VANISHING OF TRACE FORMS IN LOW CHARACTERISTICS

SKIP GARIBALDI

Abstract. A finite-dimensional representation of an algebraic group G gives a trace symmetric bilinear form on the Lie algebra of G. We give a criterion in terms of the root system data for this form to vanish. As a corollary, we show that a Lie algebra of type E_8 over a field of characteristic 5 does not have a so-called "quotient trace form", answering a question posed in the 1960s.

Let G be an algebraic group over a field F, acting on a finite-dimensional vector space V via a homomorphism $\rho\colon G \to GL(V)$. The differential $\mathrm{d}\rho$ of ρ maps the Lie algebra $\mathrm{Lie}(G)$ of G into $\mathfrak{gl}(V)$, and we put Tr_ρ for the symmetric bilinear form

$$\mathrm{Tr}_\rho(x, y) := \mathrm{trace}(\mathrm{d}\rho(x)\,\mathrm{d}\rho(y)) \quad \text{for } x, y \in \mathrm{Lie}(G).$$

We call Tr_ρ a *trace form of* G. Such forms appear, for example, in the hypotheses for the Jacobson-Morozov Theorem [Ca, 5.3.1]. We prove:

Theorem A. *Assume G is simply connected, split, and almost simple. Then the following are equivalent:*

(a) *The characteristic of F is a torsion prime for G.*
(b) *Every trace form of G is zero.*

The set of torsion primes for G is given by the following table, cf. e.g. [St 75, 1.13]:

type of G	torsion primes
A_n, C_n	none
B_n $(n \geq 3)$, D_n $(n \geq 4)$, G_2	2
F_4, E_6, E_7	2, 3
E_8	2, 3, 5

A prime p is called a torsion prime for G if the corresponding group $G(\mathbb{C})$ over $\mathbb{C}$ (or, equivalently, its compact form) is such that one of its homology groups, with coefficients in $\mathbb{Z}$, contains an element of order p.

We also prove a generalization of Theorem A that removes the hypotheses "simply connected" and "split"; it is somewhat more complicated, so we leave the statement until Th. D (and Remark 4.6). Replacing the simply connected group G with a nontrivial quotient G' changes the situation in two ways: the group G' has "fewer" representations and the Lie algebras of G and G' may be different. These two changes are reflected in the integers $N(G)$ and $E(G)$ defined below.

As a particular example of Th. A, for G of type E_8 over a field of characteristic 2, 3, or 5, Tr_ρ is zero for every representation ρ of G. One may ask whether the same is true for the representations of the Lie algebra $\mathrm{Lie}(G)$. That is, for a representation ψ of $\mathrm{Lie}(G)$, we write Tr_ψ for the bilinear form $(x, y) \mapsto \mathrm{trace}(\psi(x)\,\psi(y))$, and ask

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if Tr_ψ is necessarily zero. We do not know if this holds in characteristic 2 or 3, but it does in characteristic 5:

Corollary B. *If F has characteristic 5 and G is of type E_8, then the trace form of every representation of $\mathrm{Lie}(G)$ is zero.*

Indeed, we shall prove in §7 a slightly stronger statement, namely that $\mathrm{Lie}(G)$ has no "quotient trace form", thus answering a question posed in the early 1960s, see e.g. [Bl, p. 554], [BlZ, p. 543], or [Se, p. 48].

Trace forms on Lie algebras seem to have been studied—motivated by results from Borel-Mostow [BoM, §3]—as a way of defining a class of Lie algebras that was small enough to be tractable but large enough to be interesting, see e.g. [Se, pp. 47–49] and [BlZ]. Since then, tremendous progress has been made: Block, Premet, Strade, Wilson, and others have classified the simple Lie algebras over algebraically closed fields of characteristic ≥ 5, see [Strade]. These algebras are of "classical", Cartan, or Melikian type, and these types are distinct [Strade, §6.1]. Trace forms are only interesting for algebras of classical type—roughly, the simple algebras coming from Lie algebras of simple algebraic groups—because *every trace form on a simple algebra of Cartan or Melikian type is zero* by [Bl, Cor. 3.1]. Corollary B settles the last remaining question mark regarding the existence of nonzero trace forms on simple Lie algebras in characteristic ≥ 5, cf. [Bl, Cor. 3.1].

We do not use the Block-Premet-Strade-Wilson techniques here. Rather, we observe that it suffices to prove Theorem A, etc., for those representations "defined over $\mathbb{Z}$" and we compute the trace form on those representations (over $\mathbb{Z}$). We do need to compute the trace form and not just its discriminant; knowing the discriminant (over $\mathbb{Z}$) only tells you if the form is degenerate in prime characteristic, whereas we want to know whether the form is zero.

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1. The number $N(G)$ and the Dynkin index

1.1. Fix a simple root system R. We write P for its weight lattice and $\langle\ ,\ \rangle$ for the canonical pairing between P and its dual. Fix a long root $\alpha \in R$ and write $\alpha^\vee$ for the associated coroot. For each subset X of P that is invariant under the Weyl group, we put:

$$N(X) := \frac{1}{2} \sum_{x \in X} \langle x, \alpha^\vee \rangle^2 \quad \in \mathbb{Z}\left[\tfrac{1}{2}\right].$$

Note that $N(X)$ *does not depend on the choice of* α because the long roots are conjugate under the Weyl group.

Furthermore, $N(X)$ *is an integer.* To see this, note that the reflection s in the hyperplane orthogonal to α satisfies $\langle sx, \alpha^\vee \rangle = \langle x, s\alpha^\vee \rangle = -\langle x, \alpha^\vee \rangle$, so in the definition of $N(X)$, the sum can be taken to run over those x satisfying $x \neq sx$. For such x, we have $\langle x, \alpha^\vee \rangle^2 + \langle sx, \alpha^\vee \rangle^2 = 2\langle x, \alpha^\vee \rangle^2$, proving the claim.

