Essential dimension of Hermitian spaces

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Abstract

We compute the essential dimension of Hermitian forms in the sense of O. Izhboldin. Apart from this we investigate the Chow motives of twisted incidence varieties and prove their incompressibility in dimensions $2^r - 1$.

Let W be an n-dimensional vector space over a field L which is a quadratic extension of some subfield F with $\operatorname{char} F \neq 2$. Given a nonsingular Hermitian form h on W we can associate with it two smooth projective F-varieties:

- the $(2n-3)$-dimensional variety V_h of h-isotropic L-lines in W and
- the projective quadric Q_h of dimension $2n-2$ defined by the quadratic form q_h associated with h, i.e., $q_h(v) = h(v,v)$, $v \in W$.

The variety V_h is a projective homogeneous variety under the action of the unitary group associated with h. It is also a twisted form of the *incidence variety*, i.e., of the variety of flags consisting of a dimension one and codimension one linear subspaces in an n-dimensional vector space.

We will use the following classical result of Milnor-Husemoller (see [Le79]):

> A quadratic form q on an F-vector space V is the underlying form of a Hermitian form over a quadratic field extension $L = F(\sqrt{a})$ iff $\dim V = 2n$, q_L is hyperbolic, and $\det q = (-a)^n \mod F^2$.

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1. Essential dimension. Following O. Izhboldin we define the *essential dimension* of a Hermitian form h as $\dim_{es}(h) = \dim V_h - i(q_h) + 2$, where $i(q_h)$ stands for the first Witt index of q_h (cf. [KM03]).

The following theorem provides a *Hermitian version* of the main result of [KM03].

Theorem. *Let Y be a complete F-variety with all closed points of even degree. Suppose that Y has a closed point of odd degree over $F(V_h)$. Then $\dim_{es}(h) \leq \dim Y$ and if $\dim_{es}(h) = \dim Y$, then V_h is isotropic over $F(Y)$.*

Proof. In [Kr07] D. Krashen constructed a projective bundle of rank 1

$$Bl(Q_h) \to V_h, \tag{1}$$

where $Bl(Q_h)$ is the blow-up of the quadric Q_h along some subvariety. In particular, the function field of Q_h is a purely transcendental extension of the function field of V_h of degree 1, and therefore our theorem follows from [KM03, Theorem 3.1]. ∎

2. Incompressibility. A smooth projective F-variety X is called *incompressible* if any rational map $X \dashrightarrow X$ is dominant. The basic example of such varieties are anisotropic quadrics of dimensions $2^r - 1$.

Theorem. *Assume that V_h is anisotropic and $\dim V_h = 2^r - 1$ for some $r > 0$. Then the variety V_h is incompressible.*

Proof. We provide two independent proofs.

1. One can notice that V_h is a twisted form of the *Milnor hypersurface* H defined by the line bundle $\mathcal{O}(1) \otimes \mathcal{O}(1)$ on $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, since over a separable closure of F they are given by the same equation $\sum_{i=0}^{n-1} x_i y_i = 0$ (see [LM, 2.5.3]). The incompressibility of V_h follows now from the Rost degree formula (see [Me03, §7]) and from an explicit computation of the characteristic numbers of H provided in [Me02, Proposition 7.2].

2. As an alternative proof one can notice that the incompressibility of V_h follows from the equality $\dim_{es}(h) = \dim V_h$. The latter can be deduced from the Hoffmann conjecture (proven in [Ka03]) if V_h is anisotropic and $\dim V_h = 2^r - 1$. Indeed, if $\dim V_h = 2^r - 1$, then $\dim q_h = 2^r + 2$. Therefore $i(q_h) = 1$ or 2. But by the Milnor-Husemoller theorem $i(q_h)$ must be even. Therefore $\dim_{es}(h) = \dim V_h$. ∎

3. Chow motives. Using formula (1) D. Krashen proved the following formula relating the Chow motives of Q_h and V_h:

$$M(Q_h) \oplus \bigoplus_{i=1}^{n-2} M(\mathbb{P}_L^{n-1})\{i\} \simeq M(V_h) \oplus M(V_h)\{1\}. \tag{2}$$

Consider the subcategory of the category of Chow motives with $\mathbb{Z}_{(2)}$- or $\mathbb{Z}/2$-coefficients generated by $M(V_h)$. Since V_h is a projective homogeneous variety, the Krull-Schmidt theorem and the cancellation theorem hold in this subcategory (see [CM06, Corollary 35]). Computing the Poincaré polynomials of $M(Q_h)$, $M(V_h)$, and $M(\operatorname{Spec} L)$ over L we obtain the following explicit formulae:

$$P(Q_{h,L},t) = \tfrac{(1-t^n)(1+t^{n-1})}{1-t},\ P(V_{h,L},t) = \tfrac{(1-t^n)(1-t^{n-1})}{(1-t)^2},\ P(\operatorname{Spec} L,t) = 2. \tag{3}$$

Analyzing (2) and (3) we obtain that

Theorem. *There exists a motive N_h such that*

$$M(Q_h) \simeq \begin{cases} N_h \oplus N_h\{1\}, & \textit{if } n \textit{ is even;} \\ N_h \oplus M(\operatorname{Spec} L)\{n-1\} \oplus N_h\{1\}, & \textit{if } n \textit{ is odd;} \end{cases} \tag{4}$$

and

$$M(V_h) \simeq \begin{cases} N_h \oplus \bigoplus_{i=0}^{(n-4)/2} M(\mathbb{P}_L^{n-1})\{2i+1\}, & \textit{if } n \textit{ is even;} \\ N_h \oplus \bigoplus_{i=0}^{(n-3)/2} M(\mathbb{P}_L^{n-2})\{2i+1\}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{5}$$

Observe that by the projective bundle theorem $M(\mathbb{P}_L^m) \simeq \bigoplus_{i=0}^m M(\operatorname{Spec} L)\{i\}$.

4. Higher forms of Rost motives. In [Vi00, Theorem 5.1] A. Vishik showed that given a quadratic form q over F divisible by an m-fold Pfister form φ, that is $q = q' \otimes \varphi$ for some form q', there exists a direct summand N of the motive $M(Q_q)$ of the projective quadric Q_q associated with q such that

$$M(Q_q) \simeq \begin{cases} N \otimes M(\mathbb{P}_F^{2^m-1}), & \text{if } \dim q' \text{ is even;} \\ (N \otimes M(\mathbb{P}_F^{2^m-1})) \oplus M(Q_\varphi)\{\frac{\dim q}{2} - 2^{m-1}\}, & \text{if } \dim q' \text{ is odd.} \end{cases}$$

In view of the Milnor-Husemoller theorem mentioned in the beginning, formula (4) provides a shortened proof of Vishik's result for $m = 1$.

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