Characterizing the Multiplicative Group of a Real Closed Field in terms of its Divisible Maximal Subgroup

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Abstract

Let F be a field and M be a maximal subgroup of the multiplicative group $F^* = F \setminus \{0\}$. It is proved that if M is divisible, then F is Euclidean. Furthermore, it is shown that F^* contains a divisible maximal subgroup if and only if F^* is isomorphic to the multiplicative group of a real closed field.

1 Introduction

Given the field of real numbers $\mathbb{R}$, denote by $\mathbb{R}^*$ and $\mathbb{R}^+$ the multiplicative group of real numbers and the multiplicative group of positive real numbers, respectively. We recall that a nontrivial multiplicative abelian group G is *divisible* if and only if G has no maximal subgroup if and only if $G = G^p$ for each prime p. It is easily seen that $\mathbb{R}^+$ is a divisible maximal subgroup of $\mathbb{R}^*$ and $\mathbb{R}$ is Euclidean. The object of this note is to show that this property on the multiplicative group of a field F gives rise to F being Euclidean. Furthermore, if R is a real closed field, then it is easily seen that (cf. Theorem A below) R^* contains a unique maximal subgroup which is divisible. Here, we also characterize the multiplicative group of a real closed field in terms of its divisible maximal subgroup. To be more precise, it is proved that F^* contains a divisible maximal subgroup if and only if F^* is isomorphic to the multiplicative group of a real closed field. We begin our investigation with the following easy

Lemma 1. *Let G be a multiplicative abelian group and M be a maximal subgroup of G. If M is divisible, then M is the unique maximal subgroup of G.*

Proof. Assume that $M_1 \neq M$ is another maximal subgroup of G. Then, we have $G = MM_1$ and hence $G/M_1 \cong M/M \cap M_1$, i.e., $M \cap M_1$ is a maximal subgroup of M. Since M is divisible we conclude that $M \cap M_1 = 1$. Therefore, $G/M_1 \cong M \cong C_q$ for some prime number q, where C_q is the cyclic group of q elements. This last relation also leads to a contradiction since a finite group cannot be divisible, and so $M = M_1$ as required. $\square$

We shall also need the following theorem to prove our main result:

Theorem A [1, p. 107]. *If F is a real closed field, then $F^* \cong \mathbb{Z}_2 \times \mathbb{Q}^{|F|}$. Conversely, for any infinite cardinal λ, the group $\mathbb{Z}_2 \times \mathbb{Q}^{\lambda}$ is isomorphic to the multiplicative group of a suitable real closed field.*

Theorem 1. *Let M be a maximal subgroup of F^*. Then, we have*
(1) *If M is divisible, then the Brauer group of F is non-trivial and F is Euclidean.*
(2) *F^* contains a divisible maximal subgroup if and only if F^* is isomorphic to the multiplicative group of a real closed field.*

Proof. (1) Assume that M is a maximal subgroup of F^* with $F^*/M \cong C_p$ for some prime p. We first claim that there exists a cyclic field extension K/F of degree p such that $N(K^*) = F^{*p} = M$, where N is the norm of K to F. To see this, we know, by Lemma 1, that M is the unique maximal subgroup of F^* such that $F^*/M \cong C_p$. Since M is divisible and maximal in F^*, by 4.1.4 of [3], we have $F^* \cong M \times C_p$. This means that F contains a primitive p-th root of unity. Now, it is easily seen that there is an element $a \in F$ such that the equation $x^p - a = 0$ has no solutions in F. Since F has a primitive p-th root of unity we obtain a cyclic extension $K = F(b)$ of degree p over F with $b^p - a = 0$. Now, if the norm N of K to F is surjective, i.e., $N(K^*) = F^*$, then $K^*/K^1 \cong F^*$, where K^1 is the group of norm 1 elements. Hence K^*/K^1 contains a maximal subgroup M_K, say, containing K^1 of index p which is divisible. By Lemma 1, M_K is unique. Now, M_K is also a maximal subgroup of K^* and since it is divisible, by 4.1.4 of [3], we obtain $K^* \cong M_K \times C_p$. This means that there exists an element $1 \neq c \in C_p \subset K$ with $c^p = 1$. Since F contains a primitive p-th root of unity we conclude that

