-empty affine open subsets U of X, by [EGA IV, Corollaire 8.8.2.5] there exists U such that $f^{-1}(U) = Y\times_X U$ is isomorphic to $(\mathcal{F}\times X)\times_X U \simeq \mathcal{F}\times U$ as a scheme over U. Since G acts transitively on X and f is G-equivariant, the map f is a locally trivial fibration. $\square$

3.4 Corollary. *Let X be a projective G-homogeneous variety, Y be a projective variety such that $Y_{F(X)} \simeq \mathcal{F}_{F(X)}$ for some variety $\mathcal{F}$. Then the projection map $X \times Y \to X$ is a locally trivial fibration with the fiber $\mathcal{F}$. Moreover, if $\mathcal{F}$ is cellular, then $\mathcal{M}(X \times Y) \simeq \mathcal{M}(X) \otimes \mathcal{M}(\mathcal{F})$.*

Proof. Apply Lemma 3.3 to the projection map $X \times Y \to X$ and use Lemma 3.2. $\square$

3.5. Let G be a simple (connected) linear algebraic group over a field F, X be a projective homogeneous G-variety. Denote by $\mathcal{D}$ the Dynkin diagram of G. According to [Ti66] one can always choose a quasi-split group G_0 over F with the same Dynkin diagram, a parabolic subgroup P of G_0 and a cocycle $\xi \in H^1(F, G_0)$ such that G is isogenic to ${}_\xi G_0$ and X is isomorphic to ${}_\xi(G_0/P)$. We will use the following standard notation: If G_0 is split, then G is called to be of *inner type* over F.

3.6 Lemma. *Let G be a semisimple linear algebraic group over F, X and Y be projective homogeneous G-varieties corresponding to parabolic subgroups P and Q of G_0, $Q \subseteq P$. Denote by $f\colon Y \to X$ the map induced by the quotient map $G_0/Q \to G_0/P$. If G becomes split over $F(X)$ then f is a cellular fibration with the fiber $\mathcal{F} = P/Q$.*

Proof. Since G splits over $F(X)$, the fiber of f over $F(X)$ is isomorphic to $(P/Q)_{F(X)} = \mathcal{F}_{F(X)}$. Now apply Lemma 3.3 and note that $\mathcal{F}$ is cellular. $\square$

3.7 Example. Let $P = P_\Theta$ be the standard parabolic subgroup of a split group G_0, corresponding to a subset Θ of the respective Dynkin diagram $\mathcal{D}$ (enumeration of roots follows Bourbaki). In this notation the Borel subgroup corresponds to the empty set. Let ξ be a cocycle in $H^1(F, G_0)$. Set $G = {}_\xi G_0$ and $X = {}_\xi(G_0/P)$. Denote by q the degree of a splitting field of G_0 and by d the index of associated Tits algebra (for groups of classical types d is given in [Ti66, Table II]). Analyzing Tits indices of G we see that G becomes split over $F(X)$ and, therefore, X becomes *generically split* over F if the subset $\mathcal{D} \setminus \Theta$ contains one of the following vertices k (cf. [KR94, §7]):

G_0	${}^1\mathrm{A}_n$	B_n	C_n	${}^1\mathrm{D}_n$
k	$\gcd(k,d) = 1$	$k = n$; any k in the Pfister case	k is odd;	$k = n-1$; $k = n$ if $2 \nmid n$ or $d = 1$; any k in the Pfister case

G_0	G_2	F_4	${}^1\mathrm{E}_6$	E_7	E_8
k	any	$k = 1, 2, 3$; any k if $q = 3$	$k = 3, 5$; $k = 2, 4$ if $d = 1$; $k = 1, 6$ if q is odd	$k = 2, 5$; $k = 3, 4$ if $d = 1$; $k \neq 7$ if $q = 3$	$k = 2, 3, 4, 5$; any k if $q = 5$

(here by the Pfister case we mean the case when the cocycle ξ corresponds to a Pfister form or its maximal neighbor)

Case-by-case arguments of paper [CPSZ] show that under certain conditions the Chow motive of a twisted flag variety X can be expressed in terms of the motive of a minimal flag. These conditions cover almost all twisted flag varieties corresponding to groups of types A_n and B_n together with some examples of types C_n, G_2 and F_4. Using the following theorem we provide a uniform proof of these results as well as extend it to some other types.

3.8 Theorem. *Let Y and X be taken as in Lemma 3.6. Then the Chow motive $\mathcal{M}(Y)$ of Y is isomorphic to a direct sum of twisted copies of the motive $\mathcal{M}(X)$, i.e.,*

$$\mathcal{M}(Y) \simeq \bigoplus_{i \geq 0} \mathcal{M}(X)(i)^{\oplus c_i},$$

where $\sum c_i t^i = P(\mathrm{CH}_*(\overline{Y}), t)/P(\mathrm{CH}_*(\overline{X}), t)$.

Proof. Apply Lemmas 3.6 and 3.2. □

3.9 Remark. The explicit formula for $P(\mathrm{CH}_*(\overline{X}), t)$ involves the degrees of basic polynomial invariants of G_0 and is provided in [Hi82, Ch. IV, Cor. 4.5].

4 Varieties of complete flags

4.1. Let G_0 be a split simple linear algebraic group with a maximal split torus T and a Borel subgroup B containing T. Let $G = {}_\xi G_0$ be a twisted form of G_0 given by a cocycle $\xi \in H^1(F, G_0)$ and $X = {}_\xi(G_0/B)$ be the corresponding *variety of complete flags*. Observe that the group G splits over any field K over which X has a rational point, in particular, over the function field $F(X)$. According to [De74] the Chow ring $\mathrm{CH}(\overline{X})$ can be expressed in purely combinatorial terms and, therefore, depends only on the type of G but not on the base field F.

4.2. Let p be a prime integer. To simplify the notation we denote by $\mathrm{Ch}(X)$ the Chow ring of $X = {}_\xi(G_0/B)$ with $\mathbb{Z}/p$-coefficients and by $\overline{\mathrm{Ch}}(X)$ the image of the restriction map $\mathrm{CH}(X;\mathbb{Z}/p) \to \mathrm{CH}(\overline{X};\mathbb{Z}/p)$. Let $\hat{T}$ denote the group of characters of T and $S(\hat{T})$ be the symmetric algebra. By R we denote the image of the characteristic map $c\colon S(\hat{T}) \to \mathrm{Ch}(\overline{X})$ (see [Gr58, (4.1)]). According to [KM06, Thm.6.4] there is an embedding

$$R \subseteq \overline{\mathrm{Ch}}(X), \tag{1}$$

where the equality holds if the cocycle ξ corresponds to a generic torsor.

4.3. Let $\mathrm{Ch}(\overline{G})$ denote the Chow ring with $\mathbb{Z}/p$-coefficients of the group G_0. Consider the pull-back induced by the quotient map

$$\pi\colon\ \mathrm{Ch}(\overline{X}) \to \mathrm{Ch}(\overline{G})$$

According to [Gr58, Rem. 2°] π is surjective with the kernel generated by R^+, where R^+ stands for the subgroup of non-constant elements of R.

