TENSOR PRODUCT DECOMPOSITIONS AND OPEN ORBITS IN MULTIPLE FLAG VARIETIES

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Abstract. For a connected semisimple algebraic group G, we consider some special infinite series of tensor products of simple G-modules whose G-fixed point spaces are at most one-dimensional. We prove that their existence is closely related to the existence of open G-orbits in multiple flag varieties and address the problem of classifying such series.

Let G be a connected simply connected semisimple algebraic group. In this paper we establish a close interrelation between some special series of tensor products of simple G-modules whose G-fixed point spaces are at most one-dimensional and multiple flag varieties of G that contain open G-orbit. Motivated by this intimate connection with geometry, we then address the problem of classifying such series. Starting with the basic definition and examples in Sections 1 and 2, we introduce necessary notation in Section 3 and then formulate our main results in Section 4. Other results and proofs are contained in the remaining part of paper.

Below all algebraic varieties are taken over an algebraically closed field k of characteristic zero.

1. Basic definition

Fix a choice of Borel subgroup B of G and maximal torus $T \subset B$. Let P_{++} be the additive monoid of dominant characters of T with respect to B. Put

$$\mathrm{P}_{\gg} := \mathrm{P}_{++} \setminus \{0\}.$$

For $\lambda \in \mathrm{P}_{++}$, denote by E_λ a simple G-module of highest weight λ and by λ^* the highest weight of dual G-module E_λ^*. Let P_λ be the G-stabilizer of unique B-stable line in E_λ. If $\mu, \lambda_1, \ldots, \lambda_d \in \mathrm{P}_{++}$, denote by $c^{\mu}_{\lambda_1,\ldots,\lambda_d}$ the multiplicity of E_μ inside $E_{\lambda_1} \otimes \ldots \otimes E_{\lambda_d}$, i.e., the Littlewood–Richardson coefficient $\dim\big(\mathrm{Hom}_G(E_\mu, E_{\lambda_1} \otimes \ldots \otimes E_{\lambda_d})\big)$. Denote respectively by $\varpi_1, \ldots, \varpi_r$ and $\alpha_1, \ldots, \alpha_r$ the systems of fundamental weights of P_{++} and simple roots of G with respect to T and B enumerated as in [Bo$_2$]. Let respectively

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$\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Z}_{>0}$ be the sets of all nonnegative and all positive integers. We write $\mathrm{P}^d_{\gg}$ in place of $(\mathrm{P}_{\gg})^d$, etc.

Definition 1. We call a d-tuple $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$ *primitive* if

$$c^0_{n_1\lambda_1,\ldots,n_d\lambda_d} \leqslant 1 \quad \text{for all} \quad (n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geqslant 0}. \tag{1}$$

Schur's lemma and the isomorphism $(E_\mu \otimes E_\nu)^G \simeq \mathrm{Hom}_G(E_{\mu^*}, E_\nu)$ imply that

$$c^0_{\mu,\nu} = \begin{cases} 1 & \text{if } \mu^* = \nu, \\ 0 & \text{otherwise;} \end{cases} \tag{2}$$

whence condition (1) is equivalent to the following:

$$c^{n_j\lambda_j^*}_{n_1\lambda_1,\ldots,\widehat{n_j\lambda_j},\ldots,n_d\lambda_d} \leqslant 1 \text{ for all } (n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geqslant 0} \text{ and some (equivalently, every) } \lambda_j. \tag{3}$$

The set of primitive elements of $\mathrm{P}^d_{\gg}$ is clearly stable with respect to permutation of coordinates and automorphisms of $\mathrm{P}^d_{\gg}$ induced by automorphisms of the Dynkin diagram of G. If $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$ is primitive, then $(\lambda_{i_1}, \ldots, \lambda_{i_s}) \in \mathrm{P}^s_{\gg}$ is primitive for every subset $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, d\}$.

Remark 1. The notion of primitive d-tuple admits a natural generalization: we call a d-tuple $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$ *primitive at* $\mu \in \mathrm{P}_{++}$ if

$$c^\mu_{n_1\lambda_1,\ldots,n_d\lambda_d} \leqslant 1 \quad \text{for all} \quad (n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geqslant 0}. \tag{4}$$

Then "primitive" means "primitive at 0".

2. Examples

Clearly, for $d = 1$, every element of $\mathrm{P}^d_{\gg}$ is primitive. By (2) the same is true for $d = 2$. For $d \geqslant 3$, the existence of primitive elements in $\mathrm{P}^d_{\gg}$ is less evident.

Example 1. Let $G = \mathbf{SL}_2$. Then $\mathrm{P}_{++} = \mathbb{Z}_{\geqslant 0}\varpi_1$ and G/P_λ for $\lambda \neq 0$ is isomorphic to the projective line $\mathbf{P}^1$. From Definition 1 and the Clebsch–Gordan formula

$$E_{s\varpi_1} \otimes E_{t\varpi_1} \simeq \textstyle\bigoplus_{0 \leqslant i \leqslant t} E_{(s+t-2i)\varpi_1}, \quad s \geqslant t,$$

it is not difficult to deduce that an element of $\mathrm{P}^d_{\gg}$ is primitive if and only if $d \leqslant 3$. Theorems 1 and 4 below imply that this is equivalent to the classical fact that for the diagonal action of $\mathbf{SL}_2$ on $(\mathbf{P}^1)^d$, an open orbit exists if and only if $d \leqslant 3$.

Example 2. If, for $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$, condition (4) holds for every $\mu \in \mathrm{P}_{++}$, then by (3) the $(d+1)$-tuple $(\nu, \lambda_1, \ldots, \lambda_d)$ is primitive for every $\nu \in \mathrm{P}_{\gg}$. Such d-tuples $(\lambda_1, \ldots, \lambda_d)$ exist. For instance, if G is of type A, B, C, D, or E_6, then the explicit decomposition formulas for $E_{m_1\varpi_1} \otimes E_{m_2\varpi_1}$ (see [Li, 1.3] or, for the types A, B, C, D, [OV, pp. 300–302]) imply that (ϖ_1, ϖ_1) shares this property. For $G = \mathbf{SL}_n$, the classification of all d-tuples $(\lambda_1, \ldots, \lambda_d)$ sharing this property can be deduced from [St], where the classification of all multiplicity free tensor products of simple $\mathbf{SL}_n$-modules is obtained.

Example 3. Let G be the group of type E_6. By [Li, 1.3], for every $s, t \in \mathbb{Z}_{\geqslant 0}$, the following decomposition holds:

$$E_{s\varpi_1} \otimes E_{t\varpi_1} \simeq \bigoplus_{\begin{cases} a_1,\dots,a_4 \in \mathbb{Z}_{\geqslant 0} \\ a_1+a_3+a_4=s \\ a_2+a_3+a_4=t \end{cases}} E_{(a_1+a_2)\varpi_1+a_3\varpi_3+a_4\varpi_6}. \tag{5}$$

Since $\big((a_1+a_2)\varpi_1+a_3\varpi_3+a_4\varpi_6\big)^* = a_4\varpi_1+a_3\varpi_5+(a_1+a_2)\varpi_6$, it follows from (5) and (2) that $\dim\big(\bigotimes_{1\leqslant i\leqslant 4} E_{n_i\varpi_i}\big)^G$ is equal to the number of solutions in $\mathbb{Z}_{\geqslant 0}$ of the following system of eight linear equations in eight variables $a_1,\dots,a_4,b_1,\dots,b_4$:

$$\begin{cases} a_4 = b_1+b_2, \\ a_3 = 0, \\ b_3 = 0, \\ a_1+a_2 = b_4, \\ a_1+a_3+a_4 = n_1, \\ a_2+a_3+a_4 = n_2, \\ b_1+b_3+b_4 = n_3, \\ b_2+b_3+b_4 = n_4. \end{cases}$$

Since this system is nondegenerate, it has at most one such solution. Thus for G of type E_6, the 4-tuple $(\varpi_1,\varpi_1,\varpi_1,\varpi_1)$ is primitive. By Theorems 1 and 4 below (see also [P_4, Theorem 6]) this is equivalent to the existence of an open G-orbit in $(G/P_{\varpi_1})^4$. Observe that this example in not in the range of Example 2: for instance, $c^{\varpi_1+3\varpi_3+\varpi_5}_{4\varpi_1,4\varpi_1,3\varpi_1} = 2$ (this may be verified, e.g., utilizing LiE).

Example 4. The following definition singles out a natural subset in the set of all primitive d-tuples. Theorem 10 below shows that this subset admits a geometric characterization as well.

Definition 2. We call a d-tuple $(\lambda_1,\dots,\lambda_d) \in \mathrm{P}^d_{\gg}$ *invariant-free* if

$$c^0_{n_1\lambda_1,\dots,n_d\lambda_d} = 0 \quad \text{for all } (n_1,\dots,n_d) \in \mathbb{Z}^d_{\geqslant 0}.$$

Clearly, every 1-tuple is invariant-free. For $d=2$, it follows from (2) that

$$(\lambda_1,\lambda_2) \text{ is invariant-free} \iff \mathbb{Q}\,\lambda_1 \neq \mathbb{Q}\,\lambda_2^*.$$

3. Notation and conventions

Below we utilize the following notation, conventions, and definitions.

• $k[Y]$ and $k(Y)$ are respectively the algebra of regular functions and field of rational functions of an irreducible algebraic variety Y.

• Cl(Y) is the Weil divisor class group of an irreducible normal variety Y. For a nonconstant function $f \in k(Y)$, the Weil divisor, divisor of zeros, and divisor of poles of f are respectively (f), $(f)_0$, and $(f)_\infty$.

• If H is an algebraic group, $\mathrm{Lie}(H)$ and $\mathcal{X}(H)$ are respectively the Lie algebra and the character group $\mathrm{Hom}_{\mathrm{alg}}(H,\mathbf{G}_m)$ of H. We utilize additive notation for $\mathcal{X}(H)$ and identify in the natural way $\mathcal{X}(H)$ with the lattice in rational vector space

$$\mathcal{X}(H)_{\mathbb{Q}} := \mathcal{X}(H) \otimes \mathbb{Q}.$$

If H is a connected reductive group, B_H its Borel subgroup, and $S \subseteq B_H$ a maximal torus, we identify the set of isomorphism classes of simple algebraic H-modules with a submonoid $\mathcal{X}(S)_{++}$ of $\mathcal{X}(S)$ assigning to every simple H-module V the S-weight of unique B_H-stable line in V.

• Below all algebraic group actions are algebraic. The action of H on H/F is that by left multiplication. If H acts on $Y_1, \ldots, Y_n$, the action of H on $Y_1 \times \ldots \times Y_n$ is the diagonal one.

If H acts on a variety Y, then $H \cdot y$ and H_y are respectively the H-orbit and H-stabilizer of a point $y \in Y$, and $k[Y]^H$ and $k(Y)^H$ are the subalgebra and subfield of H-invariant elements in $k[Y]$ and $k(Y)^H$.

