THE IMAGE OF THE MAP FROM GROUP COHOMOLOGY TO GALOIS COHOMOLOGY

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Abstract. We study the image of the natural map from group cohomology to Galois cohomology by using motivic cohomology of classifying spaces.

1. Introduction

Let k be a field of $ch(k) = 0$, which contains a p-th root of unity. Let G be a split affine algebraic group over k and W a faithful representation of G. Then G acts also on the function field $k(W)$. Let $k(W)^G$ be the invariant filed. Then we have the natural quotient map of groups $q : Gal(\bar{k}(\bar{W})/k(W)^G) \to G$. This induces the map of cohomologies

$$q^* : H^*(G; \mathbb{Z}/p) \to H^*(k(W)^G; \mathbb{Z}/p).$$

The purpose of this paper is to study the image $Im(q^*)$ by using the motivic cohomology defined by Suslin and Voevodsky [Vo1,3]. The image $Im(q^*)$ is called the *stable* cohomology in [Bo], [Bo-Pe-Ts]. The kernel $Ker(q^*) = Ng$ is called the (geometric) negligible ideal [Pe],[Sa].

Let $H^{*,*'}(X; \mathbb{Z}/p)$ be the $mod(p)$ motivic cohomology. Let $0 \neq \tau \in H^{0,1}(Spec(k); \mathbb{Z}/p) \cong \mathbb{Z}/p$. Using affirmative solution of the Bloch-Kato conjecture by Voevodsky (and hence Beilinson-Lichtenbaum conjecture), the map q^* is decomposed as

$$q^* : H^{*,*}(BG; \mathbb{Z}/p) \to H^{*,*}(BG; \mathbb{Z}/p)/(\tau) \to H^{*,*}(Spec(k(W)^G); \mathbb{Z}/p).$$

where $H^{*,*}(BG; \mathbb{Z}/p)/(\tau) = H^{*,*}(BG; \mathbb{Z}/p)/(\tau H^{*,*-1}(BG; \mathbb{Z}/p))$ and BG is the classifying space of G defined by Totaro and Voevodsky ([To], [Vo1,4]).

By the Belinson-Lichtenbaum conjecture and the work of Bloch-Ogus [Bl-Og], we know

$$H^{*,*}(BG; \mathbb{Z}/p)/(\tau) \subset H^0_{Zar}(BG; H^*_{\mathbb{Z}/p}) \subset H^*(k(W)^G; \mathbb{Z}/p).$$

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Here $H^*_{\mathbb{Z}/p}$ is the Zarisky sheaf induced from the presheaf $H^*_{et}(V;\mathbb{Z}/p)$ for open subset V of BG. Therefore we see ([Or-Vi-Vo])

$$Im(q^*) = H^*(BG;\mathbb{Z}/p)/(Ng) \cong H^{*,*}(BG;\mathbb{Z}/p)/(\tau).$$

Note that the right hand side ring does not depend on the choice of W. We also note that the ideal (Ng) coincides the coniveau filtration $N^1H^*(BG;\mathbb{Z}/p)$ defined by Grothendieck.

In this paper we compute $Im(q^*)$ when $k = \mathbb{C}$ for abelian p-groups, symmetric group S_n, O_n, SO_n, $Spin_n$, PGL_p and exceptional groups. Extra special p-groups are also studied. For example, we see $H^*(BSpin_n;\mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2$ for all $n \geq 6$.

Recall that the cohomology invariant $Inv^*(G;\mathbb{Z}/p)$ is a ring of natural maps $H^1(F;G) \to H^*(F;\mathbb{Z}/p)$ for finitely generated fields F over k. This ring is very well studied for example see [Ga-Me-Se]. In particular, it is very useful to compute the essential dimension $ed(G)$ of G ([Re], [Br-Re-Vi]). Moreover, Totaro proved that

$$Inv^*(G;\mathbb{Z}/p) \cong H^0_{Zar}(BG;H^*_{\mathbb{Z}/p})$$

in a letter to Serre [Ga-Me-Se]. Hence $Im(q^*) \subset Inv^*(G;\mathbb{Z}/p)$. We use these results for some parts of this paper, however we also give new explanations of $Inv^*(G;\mathbb{Z}/p)$ for the case $k = \mathbb{C}$. For example,the image of (topological) Stiefel-Whitney class w_i of the map

$$H^*(BO_n;\mathbb{Z}/2) \to H^*(BO_n;\mathbb{Z}/2)/(Ng) \subset Inv^*(O_n;\mathbb{Z}/2)$$

is indeed the Stiefel-Whitney class w_i defined by Milnor and Serre as the natural function from quadratic forms to Milnor K-theories.

All examples stated above are detected by abelian p-subgroups A of G, i.e., the restriction map

$$Res : H^*(BG;\mathbb{Z}/p)/(Ng) \to \Pi_A H^*(BA;\mathbb{Z}/p)/(Ng)$$

is injective. (Indeed, most of the above cases are detected by only one elemenary abelian p-subgroup.)

Of course this detected property does not hold for general G. However to give examples is not so easy. Indeed, for a p-group G of exponent p, if $H^2(BG;\mathbb{Z}/p)/(Ng)$ is not detected by any $A \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$, then G is a counter example of the Noether's problem, namely, $k(W)^G$ is not purely transcendental over k. The examples of Saltman and Bogomolov are essentially of these types [Sa].

For each $n > 1$, we give an example G_n of a p-group $p \geq 3$, such that $H^{2n}(G_n;\mathbb{Z}/p)$ is not detected by abelian p-subgroups, while it does not implies a counter example of Noether's problem. Here the composition $Q_{2n-2}...Q_0$ of Milnor operations is used to see $x \notin Ng$ given $x \in H^{2n}(BG;\mathbb{Z}/p)$.

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2. MOTIVIC COHOMOLOGY

Let X be a smooth (quasi projective) variety. Let $H^{*,*'}(X;\mathbb{Z}/p)$ be the $mod(p)$ motivic cohomology defined by Voevodsky and Suslin.

Recall that the $(mod\ p)$ $B(n,p)$ condition holds if

$$H^{m,n}(X;\mathbb{Z}/p) \cong H^m_{et}(X;\mu_p^{\otimes n}) \qquad for\ all\ m \leq n.$$

It is known that the $B(n,p)$ condition holds for $p=2$ or $n=2$ by Voevodsky([Vo1,2]), and Merkurjev-Suslin respectively. Quite recently M.Rost and V.Voevodsky ([Vo5],[Su-Jo],[Ro]) announced that $B(n,p)$ condition holds for each p and n. Hence the Bloch-Kato conjecture also holds. Therefore in this paper, we *always assume* the $B(n,p)$-condition and so the Bloch-Kato conjecture for all n,p.

Moreover we always assume that k contains a primitive p-th *root of unity*. For these cases, we see the isomorphism $H^m_{et}(X;\mu_p^{\otimes n}) \cong H^m_{et}(X;\mathbb{Z}/p)$. Let τ be a generator of $H^{0,1}(Spec(k);\mathbb{Z}/p) \cong \mathbb{Z}/p$. Hence

$$colim_i \tau^i H^{*,*'}(X;\mathbb{Z}/p) \cong H^*_{et}(X;\mathbb{Z}/p).$$

Recall that $\mathbb{Z}/p(n)$ ([Vo1,2,3]) is the complex of sheaves in Zarisky topology such that $H^{m,n}(X;\mathbb{Z}/p) \cong H^m_{Zar}(X;\mathbb{Z}/p(n))$. Let α be the obvious map of sites from etale topology to Zarisky topology so that

$$H^m_{et}(X;\mathbb{Z}/p) \cong H^m_{et}(X;\mu_p^{\otimes n}) \cong H^m_{Zar}(X, R\alpha_*\alpha^*\mathbb{Z}/p(n)).$$

For $k \leq n$, let $\tau_{\leq k}R\alpha_*\alpha^*\mathbb{Z}/p(n)$ be the canonical truncation of $R\alpha_*\alpha^*\mathbb{Z}/p(n)$ of level k. Then we have the short exact sequence of sheaves

$$\tau_{\leq n-1}R\alpha_*\alpha^*\mathbb{Z}/p(n) \to \tau_{\leq n}R\alpha_*\alpha^*\mathbb{Z}/p(n) \to H^n_{\mathbb{Z}/p}[-n]$$

where $H^n_{\mathbb{Z}/p}$ is the Zarisky sheaf induced from the presheaf $H^n_{et}(V;\mathbb{Z}/p)$ for open subset V of X. The Beilinson and Lichtenbaum conjecture (hence $B(n,p)$-condition) (see [Vo2,5]) implies

$$\mathbb{Z}/p(k) \cong \tau_{\leq k}R\alpha_*\alpha^*\mathbb{Z}/p(n).$$

Hence we have ;

Lemma 2.1. *([Or-Vi-Vo], [Vo5]) There is the long exact sequence*

$$\begin{aligned} &\to H^{m,n-1}(X;\mathbb{Z}/p) \xrightarrow{\times\tau} H^{m,n}(X;\mathbb{Z}/p) \\ &\qquad \to H^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/p}) \to H^{m+1,n-1}(X;\mathbb{Z}/p) \to . \end{aligned}$$

In particular, we have

Corollary 2.2. *The graded ring* $grH^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/p})$ *is isomorphic to*

$$H^{m,n}(X;\mathbb{Z}/p)/(\tau)\oplus Ker(\tau)|H^{m+1,n-1}(X;\mathbb{Z}/p)$$

where $H^{m,n}(X;\mathbb{Z}/p)/(\tau)=H^{m,n}(X;\mathbb{Z}/p)/(\tau H^{m,n-1}(X;\mathbb{Z}/p))$.

