

# NONEXCELLENCE OF CERTAIN FIELD EXTENSIONS

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ABSTRACT. Consider towers of fields  $F_1 \subset F_2 \subset F_3$ , where  $F_3/F_2$  is a quadratic extension and  $F_2/F_1$  is an extension, which is either quadratic, or of odd degree, or purely transcendental of degree 1. We construct numerous examples of the above types such that the extension  $F_3/F_1$  is not 4-excellent. Also we show that if  $k$  is a field,  $\text{char } k \neq 2$  and  $l/k$  is an arbitrary field extension of degree 4, then there exists a field extension  $F/k$  such that the extension  $lF/F$  is not 4-excellent.

## 1. NOTATION AND TERMINOLOGY

The purpose of this paper is to construct nonexcellent field extensions of certain types. Recall that a field extension  $L/F$  is called excellent (resp.  $n$ -excellent) if for any quadratic form  $\varphi$  over  $F$  (resp. for any quadratic form  $\varphi$  over  $F$  of dimension at most  $n$ ) the anisotropic part of the form  $\varphi_L$  is defined over  $F$ . It is well known that any quadratic extension is excellent (see for example [4]). Obviously, the same is true for odd degree and purely transcendental extensions, since they can not make an anisotropic form isotropic. We consider below towers of fields  $F_1 \subset F_2 \subset F_3$ , where the extension  $F_3/F_2$  is quadratic and the extension  $F_2/F_1$  is either quadratic or of odd degree or purely transcendental of degree 1. We give numerous examples of extensions  $F_3/F_1$  of such types, which are not 4-excellent.

The main source of reference concerning quadratic forms over fields is the Scharlau book [4]. We keep the standard notation. All the fields considered in the sequel are of characteristic different from 2. By a form we mean a nonsingular quadratic form. If  $F$  is a field, then  $W(F)$  is the Witt ring of  $F$  and  $I(F)$  is the ideal of even-dimensional forms in  $W(F)$ . The anisotropic part of a form  $\varphi$  is denoted by  $\varphi_{an}$ . Slightly abusing notation we write  $\varphi = 0$  if  $\varphi$  is hyperbolic, i.e. its anisotropic part is zero. By  $D(\varphi)$  we denote the set of nonzero values of  $\varphi$ . The function field of the projective quadric related to the form  $\varphi$  is denoted by  $F(\varphi)$ . If  $L/F$  is a field extension, then  $\varphi_L$  is the extension of  $\varphi$  to  $L$ , i.e. the form  $\varphi \otimes_F L$ . For  $a, b \in F^*$  the symbol  $\langle\langle a, b \rangle\rangle$  means the 2-fold Pfisterform  $\langle 1, -a, -b, ab \rangle$ . If  $k$  is a field and  $p \in k[x]$  is an irreducible polynomial, by  $k(p)$  we denote the residue field  $k[x]/p$ . If

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$p$  is fixed,  $f \in k[x]$  and  $p$  does not divide  $f$ , then unless specified otherwise,  $\bar{f}$  is the image of  $f$  in  $k(p)^*/k(p)^{*2}$ .

The main tool in constructing of the examples in question is the notion of the second residue homomorphism  $\partial_p : I^2/I^3(k(x)) \rightarrow k(p)^*/k(p)^{*2}$  for an arbitrary irreducible polynomial  $p$  in  $x$  over a field  $k$  ([4]). Recall that the residue of the 2-fold Pfisterform  $\langle\langle f, g \rangle\rangle$  at the polynomial  $p$  is computed as follows:

$$\partial_p(\langle\langle f, g \rangle\rangle) = \begin{cases} 0 & \text{if } v_p(f) = v_p(g) = 0, \\ \bar{f} & \text{if } v_p(f) = 0, v_p(g) = 1. \end{cases}$$

( $v_p$  is the corresponding discrete valuation on  $k(x)$ ). In fact these equations determine the map  $\partial_p$  on the whole group  $I^2/I^3(k(x))$ .

## 2. NONEXCELLENCE OF FIELD EXTENSIONS OF DEGREE 4

We begin with the following

**Theorem 1.** *Let  $k$  be a field,  $a, b, c \in k$ ,  $a \in k^* \setminus k^{*2}$ ,  $b^2 - ac^2 \in k^* \setminus k^{*2}$ . Put  $l = k(\sqrt{b + c\sqrt{a}})$  and  $F = k(t, x, y)$ , where  $t, x, y$  are indeterminates. Then there exists a 4-dimensional quadratic form  $\varphi$  over  $F$  such that  $\dim(\varphi|_F)_{an} = 2$  and the form  $(\varphi|_F)_{an}$  is not defined over  $F$ . (This means that the field extension  $lF/F$  is not 4-excellent).*

*Proof.* Let  $f = ((t^2 - a)x^2 - (b^2 - ac^2))((t^2 - a)y^2 - 1)$ ,

$$\varphi_1 \simeq (t + \sqrt{a})\langle 1, -(b + c\sqrt{a}) \rangle,$$

$$\varphi_2 \simeq (t + \sqrt{a})\langle 1, -f(b + c\sqrt{a}) \rangle.$$

Let us find  $q \in F^*$  such that the forms  $(t + \sqrt{a})\langle\langle q \rangle\rangle$  and  $q(t + \sqrt{a})(b + c\sqrt{a})\langle\langle qf \rangle\rangle$  are defined over  $F$ . Provided  $q$  satisfies this condition we get that the form

$$\varphi \simeq \varphi_2 \perp -q\varphi_1 \simeq (t + \sqrt{a})\langle\langle q \rangle\rangle \perp q(t + \sqrt{a})(b + c\sqrt{a})\langle\langle qf \rangle\rangle$$

is defined over  $F$  as well.