Definition 3. For an irreducible variety Y, we call the action of H on Y *ample* if $k(Y)^H$ is algebraic over the field of fractions of $k[Y]^H$.

Recall the following definition introduced in [P_1].

Definition 4. The action of H on Y is called *stable* if H-orbits of points lying off a proper closed subset of Y are closed in Y.

• The natural action of H on $k[Y]$ is locally finite. If H is a connected reductive group, by $k[Y]_\lambda$, where $\lambda \in \mathcal{X}(S)_{++}$, we denote the λ-isotypical component of H-module $k[Y]$ and put

$$\mathcal{S}(H, Y) := \{\lambda \in \mathcal{X}(S)_{++} \mid k[Y]_\lambda \neq 0\}. \tag{6}$$

The set $\mathcal{S}(H, Y)$ is a submonoid of $\mathcal{X}(S)$ (indeed, $\mathcal{S}(H, Y)$ is the set of all weights of the natural action of S on $k[Y]^{B_H^u}$, where B_H^u is the unipotent radical of B_H; whence the claim). If Y is an affine variety, then the monoid $\mathcal{S}(H, Y)$ is finitely generated (this readily follows from the fact that in this case $k[Y]^{B_H^u}$ is a finitely generated k-algebra, see, e.g., [PV_2, 3.14]).

• If H is a reductive group and Y is an affine variety, we denote by

$$\pi_{H,Y} \colon Y \longrightarrow Y/\!\!/H \tag{7}$$

the corresponding categorical quotient, i.e., $Y/\!\!/H$ is an affine variety and $\pi_{H,Y}$ a dominant (in fact, surjective) morphism such that $\pi_{H,Y}^*(k[Y/\!\!/H]) = k[Y]^H$, see, e.g., [$\mathrm{PV}_2$, 4.4]. Utilizing $\pi_{H,Y}^*$, we identify $k[Y/\!\!/H]$ with $k[Y]^H$ and, if Y is irreducible, $k(Y/\!\!/H)$ with the field of fractions of $k[Y]^H$.

It follows from Rosenlicht's theorem [R_2, Theorem], see also, e.g., [PV_2, Cor. of Theorem 2.3], that the action of H on Y is ample if and only if $\dim(Y/\!\!/H) = \dim(Y) - \max\limits_{y \in Y} \dim(H \cdot y)$ (i.e., in the terminology of [Lu, Sect. 4], "$Y/\!\!/H$ *a la bonne dimension*"). One can also prove that if Y is normal, then the action of H on Y is ample if and only if $k(Y)^H$ is the field of fractions of $k[Y]^H$.

If the action of H on Y is stable, it is ample (indeed, since every fiber of $\pi_{H,Y}$ contains a unique closed H-orbit, see, e.g., [PV_2, Cor. of Theorem 4.7], the action of is stable if and only if every general fiber is a closed H-orbit of maximal dimension).

• $N_G(T)$ is the normalizer of T in G and $W := N_G(T)/T$ is the Weyl group of G. For every $w \in W$ we fix a choice of its representative $\dot{w}$ in $N_G(T)$. The space $\mathcal{X}(T)_{\mathbb{Q}}$ is endowed with the natural W-module structure.

For $\lambda = \sum_{i=1}^{r} a_i\varpi_i$, $a_i \in \mathbb{Z}_{\geqslant 0}$, the *support* of λ is

$$\mathrm{supp}(\lambda) := \{i \in \mathbb{Z}_{>0} \mid a_i \neq 0\}.$$

The subgroup P_λ is then generated by T and one-dimensional unipotent root subgroups of G corresponding to all positive roots and those negative roots that are linear combinations of $-\alpha_i$'s with $i \notin \mathrm{supp}(\lambda)$. We have the equivalence

$$P_\mu = P_\nu \iff \mathrm{supp}(\mu) = \mathrm{supp}(\nu). \tag{8}$$

For every subset A of $\subseteq \mathrm{P}_{++}$, we put

$$A^* := \{\lambda^* \mid \lambda \in A\}.$$

• We fix a choice of nonzero point v_λ of the unique B-stable line in E_λ and denote by $\mathcal{O}_\lambda$ the G-orbit of v_λ and by $\overline{\mathcal{O}}_\lambda$ its closure in E_λ. We put

$$\mathcal{O}_{\lambda_1,\ldots,\lambda_d} := \mathcal{O}_{\lambda_1} \times \ldots \times \mathcal{O}_{\lambda_d} \quad \text{and} \quad X_{\lambda_1,\ldots,\lambda_d} := \overline{\mathcal{O}}_{\lambda_1} \times \ldots \times \overline{\mathcal{O}}_{\lambda_d} \tag{9}$$

and identify in the natural way $X_{\lambda_1,\ldots,\lambda_d}$ with the closed subset of $E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_d}$.

• If M is a subset of a vector space, then $\mathrm{conv}(M)$ and $\mathrm{cone}(M)$ are respectively the covex hull of M and the convex cone generated by M. If M is a convex set, $\mathrm{int}(M)$ is the set its (relative) interior points.

• We put

$$\mathbb{Q}_{>0} := \{a \in \mathbb{Q} \mid a > 0\} \quad \text{and} \quad \mathbb{Q}_{\geqslant 0} := \{a \in \mathbb{Q} \mid a \geqslant 0\}.$$

• $|N|$ is the cardinality of a finite set N.

4. Main results

In this section we formulate main results of this paper.

Theorem 1 explicitly formulates the aforementioned remarkable connection of primitive tuples with geometry.

Theorem 1. *Let* $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$.

(i) *If* $G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$ *contains an open* G*-orbit, then* $(\lambda_1, \ldots, \lambda_d)$ *is primitive.*

(ii) *If* $(\lambda_1, \ldots, \lambda_d)$ *is primitive and the action of* G *on* $X_{\lambda_1,\ldots,\lambda_d}$ *is ample, then* $G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$ *contains an open* G*-orbit.*

Theorem 1 and equivalence (8) imply

Corollary. *Let* $(\lambda_1, \ldots, \lambda_d)$ *and* $(\mu_1, \ldots, \mu_d) \in \mathrm{P}^d_{\gg}$. *Assume that*

$$\mathrm{supp}(\lambda_i) = \mathrm{supp}(\mu_i) \textit{ for all } i.$$

If $(\lambda_1, \ldots, \lambda_d)$ *is primitive and the action of* G *on* $X_{\lambda_1,\ldots,\lambda_d}$ *is ample, then* $(\mu_1, \ldots, \mu_d)$ *is primitive as well.*

Theorem 1 clarifies relation of classifying primitive d-tuples to the following problems.

Problem 1. Classify multiple flag varieties $G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$ that contain open G-orbit.

Problem 2. For what d-tuples $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$ is the action of G on $X_{\lambda_1,\ldots,\lambda_d}$ ample?

Regarding Problem 1, obvious dimension reason yields the finiteness statement about length of possible d-tuples: d in Problem 1 cannot exceed a constant depending only on G. A more thorough analysis leads to the following upper bounds.

Theorem 2. *Let G be a simple group. If $G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$ contains an open G-orbit, then $d \leqslant b_G$ for the following b_G:*

TABLE 1

type of G	$\mathsf{A}_l,\ l \geqslant 1$	$\mathsf{B}_l,\ l \geqslant 3$	$\mathsf{C}_l,\ l \geqslant 2$	$\mathsf{D}_l,\ l \geqslant 4$	E_6	E_7	E_8	F_4	G_2
b_G	$l+2$	$l+1$	$l+1$	l	4	4	4	3	2

Notice that, by the Bruhat decomposition, every multiple flag variety $G/P_{\lambda_1} \times G/P_{\lambda_2}$ contains only finitely many G-orbits (one of which therefore is open).

In [P$_4$] a complete solution to Problem 1 for d-tuples of the form

$$(\lambda_1, \ldots, \lambda_d) = (m_1\varpi_i, \ldots, m_d\varpi_i), \quad 1 \leqslant i \leqslant r, \quad (m_1, \ldots, m_d) \in \mathbb{Z}^d_{>0}. \tag{10}$$

is obtained; the answer is the following.

Theorem 3 ([P$_4$])**.** *Let G be a simple group. Assume that $d \geqslant 3$ and $(\lambda_1, \ldots, \lambda_d)$ is given by equality* (10)*. Then the multiple flag variety $G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$ contains an open G-orbit if and only if the following conditions hold:*

TABLE 2

type of G	$\mathsf{A}_l,\ l \geqslant 1$	$\mathsf{B}_l,\ l \geqslant 3$	$\mathsf{C}_l,\ l \geqslant 2$	$\mathsf{D}_l,\ l \geqslant 4$	E_6	E_7
conditions	$d < \frac{(l+1)^2}{i(l+1-i)}$	$d = 3,$ $i = 1, l$	$d = 3,$ $i = 1, l$	$d = 3,$ $i = 1, l-1, l$	$d \leqslant 4,$ $i = 1, 6$	$d = 3,$ $i = 7$

The next two theorems concern Problem 2.

Theorem 4. *Let $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$. If every λ_s is a multiple of a fundamental weight,*

$$\lambda_s \in \mathbb{Z}_{>0}\varpi_{i_s}, \quad s = 1, \ldots, d, \tag{11}$$

then the action of G on $X_{\lambda_1,\ldots,\lambda_d}$ is ample.

Theorem 5. *Let G be a simple group and let $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$. If*

$$d \geqslant \mathrm{sep}(G),$$

where $\mathrm{sep}(G)$ is the separation index of the root system of G with respect to T (*see Definition* 5 *in Section* 9)*, then the action of G on $X_{\lambda_1,\ldots,\lambda_d}$ is stable* (*and hence ample*) *and the G-stabilizer of a point in general position in $X_{\lambda_1,\ldots,\lambda_d}$ is finite.*

Other results on stability of the G-action on $X_{\lambda_1,\ldots,\lambda_d}$ are obtained in Theorem 13 in Section 15; they are based on some results from [V]. We show that stability imposes some constraints on configuration of the set $\{\lambda_1,\ldots,\lambda_d\}$ and link the problem with some monoids that generalize Littelwood–Richardson semigroups [Z] whose investigation during the last decade culminated in solving several old problems, in particular, proving Horn's conjecture, cf. survey [F].

We apply Theorems 1–5 to studying primitive d-tuples. Theorem 4, Definition 1, and Corollary of Theorem 1 immediately imply the following saturation property.

Theorem 6. *Let* $(\lambda_1,\ldots,\lambda_d)\in \mathrm{P}^d_{\gg}$ *and* $(m_1,\ldots,m_d)\in\mathbb{Z}^d_{>0}$. *If condition* (11) *holds, then the following properties are equivalent:*

(i) $(\lambda_1,\ldots,\lambda_d)$ *is primitive;*
(ii) $(m_1\lambda_1,\ldots,m_d\lambda_d)$ *is primitive.*

Utilizing Theorems 1 and 2 we prove the following finiteness theorem about length of primitive d-tuples:

Theorem 7. *Let G be a simple group. Then for every primitive d-tuple in* $\mathrm{P}^d_{\gg}$,

$$d\leqslant \mathrm{sep}(G)+1.$$

In view of inequality (41) below this implies

Corollary. *Let G be a simple group. Then* $d\leqslant |W|+1$ *for every primitive d-tuple in* $\mathrm{P}^d_{\gg}$.