The explicit presentation of $\mathrm{Ch}(\overline{G})$ is known for all types of G and all torsion primes p of G (see [Gr58, Definition 3]). Namely, by [Kc85, Theorem 3] it is a quotient of the polynomial ring in r variables $x_1, \ldots, x_r$ of codimensions $d_1 \leq d_2 \leq \ldots \leq d_r$ coprime to p, modulo an ideal generated by certain p-powers $x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}$ ($k_i \geq 0$, $i = 1, \ldots, r$)

$$\mathrm{Ch}^*(\overline{G}) = (\mathbb{Z}/p)[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}). \tag{2}$$

In the case p is not a torsion prime of G we have $\mathrm{Ch}^*(\overline{G}) = \mathbb{Z}/p$, i.e., $r = 0$.

Note that the complete list of numbers $\{d_i p^{k_i}\}_{i=1\ldots r}$ called *p-exceptional degrees* of G_0 was provided in [Kc85, Table II]. Taking the p-primary and p-coprimary parts of each p-exceptional degree one immediately restores the respective k_i and d_i.

4.4. We introduce two orders on the set of additive generators of $\mathrm{Ch}(\overline{G})$, i.e., monomials $x_1^{m_1} \ldots x_r^{m_r}$. To simplify the notation, we will denote the monomial $x_1^{m_1} \ldots x_r^{m_r}$ by x^M, where M is an r-tuple of integers $(m_1, \ldots, m_r)$. The codimension of x^M will be denoted by $|M|$. Observe that $|M| = \sum_{i=1}^r d_i m_i$.

- Given two r-tuples $M = (m_1, \ldots, m_r)$ and $N = (n_1, \ldots, n_r)$ we say $x^M \preccurlyeq x^N$ (or equivalently $M \preccurlyeq N$) if $m_i \leq n_i$ for all i. This gives a partial ordering on the set of all monomials (r-tuples).

- Given two r-tuples $M = (m_1, \dots, m_r)$ and $N = (n_1, \dots, n_r)$ we say $x^M \le x^N$ (or equivalently $M \le N$) if either $|M| < |N|$, or $|M| = |N|$ and $m_i \le n_i$ for the greatest i such that $m_i \neq n_i$. This gives a well-ordering on the set of all monomials (r-tuples) known also as *DegLex* order.

4.5 Definition. Let $X = {}_\xi(G_0/B)$ be the twisted form of the variety of complete flags by means of a cocycle $\xi \in H^1(F, G_0)$. Let $\overline{\mathrm{Ch}}(G)$ denote the image of the composite

$$\mathrm{Ch}(X) \xrightarrow{\mathrm{res}} \mathrm{Ch}(\overline{X}) \xrightarrow{\pi} \mathrm{Ch}(\overline{G})$$

Since both maps are ring homomorphisms, $\overline{\mathrm{Ch}}(G)$ is a subring of $\mathrm{Ch}(\overline{G})$.

For each $1 \le i \le r$ set j_i to be the smallest non-negative integer such that the subring $\overline{\mathrm{Ch}}(G)$ contains an element a with the greatest monomial $x_i^{p^{j_i}}$ with respect to the *DegLex* order on $\mathrm{Ch}(\overline{G})$, i.e., of the form

$$a = x_i^{p^{j_i}} + \sum_{x^M \lneq x_i^{p^{j_i}}} c_M x^M.$$

The r-tuple of integers $(j_1, \dots, j_r)$ will be called *J-invariant of G modulo p* and will be denoted by $J_p(G)$.

4.6 Example. From presentation (2) we have $j_i \le k_i$ for all $i = 1, \dots, r$. According to (1) the J-invariant takes its maximal value $J_p(G) = (k_1, \dots, k_r)$ if the cocycle ξ corresponds to a generic torsor. Later on (see Corollary 6.10) it will be shown that the J-invariant takes its minimal possible value $J_p(G) = (0, \dots, 0)$ if and only if the group G splits by a finite field extension of degree coprime to p.

4.7 Example. If the Chow ring $\mathrm{Ch}(\overline{G})$ has only one generator, i.e., $r = 1$, then the J-invariant is equal to the smallest non-negative integer j_1 such that $x_1^{p^{j_1}} \in \overline{\mathrm{Ch}}(G)$.

The next example explains the terminology 'J-invariant'

4.8 Example. Let ϕ be a quadratic form with trivial discriminant. In [Vi05, Definition 5.11] A. Vishik introduced the notion of J-invariant of ϕ, the tuple of integers, which describes the subgroup of rational cycles on the respective maximal orthogonal Grassmannian. This invariant is an important tool in studying algebraic cycles on quadrics. An equivalent but 'dual' (in terms of non-rationality of cycles) definition of $J(\phi)$ was provided in [EKM, § 88]. Using Theorem 3.8 one can show that $J(\phi)$ introduced in [EKM] can be expressed in terms of $J_2(\mathrm{O}^+(\phi)) = (j_1, \dots, j_r)$ as follows:

$$J(\phi) = \{2^l d_i \mid i = 1, \dots, r,\, 0 \leq l \leq j_i - 1\}.$$

Since all d_i are odd, $J_2(\mathrm{O}^+(\phi))$ is uniquely determined by $J(\phi)$ as well.

Now we are ready to formulate and prove the main result of this section

4.9 Theorem. *Given G and p with $J_p(G) = (j_1, \dots, j_r)$ the motive of X is isomorphic to the direct sum*

$$\mathcal{M}(X; \mathbb{Z}/p) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus c_i},$$

where the motive $\mathcal{R}_p(G)$ is indecomposable, its Poincaré polynomial over a splitting field is equal to

$$P(\overline{\mathcal{R}_p(G)}, t) = \prod_{i=1}^{r} \frac{1 - t^{d_i p^{j_i}}}{1 - t^{d_i}}, \tag{3}$$

and the integers c_i are the coefficients of the polynomial

$$\sum_{i \geq 0} c_i t^i = P(\mathrm{Ch}^*(\overline{X}), t) / P(\overline{\mathcal{R}_p(G)}, t).$$

Fix preimages e_i of x_i in $\mathrm{Ch}(\overline{X})$. For an r-tuple $M = (m_1, \dots, m_r)$ set $e^M = \prod_{i=1}^r e_i^{m_i}$. Set $K = (k_1, \dots, k_r)$ and $N = p^K - 1 = (p^{k_1} - 1, \dots, p^{k_r} - 1)$. Set $d = \dim X - |N| = \deg(P(R^*, t))$.