The following lemma is well known, but in view of the absence of a convenient reference and for the sake of completeness we give the proof.

**Lemma 2.** *Suppose that  $q \notin F^{*2} \cup aF^{*2}$  and the elements  $\alpha, \beta \in F$  are such that  $\langle\langle q, \alpha^2 - a\beta^2 \rangle\rangle = 0$ . Then the form  $(\alpha + \beta\sqrt{a})\langle\langle q \rangle\rangle$  is defined over  $F$ . Moreover, if elements  $X, Y \in F$  are such that  $X^2 - qY^2 = \alpha^2 - a\beta^2$ , then  $(\alpha + \beta\sqrt{a})\langle\langle q \rangle\rangle \simeq (2X + 2\alpha)\langle\langle q \rangle\rangle$ .*

*Proof.* Since  $(X + Y\sqrt{q})(X - Y\sqrt{q}) = (\alpha + \beta\sqrt{a})(\alpha - \beta\sqrt{a})$ , we have

$$\begin{aligned} & N_{F(\sqrt{a}, \sqrt{q})/F(\sqrt{a})}(\alpha + \beta\sqrt{a} + X + Y\sqrt{q}) = \\ & = (\alpha + \beta\sqrt{a} + X + Y\sqrt{q})(\alpha + \beta\sqrt{a} + X - Y\sqrt{q}) = (\alpha + \beta\sqrt{a})(2X + 2\alpha). \end{aligned}$$

Therefore, since obviously,  $\langle\langle q, N_{F(\sqrt{a}, \sqrt{q})/F(\sqrt{a})}(\alpha + \beta\sqrt{a} + X + Y\sqrt{q}) \rangle\rangle = 0$ , we get

$$\langle\langle q, \alpha + \beta\sqrt{a} \rangle\rangle \simeq \langle\langle q, (\alpha + \beta\sqrt{a})N_{F(\sqrt{a}, \sqrt{q})/F(\sqrt{a})}(\alpha + \beta\sqrt{a} + X + Y\sqrt{q}) \rangle\rangle \simeq \langle\langle q, 2X + 2\alpha \rangle\rangle,$$

or, equivalently  $(\alpha + \beta\sqrt{a})\langle\langle q \rangle\rangle \simeq (2X + 2\alpha)\langle\langle q \rangle\rangle$ . The lemma is proved.  $\square$

Since

$$((t^2 - a)y)^2 - \frac{t^2 - a}{(t^2 - a)y^2 - 1}((t^2 - a)y^2 - 1)^2 = t^2 - a,$$

and

$$((t^2 - a)x)^2 - \frac{t^2 - a}{(t^2 - a)y^2 - 1}f = (t^2 - a)(b^2 - ac^2) = N_{F(\sqrt{a})/F}(b + c\sqrt{a})(t + \sqrt{a}),$$

we obtain that

$$\left\langle\left\langle \frac{t^2 - a}{(t^2 - a)y^2 - 1}, t^2 - a \right\rangle\right\rangle = \left\langle\left\langle \frac{t^2 - a}{(t^2 - a)y^2 - 1}f, (t^2 - a)(b^2 - ac^2) \right\rangle\right\rangle = 0.$$

Therefore, constructing the form  $\varphi$  we can put  $q = \frac{t^2 - a}{(t^2 - a)y^2 - 1}$ . By Lemma 2 we get

$$(t + \sqrt{a})\langle\langle q \rangle\rangle \simeq (2(t^2 - a)y + 2t)\langle\langle q \rangle\rangle,$$

$$q(t + \sqrt{a})(b + c\sqrt{a})\langle\langle qf \rangle\rangle \simeq (2(t^2 - a)x + 2(bt + ac))\langle q, -f \rangle,$$

hence

$$\varphi \simeq (2(t^2 - a)y + 2t)\langle 1, -q \rangle \perp (2(t^2 - a)x + 2(bt + ac))\langle q, -f \rangle.$$

We are going to prove that the form  $\varphi$  provides a counterexample to 4-excellency of the field  $F$ . However, we don't need the explicit form of the coefficients of  $\varphi$ , which look somewhat complicated.