¿From Theorem 7 we deduce that for every simple group G,

$$\mathrm{prim}(G):=\sup\{d\in\mathbb{Z}_{>0}\mid \mathrm{P}^d_{\gg}\text{ contains a primitive element}\}$$

is a natural number not exceeding $\mathrm{sep}(G)+1$. Discussion in Section 2 and the last Corollary imply that $2\leqslant \mathrm{prim}(G)\leqslant |W|+1$.

Example 5. By Example 1 we have $\mathrm{prim}(\mathbf{SL}_2)=3$. Theorem 8 below implies that if G is respectively of type A_l, B_l, C_l, D_l, E_6, and E_7, then $\mathrm{prim}(G)\geqslant l+2$, 3, 3, 3, 4, and 3.

For $G=\mathbf{SL}_n$, we can apply to our problem the representation theory of quivers. This leads to a characterization of primitive d-tuples of fundamental weights in terms of canonical decomposition of dimension vectors of representations of some graphs and yields a fast algorithm for verifying whether such a d-tuple is primitive or not (see Theorem 14 and discussion in Section 16).

¿From Theorems 1, 3, and 4 we deduce a complete classification of primitive d-tuples of form (10):

Theorem 8. *Let G be a simple group. A d-tuple*

$$(m_1\varpi_i,\ldots,m_d\varpi_i),\quad \textit{where } d\geqslant 3,\ (m_1,\ldots,m_d)\in\mathbb{Z}^d_{>0},$$

is primitive if and only if the conditions specified in Table 2 *hold.*

Combining Theorem 1 with the results of [Li], [MWZ_1], and [MWZ_2], we prove that the following 3-tuples are primitive.

Theorem 9. *Let* $(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{P}^d_{\gg}$. *Put* $s_i := \mathrm{supp}(\lambda_i)$. *Then* $(\lambda_1, \lambda_2, \lambda_3)$ *is primitive in either of the following cases:*

TABLE 3

no.	type of G	condition
1	A_l	$s_1 = \{1\}$
2		$\|s_1\| = \|s_2\| = 1$
3		$\|s_1\| = \|s_2\| = 1; \|s_3\| = 2$
4		$\|s_1\| = 1, \|s_2\| = 2; \|s_3\| = 3$
5		$\|s_1\| = 1; \|s_2\| = 2; \|s_3\| = 4$
6		$s_1 = \{2\}; \|s_2\| = 2; \|s_3\| \geqslant 2$
7		$\|s_1\| = 1; s_2 = \{i, i+1\}$ or $\{1, j\}, i < l, j \neq 1; \|s_3\| \geqslant 2$
8	B_l	$s_1 = \{1\}; s_2 = \{1, \ldots, l\}$
9		$s_1 = s_2 = \{l\}; s_3 = \{1, \ldots, l\}$
10	C_l	$s_1 = s_3 = \{l\}$
11		$s_1 = \{l\}; s_2 = \{i\}, i \neq l; s_3 = \{j\}, j \neq l$
12		$s_1 = \{l\}; s_2 = \{i\}, i \neq l; s_3 = \{j, m\}, j \neq m$
13		$s_1 = \{l\}; s_2 = \{1\}; s_3 \neq \{l\}$
14		$s_1 = \{1\}; s_2 = \{i\}, i \neq l; s_3 \neq \{l\}$
15	D_l	$s_1 = \{1\}; s_2 = \{1, \ldots, l\}$
16		$s_1 = \{l-1\}; s_2 = \{l\}; s_3 = \{1, \ldots, l\}$
17		$s_1 = \{l\}; s_2 = \{l\}; s_3 = \{1, \ldots, l\}$
18		$s_1 = \{3\}; s_2 = \{l\}; s_3 = \{1, \ldots, l\}$
19	E_6	$s_1 = \{1\}; s_2 = \{i\}, i \neq 4; s_3 = \{1, \ldots, 6\}$
20	E_7	$s_1 = \{1\}$ or $\{2\}$ or $\{7\}; s_2 = \{7\}; s_3 = \{1, \ldots, 7\}$

Finally, Theorem 10 below explains geometric meaning of invariant-freeness of d-tuples and establishes a saturation property for them. Theorem 11 shows that invariant-freeness of $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$ imposes some constraints on configuration of the set $\{\lambda_1, \ldots, \lambda_d\}$ and gives an upper bound of length of invariant-free d-tuples.

Theorem 10. *Let* $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$ *and* $(m_1, \ldots, m_d) \in \mathbb{Z}^d_{>0}$. *The following properties are equivalent:*

(i) $(\lambda_1, \ldots, \lambda_d)$ *is invariant-free;*
(ii) $(m_1\lambda_1, \ldots, m_d\lambda_d)$ *is invariant-free;*
(iii) *the closure of every* G*-orbit in* $X_{\lambda_1, \ldots, \lambda_d}$ *contains* $(0, \ldots, 0) \in E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_d}$;
(iv) $k[X_{\lambda_1, \ldots, \lambda_d}]^G = k$.

Theorem 11. *Let* $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$ *be an invariant-free* d*-tuple. Then*

(i) $\mathbb{Q}_{>0}\lambda_i^* \notin \mathrm{cone}(\{\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_d\})$ *for every* i;
(ii) $d \leqslant \mathrm{rk}(G)$ *if* G *is a simple group.*

5. Primitiveness and open orbits

Let $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$. In this section we establish a connection between the primitiveness of $(\lambda_1, \ldots, \lambda_d)$ and some properties of the G-actions on $X_{\lambda_1,\ldots,\lambda_d}$ and $G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$.

By [PV$_1$, Theorem 1] the variety X_{λ_i} is a cone in E_{λ_i}, i.e., is stable with respect to the action of $\mathbf{G}_m$ on E_{λ_i} by scalar multiplications, and

$$X_{\lambda_i} = \mathcal{O}_{\lambda_i} \cup \{0\}. \tag{12}$$

This $\mathbf{G}_m$-action commutes with the G-action and determines a G-stable k-algebra $\mathbb{Z}_{\geqslant 0}$-grading of $k[X_{\lambda_i}]$:

$$k[X_{\lambda_i}] = \textstyle\bigoplus_{n_i \in \mathbb{Z}_{\geqslant 0}} k[X_{\lambda_i}]_{n_i}, \tag{13}$$

where $k[X_{\lambda_i}]_{n_i}$ is the space of $\mathbf{G}_m$-semi-invariants of the weight $t \mapsto t^{n_i}$. By [PV$_1$, Theorem 2] there is an isomorphism of G-modules

$$k[X]_{n_i} \simeq E_{n_i \lambda_i^*}. \tag{14}$$

The group $G^d \times \mathbf{G}_m^d$ acts on $X_{\lambda_1,\ldots,\lambda_d}$ in the natural way, and $\mathcal{O}_{\lambda_1,\ldots,\lambda_d}$ is a $\mathbf{G}_m^d$-stable open G^d-orbit in $X_{\lambda_1,\ldots,\lambda_d}$. By restriction this action entails the action of $G \times \mathbf{G}_m^d$, where G is diagonally embedded in G^d. The action of $\mathbf{G}_m^d$ on $X_{\lambda_1,\ldots,\lambda_d}$ determines a G-stable k-algebra $\mathbb{Z}^d_{\geqslant 0}$-grading of $k[X_{\lambda_1,\ldots,\lambda_d}]$,

$$k[X_{\lambda_1,\ldots,\lambda_d}] = \textstyle\bigoplus_{n_1,\ldots,n_d \in \mathbb{Z}_{\geqslant 0}} k[X_{\lambda_1,\ldots,\lambda_d}]_{(n_1,\ldots,n_d)}, \tag{15}$$

where $k[X_{\lambda_1,\ldots,\lambda_d}]_{(n_1,\ldots,n_d)}$ is the space of $\mathbf{G}_m^d$-semi-invariants of the weight $(t_1, \ldots, t_d) \mapsto t_1^{n_1} \cdots t_d^{n_d}$. Since, by (9), $k[X_{\lambda_1,\ldots,\lambda_d}]$ and $\bigotimes_{i=1}^d k[X_{\lambda_i}]$ are G-isomorphic k-algebras, (13), (14), and (15) yield that, for every $(n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geqslant 0}$, there is an isomorphism of G-modules

$$k[X_{\lambda_1,\ldots,\lambda_d}]_{(n_1,\ldots,n_d)} \simeq \textstyle\bigotimes_{1 \leqslant i \leqslant d} k[X_{\lambda_i}]_{n_i} \simeq \bigotimes_{1 \leqslant i \leqslant d} E_{n_i \lambda_i^*}. \tag{16}$$

Consider now the categorical quotient (7) for $H = G$ and $Y = X_{\lambda_1,\ldots,\lambda_d}$ and denote $\pi_{H,Y}$ by $\pi_{\lambda_1,\ldots,\lambda_d}$. The field of fractions of $k[X_{\lambda_1,\ldots,\lambda_d}]^G$ is then $\pi^*_{\lambda_1,\ldots,\lambda_d}(k(X_{\lambda_1,\ldots,\lambda_d}/\!\!/G))$. Since the action $\mathbf{G}_m^d$ on $X_{\lambda_1,\ldots,\lambda_d}$ commutes with that of G, it descends to $X_{\lambda_1,\ldots,\lambda_d}/\!\!/G$. The corresponding action of $\mathbf{G}_m^d$ on the k-algebra $k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]$ determines its $\mathbb{Z}^d_{\geqslant 0}$-grading

$$\begin{gathered} k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G] = \textstyle\bigoplus_{n_1,\ldots,n_d \in \mathbb{Z}_{\geqslant 0}} k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_{(n_1,\ldots,n_d)}, \quad \text{where} \\ \pi^*_{\lambda_1,\ldots,\lambda_d}\big(k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_{(n_1,\ldots,n_d)}\big) = k[X_{\lambda_1,\ldots,\lambda_d}]^G \cap k[X_{\lambda_1,\ldots,\lambda_d}]_{(n_1,\ldots,n_d)}. \end{gathered} \tag{17}$$

From this, (15), and (16) we deduce that

$$k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_{(n_1,\ldots,n_d)} \simeq \big(\textstyle\bigotimes_{1 \leqslant i \leqslant d} E_{n_i \lambda_i^*}\big)^G. \tag{18}$$

Lemma 1. *The following properties are equivalent:*

(i) $k(X_{\lambda_1,\ldots,\lambda_d}/\!\!/G)^{\mathbf{G}_m^d} = k$;

(ii) *there is an open* $\mathbf{G}_m^d$*-orbit in* $X_{\lambda_1,\ldots,\lambda_d}/\!\!/G$;

(iii) $(\lambda_1, \ldots, \lambda_d)$ *is primitive.*

Proof. (i)⇔(ii) follows from Rosenlicht's theorem [R_2], cf., e.g., [PV_2, Cor. of Theorem 2.3].