Note that the above long exact sequence induces the τ-Bockstein spectral sequence

$$E(\tau)_1=H^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/p})\Longrightarrow colim_i\tau^iH^{*,*'}(X;\mathbb{Z}/p)\cong H^*_{et}(X;\mathbb{Z}/p).$$

On the other hand, the filtration *coniveau* is given by

$$N^cH^m_{et}(X;\mathbb{Z}/p)=\cup_ZKer\{H^m_{et}(X;\mathbb{Z}/p)\to H^m_{et}(X-Z;\mathbb{Z}/p)\}$$

where Z runs in the set of closed subschemes of X of $codim=c$. Grothendieck wrote down the E_1-term of the spectral sequence induced from the above coniveau filtration.

$$E(c)_1^{c,m-c}\cong\Pi_{x\in X^{(c)}}H^{m-c}_{et}(k(x);\mathbb{Z}/p)\Longrightarrow H^m_{et}(X;\mathbb{Z}/p)$$

where $X^{(c)}$ is the set of primes of codimension c and $k(x)$ is the residue field of x. We can regard $i_{x*}H^{m-c}_{et}(k(x);\mathbb{Z}/p)$ as a constant sheaf $H^{m-c}_{et}(k(x);\mathbb{Z}/p)$ on $\{\bar{x}\}$ and extend it by zero to X. Then the differentials of the spectral sequence give us a complex on sheaves on X

$$(2.9)\ 0\to H^q_{\mathbb{Z}/p}\to\Pi_{x\in X^{(0)}}i_{x*}H^q_{et}(k(x);\mathbb{Z}/p)\to\Pi_{x\in X^{(1)}}i_{x*}H^{q-1}_{et}(k(x);\mathbb{Z}/p)$$
$$\to\ldots\to\Pi_{x\in X^{(q)}}i_{x*}H^0_{et}(k(x);\mathbb{Z}/p)\to 0.$$

Bloch-Ogus [Bl-Og] proved that the above sequence of sheaves is exact and the E_2-term is given by

$$E(c)_2^{c,m-c}\cong H^c_{Zar}(X,H^{m-c}_{\mathbb{Z}/p}).$$

In particular, we have ;

Corollary 2.3.

$$H^0_{Zar}(X;H^m_{\mathbb{Z}/p})\cong Ker\{H^m_{et}(k(X);\mathbb{Z}/p)\to\Pi_{x\in X^{(1)}}H^{m-1}_{et}(k(x);\mathbb{Z}/p)\}.$$

By Deligne (foot note (1) in Remark 6.4 in [Bl-Og]) and Paranjape (Corollary 4.4 in [Pj]), it is proven that there is an isomorphism of the coniveau spectral sequence with the Leray spectral sequence for the map α. Hence we have ;

Theorem 2.4. *(Deligne, Parajape) There is the isomorphism* $E(c)_r^{c,m-c}\cong E(\tau)_{r-1}^{m,m-c}$ *of spectral sequences. Hence the filtrations are the same* $N^cH^m_{et}(X;\mathbb{Z}/p)=F^{m,m-c}_\tau$ *where*

$$F^{m,m-c}_\tau=Im(\times\tau^c:H^{m.m-c}(X;\mathbb{Z}/p)\to H^{m,m}(X;\mathbb{Z}/p)).$$

3. COHOMOLOGY OF GROUPS

Let G be a reductive algebraic group over k acting on an affine variety W. A point $x \in W$ is called stable if the orbit Gx is closed and the stabilizer group $Stab(x)$ is a finite group. Let us write by X^s the set of stable points in X. Then X^s is an open subset of X and there is the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\phi} & W//G \\ \uparrow{\scriptstyle open} & & \uparrow{\scriptstyle open} \\ W^s & \xrightarrow{\phi^s} & W^s/G, \end{array}$$

where $W//G = Spec(k[W]^G)$, the geometric quotient of X. We also note that the invariant field $k(W)^G$ is a quotient field of $k[W]^G$ when $X^s \neq \emptyset$.

Suppose $X^s \neq \emptyset$. Then $H^{*,*'}(W//G; \mathbb{Z}/p) = H^{*,*'}(Spec(k[W]^G); \mathbb{Z}/p)$ and $k(W^s/G) = k(W//G) = k(W)^G$. Hence we have the diagram

$$\begin{array}{ccc} H^{*,*'}(W//G; \mathbb{Z}/p) & \longrightarrow & H^{*,*'}(Spec(k(W)^G); \mathbb{Z}/p) \\ \downarrow & & {\scriptstyle =}\downarrow \\ H^{*,*'}(W^s/G; \mathbb{Z}/p) & \xrightarrow{\psi} & H^{*,*'}(Spec(k(W^s/G); \mathbb{Z}/p) \end{array}$$

Restrict W as an affine space $W = \oplus k$ and let $\rho : G \to W = \oplus k$ a faithful representation. Let $U_n = W - S$ be an open set of W such that G act freely U where $codim_W S = n$. (Of course U is an open subset of W^s.) Then the classifying space of G is defined as $colim_{n\to\infty}(U_n/G)$. Then the $mod(p)$ motivic cohomology (for degree $* < 2n$) of BG is given by ([Vo4],[To])

$$H^{*,*'}(BG; \mathbb{Z}/p) \cong lim_{n\to\infty} H^{*,*'}(U_n/G; \mathbb{Z}/p).$$

In particular, by $BL(p, *)$ condition, we have

$$H^{*,*}(BG; \mathbb{Z}/p) \cong H^*_{et}(BG; \mathbb{Z}/p) = H^*(G; \mathbb{Z}/p) \otimes H^*(k; \mathbb{Z}/p)$$

where the last group is the cohomology group of the Galois group G (when G is finite). Thus from the above diagram, we have the map

$$\psi : H^{*,*'}(BG; \mathbb{Z}/p) \to H^{*,*'}(Spec(k(W)^G); \mathbb{Z}/p).$$

This map $\psi^{*,*} = \psi^*_{et}$ is explains also as follows. Let Γ is the absolute Galois group $\Gamma = Gal(k(\bar{W})/k(W)^G)$. Then the group $G = Gal(k(W)/k(W)^G)$ is a quotient group of the absolute Galois group Γ.

Then the map ψ_{et} is the induced map from the quotient $q : \Gamma \to G$, i.e.,

$$\psi_{et}^* = q^* : H^*(G; \mathbb{Z}/p) \to H^*(\Gamma; \mathbb{Z}/p) = H^*(k(W)^G; \mathbb{Z}/p).$$

Lemma 3.1.

$$Im(\psi^*) \cong H^{*,*}(BG; \mathbb{Z}/p)/(\tau).$$

Proof. For each field F, by the Bloch-Kato conjecture, $H^*(F; \mathbb{Z}/p)$ is generated by elements in $H^1_{et}(F; \mathbb{Z}/p) \cong H^{1,1}(Spec(F); \mathbb{Z}/p)$. So

$$\psi^{*,*-1} : H^{*,*-1}(BG; \mathbb{Z}/p) \to H^{*,*-1}(Spec(k(W)^G); \mathbb{Z}/p) = 0.$$

Hence the map ψ^* is expressed as a composition

$$H^{*,*}(BG; \mathbb{Z}/p) \to H^{*,*}(BG; \mathbb{Z}/p)/(\tau) \to H^*_{et}(k(W)^G; \mathbb{Z}/p).$$

The first map is of course surjective and we only need the injectivity of the second map. Indeed, from Corollary 2.2 and 2.3, we see

$$H^{*,*}(BG; \mathbb{Z}/p)/(\tau) \subset H^0_{Zar}(BG; H^*_{\mathbb{Z}/p}) \subset H^*(k(W)^G; \mathbb{Z}/p).$$

□

Recall the coniveau filtration given in §2

$$N^c H^m_{et}(X; \mathbb{Z}/p) = \cup_Z Ker\{H^m_{et}(X; \mathbb{Z}/p) \to H^m_{et}(X - Z; \mathbb{Z}/p)\}$$

where Z runs in the set of closed subschemes of X of $codim = c$. From Theorem 2.4, we see

Corollary 3.2. $Im(q^*) \cong H^*_{et}(BG; \mathbb{Z}/p)/(N^1 H^*_{et}(BG; \mathbb{Z}/p))$.

According to Saltman (and Serre), we say an element $x \in H^*(G; \mathbb{Z}/p)$ is geometrically negligible if $\psi^*(x) = 0$. Let us write $Ng = Ng(G) = Ker(\psi^*)$. From the above lemma, it is immediate

$$Ng(G) = N^1 H^*_{et}(BG; \mathbb{Z}/p)$$

$$= Im(\times\tau | H^{*,*-1}(BG; \mathbb{Z}/p) \to H^{*,*}(BG; \mathbb{Z}/p))$$

and we have

$$Im(\psi^*) = H^*(BG; \mathbb{Z}/p)/(Ng) \cong H^*(BG; \mathbb{Z}/p)/(N^1) \cong H^{*,*}(BG; \mathbb{Z}/p)/(\tau).$$

Lemma 3.3. *For each element x in $H^*_{et}(BG; \mathbb{Z}/p)$, the images of cohomology (Bockstein, reduced) operations $\beta(x)$, $P^i(x)$ for $i > 0$ are in $Ng(G)$ (hence $x^p \in Ng(G)$). The image of Gysin map $g_*(x)$ are also in $Ng(G)$.*

Proof. The cohomology operations act as ([Vo2,4])

$$\beta : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+1,*}(X;\mathbb{Z}/p)$$

$$P^i : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2i(p-1),*+i(p-1)}(X;\mathbb{Z}/p).$$

For an element $x \in H^{m,n}(X;\mathbb{Z}/p)$, define the difference degree $d(x) = m-n$. Then if $d(x)=0$, then $d(\beta(x)) > 0$ and $d(P^i(x)) > 0$. Hence these elements are in $Im(\tau)$ as a subset of $H^{*,*}(X;\mathbb{Z}/p)$.

For the embedding $X \subset Y$ of codimension c, the Gysin map is defined on

$$g_* : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2c,*+c}(Y;\mathbb{Z}/p).$$

By the same reason as the cases of cohomology operations, we get lemma. □

Here we give a sufficient condition for $x \notin Ng(G)$. Voevodsky define the Milnor operation Q_i also in the mod p motivic cohomology

$$Q_i : H^{*,*'}(-;\mathbb{Z}/p) \to H^{*+2p^n-1,*'+p^n-1}(-;\mathbb{Z}/p).$$

Define the weight $w(x) = 2*' - *$ for element (or operation) $x \in H^{*,*'}(X;\mathbb{Z}/p)$, e.g., $w(\tau)=2$, $w(Q_i)=-1$ and $w(P^i)=0$.

Lemma 3.4. *Let $x \in H^n_{et}(BG;\mathbb{Z}/p)$ and $Q_{n-2}...Q_0(x) \neq 0$. Then $0 \neq x \in H^n_{et}(BG;\mathbb{Z}/p)/(Ng)$.*

Proof. Identify x as an element in $H^{n,n}(BG;\mathbb{Z}/2)$. Suppose that $x = \tau\bar{x}$. So $w(\bar{x}) = n-2$ since $w(\tau)=2$. Then $t_{\mathbb{C}}(\bar{x}) = t_{\mathbb{C}}(x) = x$ where $t_{\mathbb{C}} : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*,*'}(X(\mathbb{C});\mathbb{Z}/p)$ is the realization map.