It is obvious that

$$(\varphi_{lF})_{an} \simeq (\varphi_2)_{lF} \simeq (t + \sqrt{a})\langle 1, -f \rangle_{lF}.$$

In particular,  $\dim(\varphi_{lF})_{an} = 2$ . If the form  $(\varphi_{lF})_{an}$  is defined over  $F$ , then there is  $p \in F^*$  such that

$$(t + \sqrt{a})\langle 1, -f \rangle \simeq p\langle 1, -f \rangle,$$

i.e.  $\langle\langle f, (t + \sqrt{a})p \rangle\rangle_{lF} = 0$ , which is equivalent to

$$\langle\langle f, (t + \sqrt{a})p \rangle\rangle \simeq \langle\langle b + c\sqrt{a}, u + v\sqrt{a} \rangle\rangle$$

over  $F(\sqrt{a})$  for some  $u, v \in F$ . We will show that in fact this is impossible. More precisely we will prove the following a bit more general

**Lemma 3.** *For any  $u, v, p \in F$ ,  $w \in k(\sqrt{a})$  one has*

$$\langle\langle f, (t + \sqrt{a})wp \rangle\rangle \not\simeq \langle\langle b + c\sqrt{a}, u + v\sqrt{a} \rangle\rangle. \quad (1)$$

*Proof.* Put  $e = b^2 - ac^2$ . If  $e \notin ak^{*2}$ , i.e.  $l/k$  is not a Galois extension, then the residues at either  $t + \sqrt{a}$  or  $t - \sqrt{a}$  of the left hand and on the right hand parts of (1) are different, which proves the lemma in this case.

If  $e \in ak^{*2}$ , the argument is more complicated. Assume that

$$\langle\langle f, (t + \sqrt{a})wp \rangle\rangle \simeq \langle\langle b + c\sqrt{a}, u + v\sqrt{a} \rangle\rangle \quad (1')$$

for some  $u, v, p \in k(t, y)[x]$ , and that under this condition  $m = \deg p$  is minimal (in this section by  $\deg$  we mean the degree with respect to the variable  $x$ ). In particular, the polynomial  $p$  is squarefree. Moreover, we may assume that  $\deg(u + v\sqrt{a})$  is minimal provided that  $\deg p = m$ . It follows that  $(t^2 - a)x^2 - e$  does not divide  $p$ , for otherwise we would change  $p$  for  $-p \frac{(t^2 - a)y^2 - 1}{(t^2 - a)x^2 - e}$ , a contradiction to minimality of  $\deg p$ . Also  $(t^2 - a)x^2 - e$  does not divide  $u + v\sqrt{a}$ , for in the opposite case the isomorphism

$$\langle\langle f, (t + \sqrt{a})w(b + c\sqrt{a})p \rangle\rangle \simeq \langle\langle b + c\sqrt{a}, \frac{u + v\sqrt{a}}{(t^2 - a)x^2 - e}((t^2 - a)y^2 - 1) \rangle\rangle,$$

obtained from (1') by adding  $\langle\langle b + c\sqrt{a}, f \rangle\rangle$  to its both parts, would imply a contradiction to minimality of  $\deg(u + v\sqrt{a})$ .

Let  $p = \prod p_i$ , where  $p_i \in k(t, y)[x]$  are irreducible polynomials,  $\deg p_i \geq 1$ . We need the following

**Lemma 4.** *For any  $p_i$  the elements  $f$  and  $af$  are not squares in  $k(t, y)[x]/p_i$ .*

*Proof.* Suppose for instance that  $af$  is a square in  $k(t, y)[x]/p_1$ . Consider two cases: a)  $\deg p_1 \geq 2$ , b)  $\deg p_1 = 1$ .

Case a).

Since  $af$  is a square in  $k(t, y)[x]/p_1$  we get  $P^2 - af = p_1s$ , for some  $P, s \in k(t, y)[x]$ ,  $\deg P \leq \deg p_1 - 1$ . Since  $\deg p_1 \geq 2$  and  $\deg f = 2$  it follows that  $\deg(p_1s) \leq 2(\deg p_1 - 1)$ , hence  $\deg s \leq \deg p_1 - 2$ . Over the field  $k(\sqrt{a}, t, y, x)$  we have

$$\langle\langle f, p_1s \rangle\rangle \simeq \langle\langle f, P^2 - af \rangle\rangle \simeq \langle\langle af, P^2 - af \rangle\rangle = 0,$$

so

$$\begin{aligned} \langle\langle b + c\sqrt{a}, u + v\sqrt{a} \rangle\rangle &\simeq \langle\langle f, (t + \sqrt{a})wp \rangle\rangle \simeq \langle\langle af, (t + \sqrt{a})wp \rangle\rangle \simeq \\ &\langle\langle af, (t + \sqrt{a})w \frac{p}{p_1} s \rangle\rangle \simeq \langle\langle f, (t + \sqrt{a})w \frac{p}{p_1} s \rangle\rangle, \end{aligned}$$

a contradiction to minimality of  $\deg p$ , since  $\deg \frac{p}{p_1} s \leq \deg p - 2$ .

Case b).

Let  $p_1 = Qx + P$ . We may assume that  $Q, P \in k(t)[y]$ . Since  $af$  is a square in  $k(t, y)[x]/p_1$ , we conclude that  $a((t^2 - a)\frac{P^2}{Q^2} - e)((t^2 - a)y^2 - 1) \in k(t, y)^{*2}$ . This implies that  $(t^2 - a)y^2 - 1$  is a divisor of  $(t^2 - a)P^2 - eQ^2$  as a polynomial in  $y$ . It follows that there are nonzero polynomials  $P_1, Q_1 \in k(t)[y]$  and a rational function  $g \in k(t)^*$  such that  $\deg_y P_1, \deg_y Q_1 \leq 1$  and  $(t^2 - a)P_1^2 - eQ_1^2 = g((t^2 - a)y^2 - 1)$  (here by  $\deg_y$  we mean the degree with respect to  $y$ ). It is easy to see that this implies  $g\langle\langle (t^2 - a), -1 \rangle\rangle \simeq \langle\langle (t^2 - a), -e \rangle\rangle$  over  $k(t)$ , a contradiction, since the discriminants on the left-hand and on the right-hand sides of the last isomorphism are different. The case where  $f$  is a would-be square in  $k(t, y)[x]/p_1$  is treated quite similar. Lemma 4 is proved.  $\square$