Assume that $(\lambda_1, \ldots, \lambda_d)$ is not primitive. Then $\dim\big(\bigotimes_{1\leqslant i\leqslant d} E_{n_i\lambda_i}\big)^G \geqslant 2$ for some $(n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geqslant 0}$. Since for all $(\mu_1, \ldots, \mu_d) \in \mathrm{P}^d_{++}$ and $(m_1, \ldots, m_d) \in \mathbb{Z}^d_{\geqslant 0}$, we have

$$\dim\big(\textstyle\bigotimes_{1\leqslant i\leqslant d} E_{m_i\mu_i}\big)^G = \dim\big(\textstyle\bigotimes_{1\leqslant i\leqslant d} E_{m_i\mu_i^*}\big)^G, \tag{19}$$

it then follows from (18) that $\dim(k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_{(n_1,\ldots,n_d)}) \geqslant 2$. This means that the algebra $k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]$ contains two nonproportional $\mathbf{G}_m^d$-semi-invariant functions f_1 and f_2 of the same weight. Hence $f_1/f_2 \in k(X_{\lambda_1,\ldots,\lambda_d}/\!\!/G)^{\mathbf{G}_m^d}$, $f_1/f_2 \notin k$. This proves (i)⇒(iii).

Conversely, let $f \in k(X_{\lambda_1,\ldots,\lambda_d}/\!\!/G)^{\mathbf{G}_m^d}$, $f \notin k$. Since $X_{\lambda_1,\ldots,\lambda_d}/\!\!/G$ is an affine variety, $k(X_{\lambda_1,\ldots,\lambda_d}/\!\!/G)$ is the field of fractions of $k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]$. As $\mathbf{G}_m^d$ is a connected solvable group, by [PV_2, Theorem 3.3] this implies that in $k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]$ there are two $\mathbf{G}_m^d$-semi-invariant elements of the same $\mathbf{G}_m^d$-weight, say, $f_1, f_2 \in k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_{(n_1,\ldots,n_d)}$, such that $f = f_1/f_2$. Since f_1 and f_2 are nonproportional, $\dim(k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_{(n_1,\ldots,n_d)}) \geqslant 2$. By (18) and (19) this yields $\dim\big(\bigotimes_{i=1}^d E_{n_i\lambda_i}\big)^G \geqslant 2$. Hence $(\lambda_1, \ldots, \lambda_d)$ is not primitive. This proves (iii)⇒(i). □

Remark 2. By [PV_2, Theorem 3] every X_{λ_i} is a normal variety. From (9) we then conclude that $X_{\lambda_1,\ldots,\lambda_d}$ is normal as well. Hence property (ii) in Lemma 1 means that $X_{\lambda_1,\ldots,\lambda_d}/\!\!/G$ is a toric $\mathbf{G}_m^d$-variety.

Lemma 2. *The following properties are equivalent:*

(i) $k(X_{\lambda_1,\ldots,\lambda_d})^{G\times\mathbf{G}_m^d} = k$;
(ii) *there is an open* $G \times \mathbf{G}_m^d$*-orbit in* $X_{\lambda_1,\ldots,\lambda_d}$*;*
(iii) *there is an open* G*-orbit in* $G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$.

Proof. The aforementioned Rosenlicht's theorem yields the equivalencies (i)⇔(ii) and

$$k\big(G/P_{\lambda_1}\times\ldots\times G/P_{\lambda_d}\big)^G = k \iff G/P_{\lambda_1}\times\ldots\times G/P_{\lambda_d} \text{ contains an open } G\text{-orbit.} \tag{20}$$

The natural projection $\rho_{\lambda_i}\colon \mathcal{O}_{\lambda_i} \to G/P_{\lambda_i}$ is a rational quotient for the $\mathbf{G}_m$-action on $\mathcal{O}_{\lambda_i}$, cf. [PV_2, 2.4]. Hence the G-equivariant morphism $\rho_{\lambda_1} \times \ldots \times \rho_{\lambda_d}\colon \mathcal{O}_{\lambda_1,\ldots,\lambda_d} \to G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$ is a rational quotient for the $\mathbf{G}_m^d$-action on $\mathcal{O}_{\lambda_1,\ldots,\lambda_d}$. Therefore it induces an isomorphism of invariant fields

$$k\big(G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}\big)^G \xrightarrow{\simeq} k(\mathcal{O}_{\lambda_1,\ldots,\lambda_d})^{G\times\mathbf{G}_m^d}. \tag{21}$$

But $k(\mathcal{O}) = k(X_{\lambda_1,\ldots,\lambda_d})$ since $\mathcal{O}$ is open in $X_{\lambda_1,\ldots,\lambda_d}$. This, (20), and (21) now imply (i)⇔(iii). □

Lemma 3. *Let* $(\lambda_1, \ldots, \lambda_d) \in \mathrm{P}^d_{\gg}$, $s \in \mathbb{Z}_{>0}$, *and*

$$\mathcal{M}_d(s) := \{(n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geqslant 0} \mid c^0_{n_1\lambda_1,\ldots,n_d\lambda_d} \geqslant s\}.$$

Then

$$\mathcal{M}_d(1) + \mathcal{M}_d(s) \subseteq \mathcal{M}_d(s).$$

Proof. By (18) and (19)

$$(n_1, \ldots, n_d) \in \mathcal{M}_d(s) \iff \dim\bigl(k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_{(n_1,\ldots,n_d)}\bigr) \geqslant s. \tag{22}$$

Let $\alpha \in \mathcal{M}_d(1)$ and $\beta \in \mathcal{M}_d(s)$. Pick a nonzero function $f \in k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_\alpha$ and linear independent functions $h_1, \ldots, h_s \in k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_\beta$: by (22) this is possible. Then the functions $fh_1, \ldots, fh_d$ are linearly independent since $k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]$ is an integral domain. They lie $k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_{\alpha+\beta}$ since (17) is a grading. So $\dim\bigl(k[X_{\lambda_1,\ldots,\lambda_d}/\!\!/G]_{\alpha+\beta}\bigr) \geqslant s$, whence $\alpha + \beta \in \mathcal{M}_d(s)$ by (22). $\square$

6. Proof of Theorem 1

If the assumption of (i) holds, Lemma 2 implies that

$$k(X_{\lambda_1,\ldots,\lambda_d})^{G\times \mathbf{G}_m^d} = \bigl(k(X_{\lambda_1,\ldots,\lambda_d})^G\bigr)^{\mathbf{G}_m^d} = k.$$

Since

$$\pi^*_{\lambda_1,\ldots,\lambda_d} \colon k(X_{\lambda_1,\ldots,\lambda_d}/\!\!/G) \hookrightarrow k(X_{\lambda_1,\ldots,\lambda_d})^G, \tag{23}$$

this yields $k(X_{\lambda_1,\ldots,\lambda_d}/\!\!/G)^{\mathbf{G}_m^d} = k$; whence $(\lambda_1, \ldots, \lambda_d)$ is primitive by Lemma 1. This proves (i).

If the assumption of (ii) holds, let

$$\varrho\colon X_{\lambda_1,\ldots,\lambda_d} \dashrightarrow X_{\lambda_1,\ldots,\lambda_d} \therefore G$$

be a rational quotient for the action of G on $X_{\lambda_1,\ldots,\lambda_d}$. i.e., $X_{\lambda_1,\ldots,\lambda_d} \therefore G$ is an irreducible variety and ϱ a dominant rational map such that $\varrho^*\bigl(k(X_{\lambda_1,\ldots,\lambda_d} \therefore G)\bigr) = k(X_{\lambda_1,\ldots,\lambda_d})^G$, cf., e.g., [PV$_2$, 2.4]. By [PV$_2$, Prop. 2.6] the action of $\mathbf{G}_m^d$ on $X_{\lambda_1,\ldots,\lambda_d}$ induces a rational $\mathbf{G}_m^d$-action on $X_{\lambda_1,\ldots,\lambda_d} \therefore G$ such that ϱ becomes $\mathbf{G}_m^d$-equivariant. By [PV$_2$, Cor. of Theorem 1.1] replacing $X_{\lambda_1,\ldots,\lambda_d} \therefore G$ with a birationally isomorphic variety, we may (and shall) assume that the rational action of $\mathbf{G}_m^d$ on $X_{\lambda_1,\ldots,\lambda_d} \therefore G$ is regular (morphic).

Embedding (23) induces a dominant rational $\mathbf{G}_m^d$-equivariant map $\tau\colon X_{\lambda_1,\ldots,\lambda_d} \therefore G \dashrightarrow X_{\lambda_1,\ldots,\lambda_d}/\!\!/G$ such that we obtain the following commutative diagram:

$$\begin{tikzcd} & X_{\lambda_1,\ldots,\lambda_d} \arrow[dl, dashed, "\varrho"'] \arrow[dr, "\pi_{\lambda_1,\ldots,\lambda_d}"] & \\ X_{\lambda_1,\ldots,\lambda_d} \therefore G \arrow[rr, dashed, "\tau"] & & X_{\lambda_1,\ldots,\lambda_d}/\!\!/G \end{tikzcd} . \tag{24}$$

Since the action of G on $X_{\lambda_1,\ldots,\lambda_d}$ is ample, we have

$$\dim(X_{\lambda_1,\ldots,\lambda_d} \therefore G) = \dim(X_{\lambda_1,\ldots,\lambda_d}/\!\!/G). \tag{25}$$

Since $(\lambda_1, \ldots, \lambda_d)$ is primitive, Lemma 1 yields that $X_{\lambda_1,\ldots,\lambda_d}/\!\!/G$ contains an open $\mathbf{G}_m^d$-orbit. ¿From this, (24), and (25) it then follows that $X_{\lambda_1,\ldots,\lambda_d} \therefore G$ contains an open $\mathbf{G}_m^d$-orbit. Hence $\bigl(k(X_{\lambda_1,\ldots,\lambda_d} \therefore G)\bigr)^{\mathbf{G}_m^d} = k$, i.e., $\bigl(k(X_{\lambda_1,\ldots,\lambda_d})^G\bigr)^{\mathbf{G}_m^d} = k(X_{\lambda_1,\ldots,\lambda_d})^{G\times\mathbf{G}_m^d} = k$. Lemma 2 then implies that $G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$ contains an open G-orbit. This proves (ii). $\square$

7. Proof of Theorem 2

Since dimension of a multiple flag variety $G/P_{\lambda_1} \times \ldots \times G/P_{\lambda_d}$ containing an open G-orbit does not exceed $\dim(G)$, we have

$$\dim(G) \geqslant \sum_{i=1}^{d} \big(\dim(G) - \dim(P_{\lambda_i})\big) \geqslant d\big(\dim(G) - \dim(P)\big),$$

where P is a parabolic subgroup of G of maximal dimension. Since $\dim(G) = \dim(L) + 2\dim(P_u)$ and $\dim(P) = \dim(L) + \dim(P_u)$, where L and P_u are respectively a Levi subgroup and the unipotent radical of P, this yields

$$d \leqslant \frac{2\dim G}{\dim G - \dim L}. \tag{26}$$

Let L_i be a Levi subgroup of $P_i := P_{\varpi_i}$. The equality $\dim(L_i) = 2\dim(P_i) - \dim(G)$ implies that $\dim(L_i)$ and $\dim(P_i)$, as functions in i, attain their absolute maximums at the same values of i; let M be the set of these values. Then the group P is conjugate to P_{i_0} for some $i_0 \in M$.

