Motivic splitting lemma

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Abstract

Let M be a Chow motive over a field F. Let X be a smooth projective variety over F and N be a direct summand of the motive of X. Assume the motives M and N split over the generic point of X as direct sums of shifted copies of a Tate motive. The main result of the paper says that if a morphism $f : M \to N$ splits over the generic point of X then it splits over F, i.e., N is a direct summand of M. We apply this result to various examples of motives of projective homogeneous varieties.

We say a motive M is *split* if it is isomorphic to a direct sum of shifted copies of a Tate motive. We say a motive M is *generically split* if there exists a smooth projective variety X and an integer l such that M is split over the generic point of X and M is a direct summand of the shifted motive $M(X)(l)[2l]$ of X. In particular, a variety X is called *generically split* if its Chow motive $M(X)$ is split over the generic point of X.

The classical examples of such varieties are Severi-Brauer varieties, Pfister quadrics and maximal orthogonal Grassmannians. In the present paper we provide useful technical tool to study motivic decompositions of generically split varieties (motives). Namely, we prove the following

Theorem 1. *Let M be a Chow motive over a field F. Let X be a smooth projective variety over F and N be a direct summand of the motive of X. Assume the motives M and N are split over the field of fractions K of X. Then a morphism $f : M \to N$ splits, i.e., N is a direct summand of M, if it splits over K.*

To prove the theorem we use the following auxiliary facts

Chow motives over a relative base. For a smooth variety X over F we introduce the category of Chow motives over X following [5]. First, we define the category of correspondences $\mathcal{C}(X)$ whose objects are smooth projective morphisms $Y \to X$ and (for connected Y)

$$\mathrm{Hom}_X([Y \to X], [Z \to X]) = \mathrm{CH}_{\dim(Y)}(Y \times_X Z)$$

with natural composition law. Now the category of effective Chow motives $Chow^{eff}(X)$ can be defined as the Karoubian envelope of $\mathcal{C}(X)$. One has the restriction functor

$$res_{X/F} : Chow^{eff}(F) \to Chow^{eff}(X)$$

sending $[Y]$ to $[Y \times_F X \to X]$. For a motive N we denote $N_X := res_{X/F}(N)$. In particular, the image of the Tate motive $\mathbb{Z}(1)[2]$ gives us the Tate motive $\mathbb{Z}_X(1)[2]$ in $Chow^{eff}(X)$. The category $Chow^{eff}(X)$ has natural tensor structure

$$[Y \to X] \otimes [Z \to X] := [Y \times_X Z \to X].$$

Finally, $Chow(X)$ is obtained from $Chow^{eff}(X)$ by inverting $\mathbb{Z}_X(1)[2]$.

We use the following standard notation which agrees with [5]. For a motive $M \in Ob(Chow(X))$ we denote by $M(i)[2i]$ its shift $M \otimes \mathbb{Z}_X(1)[2]^{\otimes i}$. Hence, for shifts we have

$$\mathrm{Hom}_X([Y \to X](i)[2i], [Z \to X](j)[2j]) = \mathrm{CH}_{dim(Y)+i-j}(Y \times_X Z).$$

We identify the Chow group with low index $\mathrm{CH}_m(M)$ of a motive M with $\mathrm{Hom}_X(\mathbb{Z}_X(m)[2m], M)$ and the Chow group with upper index $\mathrm{CH}^m(M)$ with $\mathrm{Hom}_X(M, \mathbb{Z}_X(m)[2m])$. For a smooth projective variety X over a field F we denote by $M(X)$ the motive $[X \to \mathrm{Spec}\, F]$ of $Chow(F)$. As usual we denote by $c^t \in \mathrm{CH}(Z \times_X Y)$ the transposition of a cycle $c \in \mathrm{CH}(Y \times_X Z)$. For a morphism $f : M \to N$ the composition operation induces realization maps $R^m(f) : \mathrm{CH}^m(N) \to \mathrm{CH}^m(M)$.

For a given motive N over F and a field extension L/F we say a cycle in $\mathrm{CH}(N_L)$ is *rational* if it is in the image of the restriction map $res_{L/F}$.