The operation Q_i descends the weight one. Let $\bar{\psi} = Q_{n-2}...Q_0(\bar{x})$. Then $w(\bar{\psi}) = -1$ but $t_{\mathbb{C}}(\bar{\psi}) = Q_{n-2}...Q_0(x) \neq 0$. This is a contradiction since $w(y) \geq 0$ for each nonzero element $y \in H^{*,*'}(Y;\mathbb{Z}/p)$ and for smooth Y. □

We have the Kunneth formula for the etale cohomology of coefficient $\mathbb{Z}/p$. Since Ng is an ideal, we have the surjection

$$H^*_{et}(BG_1;\mathbb{Z}/p)/(Ng) \otimes_{H^*(k;\mathbb{Z}/p)} H^*_{et}(BG_2;\mathbb{Z}/p)/(Ng)$$

$$\to H^*_{et}(B(G_1 \times G_2);\mathbb{Z}/p)/(Ng).$$

However it does not need isomorphic, because there is the possibility that $x_1 \otimes x_2 \in Ng(G_1 \times G_2)$ but $x_1 \notin Ng(G_1),\ x_2 \notin Ng(G_2)$.

4. COHOMOLOGY INVARIANT

Recall that $H^1(k;G)$ is the first non abelian Galois cohomology set of G, which represents the set of G-torsors over k. It is very important to study $H^1(k;G)$, for example $H^1(k;O_n)$ is isomorphic to the set of isomorphism classes of non generate quadratic forms over k of rank n. (For details, see the excellent book [Ga-Me-Se].)

The cohomology invariant is defined by

$$Inv^i(G,\mathbb{Z}/p) = Func(H^1(F;G) \to H^i(F;\mathbb{Z}/p))$$

where $Func$ means natural functions for each fields F over k. The cohomology invariant is studied by many authors. The cohomology invariants $Inv^*(G;\mathbb{Z}/p)$ are computed (for example in [Ga-Me-Se]) for groups elementary abelian 2-groups, O_n,SO_n, G_2,... It is also stated in [Ga-Me-Se] that for many G (but not all) $Inv^*(G;\mathbb{Z}/p)$ are detected by $Inv^*(H;\mathbb{Z}/p)$ for elementary abelian p-subgroups H.

Let $x \in H^0(BG;H^i_{\mathbb{Z}/p})$. Given a G-torsor E over F, we can construct $x(E) \in H^i_{et}(F;\mathbb{Z}/p)$. Roughly speaking, we can identify E as the pullback of some map $f : Spec(F) \to BG$. So we can take $x(E) = f^*(x) \in H^0(Spec(F);H^i_{\mathbb{Z}/p}) = H^i_{et}(F;\mathbb{Z}/p)$. Indeed, Totaro proved [Ga-Me-Se] the following theorem in a letter to Serre.

Theorem 4.1. $Inv^*(G;\mathbb{Z}/p) \cong H^0(BG;H^*_{\mathbb{Z}/p})$.

Therefore we see

Corollary 4.2. $Im(\psi^*) \cong H^*(BG;\mathbb{Z}/p)/(Ng) \subset Inv^*(G;\mathbb{Z}/p)$.

5. ABELIAN p-GROUPS

Let us write $H^{*,*'} = H^{*,*'}(Spec(k);\mathbb{Z}/p)$ and $H^* = H^{*,*} = K^*_M(k)/p$ so that $H^{*,*'} \cong H^*[\tau]$. First consider the case $G = \mathbb{Z}/p^r$. The $mod(p)$ motivic cohomology is computed (as the case $\mathbb{Z}/p$ in [Vo])

$$H^{*,*'}(B\mathbb{Z}/p^r;\mathbb{Z}/p) \cong H^{*,*'}[y(r)] \otimes \Lambda(x(r)) \quad |y(r)| = 2,\ |x(r)| = 1.$$

(When $p = 2$ and $r = 1$, we see by Voevodsky ([Vo2,4])

$$x(1)^2 = \tau y(1) + \rho x(1) \quad with\ \rho = (-1) \in H^1 = k^*/(k^*)^2.)$$

For the inclusion $i : \mathbb{Z}/p^r \subset \mathbb{Z}/p^s$ and quotient map $q : \mathbb{Z}/p^s \to \mathbb{Z}/p^r$, for $s \geq r$, we have

$$i^*(y(s)) = y(r),\ i^*(x(s)) = 0, \quad q^*(y(r)) = 0,\ q^*(x(r)) = x(s).$$

Moreover we still know $x(r) \in H^{1,1}(BG;\mathbb{Z}/p)$ and $y(r) \in H^{2,1}(BG;\mathbb{Z}/p)$. Thus we see $((Ker(\tau)|H^{*,*'}(B\mathbb{Z}/p^r;\mathbb{Z}/p) = 0)$.

$$Inv^*(\mathbb{Z}/p^r;\mathbb{Z}/p) \cong H^{*,*}(\mathbb{Z}/p^r;\mathbb{Z}/p)/(\tau) = H^*\{1, x(r)\}.$$

Next consider their product $G = \mathbb{Z}/p^{r_1} \times ... \times \mathbb{Z}/p^{r_s}$. The cohomology $H^{*,*'}(B\mathbb{Z}/p^r; \mathbb{Z}/p)$ has the Kunneth formula. Hence the motivic cohomology is given

$$H^{*,*'}(BG; \mathbb{Z}/p) \cong H^{*,*'}[y(r_1), ..., y(r_s)] \otimes \Lambda(x(r_1), .., x(r_s))$$

where $x(r_i) \in H^{1,1}(BG; \mathbb{Z}/p)$ and $y(r_i) \in H^{2,1}(BG; \mathbb{Z}/p)$.

Recall that $H^*(BG; \mathbb{Z}/p)/(Ng) \cong H^{*,*}(BG; \mathbb{Z}/p)/(\tau)$ (Lemma 3.1). Then we get

Lemma 5.1. *Let G be an abelian p-group, i.e., $G = \oplus_i \mathbb{Z}/(p^{r_i})$. Then*

$$Inv^*(G; \mathbb{Z}/p) \cong H^*(G; \mathbb{Z}/p)/(Ng) \cong H^* \otimes \Lambda(x(r_1), ..., x(r_s))$$

$$when\ p = 2\ r_i = 1, \quad x(r_i)^2 = \rho x(r_i).$$

The elementary 2-groups cases are stated in Theorem 16.4 in [Ga-Me-Se].

The Q_i-operation acts on $H^{*,*'}(B\mathbb{Z}/p; \mathbb{Z}/p)$ by $Q_i(x) = y^{p^i}$ (while $Q_i(x(j)) = 0$ for all $j > 1$). We consider Q_i action on

$$H^{*,*'}(B(\mathbb{Z}/p)^s; \mathbb{Z}/p) \cong H^{*,*'}[y_1, ..., y_s] \otimes \Lambda(x_1, ..., x_s).$$

Each Q_i is a derivation, and hence

$$Q_0...Q_{s-1}(x_1...x_s) = \sum sgn(j_1, ..., j_s) y_1^{p^{j_1}} y_2^{p^{j_2}} ... y_s^{p^{j_s}} \neq 0$$

where $(j_1, ..., j_s)$ are permutations of $(0, ..., s-1)$. Thus we see that this case satisfies the sufficient condition of Lemma 3.3 while the other cases does not), in fact $x_1...x_s \notin Ng(G)$.

Let us say that an element $x \in H^*(BG; \mathbb{Z}/p)/(Ng)$ is *detected* by an elemntary abelian p-subgroup A if $Res(x) \neq 0$ for

$$Res : H^*(BG; \mathbb{Z}/p)/(Ng) \to H^*(BA; \mathbb{Z}/p)/(Ng).$$

The following lemma is immediate from the above arguments.

Lemma 5.2. *If $x \in H^n(BG; \mathbb{Z}/p)/(Ng)$ is detected by elementary abelian p-subgroups, then $Q_{n-1}...Q_0(x) \neq 0$ in $H^*(BG; \mathbb{Z}/p)$.*

6. Cases $G = O_n$ and SO_n

Hereafter, in this paper (except for §11), we assume that $k = \mathbb{C}$ otherwise stated.

lt is well known that

$$H^*(BO_n; \mathbb{Z}/2) \cong H^*((B\mathbb{Z}/2)^{\times n}; \mathbb{Z}/p)^{S_n} \cong \mathbb{Z}/2[w_1, ..., w_n]$$

where S_n is the n-th symmetric group, and w_i is the Stiefel-Whitney class representing the i-the elementary symmetric function. we easily see

$$Q_{i-1}...Q_0(w_i) = y_1^{p^{i-1}} y_2^{p^{p-2}} ... y_i + ... \neq 0 \in \mathbb{Z}/2[y_1, ..., y_n] \subset H^*(B(\mathbb{Z}/2)^n; \mathbb{Z}/2)$$

and hence $w_i \notin Ng(G)$. Recall the Wu formula

$$Sq^i w_k = \sum_j^i \begin{pmatrix} k-j-1 \\ i-j \end{pmatrix} w_{k+i-j} w_j \quad (0 \leq i \leq k).$$

Many cases of product of $w_i w_j$ are in $Ng(G)$, e.g.,$w_i^2 \in Ng(G)$. More precisely, the motivic cohomology of BO_n is computed for $k = \mathbb{C}$ (Theorem 8.1 in [Ya3])

$$H^{*,*'}(BO_n; \mathbb{Z}/2) \cong H^{*,*'}((B\mathbb{Z}/2)^{\times n}; \mathbb{Z}/p)^{S_n}.$$

Given $x \in H^*(BG; \mathbb{Z}/p)$, let us define the weight $w(x)$ as the smallest weight $w(x')$ such that $t_{\mathbb{C}}(x') = x$ with $x' \in H^{*,*'}(BG; \mathbb{Z}/p)$. Indeed, the weight of the symmetric polynomial

$$t = \sum x_1^{2i_1+1} ... x_k^{2k+1} x_{k+1}^{2j_1} ... x_{k+q}^{2j_q} \quad in\ H^*((B\mathbb{Z}/2)^{\times n}; \mathbb{Z}/2)$$

(with $0 \leq i_1 \leq ... \leq i_k$, $0 \leq j_1 \leq ... \leq j_q$) is given by $w(t) = k$. Hence if $t \notin Ng$, then $w(t) = deg(t)$ and this implies $t = x_1 ... x_i = w_i$.

Theorem 6.1.

$$Inv^*(O_n; \mathbb{Z}/2) \cong H^*(BO_n; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_1, ..., w_n\}.$$

In fact, $Inv(O_n; \mathbb{Z}/2)$ is well known (Theorem 17.3 in [Ga-Me-Se]) for general k ;

$$Inv^*(O_n; \mathbb{Z}/2) \cong H^*\{1, w_1, ..., w_n\}.$$

We consider the multiplicative structure of $Inv^*(O_n; \mathbb{Z}/2)$. From the Wu formula, we see

$$Sq^1(w_{2i}) = w_{2i+1} + w_{2i} w_1 \in Ng(O_n).$$

Hence $w_{2i+1} = w_{2i} w_1$ in $Inv^*(O_n; \mathbb{Z}/2)$. By Rost and Kahn [Ka], the divided power operation can be defined in $K_*^M(F)/p$ compatible with fields F over k (and hence $Inv^*(G; \mathbb{Z}/p)$) if $\sqrt{-1} \in k$. Vial showed [Via] that the divided power operations are only compatible maps (natural maps) with field extensions over k. Moreover Becher [Be] showed that $\gamma_n(w_2) = w_{2n}$. (See also Milnor p133 in [Mi].)