Since L_i is a reductive group of rank $\mathrm{rk}(G)$ and the Dynkin diagram of its commutator group is obtained from that of G by removing the ith node, finding all the $\dim(L_i)$'s and then the set M is a matter of some clear calculations. We skip them (see some details in [P$_4$, Sect. 5–13]). The results are collected in Table 4 below.

TABLE 4

type of G	A_l, $l \geqslant 1$	B_l, $l \geqslant 3$	C_l, $l \geqslant 2$	D_l, $l \geqslant 4$	E_6	E_7	E_8	F_4	G_2
M	1, l	1	1	1	1, 6	7	8	1, 4	1, 2
$\frac{2\dim G}{\dim G - \dim L}$	$l+2$	$\frac{2l^2+l}{2l-1}$	$\frac{2l^2+1}{2l-1}$	$\frac{2l^2-l}{2l-2}$	$\frac{39}{8}$	$\frac{133}{27}$	$\frac{248}{57}$	$\frac{52}{15}$	$\frac{14}{5}$

The claim now immediately follows from (26) and Table 4. $\square$

8. Proof of Theorem 4

We put, for brevity,

$$X := X_{\lambda_1,\ldots,\lambda_d},\ \ \mathcal{O} := \mathcal{O}_{\lambda_1,\ldots,\lambda_d},\ \ v := v_{\lambda_1} \times \ldots \times v_{\lambda_d} \in \mathcal{O}.$$

Clearly, $\dim(\overline{\mathcal{O}}_\lambda) \geqslant 2$ for every $\lambda \in \mathrm{P}_{\gg}$, hence by (9) and (12)

$$\mathrm{codim}_X(X \setminus \mathcal{O}) \geqslant 2. \tag{27}$$

Recall from Remark 2 that X is normal. From (27) we conclude that

$$\mathrm{Cl}(X) \simeq \mathrm{Cl}(\mathcal{O}). \tag{28}$$

Since $G^d_v = G_{v_{\lambda_1}} \times \ldots \times G_{v_{\lambda_d}}$ and G^d is a connected simply connected semisimple group, we deduce from [P$_2$, Prop. 1; Cor. of Theorem 4] that

$$\mathrm{Cl}(\mathcal{O}) \simeq \mathrm{Cl}(G^d/G^d_v) \simeq \mathcal{X}(G^d_v) \simeq \mathcal{X}(G_{v_{\lambda_1}}) \oplus \ldots \oplus \mathcal{X}(G_{v_{\lambda_d}}). \tag{29}$$

On the other hand, (11) and [PV$_1$, §1, no. 5] imply that

$$\mathcal{X}(G_{v_{\lambda_i}}) \simeq \mathbb{Z}/m_i. \tag{30}$$

¿From (28), (29), and (30) we obtain that

$$\mathrm{Cl}(X) \simeq \bigoplus_{i=1}^{d} \mathbb{Z}/m_i. \tag{31}$$

Further, since G is semisimple, we have

$$\mathcal{X}(G) = \{0\}; \tag{32}$$

whence every invertible element of $k[G]$ is constant, see [R$_1$]. Therefore the same holds for $k[\mathcal{O}]$ as well. As $\mathcal{O}$ is open in X, this yields that every invertible element of $k[X]$ is constant.

Take now a nonconstant function $f \in k(X)^G$. Then $(f) \neq 0$. For, otherwise, the normality of X would imply (see, e.g., [M, Theorem 38]) that f is invertible element of $k[X]$, hence a constant, a contradiction.

Since, by (31), $\mathrm{Cl}(X)$ is a finite group, there is $n \in \mathbb{Z}_{>0}$ such that both divisors $n(f)_0$ and $n(f)_\infty$ are principal, i.e.,

$$n(f)_0 = (h_1) \text{ and } n(f)_\infty = (h_2) \text{ for some } h_1, h_2 \in k(X). \tag{33}$$

As $n(f)_0 \geqslant 0$ and $n(f)_\infty \geqslant 0$, the normality of X and (33) imply that $h_1, h_2 \in k[X]$ (see, e.g., [M, Theorem 38]). Further, since f is G-invariant, the supports of $(f)_0$ and $(f)_\infty$ are G-stable subsets of X. By [PV$_2$, Theorem 3.1] this and (33) imply that h_1 and h_2 are G-semi-invariants. Hence by (32)

$$h_1, h_2 \in k[X]^G. \tag{34}$$

On the other hand, $(f^n h_2/h_1) = 0$ by (33), hence $f^n h_2/h_1$ is a constant. By (34) this means that f is algebraic over the field of fractions of $k[X]^G$. Hence, by Definition 3, the action of G on X is ample. This completes the proof. □

9. Separation index of irreducible root system

Let R be a root system in a rational vector space L (we assume that L is the linear span of R) and let $W(R)$ be the Weyl group of R. For any linear function $l \in L^*$, put

$$l^+ := \{x \in L \mid l(x) \geqslant 0\},\ \ l^0 := \{x \in L \mid l(x) = 0\},\ \ l^- := \{x \in L \mid l(x) \leqslant 0\}. \tag{35}$$

Given a subset S of L, denote by $\overline{S}$ the closure of S in L.

Lemma 4. *Let R be an irreducible root system. Then for every nonzero linear function $l \in L^*$, there is a Weyl chamber $C \subset L$ of R such that*

$$\overline{C} \subset l^+ \quad \textit{and} \quad \overline{C} \cap l^0 = \{0\}.$$

Proof. First, we prove that $R \cap l^+$ contains a basis of R. If $R \cap l^0 = \varnothing$, this is proved, e.g., in [Se, §8, Prop. 4]. In general case, fix a choice of Euclidean structure on L^* and let S be a ball in L^* with the center at l. We identify in the natural way every $\alpha \in R$ with a linear function on L^*. Taking S small enough, we may (and shall) assume that every $\alpha \in R \setminus l^0$ has no zeros on S. On the other hand, since R is finite, S does not lie in the union of hyperplanes defined by vanishing of the roots from $R \cap l^0$. Hence there is an element $s \in S$ such that

$$R \cap s^0 = \varnothing \quad \text{and} \quad R \cap l^+ \supseteq R \cap s^+. \tag{36}$$

According to the aforesaid, the equality in (36) implies that $R\cap s^+$ contains a basis of R. Then the inclusion in (36) yields the claim.

Let now $\beta_1,\dots,\beta_r$ be a basis of R contained in $R\cap l^+$. Then

$$\sum_i \mathbb{Q}_{>0}\beta_i \subset l^+\setminus l^0. \tag{37}$$

Let $\pi_1,\dots,\pi_r\in L$ be the basis of L dual to $\beta_1^\vee,\dots,\beta_r^\vee$ (i.e., $\pi_1,\dots,\pi_r\in L$ are the fundamental weights corresponding to $\beta_1,\dots,\beta_r$). Then

$$\pi_i=\sum_j c_{ij}\beta_j, \tag{38}$$

where c_{ij} are the elements of inverse Cartan matrix of R. Since R is irreducible,

$$c_{ij}\in\mathbb{Q}_{>0}\quad\text{for all } i \text{ and } j, \tag{39}$$

see, e.g., [OV]. Consider now the Weyl chamber $C:=\sum_i\mathbb{Q}_{>0}\pi_i$. Since $\overline{C}:=\sum_i\mathbb{Q}_{\geqslant 0}\pi_i$, it follows from (38) and (39), that $\overline{C}\setminus\{0\}\subset\sum_i\mathbb{Q}_{>0}\beta_i$. Now the claim follows from (37). □

Corollary. *Let R be an irreducible root system. Then there is a sequence $C_1,\dots,C_n\subset L$ of the Weyl chambers of R satisfying the following property:*

$$\begin{gathered}\textit{for every nonzero linear function } l\in L^*\textit{, there is a natural } i\in[1,n] \textit{ such that}\\ \overline{C_i}\subset l^+\quad\textit{and}\quad \overline{C_i}\cap l^0=\{0\}.\end{gathered} \tag{40}$$

Proof. Let $C_1,\dots,C_n$ be a sequence of all Weyl chambers of R. By Lemma 4 it satisfies property (40). □

Definition 5. Let R be an irreducible root system in a rational vector space L. The *separation index* $\operatorname{sep}(R)$ of R is the minimal length of sequences $C_1,\dots,C_n$ of Weyl chambers of R satisfying property (40).

Lemma 5. *The following inequalities hold:*

$$\operatorname{rk}(R)+1\leqslant\operatorname{sep}(R)\leqslant|W(R)|. \tag{41}$$

Proof. Let $C_1,\dots,C_{\operatorname{sep}(R)}$ be a sequence of Weyl chambers of R satisfying property (40). For every i, fix a choice of point $x_i\in C_i$. Arguing on the the contrary, assume that $\operatorname{sep}(R)\leqslant\operatorname{rk}(R)$. Then there is a nonzero linear function $l\in L^*$ such that $x_i\in l^-$ for all i. This contradicts property (40). Thus the left inequality in (41) is proved. The right one follows from the fact that $|W(R)|$ is equal to the cardinality of set of all Weyl chambers of R. □

The example below shows that all equalities and inequalities in (41) are attained for suitable R's.

Example 6. Clearly, $\operatorname{sep}(\mathsf{A}_1)=2$, and it is not difficult to verify that

$$\operatorname{sep}(\mathsf{A}_2)=6,\quad \operatorname{sep}(\mathsf{B}_2)=4,\quad\text{and}\quad \operatorname{sep}(\mathsf{G}_2)=3$$

(since $\operatorname{sep}(R)$ depends only on the type of R, the meaning of notation is clear).

Remark 3. The notion of separation index can be defined in a more general setting.

Namely, let $\mathcal{M}$ be a finite set of nonempty subsets of a finite dimensional real vector space L.

Definition 6. We call a subset $\mathcal{S}$ of $\mathcal{M}$ *separating* for $\mathcal{M}$ if for every nonzero linear function $l \in L^*$ there exists a set $M \in \mathcal{S}$ such that l is strictly positive at every nonzero point of M. If there exists a separating set for $\mathcal{M}$, we say that the *separation property* holds for $\mathcal{M}$ and call the minimum $\mathrm{sep}(\mathcal{M})$ of cardinalities of separating sets for $\mathcal{M}$ the *separation index* of $\mathcal{M}$.

Arguing as in the proof of Lemma 5, we obtain $\dim(L) + 1 \leqslant \mathrm{sep}(\mathcal{M}) \leqslant |\mathcal{M}|$.