Rost Nilpotence Theorem for generically split motives. Assume a motive N is generically split, i.e., there is a smooth projective variety X and $l \in \mathbb{Z}$ such that N is a direct summand of $M(X)(l)[2l]$, and N_K is split, where K is the field of fractions of X. We will extensively use the following version of the Rost Nilpotence Theorem (cf. [2])

Proposition 1. *Let N be a generically split motive over F. Then Rost Nilpotence Theorem holds for N. In other words, for any field extension E/F, the kernel of the map*

$$res_{E/F} : \mathrm{End}_F(N) \to \mathrm{End}_E(N_E)$$

consists of nilpotents.

Proof. We may assume that N is a direct summand of $M(X)$ (that is, $l = 0$). Since for a split motive M and for arbitrary field extension E/L, the map $\mathrm{End}_L(M_L) \to \mathrm{End}_E(M_E)$ is an isomorphism, we may assume that $E = K$ is the field of fractions of X. We have ring homomorphisms

$$\mathrm{End}_F(N) \xrightarrow{res_{X/F}} \mathrm{End}_X(N_X) \xrightarrow{res_K} \mathrm{End}_K(N_K),$$

where the last one is induced by the generic point $\mathrm{Spec}\, K \to X$.

By Lemma 1 the kernel of res_K consists of nilpotents. On the other hand, the map $res_{X/F}$ is split injective with the section induced by the composite

$$N \otimes N \xrightarrow{id \otimes \phi_N} N \otimes M(X) \xrightarrow{id \otimes \Delta_X} N \otimes M(X) \otimes M(X) \xrightarrow{id \otimes p_N \otimes id} N \otimes N \otimes M(X),$$

where $N \xrightarrow{\phi_N} M(X) \xrightarrow{p_N} N$ are the morphisms defining N as a direct summand of $M(X)$. Proposition is proven. $\square$

Lemma 1. *For any $M \in Ob(Chow(X))$, the kernel of the map induced by an open embedding $U \to X$*

$$res_U : \mathrm{End}_X(M) \to \mathrm{End}_U(M_U)$$

consists of nilpotents.

Proof. If M is a direct summand of $[Y \to X](i)[2i]$, then $\mathrm{End}_X(M)$ is a direct summand of $\mathrm{End}_X([Y \to X])$, and it is sufficient to study the case $M = [Y \to X]$.

Let f be an element from the kernel. Let $j : Z \to X$ be the reduced closed complement to U in X. Under the identification $\mathrm{End}_X([Y \to X]) = \mathrm{CH}_{\dim(Y)}(Y \times_X Y)$, f belongs to the image of the induced push-forward $j_* : \mathrm{CH}_{\dim(Y)}((Y \times_X Y) \times_X Z) \to \mathrm{CH}_{\dim(Y)}(Y \times_X Y)$. Then $f^{\circ(\dim(X)+1)}$ must be zero, since $[Z]^{\dim(X)+1}$ is zero in $\mathrm{CH}(X)$. Lemma is proven. $\square$

Now we are ready to prove the main result of the paper

Proof of Theorem. To construct a section of f we apply recursively the following procedure starting from $g = 0$ and $m = 0$.

> For a morphism $g : N \to M$ such that the realization morphism $R^i(f_K \circ g_K)$ is the identity on $\mathrm{CH}^i(N_K)$ for $i < m$, we construct a new morphism $g' : N \to M$ such that $R^i(f_K \circ g'_K)$ is the identity on $\mathrm{CH}^i(N_K)$ for $i \le m$.

Since the motive N_K splits, for the corresponding projector ρ_N over K we may write $(\rho_N)_K = \sum_l \omega_l \times \omega_l^\vee$ for certain $\omega_l \in \mathrm{CH}^*(X_K)$ and $\omega_l^\vee \in \mathrm{CH}_*(X_K)$ such that $\deg(\omega_l \cdot \omega_m^\vee) = \delta_{l,m}$. Elements ω_l form a basis of $\mathrm{CH}^*(N_K) = (\rho_N)_K \circ \mathrm{CH}^*(X_K) \subset \mathrm{CH}^*(X_K)$.

Consider the surjection $\mathrm{CH}^m(X \times X) \twoheadrightarrow \mathrm{CH}^m(K \times_F X) = \mathrm{CH}^m(X_K)$. Let Ω_l be a preimage of an element ω_l of $\mathrm{CH}^m(X_K)$.