Theorem 6.2. *(Becher [Be], [Via]) Let $\sqrt{-1} \in k$. Then $Inv^*(O_n; \mathbb{Z}/2)$ is generated by w_1 and w_2 as an H^*-ring with divided powers by*

$$\gamma_i(w_2) = w_{2i}, \quad w_{2i+1} = w_{2i} w_1.$$

Next consider the case $G = SO_n$. It is well known that

$$H^*(BSO_n; \mathbb{Z}/2) \cong H^*(BO_n; \mathbb{Z}/2)/(w_1) \cong \mathbb{Z}/2[w_2, ..., w_n].$$

Let $n = 2m + 1 = odd$. For this case, there is the isomorphism

$$O_{2m+1} \cong SO_{2m+1} \times \mathbb{Z}/2.$$

Let $p : O_n \to SO_n$ is the projection and $i : SO_n \to O_n$ the inclusion. We consider the induced map p^* and i^* on the mod 2 motivic cohomology of their classifying spaces. Since $p^*(w_1) = 0$, we see $w_{2i+1} \in Ng(G)$ from the above theorem (in fact, $Sq^1(w_{2i}) = w_{2i+1}$ in $H^*(BSO_n; \mathbb{Z}/2)$). Moreover $i^*w_{2i} = w_{2i}\ mod(Ng(G))$. Thus we have

Theorem 6.3. *For $G = SO_{2m+1}$, we have*

$$Inv^*(G; \mathbb{Z}/2) \cong H^*(BG; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_2, ..., w_{2m}\}.$$

Moreover $Inv^*(SO_{2m+1}; \mathbb{Z}/2)$ is still computed (Theorem 19.1 in [Ga-Me-Se]) for general k ;

$$Inv^*(SO_{2m+1}; \mathbb{Z}/2) \cong H^*\{1, w_2, ..., w_{2m}\}.$$

Now consider the case $n = 2m = even$. This case the mod 2 motivic cohomology is not computed even $k = \mathbb{C}$ for $n > 4$. However we compute $H^{*,*}(BSO_n; \mathbb{Z}/2)/(\tau)$ easily. Consider the inclusion

$$SO_{2m-1} \xrightarrow{i_1} SO_{2m} \xrightarrow{i_2} SO_{2m+1}.$$

Since the restriction map for $i < m$

$$i_1^*(w_{2i}) \neq 0 \in H^{*,*}(BSO_{2m-1}; \mathbb{Z}/2),$$

we see $w_{2i} \notin Ng(SO_{2m})$. Moreover we know that $w(w_n) = n - 2$ in Lemma 9.2 in [Ya3]. On the other hand, the monomial w_I of not w_{2i} are all in $Ng(SO_{2m+1})$ and so $i_2^*(w_I) \in Ng(SO_{2m})$.

Thus we have

Theorem 6.4. $H^*(BSO_{2m}; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_2, ..., w_{2m-2}\}$.

Next we study $Inv^*(SO_{2m}; \mathbb{Z}/2)$. There is an element (Lemma 9.3 in [Ya3]) in the motivic cohomology

$$u_{m-1} \in H^{n,n-2}(BSO_n; \mathbb{Z}/2) \quad with\ \tau u_{m-1} = 0.$$

So there is the nonzero element

$$u \in Ker(\tau | H^{n,n-2}(BG; \mathbb{Z}/2)) \subset H^0(BG; H^{n-1}_{\mathbb{Z}/2}).$$

On the other hand in [Ga-Me-Su], it is proved (Theorem 20.6) for general k

$$Inv^*(SO_{2m}; \mathbb{Z}/2) \cong H^*\{1, w_2, ..., w_{n-2}\} \oplus (Im(I_\delta)).$$

Here when $k = \mathbb{C}$, $Im(I_\delta) \cong \mathbb{Z}/2\{u\}$ with $deg(u) = n-1$ from Proposition 20.1 in [Ga-Me-Se]. Thus we see

Theorem 6.5. *Let $G = SO_{2m}$ and $m \geq 2$. Then for $deg(u) = 2m-1$,*

$$Inv^*(G;\mathbb{Z}/2) \cong H^*(BG;\mathbb{Z}/2)/(Ng) \oplus \mathbb{Z}/2\{u\}.$$

From 22.10 in [Ga-Me-Se], it is known that

$$Res : Inv^*(G;\mathbb{Z}/p) \to Inv^*(H;\mathbb{Z}/p)$$

is injective for $p = 2$, $G = SO_n$ and $H = (\mathbb{Z}/2)^{n-1}$. Hence for

$$Res : H^0(BSO_n; H^*_{\mathbb{Z}/2}) \to H^0(B(\mathbb{Z}/2)^{n-1}; H^*_{\mathbb{Z}/2})$$

we have $Res(u) = x_1...x_{n-1}$. Of course $u \notin H^{*,*}(BSO_n;\mathbb{Z}/2)/(\tau)$ but $x_1...x_{n-1} \in H^{*,*}(B(\mathbb{Z}/2)^{n-1};\mathbb{Z}/2)/(\tau)$.

Recall that in Corollary 2.2 $grH^0(BG; H^*_{\mathbb{Z}/p})$ is defined by the filtration $H^{*,*}(BG;\mathbb{Z}/p)/(\tau) \subset H^0(BG; H^*_{\mathbb{Z}/p})$. So note that

$$Res : grH^0(BSO_n; H^*_{\mathbb{Z}/2}) \to grH^0(B(\mathbb{Z}/2)^{n-1}; H^*_{\mathbb{Z}/2})$$

is not injective (in Corollary 2.2).

7. $Spin_n$ and exceptional groups

The $mod(2)$ cohomology of $BSpin_n$ is computed by Quillen

$$H^*(BSpin_n;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2^h}(\Delta)] \otimes \mathbb{Z}/2[w_2, ..., w_n]/(Q_i w_2 | 0 \leq i \leq h)$$

where $w_i(\Delta)$ (resp. w_i) is the Stiefel-Whitney class of a spin representation Δ (resp. usual representation $Spin_n \to SO_n$), and 2^h is the the Radon-Hurwitz number (See [Qu] p.210). By the result of Becher (Theorem 6.2), we have

Theorem 7.1. *$H^*(BSpin_n;\mathbb{Z}/2)/(Ng) = \mathbb{Z}/2$ for $n > 4$.*

Proof. Let us write representations $j : Spin_n \to SO_n$ and $\Delta : Spin_n \to SO_N$. We consider the induced map in Galois cohomolgy

$$j^* : H^*_{et}(k(W)^{SO_n};\mathbb{Z}/2) \to H^*_{et}(k(W)^{Spin_n};\mathbb{Z}/2).$$

By the Quillen's result, we see $j^*(w_2) = 0$. By Rost and Kahn [Ka], the divided powers naturally act on $K^M_*(F)/p$ for field F over k. Hence from Becher theorem (Theorem 6.2),we get

$$j^*(w_{2i}) = j^*(\gamma_i(w_2)) = \gamma_i(j^*(w_2)) = 0.$$

Similarly $w_2(\Delta) = 0$ implies $w_{2^h}(\Delta) = 0$ if $n > 4$. □

Corollary 7.2. *For $* > 0$, there is the isomorphism*

$$Inv^*(Spin_n;\mathbb{Z}/2) \cong Ker(\tau)|H^{*+1,*-1}(BSpin_n;\mathbb{Z}/2).$$

The $mod(2)$ motivic cohomology of $BSpin_7$ is computed in (Theorem 9.6 in [Ya3]). We can easily see the above theorem also from the concrete computation. Moreover there are τ-torsion elements

$$y_2 \in H^{4,2}(Spin_7;\mathbb{Z}/2), \quad y_2' \in H^{5,3}(Bspin_7;\mathbb{Z}/2).$$

Therefore we can take $u \in H^0(BSpin_7, H^3_{\mathbb{Z}/2}), v \in H^0(BSpin_7, H^4_{\mathbb{Z}/2})$.

Theorem 7.3.

$$Inv^*(BSpin_7;\mathbb{Z}/2) \cong \mathbb{Z}/2\{1,u,v\} \quad |u|=3,\ |v|=4.$$

We consider exceptional Lie group types G_2, F_4 and split E_6. In [Ga-Me-Su], it is proved (Teorem 18.1, Theorem 22.5)

$$Inv^*(G_2;\mathbb{Z}/2) \cong Inv^*(E_6;\mathbb{Z}/2) \cong H^*\{1,u\},$$
$$Inv^*(F_4;\mathbb{Z}/2) \cong H^*\{1,u,f_5\}.$$

(Unfortunately, we can not reexplain f_5 by using $H^{*,*'}(BF_4;\mathbb{Z}/2)$, which is not computed yet.)

Moreover restriction image for elementary abelin 2-subgroup of rank 3 (resp. rank 5) is injective for G_2, E_6 (resp. for F_4), see 22.10 in [Ga-Me-Se].

Theorem 7.4. *Let $G = G_2, F_4$. Then $H^*(BG;\mathbb{Z}/2)/(Ng) = \mathbb{Z}/2$.*

Proof. It is known that the inclusion $i : G_2 \to Spin_7$ induces the epimorphism $i^* : H^*(BSpin_7;\mathbb{Z}/2) \to H^*(BG_2;\mathbb{Z}/2)$. Hence the result for $G = G_2$ follows from $H^*(BSpin_7;\mathbb{Z}/2)/(Ng) = \mathbb{Z}/2$.

It is also known that the inclusion $i' : Spin_9 \to F_4$ induces the injection $i'^* : H^*(BF_4;\mathbb{Z}/2) \to H^*(BSpin_9;\mathbb{Z}/2)$. The groups $Spin_9$ and F_4 has the same maximal abelin 2-group H (of rank 5). We consider the restriction to this H. The fact $Res(H^*(BSpin_9;\mathbb{Z}/2)/(Ng)) = \mathbb{Z}/2$ implies the results for F_4, because the restriction is injective from [Ga-Me-Se]. □

Here we give an example. The $mod(2)$ cohomology is well known

$$H^*(BG_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7].$$

The motivic cohomology is given in Theorem 7.5 in [Ya3]. In particular, c_4, c_6, c_7 are the Chern classes (in $H^{2*,*}(BG_2;\mathbb{Z}/2)$) so that $\tau^i c_i = w_i^2$. Let us write simply by $Q(n)$ the exterior algebra $\Lambda(Q_0, ..., Q_n)$.