4.10 Lemma. *The Chow ring $\mathrm{Ch}(\overline{X})$ is a free R-module with a basis $\{e^M\}$, $M \preccurlyeq N$.*

Proof. Note that R^+ is a nilpotent ideal in R. Applying Nakayama Lemma we obtain that $\{e^M\}$ generate $\mathrm{Ch}(\overline{X})$. By [Kc85, (2)] $\mathrm{Ch}(\overline{X})$ is a free R-module, hence, for the Poincaré polynomials we have

$$P(\mathrm{Ch}^*(\overline{X}), t) = P(\mathrm{Ch}^*(\overline{G}), t) \cdot P(R^*, t).$$

Substituting $t = 1$ we obtain that

$$\mathrm{rk}\,\mathrm{Ch}(\overline{X}) = \mathrm{rk}\,\mathrm{Ch}(\overline{G}) \cdot \mathrm{rk}\, R.$$

To finish the proof observe that $\mathrm{rk}\,\mathrm{Ch}(\overline{G})$ coincides with the number of generators $\{e^M\}$. $\square$

4.11 Proposition. *The pairing $R \times R \to \mathbb{Z}/p$ given by $(\alpha, \beta) \mapsto \deg(e^N \alpha\beta)$ is non-degenerated, i.e., for any element $\alpha \in R$ there exists β such that $\deg(e^N\alpha\beta) \neq 0$.*

Proof. Choose a homogeneous basis of $\mathrm{Ch}(\overline{X})$. Let $\alpha^\vee$ be the Poincaré dual of α with respect to this basis. By Lemma 4.10 $\mathrm{Ch}(\overline{X})$ is a free R-module with a basis $\{e^M\}$, hence, expanding $\alpha^\vee$ we obtain

$$\alpha^\vee = \sum_{M \preccurlyeq N} e^M \beta_M, \text{ where } \beta_M \in R.$$

Note that if $M \neq N$ then $\operatorname{codim} \alpha\beta_M > d$, therefore, $\alpha\beta_M = 0$. So we can set $\beta = \beta_N$. □

From now on we fix a homogeneous $\mathbb{Z}/p$-basis $\{\alpha_i\}$ of R and the dual basis $\{\alpha_i^\#\}$ with respect to the pairing introduced in Proposition 4.11.

4.12 Corollary. *For $|M| \leq |N|$ we have*

$$\deg(e^M \alpha_i \alpha_j^\#) = \begin{cases} 1, & M = N \text{ and } i = j \\ 0, & \text{otherwise} \end{cases}$$

Proof. If $M = N$, then it follows from the definition of the dual basis. Assume $|M| < |N|$. If $\deg(e^M\alpha_i\beta_j) \neq 0$, then $\mathrm{codim}(\alpha_i\beta_j) > d$, a contradiction with the fact that $\alpha_i\beta_j \in R$. Hence, we reduced to the case $M \neq N$ and $|M| = |N|$. Since $|M| = |N|$, $\mathrm{codim}(\alpha_i\beta_j) = d$ and, hence, $R^+\alpha_i\beta_j = 0$. From the other side there exists i such that $m_i \geq p^{k_i}$ and $e^{p^{k_i}} \in \mathrm{Ch}(\overline{X}) \cdot R^+$. Hence, $e^M\alpha_i\beta_j = 0$. □

4.13 Definition. Given two pairs (L, l) and (M, m), where L, M are r-tuples and l, m are integers, we say $(L, l) \leq (M, m)$ if either $L \lneq M$, or in the case $L = M$ we have $l \leq m$. We introduce the filtration on the ring $\mathrm{Ch}(\overline{X})$ as follows:

> The (M, m)-th term $\mathrm{Ch}(\overline{X})_{M,m}$ is the subring generated by the elements $e^I\alpha$ with $I \leq M$, $\alpha \in R$, $\operatorname{codim}\alpha \leq m$.

Define the *associated graded ring* as follows:

$$A^{*,*} = \bigoplus_{(M,m)} A^{M,m}, \text{ where } A^{M,m} = \mathrm{Ch}(\overline{X})_{M,m} / \bigcup_{(L,l) \lneq (M,m)} \mathrm{Ch}(\overline{X})_{L,l}.$$

By Lemma 4.10 if $M \preccurlyeq N$ the graded component $A^{M,m}$ consists of the classes of elements $e^M\alpha$ with $\alpha \in R$ and $\operatorname{codim}\alpha = m$. In particular, $\operatorname{rk} A^{M,m} = \operatorname{rk} R^m$. Comparing the ranks we see that $A^{M,m}$ is trivial when $M \not\preccurlyeq N$.

Consider the subring $\overline{\mathrm{Ch}}(X)$ of rational cycles with the induced filtration. The associated graded subring will be denoted by $A^{*,*}_{rat}$. From the definition of J-invariant it follows that the elements $e_i^{p^{j_i}}$, $i = 1, \ldots, r$, belong to $A^{*,*}_{rat}$.

Similarly, we introduce the filtration on the ring $\mathrm{Ch}(\overline{X} \times \overline{X})$ as follows:

> The (M, m)-th term is the subring generated by the elements $e^I\alpha \times e^L\beta$ with $I+L \leq M$, $\alpha, \beta \in R$ and $\operatorname{codim}\alpha + \operatorname{codim}\beta \leq m$.

The associated graded ring will be denoted by $B^{*,*}$. By definition $B^{*,*}$ is isomorphic to the product of graded rings $A^{*,*} \otimes_{\mathbb{Z}/p} A^{*,*}$. The graded subring associated to $\overline{\mathrm{Ch}}(X \times X)$ will be denoted by $B^{*,*}_{rat}$.

4.14. The key observation is that due to Corollary 4.12 we have

$$\mathrm{Ch}(\overline{X} \times \overline{X})_{M,m} \circ \mathrm{Ch}(\overline{X} \times \overline{X})_{L,l} \subset \mathrm{Ch}(\overline{X} \times \overline{X})_{M+L-N,m+l-d} \text{ and}$$

$$(\mathrm{Ch}(\overline{X} \times \overline{X})_{M,m})_\star(\mathrm{Ch}(\overline{X})_{L,l}) \subset \mathrm{Ch}(\overline{X})_{M+L-N,m+l-d}$$

and, therefore, we have the correctly defined composition law

$$\circ\colon B^{M,m} \times B^{L,l} \to B^{M+L-N,m+l-d}$$

and realization map (see 1.5)

$$\star\colon B^{M,m} \times A^{L,l} \to A^{M+L-N,m+l-d}$$

In particular, $B^{N+*,d+*}$ can be viewed as a graded ring with respect to the composition such that $(\alpha \circ \beta)_\star = \alpha_\star \circ \beta_\star$. Note also that both operations preserve rationality of cycles.

The proof of the following result is based on the fact that the variety X is generically split

4.15 Lemma. *The elements $e_i \times 1 - 1 \times e_i$, $i = 1, \ldots, r$, belong to $B^{*,*}_{rat}$.*

Proof. The proof is based on the fact that the variety X is generically split. Fix an i. Since X splits over $F(X)$, by Lemma 1.6 there exists a cycle in $\overline{\mathrm{Ch}}^{d_i}(X \times X)$ of the form

$$\xi = e_i \times 1 + \sum_s \mu_s \times \nu_s + 1 \times \mu,$$

where $\operatorname{codim}\mu_s, \operatorname{codim}\nu_s < d_i$. Then the cycle

$$\mathrm{pr}_{13}^*(\xi) - \mathrm{pr}_{23}^*(\xi) = (e_i \times 1 - 1 \times e_i) \times 1 + \sum_s (\mu_s \times 1 - 1 \times \mu_s) \times \nu_s$$

belongs to $\overline{\mathrm{Ch}}(X \times X \times X)$. Applying Corollary 3.4 to the projection $\mathrm{pr}_3\colon X \times X \times X \to X \times X$ we conclude that the pull-back $\mathrm{pr}_3^*\colon \mathrm{Ch}(X \times X) \to \mathrm{Ch}(X \times X \times X)$ has a (non-canonical) section, say, δ_3. Since the construction of this section preserves base change it preserves rationality of cycles. Hence, passing to a splitting field we obtain a rational cycle

$$\delta_3(\mathrm{pr}_{13}^*(\xi) - \mathrm{pr}_{23}^*(\xi)) = e_i \times 1 - 1 \times e_i + \sum_s (\mu_s \times 1 - 1 \times \mu_s)\delta_3(1 \times 1 \times \nu_s)$$

whose image in $B^{*,*}_{rat}$ is $e_i \otimes 1 - 1 \otimes e_i$. $\square$

We will write $(e \times 1 - 1 \times e)^M$ for the product $\prod_{i=1}^r (e_i \times 1 - 1 \times e_i)^{m_i}$. and $\binom{M}{L}$ for the product of binomial coefficients $\prod_{i=1}^r \binom{m_i}{l_i}$. In the computations we will extensively use the following two formulas (the first follows directly from Corollary 4.12 and the second is a well-known binomial identity).