We return to the proof of Lemma 3. By Lemma 4 we get that every  $p_i$  is a divisor of the polynomial  $u + v\sqrt{a} \in k(\sqrt{a}, t, y)[x]$ , since otherwise the residues at  $p_i$  (more precisely at some irreducible divisor of  $p_i$  over  $k(\sqrt{a}, t, y)[x]$ ) on the left-hand and the right-hand sides of the equality  $\langle\langle f, (t + \sqrt{a})wp \rangle\rangle \simeq \langle\langle b + c\sqrt{a}, u + v\sqrt{a} \rangle\rangle$  would be different. So we obtain that  $u + v\sqrt{a} = p(u_1 + v_1\sqrt{a})$ , hence

$$\langle\langle f, (t + \sqrt{a})wp \rangle\rangle \simeq \langle\langle b + c\sqrt{a}, p(u_1 + v_1\sqrt{a}) \rangle\rangle \quad (2)$$

for some  $u_1, v_1 \in k(t, y)[x]$ , and, since  $p$  is squarefree, the polynomials  $u_1 + v_1\sqrt{a}$  and  $p$  are relatively prime. Multiplying by some square in  $k(t)^*$  we may assume that  $v_{t^2-a}(p)$  equals either 0 or 1. Suppose that  $v_{t^2-a}(p) = 1$ . Then applying the automorphism  $W(F(\sqrt{a})) \rightarrow W(F(\sqrt{a}))$  determined by  $t \rightarrow -t$  to the equality (2) written as

$$\langle\langle f, (t - \sqrt{a})\frac{p}{t^2 - a}w \rangle\rangle \simeq \langle\langle b + c\sqrt{a}, \frac{p}{t^2 - a}(u_1 + v_1\sqrt{a})(t^2 - a) \rangle\rangle$$

and changing  $p(t, x, y)$  for  $p(-t, x, y)(t^2 - a)^{-1}$ , we reduce the problem to the case  $v_{t^2-a}(p) = 0$ .

We claim that  $\deg(u_1 + v_1\sqrt{a}) = 0$ . Indeed, suppose  $s \in k(\sqrt{a}, t, y)[x]$  is a prime divisor of  $u_1 + v_1\sqrt{a}$ . Since  $s$  does not divide  $((t^2 - a)x^2 - e)p$ , we get that

$$\partial_s(\langle\langle b + c\sqrt{a}, p(u_1 + v_1\sqrt{a}) \rangle\rangle) = \partial_s(\langle\langle f, (t + \sqrt{a})wp \rangle\rangle) = 1.$$

By the same argument as in Lemma 4 we can diminish  $\deg(u_1 + v_1\sqrt{a})$  in (2), which contradicts minimality of  $\deg(u + v\sqrt{a})$  in (1'). Thus, we conclude that there are no prime divisors of  $u_1 + v_1\sqrt{a}$  at all, i.e.  $u_1, v_1 \in k(t, y)$ . The equality (2) implies

$$\langle\langle (b + c\sqrt{a})f, (t + \sqrt{a})wp \rangle\rangle \simeq \langle\langle b + c\sqrt{a}, (t + \sqrt{a})(u_1 + v_1\sqrt{a})w \rangle\rangle,$$

and the right-hand side of the last isomorphism is defined over  $k(\sqrt{a}, t, y)$ . Denote it by  $\pi$ . Since  $\pi$  splits by the square root of

$$(b + c\sqrt{a})((t^2 - a)y^2 - 1)((t^2 - a)x^2 - e),$$

we conclude that  $\pi_{k(\sqrt{a}, t, y)(\tau)} = 0$ , where

$$\tau \simeq \langle\langle ((t^2 - a)y^2 - 1)(b + c\sqrt{a})(t^2 - a), -((t^2 - a)y^2 - 1)(b + c\sqrt{a})e \rangle\rangle.$$

Therefore, either  $\pi = 0$ , or

$$\pi \simeq \tau \simeq \langle\langle ((t^2 - a)y^2 - 1)(b + c\sqrt{a})(t^2 - a), -((t^2 - a)y^2 - 1)(b + c\sqrt{a})e \rangle\rangle. \quad (3)$$

On the one hand, since  $v_{t^2-a}(p) = 0$ , we have

$$\partial_{(t+\sqrt{a})}(\pi) = \partial_{(t+\sqrt{a})}(\langle\langle f(b + c\sqrt{a}), (t + \sqrt{a})wp \rangle\rangle) = b - c\sqrt{a},$$

so  $\pi \neq 0$ , hence (3) holds. On the other hand,

$$\partial_{(t-\sqrt{a})}(\pi) = \partial_{(t-\sqrt{a})}(\langle\langle f(b + c\sqrt{a}), (t + \sqrt{a})wp \rangle\rangle) = 1,$$

and

$$\begin{aligned} \partial_{(t-\sqrt{a})}(\langle\langle ((t^2 - a)y^2 - 1)(b + c\sqrt{a})(t^2 - a), -((t^2 - a)y^2 - 1)(b + c\sqrt{a})e \rangle\rangle) = \\ (b + c\sqrt{a})e = b - c\sqrt{a}, \end{aligned}$$

a contradiction in view of (3). This proves Lemma 3, hence also Theorem 1.  $\square$