Example 7. Assume that

(a) $L = \bigcup_{M \in \mathcal{M}} M$;

(b) there exists a Euclidean inner product $\langle\ ,\ \rangle$ on L such that for every set $M \in \mathcal{M}$, the angle between every two nonzero vectors of M is acute.

Then the *separation property* holds for $\mathcal{M}$. Indeed, let $l \in L^*$, $l \neq 0$. Identify L^* with L by means of $\langle\ ,\ \rangle$. Then (a) implies that l lies in some $M_0 \in \mathcal{M}$, and (b) implies that l is strictly positive at every nonzero point of M_0.

Example 8. Let $K \subset \mathbf{GL}(L)$ be an irreducible finite reflection group and let $\mathcal{M}$ be the set of closures of its chambers in L. Then (a) holds. Let $\langle\ ,\ \rangle$ be a K-invariant Euclidean inner product on L. Then the irreducibility of K implies that (b) holds as well. Hence, in this case, the separation property holds for $\mathcal{M}$.

Definition 7. In this case we call $\mathrm{sep}(\mathcal{M})$ the *separation index of* K and denote it by $\mathrm{sep}(K)$.

Definitions 5 and 7 imply that if K is crystallographic, i.e., $K = W(R)$ for an irreducible root system R, then $\mathrm{sep}(R) = \mathrm{sep}(K)$. The next example illustrates the noncrystallographic case.

Example 9. It is not difficult to verify that $\mathrm{sep}(I_2(p)) = 3$ for $p \geqslant 7$ and $\mathrm{sep}(I_2(5)) = 4$.

It would be interesting to calculate $\mathrm{sep}(K)$ for every irreducible finite reflection group K and, in particular, to find $\mathrm{sep}(\Delta)$ for every irreducible root system Δ.[1]

10. Proof of Theorem 5

The proof is based on the following lemma.

Lemma 6. *Let $V = V_1 \oplus \ldots \oplus V_n$, where $V_1, \ldots, V_n$ are finite dimensional G-modules, let $v_i \in V_i$ be a T-weight vector of a weight $\mu_i \in \mathcal{X}(T)$, and let $v := v_1 + \ldots + v_n \in V$. Then the following properties are equivalent:*

(1) *$G \cdot v$ is closed;*

[1] *Added in proof.* Recently in V. Zhgoon, D. Mironov, *Separating systems of Weyl chambers*, Math. Notes, to appear, the following upper bounds have been obtained: $\mathrm{sep}(\mathsf{A}_l) \leqslant 2l! + 2$, $\mathrm{sep}(\mathsf{B}_l) = \mathrm{sep}(\mathsf{C}_l) \leqslant 2^{l+1} - 2$, $\mathrm{sep}(\mathsf{D}_l) \leqslant 2^{l-1} l!/(l-1) + 2$, $\mathrm{sep}(\mathsf{F}_4) \leqslant 30$, $\mathrm{sep}(\mathsf{E}_6) \leqslant 242$, $\mathrm{sep}(\mathsf{E}_7) \leqslant 4610$, $\mathrm{sep}(\mathsf{E}_8) \leqslant 9222$, $\mathrm{sep}(\mathsf{H}_3) \leqslant 14$, $\mathrm{sep}(\mathsf{H}_4) \leqslant 30$.

(2) $T \cdot v$ *is closed;*
(3) $0 \in \operatorname{int}\bigl(\operatorname{conv}(\{\mu_1, \ldots, \mu_n\})\bigr)$.

Proof. This is proved in [P_3, Theorem 1]. $\square$

Passing to the proof of Theorem 5, we first establish the existence of elements $w_1,\ldots,w_d$ of W such that

(i) $\dim\bigl(\operatorname{conv}(\{w_1 \cdot \lambda_1, \ldots, w_d \cdot \lambda_d\})\bigr) = r(= \operatorname{rk}(G))$;
(ii) $0 \in \operatorname{int}\bigl(\operatorname{conv}(\{w_1 \cdot \lambda_1, \ldots, w_d \cdot \lambda_d\})\bigr)$.

Let $R \subset L := \mathcal{X}(T)_{\mathbb{Q}}$ be the root system of G with respect to T and let $C_1, \ldots, C_{\operatorname{sep}(G)}$ be a sequence of Weyl chambers of R satisfying property (40). For every $i \leqslant \operatorname{sep}(G)$, let w_i be the (unique) element of W such that $w_i \cdot \lambda_i \in \overline{C_i}$. For every $i \geqslant \operatorname{sep}(G) + 1$, put $w_i = e$. If (i) or (ii) fails, then $\operatorname{conv}(\{w_1 \cdot \lambda_1, \ldots, w_d \cdot \lambda_d\}) \subset l^-$ for some linear function $l \in L^*$. But the choice of $C_1, \ldots, C_{\operatorname{sep}(G)}$ implies that there is $i \leqslant \operatorname{sep}(G)$ such that $\overline{C_i} \subset l^+$ and $\overline{C_i} \cap l^0 = \{0\}$. Since $\lambda_i \neq 0$, we have $w_i \cdot \lambda_i \in \overline{C_i} \setminus \{0\}$, hence $w_i \cdot \lambda_i \in l^+ \setminus l^0$. Therefore $w_i \cdot \lambda_i \notin l^-$, a contradiction. Thus (i) and (ii) hold, and the existence of desired w_i's is proved.

Consider now the point

$$v := \dot{w}_1 \cdot v_{\lambda_1} + \ldots + \dot{w}_d \cdot v_{\lambda_d} \in X_{\lambda_1,\ldots,\lambda_d} \subseteq E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_d}. \tag{42}$$

Since $\dot{w}_i \cdot v_{\lambda_i} \in E_{\lambda_i}$ is a weight vector of weight $w_i \cdot \lambda_i$, it follows from (ii) and Lemma 6 that the orbit $G \cdot v$ is closed in $X_{\lambda_1,\ldots,\lambda_d}$. In turn, this implies, by Matsushima's criterion, see, e.g., [PV_2, Theorem 4.17], that G_v is a reductive group. We claim that G_v is finite, i.e., that $\operatorname{Lie}(G_v) = 0$.

To prove this, observe that since $G_{\dot{w}_i \cdot v_{\lambda_i}} = \dot{w}_i G_{v_{\lambda_i}} \dot{w}_i^{-1}$, decomposition (42) implies that

$$G_v = \bigcap_{i=1}^{d} \dot{w}_i G_{v_{\lambda_i}} \dot{w}_i^{-1}. \tag{43}$$

Taking into account that $\dot{w}_i \in N_G(T)$ and $G_{v_{\lambda_i}}$ is normalized by T, we deduce from (43) that G_v is normalized by T as well. Hence, cf., e.g., [TY, 20.7],

$$\operatorname{Lie}(G_v) = \mathfrak{h} \oplus \Bigl(\bigoplus_{\alpha \in S} \mathfrak{g}_\alpha\Bigr), \tag{44}$$

where $\mathfrak{g}_\alpha$ is the Lie algebra of one-dimensional unipotent root subgroup of G corresponding to the root $\alpha \in R$, S is a subset of R, and $\mathfrak{h}$ is a maximal torus of $\operatorname{Lie}(G_v)$ contained in $\operatorname{Lie}(T)$. Since $\operatorname{Lie}(G_v)$ is reductive, the conditions $\operatorname{Lie}(G_v) = 0$ and $\mathfrak{h} = 0$ are equivalent. To prove that $\mathfrak{h} = 0$, observe that

$$\operatorname{Lie}(G_{v_{\lambda_i}}) = \operatorname{Lie}(\ker \lambda_i) \oplus \Bigl(\bigoplus_{\alpha \in S_i} \mathfrak{g}_\alpha\Bigr) \tag{45}$$

for some $S_i \subset R$, see [PV_1]. From (43) and (45) we then deduce that

$$\operatorname{Lie}(G_v) = \bigcap_{i=1}^{d} \Bigl(\operatorname{Lie}\bigl(\ker(w_i \cdot \lambda_i)\bigr) \oplus \Bigl(\bigoplus_{\alpha \in w_i \cdot S_i} \mathfrak{g}_\alpha\Bigr)\Bigr). \tag{46}$$

In turn, it follows from (44) and (46) that

$$\mathfrak{h} \subseteq \bigcap_{i=1}^{d} \operatorname{Lie}\big(\ker(w_i \cdot \lambda_i)\big). \tag{47}$$

¿From property (i) we deduce that the right-hand side of (47) is equal to 0. Hence $\mathfrak{h} = 0$, as claimed. Thus we proved that G_v is finite.

It follows from $\dim(G_v) = 0$ that $\dim(G\cdot v) = \dim(G)$. Hence maximum of dimensions of G-orbits in $X_{\lambda_1,\dots,\lambda_d}$ is equal to $\dim(G)$. But the set of point whose G-orbit has maximal dimension is open in $X_{\lambda_1,\dots,\lambda_d}$, cf., e.g., [PV$_2$, 1.4]. Hence G-stabilizer of a point in general position in $X_{\lambda_1,\dots,\lambda_d}$ is finite. Finally, since $G \cdot v$ is a closed orbit of maximal dimension, [P$_1$, Theorem 4] implies that the action of G on $X_{\lambda_1,\dots,\lambda_d}$ is stable. □

11. Proof of Theorem 7

Let $(\lambda_1, \dots, \lambda_d) \in \mathrm{P}^d_{\gg}$ be a primitive d-tuple. Assume the contrary, i.e.,

$$\operatorname{sep}(G) + 2 \leqslant d. \tag{48}$$

¿From (48) and Theorems 1, 5 we deduce that the multiple flag variety $G/P_{\lambda_1} \times \dots \times G/P_{\lambda_d}$ contains an open G-orbit. Theorem 2 then implies that

$$d \leqslant \operatorname{rk}(G) + 2. \tag{49}$$

¿From (48), (49) we obtain the inequality $\operatorname{sep}(G) \leqslant \operatorname{rk}(G)$ that contradicts (41). □

12. Proof of Theorem 9

By Theorem 1 the claim follows from the fact that in either of the cases listed in Table 2 the multiple flag variety $G/P_{\lambda_1} \times G/P_{\lambda_2} \times G/P_{\lambda_3}$ contains an open G-orbit. The latter is proved as follows.

If G is of type B_l, D_l, E_6, or E_7, then $s_3 = \{1, \dots, \operatorname{rk}(G)\}$, hence $P_{\lambda_3} = B$. Therefore $G/P_{\lambda_1} \times G/P_{\lambda_2} \times G/P_{\lambda_3}$ contains an open G-orbit if and only if $G/P_{\lambda_1} \times G/P_{\lambda_2}$ contains an open B-orbit, cf., e.g., [P$_4$, Lem. 4]. All the pairs of fundamental weights (λ_1, λ_2) for which the latter holds are classified in [Li, 1.2]. According to this classification, for these types of G, the supports of λ_1 and λ_2 are precisely (up to automorphism of the Dynkin diagram) s_1 and s_2 specified in Table 2.