4.16. Let α be an element of R^* and $\alpha^\#$ be its dual with respect to the non-degenerate pairing from 4.11, i.e., $\deg(e^N \alpha\alpha^\#) = 1$. Then we have

$$((e \times 1 - 1 \times e)^M (\alpha^\# \times 1))_\star (e^L \alpha) = \binom{M}{M+L-N} (-1)^{M+L-N} e^{M+L-N}$$

4.17 (Lucas' Theorem). The following identity holds

$$\binom{n}{k} \equiv \prod_{i \geq 0} \binom{n_i}{k_i} \quad \bmod p,$$

where $k = \sum_{i \geq 0} k_i p^i$ and $n = \sum_{i \geq 0} n_i p^i$ are base p presentations of k and n.

Set for brevity $J = J_p(G) = (j_1, \ldots, j_r)$ and recall that $K = (k_1, \ldots, k_r)$.

4.18 Proposition. *Let $\{\alpha_i\}$ be a homogeneous $\mathbb{Z}/p$-basis of R. Then the set of elements $\mathcal{B} = \{e^{p^J L}\alpha_i \mid L \preccurlyeq p^{K-J} - 1\}$ forms a $\mathbb{Z}/p$-basis of $A^{*,*}_{rat}$.*

Proof. According to Lemma 4.10 the elements from $\mathcal{B}$ are linearly independent. Assume $\mathcal{B}$ does not generate $A^{*,*}_{rat}$. Choose an element $\omega \in A^{M,m}_{rat}$ of the smallest index (M,m) which is not in the linear span of $\mathcal{B}$. By definition of $A^{M,m}$ (see Definition 4.13) ω can be written as $\omega = e^M\alpha$, where $M \preccurlyeq N$, $\alpha \in R^m$ and M can not be presented as $M = p^J L'$ for an r-tuple L'. The latter means that in the decomposition of M into p-primary and p-coprimary components $M = p^S L$, where $S = (s_1,\dots,s_r)$, $L = (l_1,\dots,l_r)$ and $p \nmid l_k$ for $k = 1,\dots,r$, we have $J \not\preccurlyeq S$. Choose an i such that $s_i < j_i$. Denote $M_i = (0,\dots,0,m_i,0,\dots,0)$ and $S_i = (0,\dots,0,s_i,0,\dots,0)$, where m_i and s_i stand at the i-th place.

Set $T = N - M + M_i$. By Lemma 4.15 and 4.16 together with observation 4.14 the element

$$((e\times 1 - 1\times e)^T(\alpha^{\#}\times 1))_\star(e^M\alpha) = \binom{p^{k_i}-1}{m_i}(-1)^{m_i}e^{m_i}$$

belongs to $A^{M_i,0}_{rat}$. By 4.17 we have $p \nmid \binom{p^{k_i}-1}{m_i}$ and, therefore, this element is non trivial. Moreover, since $s_i < j_i$, this element is not in the span of $\mathcal{B}$. Since (M,m) was chosen to be the smallest one and $(M_i,0) \le (M,m)$ we obtain that $(M,m) = (M_i,0)$. Repeating the same arguments for $T = N - M_i + p^{S_i}$ we obtain that $M_i = p^{S_i}$, i.e., $l_i = 1$.

Now let γ be a representative of $\omega = e_i^{p^{s_i}}$ in $\overline{\mathrm{Ch}}(X)$. Then its image $\pi(\gamma)$ in $\overline{\mathrm{Ch}}(G)$ has the leading term $x_i^{p^{s_i}}$ with $s_i < j_i$. This contradicts to the definition of J-invariant. $\square$

4.19 Corollary. *The elements*

$$\{(e\times 1 - 1\times e)^S(e^{p^J L}\alpha_i \times e^{p^J(p^{K-J}-1-M)}\alpha_j^{\#}) \mid L, M \preccurlyeq p^{K-J}-1,\ S \preccurlyeq p^J - 1\}$$

form a $\mathbb{Z}/p$-basis of $B^{*,*}_{rat}$. In particular, they form a basis of $B^{N,d}_{rat}$ if and only if $S = p^J - 1$ and $L = M$.

Proof. According to 4.10 these elements are linearly independent and their number is $p^{|2K-J|}(\mathrm{rk}\, R)^2$. They are rational according to Definition 4.13 and Lemma 4.15. Applying Corollary 3.4 we obtain that

$$\mathrm{rk}\, B^{*;*}_{rat} = \mathrm{rk}\,\overline{\mathrm{Ch}}(X\times X) = \mathrm{rk}\,\overline{\mathrm{Ch}}(X)\cdot \mathrm{rk}\,\mathrm{Ch}(\overline{X}),$$

where the latter coincides with $\mathrm{rk}\, A^{*;*}_{rat}\cdot p^{|K|}\,\mathrm{rk}\, R = p^{|2K-J|}(\mathrm{rk}\, R)^2$ by Lemma 4.10 and Proposition 4.18. $\square$

4.20 Lemma. *The elements*

$$\theta_{L,M,i,j} = (e \times 1 - 1 \times e)^{p^J-1}(e^{p^J L}\alpha_i \times e^{p^J(p^{K-J}-1-M)}\alpha_j^{\#}),\ L, M \preccurlyeq p^{K-J}-1,$$

belong to $B^{*,*}_{rat}$ *and satisfy the relation* $\theta_{L,M,i,j} \circ \theta_{L',M',i',j'} = \delta_{LM'}\delta_{ij'}\theta_{L',M,i',j}$.

Proof. Follows from Corollary 4.12. □

Proof of Theorem 4.9. Consider the projection map

$$f^0 \colon \overline{\mathrm{Ch}}(X \times X)_{N,d} \to B^{N,d}_{rat}.$$

By Lemma 4.20 the elements $\theta_{L,L,i,j}$ form a family of pairwise-orthogonal idempotents whose sum is the identity. The kernel of f^0 is nilpotent and, therefore, by Proposition 2.5 there exist pair-wise orthogonal idempotents $\varphi_{L,i}$ in $\overline{\mathrm{Ch}}(X \times X)$ which are mapped to $\theta_{L,L,i,i}$ and whose sum is the identity. Their components of codimension $|N| + d = \dim X$ have the same properties and, hence, we may assume that $\varphi_{L,i}$ belong to $\overline{\mathrm{Ch}^{\dim X}}(X \times X)$.