Theorem 7.5. *The motivic cohomology $H^{*,*'}(BG_2;\mathbb{Z}/2)$ is isomorphic to*

$$\mathbb{Z}/2[c_6,c_4]\otimes(\mathbb{Z}/2\{y_2\}\oplus\mathbb{Z}/2[\tau]\otimes(\mathbb{Z}/2\{1\}\oplus(\mathbb{Z}/2[c_7]Q(2)-\mathbb{Z}/2\{1\})\{a\})$$

where $deg(y) = (4,2)$, and a is a virtual element with $deg(a) = (3,3)$ so that $c_7 a = w_6 w_7 w_4$, $Q_0 a = w_4$, $Q_1 a = w_6$ and $Q_2 a = w_6 w_4$.

Let us write $Q_{i_1}...Q_{i_s}(a)$ by $Q_{i_1...i_s}$ and $w_{i_1}...w_{i_s}$ by $w_{i_1...i_s}$. Then $(\mathbb{Z}/2[c_7]Q(2) - \mathbb{Z}/2\{1\})\{a\}$ is written as

$$\mathbb{Z}/2[c_7]\{c_7a = w_{467}, Q_0 = w_4, Q_1 = w_6, Q_{01} = w_7,$$
$$Q_2 = w_{46}, Q_{02} = w_{47}, Q_{12} = w_{67}, Q_{012} = c_7\}.$$

Note that all $w_{i_1...i_s}$ (which are generators of the $\mathbb{Z}/2[c_4, c_6, c_7]$-module $H^*(G_2; \mathbb{Z}/2)$) appeared indeed in the above. Moreover it is immediate that all elements except for 1 and a are in $Ng(G_2)$, and $y \in Ker(\tau)|H^{4,2}(BG_2; \mathbb{Z}/2)$.

Next we consider odd prime case. In 22.10 in [Ga-Me-Se] it is stated that the restriction image of $Inv^*(G; \mathbb{Z}/p)$ to some (maximal) elementary abelian p-group H is injective for $(G, p) = (F_4, 3), (E_6, 3), (E_7, 3)$ or $(E_8, 5)$.

In these cases, each exceptional Lie group has two conjugacy classes of maximal elementary abelian p-groups. One is the subgroup of a maximal torus and the other is a nontoral A. Let us write $i_A : A \to G$ and $i_T : T \to B$ be the inclusions. Tezuka and Kameko proves ([Ka], [Ka-Ya]) that the following map is injective

$$i_A^* \times i_T^* : H^*(BG; \mathbb{Z}/p) \to H^*(BA; \mathbb{Z}/p) \times H^*(BT; \mathbb{Z}/p),$$

namely, $H^*(BG; \mathbb{Z}/p)$ is detected by A and T. Since $H^*(BT; \mathbb{Z}/p)$ is non-nilpotent, the above group H must be A.

Theorem 7.6. *Let $(G, p) = (F_4, 3), (E_6, 3), (E_7, 3)$ or $(E_8, 5)$. Then $H^*(BG; \mathbb{Z}/p)/(Ng) = \mathbb{Z}/p$.*

Proof. The restriciton image to A is studied in [Ka-Ya]. Images are generated as a ring by Chern classes and

$$Q_I(x_1..x_s) \in H^*(BA; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, ..., y_s] \otimes \Lambda(x_1, ..., x_s)$$

where $Q_I = Q_{i_1}...Q_{is}$ for some $I \neq \emptyset$. (Note $x_1...x_s \notin Im(i_A^*)$.) Hence $i_A^*(H^+(BG; \mathbb{Z}/p)) \in Ng(A)$. □

Here we give an example. The mod 3 cohomology of BF_4 is completely determined by Toda.

Theorem 7.7. *([Toda]) The cohomology $H^*(BF_4; \mathbb{Z}/3)$ is isomorphic to*

$$\mathbb{Z}/3[x_{36}, x_{48}] \otimes (\mathbb{Z}/3[x_4, x_8] \otimes \{1, x_{20}, x_{20}^2\} + \mathbb{Z}/3[x_{26}] \otimes \Lambda(x_9) \otimes \{1, x_{20}, x_{21}, x_{25}\})$$

where the above two terms have the intersection $\{1, x_{20}\}$.

Indeed, we see that $x_{26}|A = Q_0Q_1Q_2(u_3)$, $x_{36}|A = c_{3,1}$, $x_{48}|A = c_{3,2}$, $x_4|A = Q_0(u_3)$, $x_8|A = Q_1(u_3)$, $x_{20}|A = Q_2(u_3)$, $x_9|A = Q_0Q_1(u_3)$, $x_{21}|A = Q_0Q_2(u_3)$, $x_{25}|A = Q_1Q_2(u_3)$. Here $u_3 = x_1x_2x_3$ and x_{36} and x_{48} are represented by Chern classes.

8. $GL_n(\mathbb{F}_\ell)$ AND PGL_p

Of course, there is the isomorphism

$$H^{*,*'}(BGL_n;\mathbb{Z}/p) \cong H^{*,*'}[c_1,...,c_n]$$

where c_i is the Chern class. Hence $Inv^*(GL_n;\mathbb{Z}/p) \cong \mathbb{Z}/p$ (for $k=\mathbb{C}$). Let G be a finite group such that

$$H^*(BG;\mathbb{Z}/p) \cong \mathbb{Z}/p[c_{i_1},...,c_{i_n}] \otimes \Lambda(e_{i_1},...,e_{i_n}) \tag{8.1}$$

where $\beta_{j_s}(e_{i_s}) = c_{i_s}$ where β_{j_s} is the higher Bockstein operation, and c_{i_s} is a Chern class of some representation of G, e.g., $G = GL_n(\mathbb{F}_\ell)$ where ℓ is prime to p.

Theorem 8.1. *Let G be a finite group given as (8.1) so that $i_1 \geq 2$. Then $H^*(BG;\mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/p$.*

Proof. First note that $H^*(BG;\mathbb{Z}/p)/(Ng)$ is a quotient $\Lambda(e_{i_1},...,e_{i_n})$. Since the motivic cohomology has the transfer map, we know that each element x in $H^{*,*'}(BG;\mathbb{Z})$ has the exponent dividing $|G|$. Hence there is $e'_s \in H^{*,*-1}(BG;\mathbb{Z}/p)$ such that

$$\beta_{s'}(e'_s) = \tau^{i_s-2}c_{i_s}.$$

From Lemma 2.1, we know that $\tau : H^{*,*-1}(X;\mathbb{Z}/p) \to H^{*,*}(X;\mathbb{Z}/p)$ is injective. Hence $\tau e'_s \neq 0$. This means $\beta_{s''}(\tau e'_s) = \tau^{i_s-1}c_{i_s}$ for $s'' \leq s'$. So $s'' = i_s$ and we can take $\tau e'_s = e_{i_s}\ mod(Ideal(e_{i_1},...,e_{i_{s-1}}))$. By induction on s, we can prove all $e_{i_s} \in Ng$. □

Let p be an odd prime and denote by PGL_p the projective group which is the quotient of the general linear group GL_p by the center $\mathbb{G}_m$. Its ordinary $mod(p)$ cohomology and the Chow ring are known by Vistoli [Vi] and Kameko-Yagita [Ka-Ya].

To state the cohomology $H^*(BPGL_p;\mathbb{Z}/p)$, we recall the Dickson algebra. Let $A \cong (\mathbb{Z}/p)^n$ be an elementary abelian p-group of rank n, and $H^*(BA) \cong \mathbb{Z}/p[y_1,...,y_n] \otimes \Lambda(x_1,...,x_n)$. The Dickson algebra is

$$D_n = \mathbb{Z}/p[y_1,...,y_n]^{GL_n(\mathbb{F}_p)} \cong \mathbb{Z}/p[c_{n,0},...,c_{n,n-1}]$$

with $|c_{n,i}| = 2(p^n - p^i)$. The invariant ring under $SL_n(\mathbb{F}_p)$ is also given

$$SD_n = \mathbb{Z}/p[y_1,...,y_n]^{SL_n(\mathbb{F}_p)} \cong D_n\{1, e_n, ..., e_n^{p-2}\} \quad with\ e_n^{p-1} = c_{n,0}.$$

We also recall the Mui's result by using Q_i according to Kameko and Mimura [Ka-Mi]

$$grH^*(BA)^{SL_n(\mathbb{F}_p)} \cong SD_n/(e_n) \oplus SD_n \otimes Q(n-1)\{u_n\}$$

where $u_n = x_1...x_n$ and $e_n = Q_0...Q_{n-1}u_n$. (Here note

$$SD_n/(e_n) \cong D_n/(c_{n,0}) \cong \mathbb{Z}/p[c_{n,1},...,c_{n,n-1}].)$$

Theorem 8.2. *(*[Vi],[Ka-Ya]*) There is the isomorphism*

$$H^*(BPGL_p;\mathbb{Z}/p) \cong M \oplus N$$

where $M \cong \mathbb{Z}/p[x_4, x_6, \cdots, x_{2p}]$ *as modules (but not rings) and*

$$N \cong SD_2 \otimes Q(1)\{u_2\} \cong \mathbb{Z}/p[e_2, c_{2,1}] \otimes Q(1)\{u_2\}.$$

Theorem 8.3. *(*[Vi],[Ka-Ya]*) There is the additive isomorphism*

$$CH^*(BPGL_p)/p \cong M \oplus SD_2\{Q_0Q_1u_2\}.$$

It is also proved that $Ker(\tau)|H^{*,*'}(BPGL_p;\mathbb{Z}/p) = 0$ in Theorem 10.4 in [Ya3].

Theorem 8.4.

$$Inv^*(PGL_p;\mathbb{Z}/p) \cong H^*(G;\mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/2\{1,u\} \quad |u| = 2.$$

This fact is also shown in [Ga-Me-Se].

9. Symmetric group S_n

Let S_n be the Symmetric group generated by permutations of n-letters. The permutations induce the natural representation $S_n \to O_n$. Let us write by w_i its Stiefel-Whitney class. Then it is proved for example in [Ga-Me-Se] that for general k

Theorem 9.1. $Inv^*(S_n;\mathbb{Z}/2) \cong H^*\{1, w_1, ..., w_{[n/2]}\}$.

Let A be a subgroup of S_n generated by the transpositions $(2i-1, 2j)$ for $1 \le j \le [n/2]$ so that $A \cong \oplus^{[n/2]}\mathbb{Z}/2$. Then $Inv^*(S_n;\mathbb{Z}/2)$ is detected by the group A.