Consider the difference $\mathrm{id} - f \circ g$ and denote it by h. Assume that over K it sends a basis element ω_j to a cycle α_j. Since $R^i(h_K)$ is trivial for all $i < m$, the cycle h_K can be written as

$$h_K = h_K \circ (\rho_N)_K = \sum_{\operatorname{codim} \alpha_l = m} \alpha_l \times \omega_l^\vee + \sum_{\operatorname{codim} \alpha_j > m} \alpha_j \times \omega_j^\vee \in \mathrm{CH}^{\dim X}(X_K \times X_K). \tag{1}$$

From (1) we immediately see that

$$\alpha_l = \mathrm{pr}_{1*}(\Omega_{l,K} \cdot h_K) \in \mathrm{CH}^m(X_K) \text{ is rational.} \tag{2}$$

Also, $(\rho_N)_K \circ \alpha_l = \alpha_l$.

The realization $R^m(f_K)$ is a $\mathbb{Z}$-linear map $\mathrm{CH}^m(N_K) \to \mathrm{CH}^m(M_K)$. Let $C = (c_{ij})$ be the respective matrix of coefficients, i.e.,

$$R^m(f_K) : \omega_i \mapsto \sum_j c_{ji} \theta_j,$$

where $\{\theta_i\}$ is a $\mathbb{Z}$-basis of $\mathrm{CH}^m(M_K)$. Let $s : N_K \to M_K$ be a section of f_K. The realization map $R^m(s)$ is a left inverse to $R^m(f_K)$. Hence, for the respective matrix of coefficients $D = (d_{ij})$ we have

$$R^m(s) : \theta_i \mapsto \sum_j d_{ji} \omega_j$$

and $D \cdot C = \mathrm{id}$, i.e., $\sum_j d_{ij} c_{jk} = \delta_{ik}$. For each α_l define the morphism $u_l : N \to M$ as

$$u_l = (\mathrm{pr}_1^*(\alpha_l) \cdot \sum_i d_{li} \Theta_{i,K}^\vee) \circ (p_N)_K,$$

where $\Theta_i^\vee$ is a preimage of an element $\theta_i^\vee$ of $\mathrm{CH}_m(M_K)$ by means of the canonical surjection $\mathrm{Hom}_F(M(X)(m), M) \to \mathrm{CH}_m(M_K)$ and $p_N : N \to M(X)$ be the morphism presenting N as a direct summand of $M(X)$. By definition u_l is a rational morphism and the realization $R^m(u_l)$ is given by

$$\theta_i \mapsto d_{li}\alpha_l$$

Hence, the composite $R^m(f_K \circ u_l) = R^m(u_l) \circ R^m(f_K)$ maps ω_i to $\delta_{il}\alpha_l$.

Set $v = g_K + \sum_l u_l$. By construction, the realization $R(f_K \circ v)$ is the identity on $\mathrm{CH}^i(N_K)$ for $i \leq m$ and v is rational. Let $\tilde{g}$ be a morphism defined over the base field such that $\tilde{g}_K = v$. Consider the endomorphism $\mathrm{id} - f \circ \tilde{g}$ of N. Over K its realization $R^i(\mathrm{id} - f_K \circ v)$ is trivial for each $i \leq m$.

Recursion step is proven and we obtain map $\tilde{g} : N \to M$ such that $(f \circ \tilde{g})_K = id_{N_K}$. Let $q = id - f \circ \tilde{g}$. By the Proposition 1, $q^r = 0$, for some r. Set $g = \tilde{g} \circ (id + q + q^{\circ 2} + \ldots + q^{\circ(r-1)})$. Then $f \circ g = id_N$ and N is a direct summand of M. □

Geometric construction of a generalized Rost motive.

Let p be a prime and n be a positive integer. To each nonzero cyclic subgroup $\langle \alpha \rangle$ in $K_n^M(F)/p$ consisting of pure symbols one can assign some motive M_α in the category $Chow(F)$ with $\mathbb{Z}/p\mathbb{Z}$-coefficients, which has the property that for arbitrary field extension E/F, $(M_\alpha)_E$ is either indecomposable, which happens if and only if $\alpha|_E \neq 0$, or $(M_\alpha)_E$ is split, which happens if and only if $\alpha|_E = 0$. It follows from the results of V. Voevodsky and M. Rost that for a given subgroup such motive always exists and is unique (see [10]). Moreover, when split it is isomorphic to

$$\bigoplus_{i=0}^{p-1} \mathbb{Z}/p\mathbb{Z}(i \cdot \tfrac{p^{n-1}-1}{p-1})[2i \cdot \tfrac{p^{n-1}-1}{p-1}]$$

Such a motive is called a *generalized Rost motive (with $\mathbb{Z}/p\mathbb{Z}$-coefficients)*. A motive with integral coefficients which specializes modulo p into a generalized Rost motive and splits modulo q for every prime q different from p will be called an *integral generalized Rost motive* and denoted by $\mathcal{R}_{n,p}$.