We show that $\varphi_{L,i}$ are indecomposable. By Corollary 4.19 and Lemma 4.20 the ring $(B^{N,d}_{rat}, \circ)$ can be identified with a product of matrix rings over $\mathbb{Z}/p$

$$B^{N,d}_{rat} \simeq \prod_{s=0}^{d} \mathrm{End}((\mathbb{Z}/p)^{p^{|K-J|}\,\mathrm{rk}\,R^s}).$$

By means of this identification $\theta_{L,L,i,i} \colon e^{p^J M}\alpha_j \mapsto \delta_{L,M}\delta_{i,j}e^{p^J L}\alpha_i$ is an idempotent of rank 1 and, therefore, is indecomposable. Since f^0 preserves isomorphisms, $\varphi_{L,i}$ are indecomposable as well.

We show that $\varphi_{L,i}$ is isomorphic to $\varphi_{M,j}$. In the ring $B^{*,*}_{rat}$ mutually inverse isomorphisms between them are given by $\theta_{L,M,i,j}$ and $\theta_{M,L,j,i}$. Let

$$f \colon \overline{\mathrm{Ch}}(X \times X) \to B^{*,*}_{rat}$$

be the *leading term* map; it means that for any $\gamma \in \overline{\mathrm{Ch}}(X \times X)$ we find the smallest degree (I, s) such that γ belongs to $\overline{\mathrm{Ch}}(X \times X)_{I,s}$ and set $f(\gamma)$ to be the image of γ in $B^{I,s}_{rat}$. Note that f is not a homomorphism but satisfies the condition that $f(\xi) \circ f(\eta)$ equals either $f(\xi \circ \eta)$ or 0. Choose preimages $\psi_{L,M,i,j}$ and $\psi_{M,L,j,i}$ of $\theta_{L,M,i,j}$ and $\theta_{M,L,j,i}$ by means of f. Applying Lemma 2.6 we obtain mutually inverse isomorphisms $\vartheta_{L,M,i,j}$ and $\vartheta_{M,L,j,i}$ between $\varphi_{L,i}$ and $\varphi_{M,j}$. It remains to take their homogeneous components of the appropriate degrees.

Now applying Lemma 1.7 and Corollary 2.10 to the restriction map

$$\mathrm{res}_F\colon \mathrm{End}(\mathcal{M}(X;\mathbb{Z}/p)) = \mathrm{Ch}(X\times X) \to \overline{\mathrm{Ch}}(X\times X) = \overline{\mathrm{End}^*}(\mathcal{M}(X;\mathbb{Z}/p))$$

and the family of idempotents $\varphi_{L,i}$ we obtain the family of pair-wise orthogonal idempotents $\phi_{L,i}\in\mathrm{End}(\mathcal{M}(X;\mathbb{Z}/p))$ such that

$$\Delta_X = \sum_{L,i}\phi_{L,i}.$$

Since res_F is isomorphism preserving, for the respective motives we have $(X,\phi_{L,i})\simeq(X,\phi_{0,0})(|L|+i)$ for all L and i (see Example 2.2). Denoting $\mathcal{R}_p(G)=(X,\phi_{0,0})$ we obtain the desired motivic decomposition. □

As a direct consequence of the proof we obtain

4.21 Corollary. *Any direct summand of $\mathcal{M}(X;\mathbb{Z}/p)$ is isomorphic to a direct sum of twisted copies of $\mathcal{R}_p(G)$.*

Proof. Indeed, in the ring $B^{N,d}_{rat}$ any idempotent is isomorphic to a sum of idempotents $\theta_{L,L,i,i}$, and the map f^0 preserves isomorphisms. □

4.22 Remark. Note that Corollary 4.21 can be viewed as a particular case of the Krull-Schmidt Theorem proven by V. Chernousov and A. Merkurjev (see [CM06, Corollary 9.7]).

5 Motivic decompositions

In the present section we prove the main result of this paper

5.1 Theorem. *Let G be a simple linear algebraic group of inner type over a field F and p be a prime integer. Let X be a generically split projective homogeneous G-variety. Then the motive of X with $\mathbb{Z}/p$-coefficients is isomorphic to the direct sum*

$$\mathcal{M}(X;\mathbb{Z}/p)\simeq\bigoplus_{i\geq 0}\mathcal{R}_p(G)(i)^{\oplus a_i},$$

where $\mathcal{R}_p(G)$ is an indecomposable motive, whose Poincaré polynomial $P(\overline{\mathcal{R}_p(G)},t)$ is given by (3) and, hence, depends only on the J-invariant of G, and a_i are the coefficients of the quotient polynomial

$$\sum_{i\geq 0}a_it^i = P(\mathrm{CH}^*(\overline{X}),t)/P(\overline{\mathcal{R}_p(G)},t).$$

Proof. The variety X is generically split means that the group G becomes split over $F(X)$. Let Y be the variety of complete G-flags. According to Theorem 3.8 the motive of Y is isomorphic to a direct sum of twisted copies of the motive of X. To finish apply Theorem 4.9 and Corollary 4.21. $\square$

5.2 Lemma. *Let G be a group of inner type, X be a projective homogeneous G-variety. Then any field extension E/F is rank preserving with respect to X and $X \times X$.*

Proof. By [Pa94, Theorem 2.2 and 4.2] the restriction map $K_0(X) \to K_0(X_E)$ becomes an isomorphism after tensoring with $\mathbb{Q}$. Now the Chern character $ch\colon K_0(X) \otimes \mathbb{Q} \to \mathrm{CH}^*(X) \otimes \mathbb{Q}$ is an isomorphism and respects pull-backs, hence E is rank preserving with respect to X. It remains to note that $X \times X$ is $G \times G$-homogeneous variety. $\square$

Now we provide several properties of $\mathcal{R}_p(G)$ which will be extensively used in the applications:

5.3 Proposition. *Let G and G' be two simple groups of inner type, X and X' be the corresponding varieties of complete flags.*

- **(base change)** *For any field extension E/F we have*

$$\mathcal{R}_p(G)_E \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G_E)(i)^{\oplus a_i},$$

  where $\sum a_i t^i = P(\overline{\mathcal{R}_p(G)}, t)/P(\overline{\mathcal{R}_p(G_E)}, t)$.

- **(transfer argument)** *If E/F is a field extension of degree coprime to p then $J_p(G_E) = J_p(G)$ and $\mathcal{R}_p(G_E) = \mathcal{R}_p(G)_E$. Moreover, if $\mathcal{R}_p(G_E) \simeq \mathcal{R}_p(G'_E)$ then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$.*

- **(comparison lemma)** *If G splits over $F(X')$ and G' splits over $F(X)$ then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$.*

Proof. The first claim follows from Theorem 4.9 and Corollary 4.21. To prove the second claim note that E is rank preserving with respect to X and $X \times X$ by Lemma 5.2. Now $J_p(G_E) = J_p(G)$ by Lemma 2.12, and hence $\mathcal{R}_p(G_E) = \mathcal{R}_p(G)_E$ by the first claim. The remaining part of the claim follows from Corollary 2.13 applied to the variety $X \coprod X'$.

Now we prove the last claim. The variety $X \times X'$ is the variety of complete $G \times G'$-flags. Applying Corollary 3.4 we can express $\mathcal{M}(X \times X'; \mathbb{Z}/p)$ in terms of $\mathcal{R}_p(G)$ and $\mathcal{R}_p(G')$. Now apply Corollary 4.21. $\square$

5.4 Corollary. *We have $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G_{an})$, where G_{an} is the anisotropic kernel of G.*

Let m be a positive integer. We say a polynomial $g(t)$ is *m-positive*, if $g \neq 0$, $P(\overline{\mathcal{R}_p(G)}, t) \mid g(t)$ and the quotient polynomial $g(t)/P(\overline{\mathcal{R}_p(G)}, t)$ has non-negative coefficients for all primes p dividing m.