In this section, by using the cohomology $H^*(BS_n;\mathbb{Z}/2)$, we will re-explain above facts (for $H^*(BS_n;\mathbb{Z}/2)$) but when $k = \mathbb{C}$ and $n = 2^m$. (We assume $k = \mathbb{C}$.)

V.Voevodsky showed (for the definition of the reduced power operation)

$$H^{*,*'}(BS_p;\mathbb{Z}/p) \cong H^{*,*'}(B\mathbb{Z}/p;\mathbb{Z}/p)^{S_p} \cong \mathbb{Z}/p[Y] \otimes \Lambda(X)$$

where $Y = y^{p-1}$ and $X = y^{p-2}x$ in $H^{*,*'}(B\mathbb{Z}/p;\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau, y] \otimes \Lambda(x)$. Hence $H^*(BS_p;\mathbb{Z}/p)/(Ng) = \mathbb{Z}/p$ for $p \neq 2$ but $H^*(BS_2;\mathbb{Z}/2)/(Ng) \cong \Lambda(x)$.

Now we restrict $p = 2$ and let $n = 2^m$. Consider the natural embedding

$$g_m : V_m = \oplus^m\mathbb{Z}/2 \to \mathbb{Z}/2 \wr ... \wr \mathbb{Z}/2 \to S_{2^m}$$

where $- \wr -$ is the wreath product. Then it is known (Theorem 3.23 in [Ma-Mi]) that $H^*(BS_{2^m};\mathbb{Z}/2)$ is detected by $S_{2^{m-1}} \times S_{2^{m-1}}$ and V_m.

Let $f_m : V_m \xrightarrow{g_m} S_n \to O_n$. The the total Whitney class is written as

$$w(f_m) = 1 + w(m-1) + w(m-2) + ... + w(0)$$

where $w(i) = w_{2^m - 2^i}(f_m)$. Moreover $Sq^{2^i}(w(i+1)) = w(i)$. In fact the image of g_m^* is just the Dickson algebra

$$Im(g_m^*) = \mathbb{Z}/2[x_1, ..., x_m]^{GL_2(\mathbb{F}_2)} = \mathbb{Z}/2[w(m-1), ..., w(0)].$$

(Here $w(i)^2$ corresponds to $c_{m,i}$ for odd prime cases stated in the preceding section.)

By induction on m, we assume

$$H^*(BS_{2^{m-1}}; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_1, ..., w_{2^{m-2}}\}.$$

Then considering the restriction

$$H^*(BS_n; \mathbb{Z}/2)/(Ng) \to H^*(BS_{n-1}; \mathbb{Z}/2)/(Ng) \otimes H^*(BS_{n-1}; \mathbb{Z}/2)/(Ng),$$

we see

$$H^*(BS_n; \mathbb{Z}/2)/(Ng) \supset \mathbb{Z}/2\{1, w_1, ..., w_{2^{m-2}}, ..., w_{2^{m-1}}\}.$$

Note by construction of maps g_m, f_m, we see $g_m^* w_{2^{m-1}} = w(m-1)$. Hence

$$w(j)' = Sq^{2^j} ... Sq^{2^{m-2}} w_{2^{m-1}} \in Ng(S_n)$$

with $g_m^*(w(j)') = w(j)$. Thus we have

Proposition 9.2. $H^*(BS_n; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_1, ..., w_{2^{m-1}}\}$.

In [Ga-Me-Se], it is also shown $Inv^*(S_n; \mathbb{Z}/p) = \mathbb{Z}/p$ for odd prime p. Let $n = p^m$. The symmetric group S_n has a subgroup isomorphic to

$$S(m) = \mathbb{Z}/p \wr S_{p^{m-1}} \cong (S_{p^{m-1}})^p \rhd \mathbb{Z}/p$$

of the index prime to p. Hence $H^*(BS_n; \mathbb{Z}/p) \subset H^*(BS(m); \mathbb{Z}/p)$.

We consider the Hochshild-Serre spectral sequence

$$E_2^{*,*'} = H^*(\mathbb{Z}/p; H^*((BS_{p^{m-1}})^p; \mathbb{Z}/p)) \Longrightarrow H^*(BS(m); \mathbb{Z}/p).$$

Let us write by σ the generator of the cyclic group $\mathbb{Z}/p$. Let $T = (1-\sigma)$ and $N = (1+\sigma+...+\sigma^{p-1})$. For a $\mathbb{Z}/p$-module M, the homology is written

$$H^*(\mathbb{Z}/p; M) = \begin{cases} Ker(T) = M^{\mathbb{Z}/p} & * = 0 \\ Ker(T)/Im(N) & * = even > 0 \\ Ker(N)/Im(T) & * = odd. \end{cases}$$

Let $\{x_i\}$ be a $\mathbb{Z}/p$ basis of $H^*(BS_{p^{m-1}}; \mathbb{Z}/p)$. Then the basis of

$$S = H^*((BS_{p^{m-1}})^p; \mathbb{Z}/p) \cong \mathbb{Z}/p\{x_{i_1} \otimes x_{i_2} \otimes ....x_{i_p}\}.$$

decomposes as $I \cup F$ with $I = \{x_i \otimes ... \otimes x_i\}$ and

$$F = \{x_{i_1} \otimes ... \otimes x_{i_s} | i_k \neq i_\ell \ for \ \ some \ k \neq \ell\}.$$

The generator σ acts on F freely but invariants on I. Then cohomology is computed

$$H^*(\mathbb{Z}/p; \mathbb{Z}/p\{I\}) \cong \mathbb{Z}/p[y] \otimes \Lambda(x) \otimes \mathbb{Z}/p\{I\} \quad |x| = 1, \ |y| = 2$$

$$H^*(\mathbb{Z}/p; \mathbb{Z}/p\{F\}) \cong \mathbb{Z}/p\{F\}^{\mathbb{Z}/p}.$$

Since $S^{\mathbb{Z}/p} = \mathbb{Z}/p\{F\}^{\mathbb{Z}/p} \oplus \mathbb{Z}/p\{I\}$, we have the isomorphism

$$E_2^{*,*'} \cong S^{\mathbb{Z}/p} \oplus \mathbb{Z}/p\{I\} \otimes \mathbb{Z}/p[y]\{x, y\}.$$

It is well known that $S^{\mathbb{Z}/p} \subset H^*(BG; \mathbb{Z}/p)$ by Nakaoka (also Totaro [To])). Hence this spectral sequence collapses from the E_2-term. Thus we have the well known result ;

Theorem 9.3. *(Nakaoka)*

$$H^*(BS(m); \mathbb{Z}/p) \cong E_2^{*,*'} \cong S^{\mathbb{Z}/p} \oplus H^*(BS_{p^{m-1}}; \mathbb{Z}/p)^{[p]} \otimes \mathbb{Z}/p[y]\{x, y\}$$

where $A^{[p]}$ is the graded algebra whose degree is given by p-times of degree of A.

By induction on m, we may assume $H^*(BS_{p^{m-1}}; \mathbb{Z}/p)/(Ng) = \mathbb{Z}/p$. Hence $\mathbb{Z}/p\{I\}/(Ng) = \mathbb{Z}/p$. Therefore there is the surjection

$$\mathbb{Z}/p\{1, x\} \to H^*(BS; \mathbb{Z}/p)/(Ng).$$

But $H^2(BS_n; \mathbb{Z}/p) = 0$. Hence $H^+(BS_n; \mathbb{Z}/p)/(Ng) = 0$.

Proposition 9.4. $H^*(BS_{p^m}; \mathbb{Z}/p)/(Ng) = \mathbb{Z}/p$.

We note here about the restriction image for $g_m : V_m = \mathbb{Z}/p \to S_n$. The restriction image is contained in the Dickson algebra as stated in the preceding section

$$grH^*(BA)^{GL_n(\mathbb{F}_p)} \cong D_n/(c_{n,0}) \oplus D_n \otimes Q(n-1)\{e_n^{p-2}u_n\}$$

where $u_n = x_1...x_n$ and $e_n = Q_0...Q_{n-1}u_n$. (Here note $D_n/(c_{n,0}) = \mathbb{Z}/p[c_{n,1}, ..., c_{n,n-1}]$.) We know D_n is in the image of g_m^*. However it is known that $e_n^{p-2}u_n \notin g_m^*$ for $n \geq 3$ (p.196 in [Ad-Mi]).

10. EXTRASPECIAL p-GROUPS

We assume that p is an odd prime. The extraspecial p-group $E_n = p_+^{1+2n}$ is the group such that exponent is p, its center is $C \cong \mathbb{Z}/p$ and there is the extension

$$0 \longrightarrow C \xrightarrow{i} E_n \xrightarrow{\pi} V \longrightarrow 0$$

with $V = \oplus^{2n}\mathbb{Z}/p$. (For details of the cohomology of E_n see [Te-Ya].) We can take generators $a_1, ..., a_{2n}, c \in E_n$ such that $\pi(a_1), .., \pi(a_{2n})$ (resp. c) make a base of V (resp. C) such that

$$[a_{2i-1}, a_{2i}] = c \quad and \quad [a_{2i-1}, a_j] = 1 \;\; if \; j \neq 2i.$$

We note that E_n is also the central product of the n-copies of E_1

$$E_n \cong E_1 \cdots E_1 = E_1 \times_{\langle c \rangle} E_1 ... \times_{\langle c \rangle} E_1.$$

Take cohomologies

$$H^*(BC; \mathbb{Z}/p) \cong \mathbb{Z}/p[u] \otimes \Lambda(z), \quad \beta z = u,$$

$$H^*(BV; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, ..., y_{2n}] \otimes \Lambda(x_1, ...x_{2n}), \quad \beta x_i = y_i,$$

identifying the dual of a_i (resp.c) with x_i (resp. z). That means

$$H^1(E_n; \mathbb{Z}/p) \cong Hom(E_n; \mathbb{Z}/p) \ni x_i : a_j \mapsto \delta_{ij}.$$

The central extension is expressed by

$$f = \sum_{i=1}^{n} x_{2i-1}x_{2i} \in H^2(BV; \mathbb{Z}/p).$$

Hence $\pi^* f = 0$ in $H^2(BE_n; \mathbb{Z}/p)$. We consider the Hochshild-Serre spectral sequence

$$E_2^{*,*'} \cong H^*(BV; \mathbb{Z}/p) \otimes H^*(BC; \mathbb{Z}/p) \Longrightarrow H^*(BE_n; \mathbb{Z}/p).$$