Example 1.2. The computations in [SpSt, pp. 180, 181] show that $N(R) = 2h^\vee$, where $h^\vee$ denotes the *dual Coxeter number* of R, which is defined as follows. Fix a set of simple roots Δ of R. Write $\tilde{\alpha}$ for the highest root; the corresponding coroot $\tilde{\alpha}^\vee$ is

$$\tilde{\alpha}^\vee = \sum_{\delta \in \Delta} m_\delta^\vee \delta^\vee$$

for some natural numbers $m_\delta^\vee$. The dual Coxeter number $h^\vee$ is defined by

$$h^\vee := 1 + \sum m_\delta^\vee.$$

In case all the roots of R have the same length, it is the (usual) Coxeter number h and is given in the tables in [B Lie].

Suppose that there are two different root lengths in R; we write L for the set of long roots and S for the set of short roots. The arguments in [SpSt] are easily adapted to show that

$$N(L) = 2\left[1 + \sum_{\delta \in \Delta \cap L} m_\delta^\vee\right] \quad \text{and} \quad N(S) = 2 \sum_{\delta \in \Delta \cap S} m_\delta^\vee.$$

We obtain the following numbers:

type of R	h	$h^\vee$	$N(L)$	$N(S)$
B_n $(n \geq 2)$	$2n$	$2n-1$	$4(n-1)$	2
C_n $(n \geq 2)$	$2n$	$n+1$	4	$2(n-1)$
G_2	6	4	6	2
F_4	12	9	12	6

Definition 1.3. Fix a split almost simple linear algebraic group G over F. Fix also a pinning of G with respect to some maximal torus T; this includes a root system R and a set of simple roots Δ of G with respect to T. For a representation ρ of G over F, one defines

$$N(\rho) := \sum_{\text{dominant weights } \lambda} \begin{pmatrix} \text{multiplicity of } \lambda \\ \text{as a weight of } \rho \end{pmatrix} \cdot N(W\lambda) \quad \in \mathbb{Z}.$$

For example, the adjoint representation Ad has $N(\mathrm{Ad}) = 2h^\vee$ by Example 1.2.

We put:

$$N(G) := \gcd N(\rho),$$

where the gcd runs over the representations of G defined over F.

The map $\rho \mapsto N(\rho)$ is compatible with short exact sequences

$$0 \longrightarrow \rho' \longrightarrow \rho \longrightarrow \rho/\rho' \longrightarrow 0 \tag{1.4}$$

in the sense that

$$N(\rho) = N(\rho') + N(\rho/\rho').$$

Writing RG for the representation ring of G, we obtain a homomorphism of abelian groups $N \colon RG \to \mathbb{Z}$ with image $N(G) \cdot \mathbb{Z}$.

In the definition of $N(G)$, it suffices to let the gcd run over generators of RG, e.g., the irreducible representations of G. For an irreducible representation ρ, the highest weight λ has multiplicity 1 and all the other weights of ρ are lower in the partial ordering. Inducting on the partial ordering, we find:

$$N(G) = \gcd_{\lambda \in T^*} N(W\lambda).$$

In particular, $N(G)$ depends only on the root system R and the lattice T^*, and not on the field F.

1.5. When G is simply connected, the number $N(\rho)$ is the *Dynkin index* of the representation ρ defined in [D, p. 130] and studied in [Mer], and $N(G)$ is the *Dynkin index* of the group G. The Dynkin index of G and of the fundamental irreducible representations of G (over $\mathbb{C}$) are listed in [LS, Prop. 2.6] or [MPR, pp. 36–44], correcting some small errors in Dynkin's calculations. *For G simply connected, the primes dividing $N(G)$ are the torsion primes of G.*

Example 1.6. Write Spin_n and SO_n for the spin and special orthogonal groups of an n-dimensional nondegenerate quadratic form of maximal Witt index. For $n = 3$ or $n \geq 5$, these groups are split and almost simple of type B or D. The Dynkin index $N(\mathrm{Spin}_n)$ is 2; it obviously divides $N(\mathrm{SO}_n)$. On the other hand, the natural n-dimensional representation ρ of SO_n has $N(\rho) = 2$, so $N(\mathrm{SO}_n) = 2$.

Example 1.7. Write PSp_{2n} for the split adjoint simple group of type C_n; it can be viewed as $\mathrm{Sp}_{2n}/\boldsymbol{\mu}_2$. We claim that

$$N(\mathrm{PSp}_{2n}) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

for $n \geq 2$. The number $N(\mathrm{PSp}_{2n})$ divides 4 and $2(n-1)$ by Example 1.2. Further, $N(\mathrm{PSp}_{2n})$ is even by [Mer, 14.2]. This shows that $N(\mathrm{PSp}_{2n})$ is 2 or 4, and is 2 in case n is even.

Suppose that n is odd. We must show that $N(W\lambda)$ is divisible by 4 for every element λ of the root lattice of PSp_{2n}. We use the same notation as [Mer, §14] for the weights of PSp_{2n}: they are a sum $\sum_{i=1}^n x_i e_i$ such that $\sum x_i$ is even. The Weyl group W is a semidirect product of $(\mathbb{Z}/2\mathbb{Z})^n$ (acting by flipping the signs of the e_i) and the symmetric group on n letters (acting by permuting the e_i). Taking X for the $(\mathbb{Z}/2\mathbb{Z})^n$-orbit of $\sum x_i e_i$, we have

$$\frac{1}{2} \sum_{x \in X} \left\langle \sum_i x_i e_i, (2e_n)^\vee \right\rangle^2 = 2^{r-1} x_n^2 \tag{1.8}$$

where r denotes the number of nonzero x_i's, cf. [Mer, pf. of Lemma 14.2]. If $r = 1$, then the unique nonzero x_i is even, and we find that for $r \neq 2$, the sum—hence also $N(W \sum x_i e_i)$—is divisible by 4. Suppose that x_1, x_2 are the only nonzero x_i's; then by (1.8) we have:

$$N(W(x_1e_1 + x_2e_2)) = \begin{cases} 2(n-1)(x_1^2 + x_2^2) & \text{if } x_1 \neq \pm x_2 \\ 2(n-1)x_1^2 & \text{if } x_1 = \pm x_2. \end{cases}$$

As n is odd, $N(W(x_1e_1 + x_2e_2))$ is divisible by 4, which completes the proof of the claim.