5.5 Proposition. *Let G be a simple linear algebraic group of inner type over a field F and X be a generically split projective homogeneous G-variety. Assume that X splits by a field extension of degree m. Let $f(t)$ be an m-positive polynomial dividing $P(\mathcal{M}(\overline{X}), t)$ which can not be presented as a sum of two m-positive polynomials. Then the motive of X with integer coefficients splits as a direct sum*

$$\mathcal{M}(X) \simeq \bigoplus_i \mathcal{R}_i(c_i), \quad c_i \in \mathbb{Z},$$

where $\mathcal{R}_i$ is indecomposable and $P(\overline{\mathcal{R}}_i, t) = f(t)$ for all i. Moreover, if $m = 2, 3, 4$ or 6, then all motives $\mathcal{R}_i$ are isomorphic up to twists.

Proof. First, we apply Corollary 2.7 and Lemma 2.8 to obtain a decomposition with $\mathbb{Z}/m$-coefficients. By Lemma 5.2 our field extension is rank preserving so we can apply Theorem 2.16 to lift the decomposition over $\mathbb{Z}$. □

6 Properties of J-invariant

6.1. Recall (see [Br03]) that if the characteristic of the base field F is different from p then one can construct *Steenrod p-th power operations*

$$S^l\colon \mathrm{Ch}^*(X) \to \mathrm{Ch}^{*+l(p-1)}(X), \qquad l \geq 0$$

such that $S^0 = \mathrm{id}$, S^l restricted to $\mathrm{Ch}^l(X)$ coincides with taking to the p-th power, and the total operation $S^\bullet = \sum_{l\geq 0} S^l$ is a homomorphism of $\mathbb{Z}/p$-algebras compatible with pull-backs. In the case of varieties over the field of complex numbers S^l is compatible with its topological counterparts: reduced power operation $\mathcal{P}^l$ if $p \neq 2$ and Steenrod square Sq^{2l} if $p = 2$ (in this case $\mathrm{Ch}^*(X)$ can be viewed as a subring in $\mathrm{H}^{2*}(X, \mathbb{Z}/p)$).

When X is the variety of complete G-flags the action of Steenrod operations on $\mathrm{Ch}^*(\overline{X})$ can be described in purely combinatorial terms (see [DZ07]) and, hence, doesn't depend on the choice of a base field. Since Steenrod operations respect pull-back they respect rationality as well.

Over the field of complex numbers $\mathrm{Ch}^*(\overline{G})$ may be identified with the image of the pull-back map $\mathrm{H}^{2*}(\overline{X}, \mathbb{Z}/p) \to \mathrm{H}^{2*}(\overline{G}, \mathbb{Z}/p)$. An explicit description of this image and formulae describing the action of $\mathcal{P}^l$ on $\mathrm{H}^*(\overline{G}, \mathbb{Z}/p)$ can be found in [MT91].

The following lemma provides an important technical tool for computing the possible values of J-invariant of G.

6.2 Lemma. *Assume that in* $\mathrm{Ch}^*(\overline{G})$ *we have* $S^l(x_i) = x_m^{p^s}$ *and* $S^l(x_{i'}) < x_m^{p^s}$ *if* $i' < m$ *with respect to the* DegLex *order. Then* $j_m \leq j_i + s$.

Proof. By definition there exists a cycle $\alpha \in \overline{\mathrm{Ch}}(X)$ such that the leading term of $\pi(\alpha)$ is $x_i^{p^{j_i}}$. For the total operation we have

$$S(x_i^{p^{j_i}}) = S(x_i)^{p^{j_i}} = S^0(x_i)^{p^{j_i}} + S^1(x_i)^{p^{j_i}} + \ldots + S^{d_i}(x_i)^{p^{j_i}}.$$

In particular, $S^{lp^{j_i}}(x_i^{p^{j_i}}) = S^l(x_i)^{p^{j_i}}$. Applying $S^{lp^{j_i}}$ to α we obtain a rational cycle whose image under π has the leading term $x_m^{p^{j_i+s}}$. $\square$

6.3. We summarize information about restrictions on J-invariant which can be obtained using Lemma 6.2 into the following table (numbers d_i and k_i are taken from [Kc85, Table II]).

G_0	p	r	d_i	k_i	j_i
SL_n/μ_m, $m \mid n$	$p \mid m$	1	1	$p^{k_1} \parallel n$	any
PGSp_n, $2 \mid n$	2	1	1	$2^{k_1} \parallel n$	any
SO_n	2	$[\frac{n+1}{4}]$	$2i-1$	$[\log_2 \frac{n-1}{2i-1}]$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i-1}{l}$,
					$j_i \leq j_{2i-1}+1$
Spin_n	2	$[\frac{n-3}{4}]$	$2i+1$	$[\log_2 \frac{n-1}{2i+1}]$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i}{l}$,
					$j_i \leq j_{2i}+1$
PGO_{2n}, $n>1$	2	$[\frac{n+2}{2}]$	1, $i=1$	$2^{k_1} \parallel n$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i-2}{l}$,
			$2i-3$, $i \geq 2$	$[\log_2 \frac{2n-1}{2i-3}]$	$j_i \leq j_{2i-2}+1$
Ss_{2n}, $2 \mid n$	2	$\frac{n}{2}$	1, $i=1$	$2^{k_1} \parallel n$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i-1}{l}$
			$2i-1$, $i \geq 2$	$[\log_2 \frac{2n-1}{2i-1}]$	$j_i \leq j_{2i-1}+1$
G_2, F_4, E_6	2	1	3	1	
F_4, E_6^{sc}, E_7	3	1	4	1	
E_6^{ad}	3	2	1, 4	2, 1	
E_7^{sc}	2	3	3, 5, 9	1, 1, 1	$j_1 \geq j_2 \geq j_3$
E_7^{ad}	2	4	1, 3, 5, 9	1, 1, 1, 1	$j_2 \geq j_3 \geq j_4$
E_8	2	4	3, 5, 9, 15	3, 2, 1, 1	$j_1 \geq j_2 \geq j_3$,
					$j_1 \leq j_2+1$, $j_2 \leq j_3+1$
E_8	3	2	4, 10	1, 1	$j_1 \geq j_2$
E_8	5	1	6	1	

We give some applications of J-invariant. First, as a by-product of the proof of Theorem 4.9 we obtain the following expression for the canonical p-dimension of the variety of complete flags (cf. [EKM, Theorem 90.3] for the case of quadrics).

6.4 Proposition. *In the notation of Theorem 4.9 we have*

$$cd_p(X) = \sum_{i=1}^{r} d_i(p^{j_i}-1).$$

Proof. Follows from Proposition 4.18 and [KM06, Theorem 5.8]. $\square$

Let X be a smooth projective variety which has a splitting field.