Hence the first nonzero differential is $d_2(z) = f$ and the next differential is

$$d_3(u) = d_3(Q_0(z)) = Q_0(f) = \sum y_{2i-1}x_{2i} - y_{2i}x_{2i-1}.$$

In particular

$$E_4^{0,*} \cong \mathbb{Z}/p[y_1, ..., y_{2n}] \otimes \Lambda(x_1, ...x_{2n})/(f, Q_0(f)).$$

Lemma 10.1. *We have the inclusion*

$$\Lambda(x_1, ..., x_{2n})/(f) \subset H^*(BE_n; \mathbb{Z}/p).$$

Proof. We consider similar group E'_n such that its center is $C \cong \mathbb{Z}/p$ and there is the extension

$$0 \longrightarrow C \xrightarrow{i} E'_n \xrightarrow{\pi} V' \longrightarrow 0$$

but $V' = \oplus^{2n}\mathbb{Z}_p$ such that there is the quotient map $q : E'_n \to E_n$. We also consider the spectral sequence

$$E_2^{*,*'} \cong H^*(BV';\mathbb{Z}/p) \otimes H^*(BC;\mathbb{Z}/p) \Longrightarrow H^*(BE'_n;\mathbb{Z}/p).$$

Here $H^*(BV';\mathbb{Z}/p) \cong \Lambda(x_1, ...x_{2n})$. The first nonzero differential is $d_2(z) = f$ but the second differential is

$$d_3(u) = \sum y_{2i-1}x_{2i} - y_{2i}x_{2i-1} = 0.$$

Hence we have

$$H^*(BE'_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[u] \otimes \Lambda(x_1, ..., x_{2n})/(f).$$

From the map $q^* : H^*(BE_n;\mathbb{Z}/p) \to H^*(BE'_n;\mathbb{Z}/p)$, we get the result. □

However $H^*(BE_n;\mathbb{Z}/p)/(Ng) \ncong \Lambda(x_1, ..., x_{2n})/(f)$, infact, when $n = 1$, from Theorem 3.3 in [Ya3] we see

Proposition 10.2.

$$H^*(Bp_+^{1+2};\mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/p\{1, x_1, x_2, a'_1, a'_2\} \quad deg(a'_i) = 2.$$

When $p = 2$, the situation becomes well. The extraspecial 2-group $D(n) = 2_+^{1+2n}$ in the n-th central extension of the dihedral group D_8 of order 8. It has the central extension

$$0 \to \mathbb{Z}/2 \to D(n) \to V \to 0$$

with $V = \oplus^{2n}\mathbb{Z}/2$. Hence $H^*(BV;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, ..., x_{2n}]$. Then using the Hochschild-Serre spectral sequence, Quillen proved [Qu]

$$H^*(BD(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, ..., x_{2n}]/(f, Q_0(f), ..., Q_{n-2}(f)) \otimes \mathbb{Z}[w_{2^n}(\Delta)].$$

(In fact when n is the real case (i.e., $n = -1, 0, 1\ mod(8)$), the cohomology $H^*(BSpin_n;\mathbb{Z}/2)$ injects into $H^*(BD(n);\mathbb{Z}/2)$.) Here w_i (resp. $w_{2^n}(\Delta)$) is the Stiefel-Whitney class of usual representation from the above extension (resp. 2^n-dimensional representation which restrict nonzero on the center). Moreover Quillen proves following two theorems (Theorem 5.10-11 in [Qu])

Theorem 10.3. *([Qu]) $H^*(BD(n);\mathbb{Z}/2)$ is detected by the product of cohomoloogy of maximal elementary abeian groups.*

Theorem 10.4. *([Qu]) The nonzero Stiefel-Whitney $w_i(\Delta)$ are those of degrees 2^n and $2^n - 2^i$ for $0 \le i < n$.*

In fact $w_i(\Delta)$ generates the Dickson algebra in the cohomology of the maximal elementary abelian 2-group as stated in the preceding section.

Proposition 10.5. *When $n > 2$, there is the surjection*

$$\Lambda(x_1, ..., x_{2n})/(f) \to H^*(BD(n); \mathbb{Z}/2)/(Ng).$$

Proof. By the same arguments with $p = odd$, we see

$$\Lambda = \Lambda(x_1, ..., x_{2n})/(f) \subset H^*(BD(n); \mathbb{Z}/2).$$

The fact $w_2(\Delta) = 0$ follows from the above second Quillen's theorem. Hence we have $w_{2^n}(\Delta) \in Ng$ from Becher's theorem (Theorem 6.2). Thus we get the theorem. □

Let $0 \neq x \in \Lambda$. Then by the Quillen's theorem, $i_A^*(x) \neq 0$ for

$$i_A^* : H^*(BD(n); \mathbb{Z}/2) \to H^*(BA; \mathbb{Z}/2)$$

with $A \cong \mathbb{Z}/2 \oplus ... \oplus \mathbb{Z}/2$. Let $H^*(BA; \mathbb{Z}/2)/(Ng) \cong \Lambda(x'_1, ..., x'_m)$. Then $i_A^*(x_i) \in \mathbb{Z}/2\{x'_1, ..., x'_m\}$ and hence the map is restricted

$$i_A^* : \Lambda(x_1, ..., x_{2n})/(f) \to \Lambda(x'_1, ..., x'_m).$$

However this map is not need injective. In fact there is a possibility of $i_A^*(x) \in Ng(A)$, e.g., $i_A^*(x_1x_2) = (x'_1)^2 \in Ng(A)$.

11. UNRAMIFIED THEORY

In this section, we assume that k is an *algebraic closed field* of $ch(k) = 0$.

Let K be a function field of k, that is finitely generated as a field over k. Here we recall the definition of unramifed cohomology of $H^*(K; \mathbb{Z}/p)$ according to Saltman, Peyre and Colliot-Thelene. We denote by $P(K/k)$ the set of discrete valuation rings A of rank one such that $k \subset A \subset K$ and that the fraction field $Fr(A)$ of A is K. If A belongs to $P(K/k)$, then for the residue field κ_A, we can define the residue map $\partial_A : H^*(K; \mathbb{Z}/p) \to H^{*-1}(\kappa_A; \mathbb{Z}/p)$ as follows.

Let $\hat{K}_A$ be the completion, $\hat{K}_A^{nr}$ the maximal unramified extension of K_A and $\bar{K}_A$ is an algebraic closure of K_A. Put $I_A = Gal(\bar{K}_A/\hat{K}_A^{nr})$ and $G_A = Gal(\bar{K}_A/\hat{K}_A)$.

$$\bar{K}_A \overset{I_A}{--} \hat{K}_A \overset{G_A/I_A}{--} \hat{K}_A -- K.$$

Then ∂_A is defined as the composition of maps

$$\partial : H^*(K; \mathbb{Z}/p) \to H^*(\hat{K}_A; \mathbb{Z}/p)$$

$$\overset{proj.}{\to} H^{*-1}(G_A/I_A) \otimes H^1(I_A; \mathbb{Z}/p) \cong H^{*-1}(\kappa_A; \mathbb{Z}/p).$$

Here we used that $\hat{K}_A \cong I_A \oplus (G_A/I_A)$ ([Sa]) and $I_A \cong \hat{\mathbb{Z}}$. Moreover $H^*(\hat{\mathbb{Z}};\mathbb{Z}/p) \cong \mathbb{Z}/p$ if $* = 0,1$ and $\cong 0$ otherwise.

Then we can define the unramified cohomology

$$H^*_{nr}(K;\mathbb{Z}/p) = \cap_{A\in P(K/k)} Ker(H^*(K;\mathbb{Z}/p) \xrightarrow{\partial_A} H^{*-1}(\kappa_A;\mathbb{Z}/p)).$$

Namely, when X is complete, the residue map is the same as the differential d_1 of the coniveau spectral sequence given in §2, and hence the unramified cohomology is just the E_2-term $H^0_{Zar}(X;H^m_{\mathbb{Z}/p})$ of the coniveau spectral sequence.

Corollary 11.1. *When X is complete, there is the isomorphism*

$$H^*_{nr}(k(X);\mathbb{Z}/p) \cong H^{*,*}(X;\mathbb{Z}/p)/(\tau) \oplus Ker(\tau)|H^{*+1,*-1}(X;\mathbb{Z}/p).$$

Now we consider the case $X = BG$; non complete cases. Let $W//G$ be the scheme which has the coordinate ring $k[W]^G$. Then it is known that $W//G$ contains U_n/G as an open set. Here U_n is the open set given in §3, where G acts freely. Hence we have for $* < n$

$$H^0_{Zar}(W//G;H^*_{\mathbb{Z}/p}) \subset H^0_{Zar}(U_n/G;H^*_{\mathbb{Z}/p}).$$

Since G is reductive, it is also known that the fraction field of $k[W]^G$ is $k(W)^G$, that is $k(W//G) = k(W)^G$. Thus we get

Lemma 11.2. *For $* < n$, we have*

$$\begin{aligned} H^*_{nr}(K(W)^G;\mathbb{Z}/p) &= H^0_{Zar}(W//G;H^*_{\mathbb{Z}/p}) \\ &\subset H^0(U_n/G;H^*_{\mathbb{Z}/p}) = H^0(BG;H^*_{\mathbb{Z}/p)}). \end{aligned}$$

According to Peyre [Pe], we define a subring $H^*_{nr}(G;\mathbb{Z}/p)$ of $H^*_{et}(BG;\mathbb{Z}/p)$ as follows. Let $P(G)$ be the set of elements $g \in G$ such that $I = \langle g\rangle \cong \mathbb{Z}/p^s$ for some $s \geq 1$ but $g \neq h^p$ for any $h \in G$. Then the centralizer is written as

$$Z_G(I) \cong I \oplus H \qquad with\ H = Z_G(I)/I.$$

Let us write by ∂_g the composition map

$$\partial_g : H^*(BG;\mathbb{Z}/p) \to H^*(BZ_G(I);\mathbb{Z}/p)$$

$$\cong H^{*-*'}(BH;\mathbb{Z}/p) \otimes H^{*'}(BI;\mathbb{Z}/p) \xrightarrow{proj.} H^{*-1}(BH;\mathbb{Z}/p)$$

using $H^1(BI;\mathbb{Z}/p) \cong \mathbb{Z}/p$. Then define the unramified cohomology by

$$H^*_{nr}(G;\mathbb{Z}/p) = \bigcap_{g\in P(G)} Ker(H^*(BG;\mathbb{Z}/p) \xrightarrow{\partial_g} H^{*-1}(BH;\mathbb{Z}/p)).$$

Remark. The restriction map $H^1(B\langle g\rangle;\mathbb{Z}/p) \to H^1(B\langle g^p\rangle;\mathbb{Z}/p)$ is always zero. Hence we need not consider the case $I = \langle g^p\rangle$.