The integral generalized Rost motives, hypothetically, should be parameterized not by the pure cyclic subgroups of $\mathrm{K}_n^M(F)/p$, but by the pure symbols of $\mathrm{K}_n^M(F)/p$ up to a sign. The existence of integral generalized Rost motives is known for $n = 2$ and arbitrary p, for $p = 2$ and arbitrary n, and for the pair $n = 3, p = 3$. All these examples are essentially due to M. Rost.

Corollary 1. *Let X be a hyperplane section of a n-fold Pfister quadric Y. Then $M(Y) \simeq M(X)(1)[2] \oplus \mathcal{R}_{n,2}$, where $\mathcal{R}_{n,2}$ is an integral Rost motive (cf. [8]).*

Proof. It is known that the variety X (a maximal Pfister neighbor of Y) and Y become cellular over the generic point of X, i.e., the motives $M(X)$ and $M(Y)$ are split over K. Let Γ_i be the graph of the closed embedding $i : X \hookrightarrow Y$. Its realization $R(\Gamma_i)$ over K coincides with the induced pull-back i_K^* which maps an additive generator of $\mathrm{CH}(Y_K)$ to an additive generator of $\mathrm{CH}(X_K)$ and, hence, splits. The latter means that correspondences Γ_i and the transposed Γ_i^t split over K. Take $f : M(Y) \to M(X)(1)[2]$ to be the morphism induced by Γ_i^t and apply the theorem. $\square$

Corollary 2. *Let X be a hyperplane section of a twisted form Y of a Cayley plane which splits by a cubic field extension. Then $M(Y) \simeq M(X)(1)[2] \oplus \mathcal{R}_{3,3}$, where $\mathcal{R}_{3,3}$ is an integral generalized Rost motive.*

Proof. Consider the closed embedding $X \hookrightarrow Y$, where $X = \mathrm{F}_4(J)/P_4$ and $Y = \mathbb{OP}^2(J)$ are the twisted forms of F_4/P_4 and the Cayley plane $\mathbb{OP}^2 = \mathrm{E}_6/P_6$ corresponding to a Jordan algebra J defined by means of the first Tits construction (i.e., which splits by a cubic field extension) and proceed as in the previous proof. Finally, observe that the specialization of $\mathcal{R}_{3,3}$ with $\mathbb{Z}/3\mathbb{Z}$-coefficients is a generalized Rost motive corresponding to a symbol given by the Rost-Serre invariant $\mathfrak{g}_3$. $\square$

Remark 1. Observe that in view of the main result of [7] we obtain

$$M(\mathbb{OP}^2(J)) \simeq \bigoplus_{i=0}^{8} \mathcal{R}_{3,3}(i)[2i].$$

So from the motivic point of view the variety $\mathbb{OP}^2(J)$ is a 3-analog of a Pfister quadric and $\mathrm{F}_4(J)/P_4$ is a 3-analog of a maximal Pfister neighbor.

Twisted forms of Grassmannians.

PGL_n**:** Consider a Grassmannian $\mathbb{G}(d, n)$ of d-dimensional planes in a n-dimensional affine space. Its twisted form is called a generalized Severi-Brauer variety and denoted by $\mathrm{SB}_d(A)$, where A is the respective c.s.a. of degree n. The next corollary relates the motive of a generalized Severi-Brauer variety with the motive of usual Severi-Brauer variety (cf. [11]).

Corollary 3. *Let A and B be two central division algebras of degree n with $[A] = \pm d[B]$ in the Brauer group $\mathrm{Br}(F)$, where d and n are coprime. Then the motive of the Severi-Brauer variety $\mathrm{SB}(A)$ is a direct summand in the motive of the generalized Severi-Brauer variety $\mathrm{SB}_d(B)$.*