Example 1.9. *For G adjoint of type E_7, we have $N(G) = 12$.* To see this, we note that $N(G)$ is divisible by N of the universal covering $\widetilde{G}$ of G (which is 12) and that $N(G)$ divides $2h^\vee = 36$ by Example 1.2. For the minuscule representation ρ of $\widetilde{G}$, we have $\dim \rho = 56$ and $N(\rho) = 12$. The representation $\rho^{\otimes 2}$ of $\widetilde{G}$ factors through G and by the "derivation formula"

$$N(\rho_1 \otimes \rho_2) = (\dim \rho_1) \cdot N(\rho_2) + (\dim \rho_2) \cdot N(\rho_1)$$

(see e.g. [Mer, p. 122]) we have

$$N(\rho^{\otimes 2}) = 2(\dim \rho) N(\rho) = 2^6 \cdot 3 \cdot 7.$$

It follows that $N(G)$ equals 12, as claimed.

2. The Lie algebra of G

2.1. Let G be a split almost simple algebraic group over F. We fix a pinning for G; it gives a split form $G_{\mathbb{Z}}$ of G over $\operatorname{Spec}\mathbb{Z}$ such that the base change $\mathbb{Z} \to F$ sends $G_{\mathbb{Z}}$ to G. Similarly the pinning gives a $\mathbb{Z}$-form $T_{\mathbb{Z}}$ of the maximal torus T. We have a Cartan decomposition of the Lie algebra of $G_{\mathbb{Z}}$:

$$\operatorname{Lie}(G_{\mathbb{Z}}) = \operatorname{Lie}(T_{\mathbb{Z}}) \oplus \bigoplus\nolimits_{\alpha\in R} \mathbb{Z}x_\alpha \tag{2.2}$$

and

$$\operatorname{Lie}(T_{\mathbb{Z}}) = \{h \in \operatorname{Lie}(T_{\mathbb{C}}) \mid \mu(h) \in \mathbb{Z} \text{ for all } \mu \in T^*\}. \tag{2.3}$$

see [St 68, p. 64]. Note that $\operatorname{Lie}(G)$ is naturally identified with $\operatorname{Lie}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} F$ and similarly for $\operatorname{Lie}(T)$.

2.4. Write $\widetilde{G}$ for the universal covering of G; we use the obvious analogues of the notations in 2.1 for $\widetilde{G}$. The group G acts on $\widetilde{G}$ by conjugation, hence also on $\operatorname{Lie}(\widetilde{G})$. If the kernel of the map $\widetilde{G} \to G$ is étale, then the representation $\operatorname{Lie}(\widetilde{G})$ is equivalent to the adjoint representation on $\operatorname{Lie}(G)$. But in prime characteristic, this need not hold. In any case, the natural map $\operatorname{Lie}(\widetilde{G}) \to \operatorname{Lie}(G)$ is an isomorphism on the F-span of the x_α's.

2.5. We claim that $\operatorname{Lie}(\widetilde{G})$ *is a Weyl module for* G in the sense of [J, p. 183], i.e., its character is given by Weyl's formula and it is generated as a G-module by a highest weight vector. The first condition holds by (2.2), so it suffices to check the second.

To check that the submodule $Gx_{\tilde\alpha}$ generated by the highest weight vector $x_{\tilde\alpha}$ is all of $\operatorname{Lie}(\widetilde{G})$, one quickly reduces to checking that $Gx_{\tilde\alpha}$ contains $\operatorname{Lie}(\widetilde{T}_{\mathbb{Z}})$. Equation (2.3) gives a natural isomorphism $\mathbb{Z}[R^\vee] \xrightarrow{\sim} \operatorname{Lie}(\widetilde{T}_{\mathbb{Z}})$ where $\widetilde{T}_{\mathbb{Z}}$ is the maximal torus in $\widetilde{G}_{\mathbb{Z}}$ mapping onto $T_{\mathbb{Z}}$. We write (as is usual) h_α for the image of $\alpha^\vee$ under this map. As $[x_\alpha, x_{-\alpha}] = h_\alpha$, the claim is proved.

See [Hi] or [Ho] for descriptions of the composition series of $\operatorname{Lie}(\widetilde{G})$.

3. The number $E(G)$

Definition 3.1. Maintain the notation of the preceding section. The Killing form on $\operatorname{Lie}(\widetilde{G}_{\mathbb{Z}})$ is divisible by $2h^\vee$ [GN] and dividing by $2h^\vee$ gives an indivisible even symmetric bilinear form $\widetilde{b}$ on $\operatorname{Lie}(\widetilde{G}_{\mathbb{Z}})$ such that

$$\widetilde{b}(h_\alpha, h_\alpha) = 2 \quad\text{and}\quad \widetilde{b}(x_\alpha, x_{-\alpha}) = 1 \tag{3.2}$$

for long roots α, see [SpSt, p. 181] or [B Lie, Lemma VIII.2.4.3]. For a short root β, we have: $\widetilde{b}(h_\beta, h_\beta) = 2c$ and $\widetilde{b}(x_\beta, x_{-\beta}) = c$, where c is the square-length ratio of α to β. For example, $G = \mathrm{SL}_n$ has Lie algebra the trace zero n-by-n matrices, and the form $\widetilde{b}$ is the usual trace bilinear form $(x, y) \mapsto \operatorname{trace}(xy)$, cf. [B Lie, Exercise VIII.13.12].

The natural map $\operatorname{Lie}(\widetilde{G}_{\mathbb{Z}}) \to \operatorname{Lie}(G_{\mathbb{Z}})$ is an inclusion and extending scalars to $\mathbb{Q}$ gives an isomorphism. Therefore, $\widetilde{b}$ gives a rational-valued symmetric bilinear form on $\operatorname{Lie}(G_{\mathbb{Z}})$. We define $E(G)$ to be the smallest positive rational number such that

$E(G) \cdot \widetilde{b}$ is integer-valued on $\mathrm{Lie}(G_{\mathbb{Z}})$; we write b for this form. Note that $E(G)$ is an integer by (3.2).

Clearly, $E(G)$ depends only on the root system of G and the character lattice T^* viewed as a sublattice of the weight lattice, and not on the field F.