6.5 Lemma. *For any* $\phi, \psi \in \mathrm{CH}^*(\overline{X} \times \overline{X})$ *one has*

$$\deg((\mathrm{pr}_2)_*(\phi \cdot \psi^t)) = \mathrm{tr}((\phi \circ \psi)_*).$$

Proof. Choose a homogeneous basis $\{e_i\}$ of $\mathrm{CH}^*(\overline{X})$. Let $\{e_i^\vee\}$ be its Poincaré dual. Since both sides are bilinear, it suffices to check the assertion for $\phi = e_i \times e_j^\vee$ and $\psi = e_k \times e_l^\vee$. In this case both sides are equal to $\delta_{il}\delta_{jk}$. $\square$

Denote by $d(X)$ the greatest common divisor of the degrees of all zero cycles on X and by $d_p(X)$ its p-primary component.

6.6 Corollary. *For any $\phi \in \overline{\mathrm{CH}}(X \times X; \mathbb{Z}/m)$ we have*

$$g.c.d.(d(X), m) \mid \mathrm{tr}(\phi_*).$$

Proof. Set $\psi = \Delta_{\bar{X}}$ and apply Lemma 6.5. $\square$

6.7 Corollary. *Assume that $\mathcal{M}(X; \mathbb{Z}/p)$ has a direct summand M. Then*

1. *$d_p(X) \mid P(\overline{M}, 1)$;*
2. *if $d_p(X) = P(\overline{M}, 1)$ and the kernel of the restriction $\mathrm{End}(\mathcal{M}(X)) \to \mathrm{End}(\mathcal{M}(\overline{X}))$ consists of nilpotents, then M is indecomposable.*

Proof. Set $q = d_p(X)$ for brevity. Let $M = (X, \phi)$. By Corollary 2.7 there exists an idempotent $\varphi \in \mathrm{End}(\mathcal{M}(X); \mathbb{Z}/q)$ such that $\varphi \mod p = \phi$. Then $\mathrm{res}(\varphi) \in \mathrm{End}(\mathcal{M}(\overline{X}); \mathbb{Z}/q)$ is a rational idempotent. Since every projective module over $\mathbb{Z}/q$ is free, we have

$$\mathrm{tr}(\mathrm{res}(\varphi)_*) = \mathrm{rk}_{\mathbb{Z}/q}(\mathrm{res}(\varphi)_*) = \mathrm{rk}_{\mathbb{Z}/p}(\mathrm{res}(\phi)_*) = P(\overline{M}, 1) \mod q,$$

and the first claim follows by Corollary 6.6. The second claim follows from the first, since the second assumption implies that for any nontrivial direct summand M' of M we have $P(\overline{M'}, 1) < P(\overline{M}, 1)$. $\square$

6.8. Let G be a group of inner type. Denote by $n(G)$ the greatest common divisor of degrees of all finite splitting fields of G and by $n_p(G)$ its p-primary component. Note that $n(G) = d(X)$ and $n_p(G) = d_p(X)$, where X is the variety of complete G-flags.

We obtain the following estimate on $n_p(G)$ in terms of J-invariant (cf. [EKM, Prop. 88.11] in the case of quadrics).

6.9 Proposition. *For a group G of inner type with $J_p(G) = (j_1, \ldots, j_r)$ we have*

$$n_p(G) \leq p^{\sum_i j_i}.$$

Proof. Follows from Theorem 4.9 and Corollary 6.7. ☐

6.10 Corollary. *The following statements are equivalent:*

- $J_p(G) = (0, \ldots, 0)$;
- $n_p(G) = 1$;
- $\mathcal{R}_p(G) = \mathbb{Z}/p$.

Proof. If $J_p(G) = (0, \ldots, 0)$ then $n_p(G) = 1$ by Proposition 6.9. If $n_p(G) = 1$ then there exists a splitting field L of degree m prime to p and, therefore, $\mathcal{R}_p(G) = \mathbb{Z}/p$ by the transfer argument (see Proposition 5.3). The remaining implication is obvious. ☐

Finally, we obtain the following reduction formula (cf. [EKM, Cor. 88.7] in the case of quadrics).

6.11 Proposition. *Let G be a group of inner type, X be the variety of complete G-flags, Y be a projective variety such that the map $\mathrm{CH}^l(Y) \to \mathrm{CH}^l(Y_{F(x)})$ is surjective for all $x \in X$ and $l \leq n$. Then $j_i(G) = j_i(G_{F(Y)})$ for all i such that $d_i p^{j_i(G_{F(Y)})} \leq n$.*

Proof. Indeed, by [EKM, Lemma 88.5] the map $\mathrm{CH}^l(X) \to \mathrm{CH}^l(X_{F(Y)})$ is surjective for all $l \leq n$, and therefore $j_i(G) \leq j_i(G_{F(Y)})$. The converse inequality is obvious. ☐

6.12 Corollary. $J_p(G) = J_p(G_{F(t)})$.

Proof. Take $Y = \mathbb{P}^1$ and apply Proposition 6.11. ☐

7 Examples

In the present section we provide examples of motivic decompositions of projective homogeneous varieties obtained by applying Theorem 5.1.

The case $r = d_1 = 1$. According to Table 6.3 this corresponds to the case when G is of type A_n or C_n. Let A be a central simple algebra corresponding to G. We have $A = \mathrm{M}_m(D)$, where D is a division algebra of index $d \geq 1$ over a field F. Let p be a prime divisor of d ($p = 2$ in the case of C_n). Observe that according to Table 6.3 $J_p(G) = (j_1)$ for some $j_1 \geq 0$. Let X_Θ be the projective homogeneous G-variety given by a subset Θ of vertices of the respective Dynkin diagram such that $p \nmid j$ for some $j \notin \Theta$ (cf. Example 3.7). Then by Theorem 5.1 we obtain that

$$\mathcal{M}(X_\Theta; \mathbb{Z}/p) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i}, \tag{4}$$

where $\mathcal{R}_p(G)$ is indecomposable and

$$\overline{\mathcal{R}_p(G)} \simeq \bigoplus_{i=0}^{p^{j_1}-1} (\mathbb{Z}/p)(i).$$

Now we identify $\mathcal{R}_p(G)$. Using the comparison lemma (see Proposition 5.3) we conclude that $\mathcal{R}_p(G)$ depends only on D, so we may assume $m = 1$. By Table 6.3 we have $p^{j_1} \mid d$, but on the other hand by Proposition 6.9 we have $n_p(G) \leq p^{j_1}$. Therefore, p^{j_1} is a p-primary part of d.

We have $D \simeq D_p \otimes_F D'$, where $p^{j_1} = \mathrm{ind}(D_p)$ and $p \nmid \mathrm{ind}(D')$. Passing to a splitting field of D' of degree prime to p and using Proposition 5.3 we conclude that the motives of X_Θ and $\mathrm{SB}(D_p)$ are direct sums of twisted $\mathcal{R}_p(G)$. Comparing the Poincaré polynomials we conclude that

7.1 Lemma. $\mathcal{M}(\mathrm{SB}(D_p); \mathbb{Z}/p) \simeq \mathcal{R}_p(G)$.

Applying Proposition 5.5 to $X = \mathrm{SB}(D)$ and comparing the Poincaré polynomials of $\mathcal{M}(X)$ and $\mathcal{R}_i$ we obtain that

7.2 Corollary. *The motive of* $\mathrm{SB}(D)$ *with integer coefficients is indecomposable.*

7.3 Remark. Indeed, we provided a uniform proof of the results of paper [Ka96]. Namely, the decomposition of $\mathcal{M}(\mathrm{SB}(A); \mathbb{Z}/p)$ (see [Ka96, Cor. 1.3.2]) and indecomposability of $\mathcal{M}(\mathrm{SB}(D); \mathbb{Z})$ (see [Ka96, Thm. 2.2.1]).