$\square$

**Corollary 5.** *Let  $k$  be a field, let  $l/k$  be an arbitrary field extension of fourth degree. Then there is a field extension  $F/k$  such that the extension  $lF/F$  is not 4-excellent.*

*Proof.* There is a field extension  $k_1/k$ , which is either cubic or trivial, such that  $lk_1/k_1$  is the tower of quadratic extensions. If this tower is not the composite of two quadratic extensions of  $k_1$  we can apply Theorem 1. The opposite case is considered in [5].  $\square$

If  $k(\sqrt{b+c\sqrt{a}})/k$  is not a Galois extension i.e.  $e \notin k^{*2} \cup ak^{*2}$ , then Theorem 1 can be strengthened and the proof becomes much simpler. Namely, we have the following

**Proposition 6.** *Let  $k$  be a field,  $a, b, c \in k$ ,  $a \in k^* \setminus k^{*2}$ ,  $e = b^2 - ac^2 \in k^* \setminus (k^{*2} \cup ak^{*2})$ ,  $l = k(\sqrt{b+c\sqrt{a}})$ . Let  $t$  be an indeterminate,  $F = k(t)$ . Then there exists a 4-dimensional form  $\varphi$  over  $F$  such that the following holds:*

*For any field  $\tilde{l} \supset l$  such that  $b - c\sqrt{a}$  is not a square in  $\tilde{l}$  and  $t$  is transcendental over  $\tilde{l}$  the form  $(\varphi_{\tilde{l}F})_{an}$  is not defined over  $F$ . In particular, the field  $l$  itself satisfies these conditions.*

*Proof.* Let

$$\varphi \simeq \langle 2t, 2t(t^2 - a), -2(bt + ac), -2e(bt + ac)(t^2 - a) \rangle.$$

It is easy to see that

$$\varphi_{F(\sqrt{a})} \simeq (t - \sqrt{a})\langle 1, -(b - c\sqrt{a}) \rangle \perp (t + \sqrt{a})\langle 1, -(b + c\sqrt{a}) \rangle,$$

hence  $\dim(\varphi_{\tilde{l}F})_{an} = 2$ . Assume that the form  $(\varphi_{\tilde{l}F})_{an}$  is defined over  $F$ . Then there is  $z \in F^*$  such that

$$(t - \sqrt{a})\langle 1, -(b - c\sqrt{a}) \rangle \simeq z\langle 1, -(b - c\sqrt{a}) \rangle,$$

i.e.  $\langle\langle b - c\sqrt{a}, (t - \sqrt{a})z \rangle\rangle_{\tilde{l}F} = 0$ . On the other hand, since  $b - c\sqrt{a}$  is not a square in  $\tilde{l}$  and  $z \in F^*$ , the residue of this 2-fold Pfisterform at either  $t - \sqrt{a}$  or  $t + \sqrt{a}$  is nontrivial, a contradiction.  $\square$

### 3. NONEXCELLENCE OF $2n$ -DEGREE EXTENSIONS FOR AN ODD $n$

In this section we consider towers of two field extensions one of which is a Galois extension of odd degree and the other is quadratic.

**Proposition 7.** *Let  $k$  be a field,  $l = k(\alpha, \beta)$  is a finite Galois field extension of odd degree  $n > 1$ . Assume that  $\alpha \notin k$  and  $\beta \notin k(\alpha)$ . Let further  $t, x, y$  be indeterminates,  $a, b, d \in k^*$  such that  $d, -ab, -abd \notin k^{*2}$ . Set  $\varphi \simeq \langle a, b, x, abdx \rangle$ ,  $K = k(t, x, y)$ ,  $L = lK$ ,  $E = L(\sqrt{-(xt^2 + a\alpha^2)(by^2 + abdx\beta^2)})$ . Then*

- 1)  $\dim(\varphi_E)_{an} = 2$ .
- 2) The form  $(\varphi_E)_{an}$  is not defined over  $K$ .

*Proof.* The argument is somewhat similar to the one in Theorem 1.

1) Obviously,

$$\varphi_L \simeq \langle xt^2 + a\alpha^2, ax(xt^2 + a\alpha^2), by^2 + abdx\beta^2, adx(by^2 + abdx\beta^2) \rangle,$$

hence  $\varphi_E$  is isotropic and  $(\varphi_E)_{an} \simeq ax(xt^2 + a\alpha^2)\langle 1, -d \rangle$ .

2) Assume that the form  $(\varphi_E)_{an}$  is defined over  $K$ . Then there is  $p \in K^*$  such that

$$ax(xt^2 + a\alpha^2)\langle 1, -d \rangle_E \simeq p\langle 1, -d \rangle_E,$$

i.e.  $\langle\langle d, ax(xt^2 + a\alpha^2)p \rangle\rangle_E = 0$ . This means that

$$\langle\langle d, ax(xt^2 + a\alpha^2)p \rangle\rangle_L \simeq \langle\langle -(xt^2 + a\alpha^2)(by^2 + abdx\beta^2), q \rangle\rangle_L \quad (4)$$

for some  $q \in L^*$ .