3.3. Write $\bar{G}$ for the adjoint group of G; we use the obvious analogues of the notations in 2.1 for $\bar{G}$. We have a commutative diagram

$$\begin{array}{ccc} Q^\vee & \xrightarrow{\sim} & \mathrm{Lie}(\widetilde{T}_{\mathbb{Z}}) \\ \downarrow & & \downarrow \\ P^\vee & \xrightarrow{\sim} & \mathrm{Lie}(T_{\mathbb{Z}}) \end{array}$$

where $Q^\vee$ and $P^\vee$ are the root and weight lattices of the dual root system. The form $\widetilde{b}$ restricts to be an inner product on $Q^\vee$ such that the square-length of a short coroot $\alpha^\vee$ is 2. This inner product extends to a rational-valued inner product on $P^\vee$, and $E(\bar{G})$ is the smallest positive integer such that $E(\bar{G}) \cdot \widetilde{b}$ is integer-valued on $P^\vee$.

Example 3.4. Consider the case where G is PSp_{2n} for some $n \geq 2$, i.e., adjoint of type C_n. In the notation of the tables in [B Lie], the form $\widetilde{b}$ is twice the usual scalar product, i.e., $\widetilde{b}(e_i, e_j) = 2\delta_{ij}$ (Kronecker delta). The fundamental weight ω_n has $\widetilde{b}(\omega_n, \omega_n) = n/2$. Checking $\widetilde{b}(\omega_i, \omega_j)$ for all i, j, shows that $E(\bar{G})$ is 1 if n is even and 2 if n is odd.

Example 3.5. Suppose that all the roots of G have the same length, so that we may identify the root system R with its dual and normalize lengths so that $\langle\, , \rangle$ is symmetric and equals $\widetilde{b}$.

(1): *$E(\bar{G})$ is the exponent of P/Q*, the weight lattice modulo the root lattice. Indeed, the natural isomorphism between P and $\mathrm{Lie}(\bar{T}_{\mathbb{Z}})$ shows that $E(\bar{G})$ is the smallest natural number such that $E(\bar{G}) \cdot \langle\, , \rangle$ is integer-valued on $P \times P$, equivalently, the smallest natural number e such that eP is contained in Q; this is the exponent of P/Q.

(2): The bilinear form

$$\widetilde{b} \colon \mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}) \times \mathrm{Lie}(\bar{G}_{\mathbb{Z}}) \to \mathbb{Q}$$

has image $\mathbb{Z}$ and identifies $\mathrm{Lie}(\bar{G}_{\mathbb{Z}})$ with $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}), \mathbb{Z})$. (On the span of the x_α's, this is clear from (3.2). On the Cartan subalgebras, it amounts to the statement that $\langle\, , \rangle$ identifies P with $\mathrm{Hom}(Q, \mathbb{Z})$.) It follows that $\mathrm{Lie}(\bar{G})$, as a G-module, is the dual of $\mathrm{Lie}(\widetilde{G})$, i.e., $\mathrm{Lie}(\bar{G})$ is the module denoted by $H^0(\tilde{\alpha})$ in [J].

Example 3.6. For $n = 3$ or $n \geq 5$, we claim that $E(\mathrm{SO}_n) = 1$.

For n odd, SO_n is adjoint of type B_ℓ for $\ell = (n-1)/2$, and we compute as in 3.3 and Example 3.4. The dual root system is of type C_ℓ, and the form $\widetilde{b}$ is the usual scalar product, i.e., $b(e_i, e_j) = \delta_{ij}$. The fundamental weight ω_i is $e_1 + e_2 + \cdots + e_i$, so $E(\mathrm{SO}_{2\ell+1}) = 1$.

For n even, SO_n has type D_ℓ for $\ell = n/2$. The character group T^* of a maximal torus in SO_n consists of the weights whose restriction to the center of Spin_n is 0 or agrees with the vector representation, i.e., the weights $\sum c_i \omega_i$ such that $c_{\ell-1} + c_\ell$

is even. It follows that that the cocharacter lattice T_* is generated by the (co)root lattice and

$$\omega_{\ell-1} + \omega_\ell = \alpha_1 + 2\alpha_2 + \cdots + (\ell-2)\alpha_{\ell-2} + \frac{\ell-1}{2}(\alpha_{\ell-1} + \alpha_\ell).$$

We have:

$$\widetilde{b}(\omega_{\ell-1} + \omega_\ell, \omega_{\ell-1} + \omega_\ell) = \langle \omega_{\ell-1} + \omega_\ell, \omega_{\ell-1} + \omega_\ell \rangle = \ell - 1 \in \mathbb{Z},$$

so the form $\widetilde{b}$ is integer-valued on T_* and $E(\mathrm{SO}_{2\ell}) = 1$.

Example 3.7. For $n \geq 3$, write HSpin_{4n} for the nontrivial quotient of Spin_{4n} that is neither SO_{4n} nor adjoint. A calculation like the one in Example 3.6 gives:

$$E(\mathrm{HSpin}_{4n}) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

4. Main results

The integer-valued symmetric bilinear form b on $\mathrm{Lie}(G_{\mathbb{Z}})$ defined in 3.1 gives by scalar extension a symmetric bilinear form on $\mathrm{Lie}(G)$ which we denote by $b_{/F}$.

Proposition 4.1. *Let ρ be a representation of a split and almost simple algebraic group G over F. Then:*

(1) *$E(G)$ divides $N(\rho)$.*
(2) $\mathrm{Tr}_\rho = \frac{N(\rho)}{E(G)} b_{/F}$.

The following example is really the crux of the proof of the proposition.

Example 4.2. Suppose that ρ is a Weyl module of G. There is a $\mathbb{Z}$-form $\rho_{\mathbb{Z}}$ of ρ and composing $\rho_{\mathbb{Z}}$ with the natural homomorphism $\widetilde{G}_{\mathbb{Z}} \to G_{\mathbb{Z}}$ gives a representation $\widetilde{\rho}_{\mathbb{Z}}$ of $\widetilde{G}_{\mathbb{Z}}$.