The case $r = 1$ and $d_1 > 1$. According to Table 6.3 this holds if

$p = 2$: G is a group of type Spin_n, $n = 7, 8, 9, 10$, G_2, F_4 or E_6;

$p = 3$: G is a group of type F_4, E_7 or strongly inner form of type E_6;

$p = 5$: G is a group of type E_8.

Observe that in all these cases $J_p(G) = (0)$ or (1). Let X be a generically split projective homogeneous G-variety (cf. Example 3.7). By Theorem 5.1 we obtain the decomposition

$$\mathcal{M}(X; \mathbb{Z}/2) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i}, \tag{5}$$

where the motive $\mathcal{R}_p(G)$ is indecomposable and (cf. [Vo03, (5.4-5.5)])

$$\overline{\mathcal{R}_p(G)} \simeq \bigoplus_{i=1}^{p-1} (\mathbb{Z}/p)(i \cdot (p+1)).$$

Now we identify $\mathcal{R}_p(G)$. Let $\mathfrak{r}$ be the Rost invariant as defined in [Me03] and $\mathfrak{r}_p$ denote its p-part.

7.4 Lemma. *Let G be a simple linear algebraic group over F satisfying $r = 1$ and $d_1 > 1$ and p be its torsion prime. Then $\mathfrak{r}_p(G)$ is trivial iff $\mathcal{R}_p(G) \simeq \mathbb{Z}/p$.*

Proof. According to [Ga01, Theorem 0.5], [Ch94] and [Gi00, Theoreme 10] the invariant $\mathfrak{r}_p(G)$ is trivial iff there exists a field extension E/F of degree coprime with p such that the group G splits over E. By Corollary 6.10 the latter is equivalent to the fact that $\mathcal{R}_p(G) \simeq \mathbb{Z}/p$. □

7.5 Lemma. *Let G and G' be simple linear algebraic groups over F satisfying $r = 1$ and $d_1 > 1$ and p be its common torsion prime. For groups of type E_8 we assume in addition that they are twisted forms by means of cocycles taken from the image of the twisted map defined in [Ga06, 13.12] (examples of such groups are provided in [Ga06, 14.7]).*

If $\mathfrak{r}_p(G) = \mathfrak{r}_p(G')c$ for some $c \in (\mathbb{Z}/p)^\times$, then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$.

Proof. By transfer arguments (see Proposition 5.3) it is enough to prove this over a p-primary closure of F. Let X and X' be the respective varieties of complete flags. Observe that the invariant $\mathfrak{r}_p(G)$ becomes trivial over the function field $F(X)$. Since $\mathfrak{r}_p(G) = \mathfrak{r}_p(G')c$, it becomes trivial over $F(X')$ as well. By triviality of the kernel of the Rost invariant (see [Ga01, Theorem 0.5] and [Ga06, Theorem 13.14]) this implies that G splits over $F(X')$. To finish the proof we apply the comparison lemma (see Proposition 5.3) to the groups G and G'. □

$\mathbb{Z}$-coefficients. Let G be a group of type F_4 or strongly inner form of type E_6 which doesn't split by field extensions of degrees 2 and 3. Observe that such a group splits by an extension of degree 6. Let X be a generically split projective homogeneous G-variety. Then according to Proposition 5.5 the Chow motive of X with integer coefficients splits as a direct sum of twisted copies of an indecomposable motive $\mathcal{R}(G)$ such that

$$\mathcal{R}(G)\otimes\mathbb{Z}/2 = \bigoplus_{i=0,1,2,6,7,8} \mathcal{R}_2(G)(i), \qquad P(\overline{\mathcal{R}_2(G)},t) = 1+t^3,$$
$$\mathcal{R}(G)\otimes\mathbb{Z}/3 = \bigoplus_{i=0,1,2,3} \mathcal{R}_3(G)(i), \qquad P(\overline{\mathcal{R}_3(G)},t) = 1+t^4+t^8,$$
$$P(\overline{\mathcal{R}(G)},t) = 1+t+t^2+\ldots+t^{11}.$$

7.6 Remark. In particular, we provided a uniform proof of the main results of papers [Bo03] and [NSZ], where the cases of some G_2- and F_4-varieties are considered.

The case $r > 1$. According to Table 6.3 this holds for groups G of types B_n and D_n and exceptional types E_7, E_8 for $p = 2$ and E_6^{ad}, E_8 for $p = 3$.

Pfister case. Let $G = \mathrm{O}^+(\phi)$, where ϕ is a k-fold Pfister form or its maximal neighbor. Assume $J_2(G) \neq (0,\ldots,0)$. In view of Corollary 6.10 this holds iff $n_2(G) \neq 1$. By Springer Theorem the latter holds iff ϕ is not split. According to Theorem 5.1 we obtain the decomposition

$$\mathcal{M}(X;\mathbb{Z}/2) \simeq \bigoplus_{i\geq 0} \mathcal{R}_2(G)(i)^{\oplus a_i}$$

where $\mathcal{R}_2(G)$ is indecomposable. Moreover, according to Theorem 2.16 the similar decomposition exists with $\mathbb{Z}$-coefficients.

Now we compute $J_2(G)$. Let Y be a projective quadric corresponding to ϕ. Then G splits over $F(Y)$ and Y splits over $F(x)$ for any $x \in X$.

It is known that $\mathrm{CH}^l(\overline{Y})$ for $l < 2^{k-1} - 1$ is generated by $\mathrm{CH}^1(\overline{Y})$ and, therefore, is rational. By Proposition 6.11 and Table 6.3 we see that $j_i(G) = 0$ for $0 \leq i < r$, where $r = 2^{k-2}$. Therefore, $J_2(G) = (0, \dots, 0, 1)$ and $P(\overline{\mathcal{R}_2(G)}, t) = 1 + t^{2^{k-1}-1}$. Finally, by Corollary 4.21 the motive $\mathcal{R}_2(G)$ coincides with the motive introduced in [Ro98] which is called Rost motive.

In this way we obtain Rost decomposition of the motive of a Pfister quadric and its maximal neighbor.

Maximal orthogonal Grassmannian Let $G = \mathrm{O}^+(q)$, where $q\colon V \to F$ is an arbitrary anisotropic regular quadratic form and X is the respective maximal orthogonal Grassmannian. The variety X is generically split, hence, by Theorem 5.1 we have the decomposition

$$\mathcal{M}(X; \mathbb{Z}/2) \simeq \bigoplus_{i \geq 0} \mathcal{R}_2(G)(i)^{\oplus a_i},$$

where the motive $\mathcal{R}_2(G)$ is indecomposable. Comparing the Poincaré polynomials of $\mathcal{M}(X; \mathbb{Z}/2)$ and $\mathcal{R}_2(G)$ we obtain the following particular cases:

- If the group G corresponds to a generic cocycle (see 4.2), the motive $\mathcal{M}(X; \mathbb{Z}/2)$ is isomorphic to $\mathcal{R}_2(G)$ and, hence, is indecomposable. This corresponds to the maximal value of the J-invariant.

- If q is a Pfister form or its maximal neighbor, by the previous example $\mathcal{R}_2(G)$ coincides with the Rost motive. This corresponds to the minimal non-trivial value of the J-invariant.

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