We may assume that  $p \in k(t, x)[y]$  and  $m = \deg p$  is minimal (in this section we mean by  $\deg$  the degree with respect to the variable  $y$ ). Notice also that  $by^2 + abdx\beta^2 \nmid p$ , for otherwise  $by^2 + abdx(\sigma\beta)^2 \mid p$  for any  $\sigma \in G(l/k)$  and comparing the residues of both parts of (4) at  $by^2 + abdx(\sigma\beta)^2$ , with  $\sigma\beta \neq \beta$  would give us a contradiction.

Now let  $p = \prod p_i$ , where  $p_i \in k(t, x)[y]$  are irreducible polynomials in  $y$ .

**Lemma 8.**  $d$  is not a square in  $k(t, x)[y]/p_i$  for any  $p_i$ .

*Proof.* Suppose for instance that  $d$  is a square in  $k(t, x)[y]/p_1$ . We get  $P^2 - d = p_1s$  for some  $P, s \in k(t, x)[y]$ ,  $\deg P \leq \deg p_1 - 1$ . Hence

$$\deg s \leq 2(\deg p_1 - 1) - \deg p_1 = \deg p_1 - 2.$$

Since  $\langle\langle d, p_1s \rangle\rangle \simeq \langle\langle d, P^2 - d \rangle\rangle = 0$ , we can change the divisor  $p_1$  of  $p$  for  $s$  in (4), a contradiction to minimality of  $\deg p$ . The lemma is proved.  $\square$

Let  $p_i = \prod p_{ij}$ , where  $p_{ij} \in l(t, x)[y]$  are irreducible polynomials. Since  $l/k$  is an odd degree Galois extension, we have  $l(t, x)[y]/p_{ij} \simeq l(k(t, x)[y]/p_i)$  and, moreover,  $\deg[l(k(t, x)[y]/p_i) : k(t, x)[y]/p_i]$  is odd. Therefore, by Lemma 8  $d$  is not a square in  $l(k(t, x)[y]/p_i)$ , hence in  $l(t, x)[y]/p_{ij}$ . This means that the form (4) has nonzero residues at each  $p_{ij}$ . This implies that  $q = pf$ , where  $f \in l(t, x)[y]$  and the polynomials  $f$  and  $p$  are relatively prime, so we can represent (4) as

$$\langle\langle d, ax(xt^2 + a\alpha^2)p \rangle\rangle_L \simeq \langle\langle -(xt^2 + a\alpha^2)(by^2 + abdx\beta^2), pf \rangle\rangle_L. \quad (5)$$

Just as in Lemma 2 we may assume that in equality (5), where  $\deg p = m$  is fixed,  $\deg f$  is minimal. Denote the Pfisterform in (5) by  $\pi$ . Notice that  $by^2 + abdx\beta^2$  does not divide  $f$ , since otherwise we could change  $f$  for  $\frac{f}{(xt^2 + a\alpha^2)(by^2 + abdx\beta^2)}$ , a contradiction to minimality of  $\deg f$ .

**Lemma 9.**  $\deg f = 0$ .

*Proof.* Let  $f = \prod f_i$ , where  $f_i \in l(t, x)[y]$  are irreducible polynomials. Since  $f_i$  do not appear on the left- hand side of (5), we have  $\partial_{f_i}(\pi) = 1$ , for any  $f_i$ . Assume first that, say  $\deg f_1 \geq 2$ . Since  $\partial_{f_1}(\pi) = 1$ , we get that

$$P^2 + (xt^2 + a\alpha^2)(by^2 + abdx\beta^2) = f_1s$$

for some  $P, s \in l(t, x)[y]$  such that  $\deg P \leq \deg f_1 - 1$ . Since  $\deg f_1 \geq 2$  we get that

$$\deg(P^2 + (xt^2 + a\alpha^2)(by^2 + abdx\beta^2)) \leq 2(\deg f_1 - 1),$$

hence  $\deg s \leq \deg f_1 - 2$ , and we are done just as in Lemma 8. Now suppose that  $\deg f_1 = 1$ , i.e.  $f_1 = R(y - \frac{P}{Q})$ , where  $P, Q \in l(x)[t]$ ,  $R \in l(x, t)$ . Since  $\partial_{f_1}(\pi) = 1$ , we get that  $-(xt^2 + a\alpha^2)(bP^2 + abdx\beta^2Q^2)$  is a square in  $l(x)[t]$ . In particular,

$$xt^2 + a\alpha^2 | bP^2 + abdx\beta^2Q^2.$$

Therefore, there are some nonzero polynomials  $P_1, Q_1 \in l(x)[t]$  such that  $\deg_t P_1, \deg_t Q_1 \leq 1$  and  $xt^2 + a\alpha^2 | bP_1^2 + abdx\beta^2Q_1^2$ , i.e.

$$bP_1^2 + abdx\beta^2Q_1^2 = g(xt^2 + a\alpha^2)$$

for some  $g \in l(x)$ . It is easy to see that this implies  $\langle b, abdx \rangle_{l(x)} \simeq \langle gx, ga \rangle$ , a contradiction, since the discriminants on the left-hand and on the right-hand sides of the last isomorphism are different. The lemma is proved.  $\square$