We first compute $\mathrm{Tr}_{\widetilde{\rho}}$ over $\mathbb{C}$. If we decompose the representation ρ with respect to the action of $\widetilde{T}$ and write V_μ for the eigenspace relative to the weight μ, then h_α acts on V_μ by scalar multiplication by $\langle \mu, \alpha^\vee \rangle$, hence $\mathrm{Tr}_\rho(h_\alpha, h_\alpha) = \sum \dim(V_\mu) \langle \mu, \alpha^\vee \rangle^2$. By putting together the μ in an orbit $W\lambda$ (where λ is dominant) and taking α to be a long root, one gets:

$$\mathrm{Tr}_{\widetilde{\rho}}(h_\alpha, h_\alpha) = 2\, N(\rho). \tag{4.3}$$

The represenation $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}) \otimes \mathbb{C}$ is irreducible and has a nondegenerate $\widetilde{G}_{/\mathbb{C}}$-invariant symmetric bilinear form, so by Schur's Lemma we have:

$$\mathrm{Hom}_{\widetilde{G}_{/\mathbb{C}}}(\mathrm{Lie}(\widetilde{G}) \otimes \mathbb{C}, (\mathrm{Lie}(\widetilde{G})^*) \otimes \mathbb{C}) = \mathbb{C}.$$

In particular, $\mathrm{Tr}_{\widetilde{\rho}}$ equals $z\,\widetilde{b}$ for some complex number z and

$$2N(\rho) = \mathrm{Tr}_{\widetilde{\rho}}(h_\alpha, h_\alpha) = z\,\widetilde{b}(h_\alpha, h_\alpha) = 2z.$$

Hence $\mathrm{Tr}_{\widetilde{\rho}} = N(\rho)\,\widetilde{b}$.

Now $\mathrm{Lie}(G_{\mathbb{Z}}) \otimes \mathbb{C}$ is naturally identified with $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}) \otimes \mathbb{C}$, so the form Tr_ρ on $\mathrm{Lie}(G_{\mathbb{Z}})$ is exactly $N(\rho)\,\widetilde{b}$, i.e.,

$$\mathrm{Tr}_\rho = \frac{N(\rho)}{E(G)}\, b. \tag{4.4}$$

As the representation ρ of G is defined over $\mathbb{Z}$, the form Tr_ρ is integer-valued. But b is indivisible, and it follows that $E(G)$ divides $N(\rho)$.

By extending scalars, equation (4.4) holds with b replaced by $b_{/F}$. This verifies Prop. 4.1 for the Weyl module ρ.

Proof of Prop. 4.1. The number $N(\rho)$ depends only on the class of ρ in the representation ring RG. As the Weyl modules generate RG as an abelian group and $E(G)$ divides $N(\psi)$ for every Weyl module ψ by Example 4.2, (1) follows.

For (2), we note that the map $\rho \mapsto \mathrm{Tr}_\rho - (N(\rho)/E(G))\, b_{/F}$ is compatible with exact sequences like (1.4) in the sense that $\mathrm{Tr}_\rho = \mathrm{Tr}_{\rho'} + \mathrm{Tr}_{\rho/\rho'}$. We obtain a homomorphism of abelian groups

$$RG \to \boxed{\text{symmetric bilinear forms on } \mathrm{Lie}(G)}$$

that vanishes on the Weyl modules by Example 4.2, hence is zero. □

The form $b_{/F}$ is not zero, because b is indivisible (as a form over $\mathbb{Z}$). Proposition 4.1 immediately gives:

Theorem C. *Let ρ be a representation of a split and almost simple algebraic group G over F. Then Tr_ρ is zero if and only if the characteristic of F divides $N(\rho)/E(G)$.* □

Theorem D. *Assume G is split and almost simple. Then the following are equivalent:*

(a) *The characteristic of F divides the integer $N(G)/E(G)$.*
(b) *Every trace form of G is zero.*

Proof of Th. D. The number $N(G)$ is defined to be $\gcd N(\rho)$ as ρ varies over the representations ρ of G defined over F. Therefore,

$$N(G)/E(G) = \gcd_\rho \{N(\rho)/E(G)\}.$$

The theorem now follows from Th. C. □

Theorem A is the special case of Th. D where G is simply connected. (Indeed, for G simply connected, $E(G)$ is 1 and the primes dividing $N(G)$ are the torsion primes of G by 1.5.) Although Th. A is weaker, it has a much simpler condition (a).

Example 4.5. Suppose that the characteristic of F is an odd prime p, and let n be a natural number divisible by $\mathrm{char}\, F$.

(1): The groups SL_n and PGL_n act naturally by conjugation on the n-by-n matrices $M_n(F)$. For this representation ρ, the number $N(\rho)$ is $2n$ (by Example 1.2), hence the corresponding trace form is zero on $\mathrm{Lie}(\mathrm{SL}_n)$. But the trace form is nonzero on $\mathrm{Lie}(\mathrm{PGL}_n)$, as one finds by checking directly or looking ahead to Prop. 5.1 below; the radical is the (codimension 1) image of $\mathrm{Lie}(\mathrm{SL}_n)$.

(2): If p^2 divides n, then p divides $N(G)/E(G)$ for $G = \mathrm{SL}_n / \boldsymbol{\mu}_p$ by Prop. 5.1. Theorem D says that every trace form of G is zero, even though the universal covering SL_n and adjoint group PGL_n have representations with nonzero trace forms. (I thank George McNinch for suggesting $\mathrm{Lie}(G)$ as an interesting example. The representation $\mathrm{Lie}(G)$ of G is not only reducible, it is a direct sum of the image of $\mathrm{Lie}(\mathrm{SL}_n)$ and a 1-dimensional subspace.)

Remark 4.6 (Non-split groups). One can extend Prop. 4.1 and Theorems A, C, and D to the case where G is nonsplit as follows. Assume G is absolutely almost simple and fix a pinning of G over a separable closure F_{sep} of F. For a representation ρ of G (over F), define $N(\rho)$ to be the integer calculated as in 1.3 relative to the pinning over F_{sep}. We define $N(G)$ to be $\gcd N(\rho)$ as ρ varies over the representations of G defined over F. We take $E(G)$ to be the number given by the pinning over F_{sep} as in §3. (Note that with these extended definitions, $N(G)$ now depends on the field F, but $E(G)$ does not.) Proposition 4.1 (applied to the split group over F_{sep}) implies that $E(G)$ divides $N(G)$. Similarly, Theorems A, C, and D hold with the hypothesis "absolutely almost simple" instead of "split and almost simple".