Proof. We construct the morphism $f : M(\mathrm{SB}_d(B)) \to M(\mathrm{SB}(A))$ as follows. Consider the Plücker embedding $pl : \mathrm{SB}_d(B) \to \mathrm{SB}(\Lambda^d B)$. It induces the morphism $M(\mathrm{SB}_d(B)) \to M(\mathrm{SB}(\Lambda^d B))$, where $\Lambda^d B$ is the d-th lambda power of B [6, II.10.A]. By the result of Karpenko [4, Cor. 1.3.2] the motive $M(\mathrm{SB}(\Lambda^d B))$ splits as a direct sum of shifted copies of $M(\mathrm{SB}(A))$, where $[A] = d[B]$ in $\mathrm{Br}(F)$. Take f to be the composite of the Plücker embedding and the projection $M(\mathrm{SB}(\Lambda^d B)) \to M(\mathrm{SB}(A))$. Then the condition that f splits over the generic point of $\mathrm{SB}(A)$ is equivalent to the fact that for each $m = 0, \ldots, n-1$

$$\underset{i}{g.c.d.}(c_i^{(m)}) = 1$$

where $c_i^{(m)}$ are degrees of Schubert varieties generating $\mathrm{CH}^m(\mathbb{G}(d,n))$. The latter can be easily computed using explicit formulas for degrees of Schubert varieties provided for instance in [3, Ch. 14, Ex. 14.7.11.(ii)]. Finally, observe that the motives $M(\mathrm{SB}(A))$ and $M(\mathrm{SB}(A^{\mathrm{op}}))$ are isomorphic. So replacing A by A^{op} doesn't change anything. □

G_2**:** Let G_2/P_1 and G_2/P_2 denote projective homogeneous varieties of a split group of type G_2 and maximal parabolic subgroups P_i corresponding to the respective vertices $i = 1, 2$ of the Dynkin diagram. These are non-isomorphic varieties of dimension 5. The following corollary provides a shortened proof of Bonnet result [1].

Corollary 4. *Let X and Y be twisted forms (by means of the same cocycle) of projective homogeneous varieties G_2/P_1 and G_2/P_2 respectively. Then $M(X) \simeq M(Y)$.*

Proof. It is known that X is a maximal Pfister neighbor of a 3-fold Pfister quadric, both X and Y are cellular over the generic points of each other and split by a quadratic field extension L/F. Since X is a maximal Pfister neighbor, it splits as a direct sum of shifted copies of a Rost motive $\mathcal{R}_{3,2}$. To construct a motivic isomorphism $f : M(X) \xrightarrow{\simeq} M(Y)$ we construct certain morphisms $f_i : \mathcal{R}_{3,2}(i)[2i] \to M(Y)$ on each component $\mathcal{R}_{3,2}(i)[2i]$ of the motivic decomposition of X. Then we prove that all these f_i satisfy the conditions of the theorem and, hence, split. Finally, we define f to be the direct sum $f = \oplus_i f_i$.

To construct such f_i we proceed as follows. Consider the twisted form Z of the variety of complete flags G_2/B. Observe that the variety Z has dimension 6. Let $\alpha \in \mathrm{Pic}(Z_L)$ be an element of the Picard group of Z. Since Tits algebras for G_2 are trivial, $\mathrm{Pic}(Z) \simeq \mathrm{Pic}(Z_L)$, i.e., the cycle α is defined over the base field. Set $\alpha' \in \mathrm{CH}_5(X \times Y)$ to be the image of α by means of the push-forward $(\mathrm{pr}_{X*}, \mathrm{pr}_{Y*}) : Z \to X \times Y$ induced by the canonical quotient maps $Z \to X$ and $Z \to Y$. Define f_i to be the composite

$$f_i : \mathcal{R}_{3,2}(i)[2i] \to M(X) \xrightarrow{\alpha'} M(Y).$$

So the problem reduces to finding a cycle $\alpha \in \mathrm{Pic}(Z_L)$ such that the map

$$R(f)_L = \oplus_i R(f_i)_L : \mathrm{CH}(Y_L) \to \bigoplus_i \mathrm{CH}(\mathcal{R}_{3,2}(i)[2i]_L)$$

is surjective. Since the Chow groups of X_L and Y_L have only one additive generator in each codimension, it is enough to show that for each i the map $R(f_i)_L : \mathrm{CH}(Y_L) \to \mathrm{CH}(\mathcal{R}_{3,2}(i)[2i]_L)$ is surjective.

The latter can be done easily by writing down the surjectivity conditions in terms of $\mathbb{Z}$-bases of the respective Chow groups and, then, solving arising system of $\mathbb{Z}$-linear equations.