Adding to the both parts of (5) the Pfisterform  $\langle\langle d, pf \rangle\rangle$  we can represent (5) as

$$\langle\langle d, ax(xt^2 + a\alpha^2)f \rangle\rangle_L \simeq \langle\langle -d(xt^2 + a\alpha^2)(by^2 + abdx\beta^2), pf \rangle\rangle_L. \quad (6)$$

By Lemma 8 the left-hand side of the last isomorphism is defined over  $l(t, x)$ . Moreover, the right-hand side splits over  $l(t, x)(\sqrt{-d(xt^2 + a\alpha^2)(by^2 + abdx\beta^2)})$ . So, just as for the form  $\pi$  from section 2, we have that either

- a)  $\langle\langle d, ax(xt^2 + a\alpha^2)f \rangle\rangle_{l(t, x)} = 0$ , or
- b)  $\langle\langle d, ax(xt^2 + a\alpha^2)f \rangle\rangle_{l(t, x)} \simeq \langle\langle -bd(xt^2 + a\alpha^2), -abx(xt^2 + a\alpha^2) \rangle\rangle_{l(t, x)}$ .

Assume that case a) holds. We may suppose that  $f \in l(x)[t]$ ,  $p \in k(x, y)[t]$  and that  $f$  and  $p$  are squarefree as polynomials in  $t$ . Obviously,  $xt^2 + a\alpha^2 | f$ , for otherwise we would have

$$d = \partial_{xt^2 + a\alpha^2}(\langle\langle d, ax(xt^2 + a\alpha^2)f \rangle\rangle) = \partial_{xt^2 + a\alpha^2}(0) = 1,$$

a contradiction. By the similar reason if  $\sigma\alpha \neq \alpha$  (such  $\sigma$  exists, since  $\beta \notin k(\alpha)$ ), then  $xt^2 + a(\sigma\alpha)^2 \nmid f$ . Hence from (6) it follows that  $xt^2 + a(\sigma\alpha)^2 \nmid p$ , if  $\sigma\alpha \neq \alpha$ , or equivalently  $xt^2 + a\alpha^2 \nmid p$ .

Let  $f = f_1(xt^2 + a\alpha^2)$ . Then

$$1 = \partial_{xt^2 + a\alpha^2}(\langle\langle -d(xt^2 + a\alpha^2)(by^2 + abdx\beta^2), pf \rangle\rangle) = \overline{df_1(by^2 + abdx\beta^2)p \text{ mod}(xt^2 + a\alpha^2)} \in l(t)(u)^*/l(t)(u)^{*2}. \quad (7)$$

Since  $\beta \notin k(\alpha)$  and  $[k(\alpha, \beta) : k]$  is odd we get that  $\beta^2 \notin k(\alpha)$ , and so there is  $\sigma \in G(l/k)$  such that  $\sigma\alpha = \alpha$  and  $\sigma(\beta^2) \neq \beta^2$ . Applying such  $\sigma$  to (7) we get

$$\overline{d\sigma f_1(by^2 + abdx(\sigma\beta)^2)p \text{ mod}(xt^2 + a\alpha^2)} = 1. \quad (8)$$

Combining (7) and (8) we get that

$$\overline{f_1 \sigma f_1 (by^2 + abdx\beta^2)(by^2 + abdx(\sigma\beta)^2) \bmod (xt^2 + a\alpha^2)} = 1.$$

Since  $\overline{f_1 \sigma f_1 \bmod (xt^2 + a\alpha^2)} \in l(t, x, \sqrt{-ax})$  does not depend on  $y$ , and  $\sigma(\beta^2) \neq \beta^2$ , we come to a contradiction. This shows that case  $a$ ) is impossible.

Now assume that case  $b$ ) holds, i.e.

$$\langle\langle d, ax(xt^2 + a\alpha^2)f \rangle\rangle_{l(t,x)} \simeq \langle\langle -bd(xt^2 + a\alpha^2), -abx(xt^2 + a\alpha^2) \rangle\rangle_{l(t,x)}.$$

Comparing residues at  $x$  on the left-hand and on the right-hand sides of the last isomorphism we obtain that either  $-ab \in k^{*2}$ , or  $-dab \in k^{*2}$ , which is not so by the hypothesis. This shows that case  $b$ ) is also impossible, which finishes the proof of Proposition 7.  $\square$

**Corollary 10.** *Let  $n \geq 2$  be an integer, which is not an odd prime. Then there exists a finite 4-nonexcellent field extension of degree  $2n$ .*

*Proof.* If  $n$  is even, then  $2n = 2^k m$ , where  $k \geq 2$  and  $m$  is odd. Let  $L/F$  be a 4-nonexcellent multiquadratic extension of degree  $2^k$  (the existence of such extensions has been proven in [5]). Let further  $\varphi$  be a 4-dimensional form providing a corresponding counterexample. Consider the extension  $L(t^{\frac{1}{m}})/F(t)$  of degree  $2n$ . Applying the specialization  $t = 0$  (the first residue map) it is easy to see that the extension  $L(t)/F(t)$  is 4-nonexcellent with the form  $\varphi_{F(t)}$  as a counterexample. Since the extension  $L(t^{\frac{1}{m}})/L(t)$  is of odd degree, we conclude that  $\varphi_{F(t)}$  is a counterexample for the extension  $L(t^{\frac{1}{m}})/F(t)$  as well.