Remark 4.7 (char. 2). Readers familiar with characteristic 2 might prefer to consider the quadratic form

$$s_\rho \colon x \mapsto -\operatorname{trace}\left(\wedge^2 \mathrm{d}\rho(x)\right)$$

instead of the symmetric bilinear form Tr_ρ. The form s_ρ gives the negative of the "degree 2" coefficient of the characteristic polynomial of $\mathrm{d}\rho(x)$. (Because $\mathrm{d}\rho(\mathrm{Lie}(G))$ consists of trace zero matrices, s_ρ is the map $x \mapsto \operatorname{trace}(\mathrm{d}\rho(x)^2)/2$; our definition has the advantage that it obviously makes sense also in characteristic 2.) The bilinear form derived from s_ρ—i.e., $(x, y) \mapsto s_\rho(x + y) - s_\rho(x) - s_\rho(y)$—is Tr_ρ.

Theorem A is easy to extend. In case G is simply connected, $\mathrm{Lie}(G)$ is a Weyl module by 2.5 and s_ρ is zero if and only if Tr_ρ is zero by [Ga, Prop. 6.4(1)]. That is, conditions (a) and (b) in Th. A are equivalent to:

(c) For every representation ρ of G, the quadratic form s_ρ is zero.

Alternatively, one can proceed as follows. The bilinear form $\widetilde{b}$ on $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})$ is even [GN, Prop. 4], so it is the bilinear form derived from a unique quadratic form $\widetilde{q}$ on $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})$. The form $\widetilde{q}$ extends to a rational-valued quadratic form on $\mathrm{Lie}(G_{\mathbb{Z}})$ and we write $E_q(G)$ for the smallest positive rational number such that $E_q(G)\,\widetilde{q}$ is integer-valued on $\mathrm{Lie}(G_{\mathbb{Z}})$. It is easy to see that $E_q(G)$ is $E(G)$ or $2E(G)$, and both cases can occur. (E.g., take $G = \mathrm{SO}_{2\ell}$ with ℓ odd or even, respectively.) The statements and proofs of Theorems C and D go through if we replace Tr_ρ, $E(G)$, and b with s_ρ, $E_q(G)$, and $E_q(G)\,\widetilde{q}$ respectively.

5. The ratio $N(G)/E(G)$ for $G = \mathrm{SL}_n/\boldsymbol{\mu}_m$

With Theorem D in hand, it remains to determine the primes dividing $N(G)/E(G)$ for each group G. In this section, we fix natural numbers m and n with m dividing n, and we prove:

Proposition 5.1. *For $G = \mathrm{SL}_n/\boldsymbol{\mu}_m$, the primes dividing $N(G)/E(G)$ are precisely the primes dividing*

$$\begin{cases} \gcd(m, n/m) & \text{if } m \text{ is odd} \\ 2\gcd(m, n/m) & \text{if } m \text{ is even.} \end{cases}$$

Here $\boldsymbol{\mu}_m$ denotes the group scheme of m-th roots of unity, identified with the corresponding scalar matrices in SL_n.

In the important special cases where G is simply connected ($m = 1$), G is adjoint ($m = n$), or n is square-free, the gcd in the proposition is 1, and we have that $N(G)/E(G)$ *is* 1 *if m is odd and* 2 *if m is even.*

Lemma 5.2.
$$E(\mathrm{SL}_n/\boldsymbol{\mu}_m) = \frac{m}{\gcd(m, n/m)}\,.$$

Proof. Use the notation of [B Lie] for the simple roots and fundamental weights of the root system A_{n-1} of SL_n. Let Λ denote the lattice generated by the root lattice Q and
$$\beta := \frac{n}{m}\,\omega_{n-1} = \frac{1}{m}\,(\alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1})\,.$$
We claim that Λ is identified with the cocharacter lattice T_* for a pinning of $\mathrm{SL}_n/\boldsymbol{\mu}_m$. Certainly, Λ/Q is cyclic of order m, so it suffices to check that the set of inner products (Λ, T^*) consists of integers. But T^* is the collection of weights $\sum c_i\omega_i$ with $c_i \in \mathbb{Z}$ such that $\sum_{i=1}^{n-1} ic_i$ is divisible by m. We have:
$$\left\langle \beta, \sum c_i\omega_i \right\rangle = \sum_i \frac{1}{m} ic_i \in \mathbb{Z} \quad \left(\sum c_i\omega_i \in T^*\right),$$
which proves that $T_* = \Lambda$ as claimed.

Finally, we compute:
$$\langle \beta, \alpha_{n-1}\rangle = \frac{n}{m} \in \mathbb{Z} \quad \text{and} \quad \langle \beta, \beta\rangle = \left\langle \frac{1}{m}\sum i\alpha_i, \frac{n}{m}\omega_{n-1}\right\rangle = \frac{n(n-1)}{m^2}.$$
Since m divides n, it is relatively prime to $n-1$, so the minimum multiplier of $\langle\, ,\rangle$ that takes integer values on T_* is $m/\gcd(m, n/m)$, as claimed. $\square$

5.3. Weights of representations of $\mathrm{SL}_n/\boldsymbol{\mu}_m$. Fix the "usual" pinning of SL_n, where the torus T consists of diagonal matrices and the dominant weights are the maps
$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \prod_{i=1}^{n-1} t_i^{e_i}$$
where $e_1 \ge e_2 \ge \cdots \ge e_{n-1} \ge 0$. Such a weight restricts to $x \mapsto x^{\sum e_i}$ on the center of SL_n; in particular, m divides $\sum e_i$ for every dominant weight λ of a representation of $\mathrm{SL}_n/\boldsymbol{\mu}_m$. The proof of [Mer, Lemma 11.4] shows that m divides $N(W\lambda)$, hence *m divides $N(\mathrm{SL}_n/\boldsymbol{\mu}_m)$.*