For G of types A_l and C_l, in [MWZ$_1$] and [MWZ$_2$] it is given a classification of all the products $G/P_{\lambda_1} \times G/P_{\lambda_2} \times G/P_{\lambda_3}$ that contain only finitely many G-orbits. One of these orbits is then open in $G/P_{\lambda_1} \times G/P_{\lambda_2} \times G/P_{\lambda_3}$. The triples $(\lambda_1, \lambda_2, \lambda_3)$ arising in these classifications are precisely (up to automorphism of the Dynkin diagram) the ones whose supports satisfy the conditions of cases listed in Table 2 for these types of G. (Actually, in [MWZ$_1$] and [MWZ$_2$], flag varieties are described in terms of "compositions", i.e., essentially, dimension vectors of corresponding flags. The information in Table 2 is obtained by reformulating results of [MWZ$_1$] and [MWZ$_2$] in terms of supports of the corresponding dominant weights; obtaining this reformulation is not difficult: for instance, for G of type A_l, one deduces it from the fact that cardinality of the set of nonzero parts of a composition is equal to cardinality of the support of corresponding dominant weight plus 1.) □

13. Proof of Theorem 10

Since $(0,\dots,0)$ is a fixed point for the action of G on $X_{\lambda_1,\dots,\lambda_d}$, the equivalence (iii)$\Leftrightarrow$(iv) follows from the property that for every reductive group action on affine variety, disjoint invariant closed subsets are separated by the algebra of invariants, see, e.g., [PV$_2$, Theorem 4.7].

The equivalence (i)$\Leftrightarrow$(iv) follows from (17), (18), (19), and Definition 2.

The implication (i)$\Rightarrow$(ii) follows from Definition 2.

Arguing on the contrary, assume that (ii) holds, but (i) does not. The latter means that $c^0_{s_1\lambda_1,\dots,s_d\lambda_d}\geqslant 1$ for some $(s_1,\dots,s_d)\in\mathbb{Z}^d_{\geqslant 0}$. Lemma 3 then implies that $c^0_{ms_1,\dots,ms_d}\geqslant 1$ for every $m\in\mathbb{Z}_{>0}$. Taking $m=m_1\dots m_d$, we obtain

$$c^0_{n_1m_1\lambda_1,\dots,n_dm_d\lambda_d}\geqslant 1 \quad \text{where } n_i=m_1\dots\widehat{m_i}\dots m_ds_i. \tag{50}$$

Definition 2 now shows that property (ii) contradicts (50). □.

14. Proof of Theorem 11

We utilize the following lemma.

Lemma 7. *Let R be an irreducible reduced root system in an n-dimensional rational vector space L. Then there are the Weyl chambers $C_1,\dots,C_{n+1}\subset L$ of R such that*

$$0\in\operatorname{conv}(\{x_1,\dots,x_{n+1}\}) \textit{ for every choice of points } x_1\in\overline{C_1},\dots,x_{n+1}\in\overline{C_{n+1}}. \tag{51}$$

Proof (R. Suter). Let $R^\vee\subset L^*$ be the dual root system. Take a basis $l_1,\dots,l_n$ of $R^\vee$ and let $-l_{n+1}$ be the corresponding maximal root of $R^\vee$. Utilizing notation (35), put

$$Z_i:=\bigcap_{j\in[1,n+1],\,j\neq i} l_j^+. \tag{52}$$

We claim that $0\in\operatorname{conv}(\{x_1,\dots,x_{n+1}\})$ for every choice of points $x_i\in Z_i$, $i=1,\dots,n+1$. Indeed, if $0\notin\operatorname{conv}(\{x_1,\dots,x_{n+1}\})$ for some $x_i\in Z_i$, $i=1,\dots,n+1$, then there is a nonzero linear function $l\in L^*$ such that

$$l(x_i)<0 \quad \text{for every } i=1,\dots,n+1. \tag{53}$$

Since $L^*=\bigcup_{j=1}^{n+1}\operatorname{cone}(\{l_1,\dots,\widehat{l_j},\dots,l_{n+1}\})$, there is i_0 such that

$$l\in\operatorname{cone}(\{l_1,\dots,\widehat{l_{i_0}},\dots,l_{n+1}\}). \tag{54}$$

¿From (52) and (54) we deduce that $l(x_{i_0})\geqslant 0$, contrary to (53). A contradiction.

Now, since every Z_i is a union of the closures of Weyl chambers, we can choose a Weyl chamber C_i lying in Z_i. Then required property (51) holds for $C_1,\dots,C_{n+1}$. □

Passing to the proof of Theorem 11 and arguing on the contrary, assume that (i) fails, i.e., for some i,

$$m_i\lambda_i^*=m_1\lambda_1+\ldots+\widehat{m_i\lambda_i}+\ldots+m_d\lambda_d, \tag{55}$$

where $(m_1,\dots,m_d)\in\mathbb{Q}^d_{\geqslant 0}$ and $m_i>0$. Multiplying both sides of (55) by an appropriate natural number, we may (and shall) assume that $(m_1,\dots,m_d)\in\mathbb{Z}^d_{\geqslant 0}$. The Cartan component of $E_{m_1\lambda_1}\otimes\ldots\otimes\widehat{E_{m_i\lambda_i}}\otimes\ldots\otimes E_{m_d\lambda_d}$ is $E_{m_1\lambda_1+\ldots+\widehat{m_i\lambda_i}+\ldots+m_d\lambda_d}$. Hence

$$E_{m_1\lambda_1}\otimes\ldots\otimes\widehat{E_{m_i\lambda_i}}\otimes\ldots\otimes E_{m_d\lambda_d}\simeq E_{m_1\lambda_1+\ldots+\widehat{m_i\lambda_i}+\ldots+m_d\lambda_d}\oplus\cdots, \tag{56}$$

where the right-hand side of (56) is a direct sum of simple G-modules. It follows from (55), (56), and (2) that

$$c^0_{m_1\lambda_1,\ldots,m_d\lambda_d} \geqslant 1. \tag{57}$$

Since (57) contradicts the assumption that $(\lambda_1,\ldots,\lambda_d)$ is invariant-free, this proves (i).

Again arguing on the contrary, assume that (ii) fails, i.e.,

$$d \geqslant r+1, \quad \text{where } r = \mathrm{rk}(G). \tag{58}$$

By Lemma 7 there are the Weyl chambers $C_1,\ldots,C_{r+1} \subset \mathcal{X}(T)_{\mathbb{Q}}$ of the root system of G with respect to T such that property (51) (with $n = r$) holds. Inequality (58) implies that there are (unique) elements $w_1,\ldots,w_{r+1} \in W$ such that $w_i \cdot \lambda_i \in \overline{C_i}$ for every i. By (51) we have

$$0 \in \mathrm{conv}(\{w_1\cdot\lambda_1,\ldots,w_{r+1}\cdot\lambda_{r+1}\}).$$

Hence 0 is an interior point of some face of the potytope $\mathrm{conv}(\{w_1\cdot\lambda_1,\ldots,w_{r+1}\cdot\lambda_{r+1}\})$; whence

$$0 \in \mathrm{int}\big(\mathrm{conv}(\{w_{i_1}\cdot\lambda_{i_1},\ldots,w_{i_m}\cdot\lambda_{i_m}\})\big) \tag{59}$$

for some $i_1,\ldots,i_m$. Since $\dot{w}_i \cdot v_{\lambda_i} \in E_{\lambda_i}$ is a weight vector of weight $w_i\cdot\lambda_i$, it follows from (59) and Lemma 6 that the G-orbit of point $\dot{w}_{i_1}\cdot v_{\lambda_{i_1}} + \ldots + \dot{w}_{i_m}\cdot v_{\lambda_{i_m}}$ is closed in $X_{\lambda_{i_1},\ldots,\lambda_{i_m}}$. But $X_{\lambda_{i_1},\ldots,\lambda_{i_m}}$ clearly admits a closed G-invariant embedding in $X_{\lambda_1,\ldots,\lambda_d}$, so this gives a closed G-orbit in $X_{\lambda_1,\ldots,\lambda_d}$ as well. Since this orbit is different from $(0,\ldots,0)$, Theorem 10 yields a contradiction with the assumption that $(\lambda_1,\ldots,\lambda_d)$ is invariant-free. This proves (ii). $\square$

15. Stability of G-action on $X_{\lambda_1,\ldots,\lambda_d}$

In this section we prove that several other conditions are sufficient for stability of the action of G on $X_{\lambda_1,\ldots,\lambda_d}$.

Consider P^d_{++} as a submonoid of the group $\mathcal{X}(T)^d$ that, in turn, is considered as a lattice in the rational vector space $\mathcal{X}(T)^d_{\mathbb{Q}} := \mathcal{X}(T)^d \otimes \mathbb{Q}$. Notice that if A and B are submonoids of P^d_{++}, then the condition

$$\mathrm{int}\big(\mathrm{cone}(A)\big) \cap \mathrm{int}\big(\mathrm{cone}(B)\big) \neq \varnothing$$

is equivalent to the property that $A - B := \{a - b \mid a \in A, b \in B\}$ is a group.

For $(\lambda_1,\ldots,\lambda_d) \in \mathrm{P}^d_{++}$, consider the submonoid $\langle\lambda_1,\ldots,\lambda_d\rangle$ of P^d_{++} generated by $(\lambda_1,0,\ldots,0),\ldots,(0,\ldots,0,\lambda_d)$,

$$\langle\lambda_1,\ldots,\lambda_d\rangle := \{(n_1\lambda_1,\ldots,n_d\lambda_d) \mid (n_1,\ldots,n_d) \in \mathbb{Z}^d_{\geqslant 0}\}.$$

Put

$$\Gamma(G,d) := \{(\mu_1,\ldots,\mu_d) \in \mathrm{P}^d_{++} \mid (E_{\mu_1}\otimes\ldots\otimes E_{\mu_d})^G \neq 0\}. \tag{60}$$

Example 10. $\Gamma(G,1) = \{0\}$, and, by (2), we have $\Gamma(G,2) = \{(\mu,\mu^*) \mid \mu \in \mathrm{P}_{++}\}$.

Example 11. Put $\mathrm{LR}(G,3) := \{(\lambda_1,\lambda_2,\lambda_3) \mid (\lambda_1,\lambda_2,\lambda_3^*) \in \Gamma(G,3)\}$. Then $\mathrm{LR}(\mathbf{SL}_n,3)$ is the Littelwood–Richardson semigroup of order n, [Z]. It has been intensively studied during the last decade and is now rather well understood. For instance, a minimal system of linear inequalities cutting out $\mathrm{cone}\big(\mathrm{LR}(\mathbf{SL}_n,3)\big)$ in $\mathcal{X}(T)^3_{\mathbb{Q}}$ is found, the walls of $\operatorname{cone}\bigl(\mathrm{LR}(\mathbf{SL}_n,3)\bigr)$ are described, and it is proved that $\mathrm{LR}(\mathbf{SL}_n,3)$ is the intersection of $\operatorname{cone}\bigl(\mathrm{LR}(\mathbf{SL}_n,3)\bigr)$ with the corresponding lattice in $\mathcal{X}(T)^3_{\mathbb{Q}}$ (saturation conjecture), see survey [F] and [Be$_1$], [Be$_2$]. This immediately implies analogous results about $\Gamma(\mathbf{SL}_n,3)$. In [KM] some general structural results for $\Gamma(G,3)$ are obtained and $\Gamma(\mathbf{Sp}_4,3)$ and $\Gamma(\mathbf{G}_2,3)$ are computed. $\Gamma(\mathbf{Spin}_8,3)$ is studied in [KKM].