Now assume that a number  $n$  is odd, a prime  $p \neq n$  divides  $n$ , and  $l/k$  is a Galois extension of degree  $n$ . Let  $k \subset k_1 \subset l$ , where the Galois group  $G(l/k_1)$  is cyclic of order  $p$ ,  $l = k(\beta)$  and  $\alpha \in k_1 \setminus k$ . Then the extension  $l/k$  and the elements  $\alpha$  and  $\beta$  satisfy the hypothesis of Proposition 7, which implies an existence of a 4-nonexcellent extension of degree  $2n$ .  $\square$

#### 4. NONEXCELLENCE OF EXTENSIONS DETERMINED BY HYPERELLIPTIC CURVES

In the last section of the present work we give examples of 4-nonexcellent extensions  $L/F$ , where  $L$  is the function field of a hyperelliptic curve over  $F$ . In this case the extension  $L/F$  is a tower of a purely transcendental extension of degree 1 and a quadratic extension. This question is of some interest, since the function field of a conic always determines an excellent extension ([1], [3]). More precisely the following statement holds.

**Proposition 11.** *Let  $k$  be a field,  $a, b \in k^*$  and the Pfisterform  $\langle\langle a, b \rangle\rangle$  is anisotropic. Suppose  $n > m \geq 1$  are positive integers. Let further  $F = k((t))$ ,  $t$  being an indeterminate, and let  $X$  be a hyperelliptic curve over  $F$  defined by the equation  $y^2 = ax^{2n} + bx^{2m} - t$ . Then the form  $(\langle 1, -a, -b, t \rangle_{F(X)})_{an}$  is not defined over  $F$ .*

*Proof.* Since the hyperbolic plane  $\langle 1, -(ax^{2n} + bx^{2m} - t) \rangle_{F(X)}$  is a subform of  $\langle 1, -a, -b, t \rangle_{F(X)}$ , we get that the form  $\langle 1, -a, -b, t \rangle_{F(X)}$  is isotropic, hence

$$\dim (\langle 1, -a, -b, t \rangle_{F(X)})_{an} = 2.$$

Assume for a moment that there are  $d, e \in F^*$  such that

$$\langle 1, -a, -b, t \rangle_{F(X)} \simeq \langle d, e \rangle_{F(X)},$$

or, equivalently that the form  $\varphi \simeq \langle 1, -a, -b, t, -d, -e \rangle$  becomes hyperbolic over  $F(X)$ . Then we have

$$(\varphi_{an})_{F(x)} \simeq \langle \langle ax^{2n} + bx^{2m} - t \rangle \rangle \otimes \tau \quad (8)$$

for some form  $\tau$ . Assume that  $\dim \tau$  is odd. Then  $\text{disc}(\varphi_{an}) = ax^{2n} + bx^{2m} - t$ . On the other hand,

$$\text{disc}(\varphi_{an}) = -abdet \neq ax^{2n} + bx^{2m} - t \pmod{F(x)^{*2}},$$

a contradiction. Hence  $\dim \tau$  is even, and so  $\dim \varphi_{an} = 4$ . Applying the first residue map (in fact the specialization  $x = 0$ ) to (8) we conclude that  $\varphi_{an}$  is similar to the form  $\langle \langle -t, -c \rangle \rangle$  for some  $c \in F^*$ . Since  $\langle \langle -t, t \rangle \rangle = 0$ , we can assume that  $c \in k^*$  and  $-c \notin k^{*2}$ . Furthermore, since  $\varphi_{F(X)} = 0$ , we have

$$ax^{2n} + bx^{2m} - t \in D(\langle -c, -t, -ct \rangle_{F(x)}).$$

Let  $r$  be a minimal nonnegative integer such that

$$t^{2r}(ax^{2n} + bx^{2m} - t) = -(cp_1^2 + tp_2^2 + ctp_3^2)$$

for some  $p_1, p_2, p_3 \in k[[t]][x]$  (such  $r$  exists by the Cassels-Pfister theorem, (see, for example [4])). Suppose  $r \geq 1$ . Then  $p_1 = tq_1$ ,  $q_1 \in k[[t]][x]$ , and so

$$t^{2r-1}(ax^{2n} + bx^{2m} - t) = -(p_2^2 + cp_3^2 + ctq_1^2).$$

Hence  $\overline{p_2^2} + c\overline{p_3^2} = 0$ , where  $\overline{p_2}, \overline{p_3} \in k[x]$  are the reductions of  $p_2, p_3$  modulo the ideal  $(t) \subset k[[t]][x]$ . Since  $-c \notin k^{*2}$ , we have  $\overline{p_2} = \overline{p_3} = 0$  and so

$$p_2 = tq_2, p_3 = tq_3, q_2, q_3 \in k[[t]][x].$$

Therefore,

$$t^{2r-2}(ax^{2n} + bx^{2m} - t) = -(cq_1^2 + tq_2^2 + ctq_3^2),$$

a contradiction to minimality of  $r$ . Thus, we conclude that  $r = 0$ , hence

$$ax^{2n} + bx^{2m} - t = -(cp_1^2 + tp_2^2 + ctp_3^2),$$

which implies  $ax^{2n} + bx^{2m} = -\overline{cp_1^2}$ . On the other hand, obviously,  $\frac{-(ax^{2n} + bx^{2m})}{c} \notin k[x]^2$ , a contradiction, which proves Proposition 9.  $\square$

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