We recall how to compute $N(W\lambda)$ from [Mer, p. 136]. Write $a_1 > a_2 > \cdots > a_{k-1} > a_k = 0$ for the distinct values of the exponents e_i in λ, where a_i appears r_i times, so that $n = \sum r_i$. We have:
$$N(W\lambda) = \frac{(n-2)!}{r_1!\,r_2!\cdots r_k!}\left[n\left(\sum_i r_i a_i^2\right) - \left(\sum_i r_i a_i\right)^2\right]. \tag{5.4}$$

Example 5.5. Let λ be a weight of G and let r_i, a_i be as in 5.3. Suppose that
$$v_2\left(\sum r_i a_i\right) \ge v_2(n) > 0,$$
where $v_2(x)$ is the 2-adic valuation of x, i.e., the exponent of the largest power of 2 dividing x. We claim that
$$v_2(N(W\lambda)) > v_2(n). \tag{5.6}$$

Write $\sum r_i a_i = 2^\theta t$ and $n = 2^\nu u$ where $\theta = v_2(\sum r_i a_i)$ and $\nu = v_2(n)$. Our hypothesis is that $0 < \nu \leq \theta$. We may rewrite (5.4) as:

$$N(W\lambda) = \frac{(n-2)!}{r_1!\, r_2! \cdots r_k!} \left[u \left(\sum_i r_i a_i^2 \right) - 2^{2\theta-\nu} t^2 \right] \cdot 2^\nu. \tag{5.7}$$

Write ℓ for the minimum of $v_2(r_i)$, and fix an index j such that $v_2(r_j) = \ell$. Note that since $\sum r_i = n$, we have $\ell \leq \nu \leq 2\theta - \nu$.

The first term on the right side of (5.7) has 2-adic value $\geq -\ell$ [Mer, p. 137]. The term in brackets has value $\geq \ell$. Therefore, to prove claim (5.6), it suffices to consider the case where $v_2(\sum r_i a_i^2) = \ell$ and the first term on the right side of (5.7) has value $-\ell$; this latter condition implies that

$$s_2(n-1) = s_2(r_1) + \cdots + s_2(r_{j-1}) + s_2(r_j - 1) + s_2(r_{j+1}) + \cdots + s_2(r_k), \tag{5.8}$$

where s_2 denotes the number of 1's appearing in the binary representation of the integer [Mer, p. 137]. That is, when adding up the numbers $r_1, \ldots, r_{j-1}, r_j - 1, r_{j+1}, \ldots, r_k$ in base 2 (to get $n-1$), there are no carries.

Suppose first that $\ell < \nu$. Equation (5.8) implies that there are exactly two indices, say, j, j' with $v_2(r_j) = v_2(r_{j'}) = \ell$. As $2^{\ell+1}$ divides $\sum r_i a_i$, it also divides $r_j a_j + r_{j'} a_{j'}$, hence a_j and $a_{j'}$ have the same parity. It follows that $2^{\ell+1}$ divides $r_j a_j^2 + r_{j'} a_{j'}^2$ and the term in brackets in (5.7) has 2-adic valuation $> \ell$ and we are done in this case.

We are left with the case where $\ell = \nu$. By (5.8), r_j is the unique r_i with 2-adic valuation ℓ. As $v_2(\sum r_i a_i^2) = \ell$, the number a_j is odd and we have:

$$\ell = v_2 \left(\sum r_i a_i \right) = \theta \geq \nu = \ell.$$

Hence both $u(\sum r_i a_i^2)$ and $2^{2\theta-\nu} t$ have 2-adic valuation ℓ. It follows that the term in brackets in (5.7) has 2-adic valuation strictly greater than ℓ, and claim (5.6) is proved.

Proof of Prop. 5.1. We write G for $\mathrm{SL}_n / \boldsymbol{\mu}_m$. For an upper bound, $N(G)$ divides $2n$ by Example 1.2. Also, the dominant weight λ with $e_1 = m$ and $e_i = 0$ for $i > 1$ belongs to T^* and has $N(W\lambda) = m^2$ by (5.4), so $N(G)$ divides m^2. Applying Lemma 5.2 gives:

$$\gcd(m, n/m) \text{ divides } N(G)/E(G) \text{ divides } \gcd(m, n/m)\gcd(m, 2n/m).$$

This completes the proof for m odd.

Clearly, an odd prime divides $N(G)/E(G)$ if and only if it divides $\gcd(m, n/m)$. So suppose that m is even and 2 does not divide $\gcd(m, n/m)$, i.e., $v_2(m) = v_2(n)$. Then every weight of a representation of G satisfies the hypotheses of Example 5.5, hence $v_2(N(G)) > v_2(n) = v_2(m)$. By Lemma 5.2, $v_2(E(G)) = v_2(m)$, so 2 divides $N(G)/E(G)$. This completes the proof of Prop. 5.1. □

6. The ratio $N(G)/E(G)$ for simple G

The purpose of this section is to compute the primes dividing $N(G)/E(G)$ for all almost simple split groups G. The results are given in Table I. We write PSO_n for the adjoint group of SO_n; when n is odd it is the same as SO_n.

6.1. Justification of Table I. We now justify the claims about $N(G)/E(G)$ given in Table I. For G simply connected, $E(G)$ is 1 and $N(G)$ is divisible precisely

G	primes dividing $N(G)/E(G)$
$\mathrm{SL}_n/\boldsymbol{\mu}_m$	see Prop. 5.1
Sp_{2n}	none
SO_n, Spin_n, and PSO_n for $n = 3$ and $n \geq 5$	2
HSpin_{4n} for $n \geq 3$, PSp_{2n}, E_6 adjoint	2
E_6 simply connected, E_7, F_4, G_2	2, 3
E_8	2, 3, 5

Table I. The primes dividing $N(G)/E(G)$.

by the torsion primes of G, see 1.5. We assume that G is not simply connected and write $\widetilde{G}$ for the universal covering of G; obviously $N(\widetilde{G})$ divides $N(G)$.

For $G = \mathrm{PSp}_{2n}$, SO_n, or adjoint of type E_7, one combines Examples 1.7 and 3.4; 1.6 and 3.6; or 1.9 and 3.5, respectively.