$c \in F$. Therefore, $N(c) = c^p = 1$, i.e., $c \in K^1 \subset M_K$. But this contradicts the fact that $M_K \cap C_p \neq 1$. Therefore, $N(K^*) \neq F^*$. Since $F^*/N(K^*)$ is torsion of bounded exponent p and M is unique, by Prũfer-Baer Theorem (cf. [3, p. 105]), we conclude that $N(K^*) = F^{*p} = M$, as claimed. Now, assume that the Galois group of K/F is generated by the automorphism σ of order $p = [K : F]$. Fix an element $\lambda \in F^* \setminus M$ and a symbol y. We set $D = K1 \oplus Ky \oplus \cdots \oplus Ky^{p-1}$, and multiply elements of D by using distributive law, and the rules $y^p = \lambda$, $yk = \sigma(k)y$ for all $k \in K$. In this way, we obtain the cyclic algebra $(K/F, \sigma, \lambda)$. Now, since $M = N(K^*)$ we conclude that $\lambda \notin N(K^*)$. Thus, by Corollary 14.8 of [2], D is a division algebra and hence $Br(F) \neq 0$. Finally, since $K = F(b)$ with $b^p = a \in F$ we obtain $N(b) = (-1)^{p+1}a$. Because $N(K^*) = F^{*p} = M$ there is $\lambda \in F$ such that $N(b) = \lambda^p$. Thus, $(-1)^{p+1}b^p = \lambda^p$. If p is odd, then in the presence of the primitive p-th root of unity in F one concludes that $b \in F$ which is a contradiction and so $p = 2$. Thus, we have $F^* \cong M \times C_2$, which shows that $-1 \notin M$. The equation $x^2 + 1 = 0$ over F has no root in F since $a^2 = -1$ with $a \in F$ implies that $-1 \in M$ which is false. Now, consider the extension $L = F(i)$ with $i^2 = -1$. The above proof shows that $N_{L/F}(L^*) = M$. We claim that M defines a positive cone for F. It is clear that $M \cap -M = \emptyset$, $MM \subseteq M$, and $M \cup -M \cup \{0\} = F$. To show that $M + M \subseteq M$, take $\alpha, \beta \in M$. Since $M = F^{*2}$, there exist $\lambda, \mu \in F^*$ such that $\alpha = \lambda^2, \beta = \mu^2$. Now, consider the element $x = \lambda + \mu i \in L$. We have $N_{L/F}(x) = \lambda^2 + \mu^2 = \alpha + \beta \in M$ since $N(L^*) = M$. Therefore, F is formally real and since $M = F^{*2}$ we conclude that F is Euclidean.

(2) One way is clear from Theorem A. If M is the divisible maximal subgroup of F^*, then from the proof of (1) we have $F^* \cong M \times C_2$. Since M is divisible from the theory of divisible abelian groups we know that M is a direct product of quasi-cyclic and full rational groups (cf. [1, p. 96]). We claim that M contains no primitive p-th root of unity. Since -1 is not in M it suffices to consider $p > 2$. If ω is a primitive p-th root of unity, then for $p \neq 2$ we have $1^2+\omega^2+\cdots+\omega^{2(p-1)} = (\omega^{2p}-1)/(\omega^2-1) = 0$, which is not possible in a formally real field. Thus, we cannot have any copy of a quasi-cyclic group in our decomposition of M and hence $M \cong \mathbb{Q}^\lambda$ for some cardinal λ. Since $\mathbb{Q}$ is of torsion-free rank 1, λ is the torsion-free rank of F^*. Now, because $Char F = 0$ we have $\mathbb{Q}^* \subset F^*$, and hence λ is infinite by Lemma 4.1.16 of [1] which asserts that $Q^* \cong \mathbb{Z}_2 \times \mathbb{Z}^{\aleph_0}$. Therefore, we have $M \cong \mathbb{Q}^\lambda$ for some infinite cardinal

λ. Now, by Theorem A, we obtain the result. $\square$

We observe that in the conclusion of the theorem F need not necessarily be real closed. In fact, if F is obtained from the rationals $\mathbb{Q}$ by iteratively adjoining roots of positive real algebraic numbers, the positive cone of the resulting field F is such a maximal subgroup. But F is not real closed.

Acknowledgments. The author thanks the Research Council of Sharif University of Technology for support. Also, he wishes to thank M. G. Mahmudi for a useful discussion on the proof of the main theorem.

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