These examples show that $\Gamma(G,d)$ for $d\leqslant 3$ is a finitely generated submonoid of P^d_{++}. Actually this is true for every d, see Corollary of Theorem 12 below. It would be interesting to understand the structure of this monoid in general case. What are the inequalities cutting out $\operatorname{cone}\bigl(\Gamma\,(G,d)\bigr)$ in $\mathcal{X}(T)^d_{\mathbb{Q}}$? What are the generators of $\Gamma(G,d)$?

Theorem 12. *Consider G as the diagonal subgroup of G^d. Then* (*see* (6))

$$\Gamma(G,d)=\mathcal{S}(G^d,G^d/G). \tag{61}$$

Proof. We can (and shall) identify in the natural way P^d_{++} with the monoid of dominant weights of the semisimple group G^d with respect to maximal torus T^d and Borel subgroup B^d. Simple G^d-modules are tensor products $E_{\mu_1}\otimes\ldots\otimes E_{\mu_d}$, where E_{μ_i} is considered as the G^d-module via the ith projection $G^d\to G$, cf., e.g., [OV, Ch. 4, §3]. This, Frobenius duality (cf., e.g., [PV$_2$, Theorem 3.12]), and formulas (19), (60), (6) now imply the claim. □

Corollary. *$\Gamma(G,d)$ is a finitely generated submonoid of P^d_{++}.*

Proof. Equality (61) implies that $\Gamma(G,d)$ is a submonoid of P^d_{++}. Since G is a reductive group, Matsushima's criterion implies that G^d/G is an affine variety; whence $\Gamma(G,d)$ is finitely generated (see the arguments right after formula (6)). □

Theorem 13. *Let $(\lambda_1,\ldots,\lambda_d)\in\mathrm{P}^d_{\gg}$. If either of the following conditions holds, then the action of G on $X_{\lambda_1,\ldots,\lambda_d}$ is stable:*

(i) $\operatorname{int}\bigl(\operatorname{cone}\bigl(\Gamma(G,d)\bigr)\bigr)\cap\operatorname{int}\bigl(\operatorname{cone}\bigl(\{(\lambda_1,0,\ldots,0),\ldots,(0,\ldots,0,\lambda_d)\}\bigr)\bigr)\neq\varnothing$;

(ii) *there is i such that*

$$\begin{gathered}\dim\bigl(\operatorname{cone}(\{\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_d\})\bigr)=\operatorname{rk}(G),\\ \lambda_i^*\in\operatorname{int}\bigl(\operatorname{cone}(\{\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_d\})\bigr);\end{gathered}\tag{62}$$

(iii) $\{1,\ldots,d\}$ *is a disjoint union of subsets $\{i_1,\ldots,i_s\}$ and $\{j_1,\ldots,j_t\}$ such that*

$$\begin{gathered}\dim\bigl(\operatorname{cone}(\{\lambda_{i_1},\ldots,\lambda_{i_s}\})\bigr)=\dim\bigl(\operatorname{cone}(\{\lambda_{j_1},\ldots,\lambda_{j_t}\})\bigr)=\operatorname{rk}(G),\\ \operatorname{int}\bigl(\operatorname{cone}(\{\lambda_{i_1},\ldots,\lambda_{i_s}\})\bigr)\cap\operatorname{int}\bigl(\operatorname{cone}(\{\lambda^*_{j_1},\ldots,\lambda^*_{j_t}\})\bigr)\neq\varnothing.\end{gathered}\tag{63}$$

Proof. (1) Discussion in Section 5 (see formula (16)) implies that $k[X_{\lambda_1,\ldots,\lambda_d}]_{(n_1,\ldots,n_d)}$ is a simple G^d-module with highest weight $(n_1\lambda_1^*,\ldots,n_d\lambda_d^*)$. This and (15) imply that

$$\mathcal{S}(G^d,X_{\lambda_1,\ldots,\lambda_d})=\langle\lambda_1^*,\ldots,\lambda_d^*\rangle. \tag{64}$$

By [V, Theorem 10] the action of G on $X_{\lambda_1,\ldots,\lambda_d}$ is stable if $\mathcal{S}(G^d,X_{\lambda_1,\ldots,\lambda_d})-\mathcal{S}(G^d,G^d/G)$ is a group. But $\Gamma(G,d)^*=\Gamma(G,d)$ by (19) and (60). Hence (64) and Theorem 12 imply that the action of G on $X_{\lambda_1,\ldots,\lambda_d}$ is stable if (i) holds.

(2) The variety $X_{\lambda_1,\ldots,\lambda_d}$ is G-isomorphic to $Y\times Z$, where $Y:=X_{\lambda_1,\ldots,\widehat{\lambda}_i,\ldots,\lambda_d}$ and $Z:=X_{\lambda_i}$. Discussion in Section 5 shows that $\mathcal{S}(G,Y)^*\ni\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_d$ and $\mathcal{S}(G,Z)=\mathbb{Z}_{\geqslant 0}\lambda_i^*$. Hence

$$\operatorname{cone}\big(\mathcal{S}(G,Y)^*\big)\supseteq\operatorname{cone}(\{\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_d\})\quad\text{and}\quad\operatorname{cone}\big(\mathcal{S}(G,Z)\big)=\mathbb{Q}_{\geqslant 0}\lambda_i^*. \tag{65}$$

If (ii) holds, we deduce from (62) and (65) that

$$\operatorname{int}\big(\operatorname{cone}\big(\mathcal{S}(G,Y)^*\big)\big)\cap\operatorname{int}\big(\operatorname{cone}\big(\mathcal{S}(G,Z)\big)\big)\neq\varnothing. \tag{66}$$

By [V, Theorem 9] inequality (66) implies that the action of G on $Y\times Z$ is stable.

(3) Assume now that (iii) holds. The variety $X_{\lambda_1,\ldots,\lambda_d}$ is isomorphic to $Y\times Z$, where $Y:=X_{\lambda_{i_1},\ldots,\lambda_{i_s}}$ and $Z:=X_{\lambda_{j_1},\ldots,\lambda_{j_t}}$. Hence

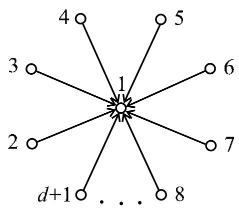
$$\begin{aligned}\operatorname{cone}\big(\mathcal{S}(G,Y)^*\big)&\supseteq\operatorname{cone}(\{\lambda_{i_1},\ldots,\lambda_{i_s}\}),\\ \operatorname{cone}\big(\mathcal{S}(G,Z)\big)&\supseteq\operatorname{cone}(\{\lambda_{j_1}^*,\ldots,\lambda_{j_t}^*\}).\end{aligned} \tag{67}$$

It follows from (63) and (67) that, as above, (66) holds and hence the action of G on $Y\times Z$ is stable. $\square$

16. Case of $\mathbf{SL}_n$

Let $G=\mathbf{SL}_n$. In this case, combining the above results with that of the representation theory of quivers (we refer to [K$_1$], [K$_2$], [DW], [Sc] for the notions of this theory) leads to a characterization of primitive d-tuples of fundamental weights in terms of canonical decomposition of dimension vectors of representations of some graphs and to an algorithmic way of solving, for every such d-tuple $(\varpi_{i_1},\ldots,\varpi_{i_d})$, whether it is primitive or not.

Namely, in this case, G/P_{ϖ_i} is the Grassmannian variety of i-dimensional linear subspaces on k^n, and the existence of an open G-orbit in $G/P_{\varpi_{i_1}}\times\ldots\times G/P_{\varpi_{i_d}}$ admits the following reformulation in terms of the representation theory of quivers. Let $\mathcal{V}_d$ be the quiver with $d+1$ vertices, d outside, one inside, and the arrows from each vertex outside to a vertex inside (the vertices are enumerated by $1,\ldots,d+1$ so that the inside vertex is enumerated by 1):



Given a vector

$$\alpha:=(a_1,\ldots,a_{d+1})\in\mathbb{Z}^{d+1}_{\geqslant 0},$$

put $\mathbf{GL}_\alpha:=\mathbf{GL}_{a_1}\times\ldots\times\mathbf{GL}_{a_{d+1}}$ (we set $\mathbf{GL}_0:=\{e\}$). Let

$$\operatorname{Rep}(\mathcal{V}_d,\alpha):=\operatorname{Mat}_{a_1\times a_2}\oplus\ldots\oplus\operatorname{Mat}_{a_1\times a_{d+1}}$$

be the space of α-dimensional representations of $\mathcal{V}_d$ endowed with the natural $\mathbf{GL}_\alpha$-action. For $\mathcal{V}_d$, the Euler inner product $\langle\ |\ \rangle$ on $\mathbb{Z}^{d+1}$ is given by

$$\langle(x_1,\ldots x_{d+1})\,|\,(y_1,\ldots,y_{d+1})\rangle=(x_1y_1+\ldots+x_{d+1}y_{d+1})-y_1(x_2+\ldots+x_{d+1}). \tag{68}$$

It then follows from the basic definitions that the following properties are equivalent:

(a) the multiple flag variety $G/P_{\varpi_{i_1}} \times \ldots \times G/P_{\varpi_{i_d}}$ contains an open G-orbit;

(b) the space $\mathrm{Rep}(\mathcal{V}_d, \gamma)$, where

$$\gamma := (n, i_1, \ldots, i_d),$$

contains an open $\mathbf{GL}_\gamma$-orbit.

Theorem 14. *Let $G = \mathbf{SL}_n$. The following properties are equivalent:*

(i) *$(\varpi_{i_1}, \ldots, \varpi_{i_d})$ is primitive;*

(ii) *all the roots β_i appearing in the canonical decomposition of γ,*

$$\gamma = \beta_1 + \ldots + \beta_s, \tag{69}$$

are real, i.e., $\langle \beta_i \mid \beta_i \rangle = 1$.

Proof. By Theorems 1 and 4 properties (i) and (a) are equivalent. On the other hand, properties (ii) and (b) are equivalent by [K$_2$, Cor. 1 of Prop. 4]. $\square$

Note that there are combinatorial algorithms for finding decomposition (69) (see [Sc], [DW]; the algorithm in [DW] is fast). Hence they, Theorem 14, and formula (68) yield algorithms verifying, for every concrete d-tuple $(\varpi_{i_1}, \ldots, \varpi_{i_d})$, whether it is primitive or not.

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