For G adjoint of type D_n, we have $E(G) = 2$ by Example 3.5. Also, 4 divides $N(G)$ by [Mer, 15.2]. On the other hand, the spinor representations of $\widetilde{G}$ have Dynkin index 2^{n-3} [LS], and it is easy to use this as in Example 1.9 to construct a representation ρ of G with $N(\rho)$ a power of 2. This shows that $N(G)/E(G)$ is a power of 2 and is not 1.

Now let $G = \mathrm{HSpin}_{4n}$ for some $n \geq 3$. The dual of the center of Spin_{4n} is the Klein four-group, and we write χ for the unique element that vanishes on the kernel of the map $\mathrm{Spin}_{4n} \to \mathrm{HSpin}_{4n}$. The gcd of $N(W\lambda)$ as λ varies over the weights that restrict to χ (respectively, 0) on the center of Spin_{4n} is 2^{2n-3} (resp., divisible by 4) by [Mer, p. 146], hence $N(G)$ is a power of 2 and at least 4. On the other hand, $E(\mathrm{HSpin}_{4n})$ is 1 or 2. We conclude that $N(G)/E(G)$ is a power of 2 and is not 1.

For G adjoint of type E_6, the number $N(G)$ is divisible by $N(\widetilde{G}) = 6$ and divides $2h^\vee = 24$ by Example 1.2. By Example 3.5, $N(G)/E(G)$ is 2, 4, or 8.

7. Trace forms and Lie algebras

We assume in this section that G is absolutely almost simple, split, and *simply connected* and that *the characteristic of* F *is neither* 2 *nor* 3. Write $\mathfrak{c}$ for the center of $\mathrm{Lie}(G)$; the quotient $\overline{\mathfrak{g}} := \mathrm{Lie}(G)/\mathfrak{c}$ is a simple Lie algebra [St 61, 2.6(5)]. Over an algebraically closed field, the algebras $\overline{\mathfrak{g}}$ arising in this way are sometimes called "simple Lie algebras of classical type" (even when the root system R is exceptional).

Proposition 7.1. *If $\overline{\mathfrak{g}}$ has a representation ψ over F with Tr_ψ nonzero, then G has an irreducible representation ρ over F such that Tr_ρ is not zero and whose differential vanishes on $\mathfrak{c}$.*

Proof. Replacing ψ with one of the irreducible quotients in its composition series, we may assume that ψ is irreducible. Then ψ is restricted by [Bl, Th. 5.1] (using that F has characteristic $\neq 2, 3$). Because the projection $\mathrm{Lie}(G) \to \overline{\mathfrak{g}}$ is restricted, the composition gives a restricted irreducible representation of $\mathrm{Lie}(G)$, which is the differential of a representation of G by [Cu] and [St 63]. (These references only give a representation of G defined over an algebraic closure of F, but G is split, so the irreducible representations of G over F are in natural one-to-one correspondence with those over an algebraic closure.) ∎

Because of our hypothesis on the characteristic, $\mathrm{Lie}(G)$ is not simple only for groups of type A_{n-1} where n is divisible by the characteristic of F. In that case, $\mathrm{Lie}(G)$ is $\mathfrak{sl}_n$ and its center $\mathfrak{c}$ consists of the scalar matrices $F \cdot 1$.

Corollary 7.2 (Block [Bl, Th. 6.2])**.** *If* $\mathrm{char}\, F$ *divides* n*, then every representation of* $\mathfrak{sl}_n/\mathfrak{c}$ *has zero trace form.*

Proof. Suppose that $\mathfrak{sl}_n/\mathfrak{c}$ has a representation with a nonzero trace form. Then SL_n has an irreducible representation ρ such that Tr_ρ is not zero and $\mathrm{d}\rho$ vanishes on the scalar matrices. Identifying the center of SL_n with the (non-reduced) group scheme $\boldsymbol{\mu}_n$ identifies the restriction of ρ to $\boldsymbol{\mu}_n$ with a map $x \mapsto x^\ell$. Our hypothesis on $\mathrm{d}\rho$ says that ℓ is divisible by the characteristic p of F, hence ρ factors through the natural map $\mathrm{SL}_n \to \mathrm{SL}_n / \boldsymbol{\mu}_p$. It follows from 5.3 that $N(\rho)$ is divisible by p. Hence Tr_ρ vanishes by Th. C, a contradiction. $\square$

We close by proving a stronger version of Cor. B from the introduction. For a Lie algebra L over F and a representation ψ of L, write $\mathrm{rad}\, \psi$ for the radical of the trace bilinear form Tr_ψ; it is an ideal of L. We prove:

Corollary B′. *For every representation* ψ *of every Lie algebra* L *over a field of characteristic* 5*, the quotient* $L/\mathrm{rad}\, \psi$ *is* not *isomorphic to the Lie algebra of an algebraic group of type* E_8*.*

That is, over a field of characteristic 5, the Lie algebra of a group of type E_8 "has no quotient trace form".

Proof of Cor. B′. Suppose the corollary is false. That is, suppose that there is a group G of type E_8 and a Lie algebra L with a representation ψ and a surjection $\pi \colon L \to \mathrm{Lie}(G)$ with kernel the radical of Tr_ψ.

By [Bl, Lemma 2.1], we may assume that the radical of Tr_ψ is contained in the center of L, i.e., L is a central extension of $\mathrm{Lie}(G)$. It follows that there is a map $f : \mathrm{Lie}(G) \to L$ such that πf is the identity [St 62, Th. 6.1(c)]. Clearly, the representation ψf of $\mathrm{Lie}(G)$ has nonzero trace form.

As $\mathrm{Lie}(G)$ is simple, we can apply Prop. 7.1 and deduce that the algebraic group of type E_8 over F has a representation ρ such that Tr_ρ is not zero, but this is impossible by Theorem A. $\square$

Note that in the course of proving Corollary B′ we have also proved Cor. B.

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Department of Mathematics & Computer Science, Emory University, Atlanta, GA 30322, USA

E-mail address: `skip@member.ams.org`

URL: `http://www.mathcs.emory.edu/~skip/`