Observe that the map induced by α' never splits. So it is not possible to construct a map from $M(X) \to M(Y)$ in this way without decomposing $M(X)$. ☐

F_4**:** Let F_4/P_4 and F_4/P_3 denote projective homogeneous varieties of a split group of type F_4 and maximal parabolic subgroups corresponding to the 4-th and the 3-rd vertices of the Dynkin diagram. The first variety has dimension 15 and the second - 21.

Corollary 5. *Let X and Y be twisted forms of varieties F_4/P_4 and F_4/P_3 by means of the (same) cocycle which splits by a cubic field extension. Then the motive $M(X)$ is a direct summand of the motive $M(Y)$.*

Proof. The proof is the same as in the case of G_2 and uses the motivic decomposition of the variety X provided in [7]. ☐

Compactifications of a Merkurjev-Suslin variety

Here we follow definitions and notation of [9]. Let A be a cubic division algebra over F. Recall that a smooth compactification D of a Merkurjev-Suslin variety $\mathcal{MS}(A,c)$ can be identified with the twisted form $X = \mathrm{SB}_3(M_2(A))$ of a smooth hyperplane section of Grassmannian $\mathbb{G}(3,6)$. Using Thm. 1 we can simplify the proof of the following result proven by N. Semenov in [9]

Corollary 6. *Let D be a twisted form (corresponding to a cubic division algebra) of a smooth hyperplane section of $\mathbb{G}(2,6)$. Then*

$$M(D) \simeq \bigoplus_{i=1}^{5} M(\mathrm{SB}(A))(i) \oplus N,$$

where N is ismorphic to $\mathcal{R}_{3,3}$ up to a 'phantom' motive in the category $Chow(F)$. In other words from the motivic point of view the variety D can be viewed as a 3-analog of a Norm quadric (cf. [9]).

Proof. Let $i : D \hookrightarrow X$ denote the closed embedding. It induces the map $\Gamma_i : M(D) \to M(X)$. The variety X is a projective homogeneous PGL_6-variety corresponding to a maximal parabolic subgroup of type P_3 corresponding to the third vertex of the respective Dynkin diagram. According to the Tits diagrams for the group $\mathrm{PGL}_{M_2(A)}$ the parabolic subgroup P_3 is defined over F and X is isotropic. By [2, Thm. 7.5] the motive of X splits as

$$M(X) = \mathbb{Z} \oplus Q(1)[2] \oplus Q(4)[8] \oplus \mathbb{Z}(9)[18],$$

where $Q = M(\mathrm{SB}(A) \times \mathrm{SB}(A^{\mathrm{op}}))$ and $Q = \bigoplus_{i=0}^{2} M(\mathrm{SB}(A))(i)[2i]$ by the projective bundle theorem. Hence, we obtain

$$M(X) = \mathbb{Z} \oplus \bigoplus_{i=1}^{6} M(\mathrm{SB}(A))(i)[2i] \oplus \mathbb{Z}(9)[18]. \qquad (3)$$

Now define f to be the composite of Γ_i and the canonical projection from $M(X)$ to the direct summand $\bigoplus_{i=1}^{5} M(\mathrm{SB}(A))(i)[2i]$ of (3). Observe that the motive $M(D)$ splits over the generic point of $\mathrm{SB}(A)$. The direct computations (using multiplication tables provided in [9]) show that f splits over $F(\mathrm{SB}(A))$. By Thm. 1 we conclude that $M(D) \simeq \bigoplus_{i=1}^{5} M(\mathrm{SB}(A))(i)[2i] \oplus N$ for some motive N which splits over $F(\mathrm{SB}(A))$ as a direct sum $\mathbb{Z} \oplus \mathbb{Z}(4)[8] \oplus \mathbb{Z}(8)[16]$.

Hence, it remains to identify N with the motive $\mathcal{R}_{3,3}$. To do this recall that both D and the twisted form of F_4/P_4 (given by the first Tits construction) split the same symbol $\mathfrak{g}_3$ in $K_3^M(F)/3$. This implies that there is a morphism $f : N \to \mathcal{R}_{3,3}$ which becomes an isomorphism over the separable closure of F. Since N is split over the generic point of the twisted form of F_4/P_4, by Thm. 1 we conclude that $\mathcal{R}_{3,3}$ is a direct summand of N. Hence, $N \simeq \mathcal{R}_{3,3} \oplus S$, where S is a *phantom* motive, i.e., the one which becomes trivial over the separable closure of F. $\square$

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