Theorem 11.3. *(Peyre [Pe]) Let W be a faithful representaion of G. Let q is the quotient map $q : Gal(k(\bar{W})/k(W)^G) \to Gal(k(W)/k(W)^G) = G$. Then*

$$q^*(H^*_{nr}(G;\mathbb{Z}/p)) \subset H^*_{nr}(k(W)^G;\mathbb{Z}/p).$$

Proof. Arguments of pages 203 to 206 in the proof of Proposition 3 in [Pe] work exchanging $H^3(-;\mathbb{Q}/\mathbb{Z}(-))$ to $H^*(-;\mathbb{Z}/p)$. Indeed we have the commutative diagram ((12) in [Pe])

$$\begin{array}{ccc} H^*(G;\mathbb{Z}/p) & \xrightarrow{\partial_{D,g}} & H^{*-1}(D;\mathbb{Z}/p) \\ \downarrow & & \downarrow \\ H^*(k(W)^G;\mathbb{Z}/p) & \xrightarrow{\partial_A} & H^{*-1}(\kappa_A;\mathbb{Z}/p) \end{array}$$

where D is the decomposition subgroup of G. If $x \in H^*_{nr}(G;\mathbb{Z}/p)$, then $\partial_{D,g}(x) = 0$ and hence $\partial_A(\rho^*(x)) = 0$. □

It is well known (Theorem 4.1.5 in [Co-Te]) that if K is purely transcendental over k (i.e., $K \cong k(x_1, ..., x_n)$ for indeterminate x_i), then

$$H^*(k;\mathbb{Z}/p) \cong H^*_{nr}(K;\mathbb{Z}/p) \quad for * > 0.$$

When k is algebraic closed field, it is immediate $H^+(k;\mathbb{Z}/p) = 0$.

Corollary 11.4. *Suppose that*

$$0 \neq x \in H^*(BG;\mathbb{Z}/p)/(Ng) \quad and \quad x \in H^*_{nr}(G;\mathbb{Z}/p).$$

Then $q^*(x) \neq 0$ in $H^*_{nr}(k(W)^G;\mathbb{Z}/p)$. Hence $k(W)^G/k$ is not purely trascendental.

Proof. From the preceding theorem, we have

$$q^*(x) \in H^*_{nr}(k(W)^G;\mathbb{Z}/p) \subset H^*(k(W)^G;\mathbb{Z}/p).$$

It is nonzero since

$$H^*(BG;\mathbb{Z}/p)/(Ng) \subset H^0_{Zar}(BG;H^*_{\mathbb{Z}/p}) \subset H^*(k(W)^G;\mathbb{Z}/p).$$

□

Corollary 11.5. *Let G be a $p-$group of exponent p. If $H^2(BG;\mathbb{Z}/p)/(Ng)$ is not detected by $\mathbb{Z}/p \times \mathbb{Z}/p$, then $k(W)^G$ is not purely transcendental.*

Proof. Suppose that $x \notin H^2_{un}(G;\mathbb{Z}/p)$. Then we can take $x = \sum x_1 x_2$ such that $x_1 | H^1(BI;\mathbb{Z}/p) \neq 0$ with $I \cong \mathbb{Z}/p$ and for $H = Z_G(I)/I$, $x_2 | H^1(BH;\mathbb{Z}/p) \neq 0$. Here

$$H^1(BH;\mathbb{Z}/p)/(Ng) = H^1(BH;\mathbb{Z}/p) \cong Hom(H,\mathbb{Z}/p).$$

Hence x_2 defines subgroup $J \cong \mathbb{Z}/p$ of H such that $x_2 | H^1(BJ;\mathbb{Z}/p) \neq 0$ (since H is exponent p). Thus the element x is detected by the subgroup $I \oplus J \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$. □

12. Saltman's example

Let G be the group defined by

$$0 \to \langle c_3, c_4 \rangle \to G \to E_2 = p_+^{1+4} \to 0$$
$$with \quad [a_1, a_3] = c_3, \quad and \quad [a_1, a_4] = c_4.$$

Saltman and Bogomolov showed that $H^2_{nr}(k(W)^G; \mathbb{Q}/\mathbb{Z}) \neq 0$ for this group. We will see the $\mathbb{Z}/p$-coefficient case.

Lemma 12.1. *The 3-dimensional cohomology $H^3(BG; \mathbb{Z}/p)$ contains the $\mathbb{Z}/p$-module*

$$A = \mathbb{Z}/p\{y_i x_j | 1 \leq i, j \leq 4\}/(Q_0(f), Q_0(x_1x_3), Q_0(x_1x_4)).$$

Proof. Consider the central extension

$$0 \to \langle c, c_3, c_4 \rangle \to G \to V = \oplus^4 \mathbb{Z}/p \to 0$$

and induced spectral sequence

$$E_2^{*,*'} \cong \mathbb{Z}/p[y_1, ..., y_4, u, u_3, u_4] \otimes \Lambda(x_1, ..., x_4, z, z_3, z_4)$$

converging to $H^*(BG; \mathbb{Z}/p)$. The first differential is

$$d_2(z) = f, \quad d_2(z_3) = x_1x_3, \quad d_2(z_4) = x_1x_4.$$

The second nonzero differential is

$$d_3(u) = Q_0(f), \quad d_3(u_3) = Q_0(x_1x_3), \quad d_3(u_4) = Q_0(x_1x_4).$$

Thus we see

$$A \cong E_4^{3,0} \cong E_\infty^{3,0} \subset H^3(BG; \mathbb{Z}/p).$$

□

Theorem 12.2. *(Saltman [Sa]) We have*

$$0 \neq x_1x_2 \in H^2_{nr}(G; \mathbb{Z}/p) \cap H^{2,2}(BG; \mathbb{Z}/p)/(\tau).$$

Hence $k(W)^G$ is not purely transcendental.

Proof. From the preceding lemma, we see $Q_0(x_1x_2) \neq 0$ in $H^3(BG; \mathbb{Z}/p)$. Hence we see $x_1x_2 \notin Ng(G)$ from Lemma 3.4.

Next we will show $x_1x_2 \in H_{nr}(G; \mathbb{Z}/p)$. Suppose this is not the case. By the definition, this means that there is an element g and $h \in Z_G(\langle g \rangle)$ such that

$$x_i|\langle g \rangle \neq 0, \quad x_j|\langle h \rangle \neq 0 \quad for\ \{i, j\} = \{1, 2\}.$$

Let us write

$$a^\Lambda = a_1^{\lambda_1} ... a_4^{\lambda_4} \quad for\ \Lambda = (\lambda_1, ..., \lambda_4),$$
$$a^M = a_1^{\mu} ... a_4^{\mu_4} \quad for\ M = (\mu_1, ..., \mu_4).$$

Note that $x_i|\langle a^\Lambda\rangle = \lambda_i$. The commutator is given by the definition

$$[a^\Lambda, a^M] = c^d c_3^{d_3} c_4^{d_4} \quad where \;\; d = \lambda_1\mu_2 - \lambda_2\mu_1 + \lambda_3\mu_4 - \lambda_4\mu_3,$$
$$d_3 = \lambda_1\mu_3 - \lambda_1\mu_3, \qquad d_4 = \lambda_1\mu_4 - \lambda_4\mu_1.$$

Take $g = a^\Lambda$. Exchanging $x_1 \mapsto x_1 + \lambda x_2$ or $x_1 \mapsto x_2$ (if necessary), we can take

$$\lambda_1 = 1, \lambda_2 = 0 \quad \Lambda = (1, 0, \lambda_3, \lambda_4).$$

(Hence $x_1\langle g\rangle = 1$ and $x_2|\langle g\rangle = 0$.) Then we can take $h = a^M$ so that $x_2|\langle h\rangle = 1$ and $h \notin \langle g\rangle$, that means

$$\mu_1 = 0, \;\; \mu_2 = 1 \quad M = (0, 1, \mu_3, \mu_4).$$

From $d_3 = 0$ and $d_4 = 0$, we see

$$\mu_3 = \lambda_3\mu_1 = 0 \quad and \quad \mu_4 = \lambda_4\mu_1 = 0$$

(and hence $M = (0, 1, 0, 0)$). Therefore $d = 1 \times 1 - 0 + 0 - 0 = 1$. This is a contradiction. Hence we have proved $x_1x_2 \in H_{nr}(G; \mathbb{Z}/p)$. □

13. NON DETECTED EXAMPLES

We consider group G'_m and element $\xi_m \in H^*(BG'_m; \mathbb{Z}/p)/(Ng)$ which is not detected by elementary abelina p-groups, while it is not in $H_{nr}(G'_m; \mathbb{Z}/p)$.

Let G' be the group defined by

$$0 \to \langle c_3, c_4\rangle \to G' \to E_2 = p_+^{1+4} \to 0$$
$$with \quad [a_1, a_3] = c_3, \quad and \quad [a_2, a_4] = c_4.$$

Difference of the definitions of G and G' is just $[a_2, a_4] = c_4$. However note G' has the rank 2 elementary abeilan p-subgroup

$$\langle a_1a_3^{-1}, a_2a_4^{-1}\rangle \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

while G does not.

Let G'_m be the central product of $m-1$-th copies of G' and one G

$$G'_m = G \cdot G' \cdot ... \cdot G' = G \times_{\langle c\rangle} G' \times_{\langle c\rangle} ... \times_{\langle c\rangle} G'$$

which is generated by $a_1, ..., a_{4m}$, c, $c_3, c_4, c_7, ..., c_{4m-1}, c_{4m}$. The comutators are given for $1 \le i \le m$

$$[a_{4i-3}, a_{4i-2}] = [a_{4i-1}, a_{4i}] = c,$$
$$[a_{4i-3}, a_{4i-1}] = c_{4i-1}, \quad and \quad [a_{4i-2}, a_{4i}] = c_{4i} \;\; for \;\; i \neq 1,$$
$$but \quad [a_1, a_4] = c_4.$$

Let us write $a^\Lambda = a_1^{\lambda_1}...a_{4m}^{\lambda_{4m}}$ and $\Lambda = (\lambda_1, ..., \lambda_{4m})$. Note also $x_i|\langle a^\Lambda\rangle = \lambda_i$. The commutator is given by the definition

$$[a^\Lambda, a^M] = c^d c_3^{d_3} c_4^{d_4} .... c_{4m}^{d_{4m}}$$

$$where \;\; d = \sum_i \lambda_{4i-3}\mu_{4i-2} - \lambda_{4i-2}\mu_{4i-3} + \lambda_{4i-1}\mu_{4i} - \lambda_{4i}\mu_{4i-1},$$

$$d_{4i-1} = \lambda_{4i-3}\mu_{4i-1} - \lambda_{4i-1}\mu_{4i-3}, \qquad d_{4i} = \lambda_{4i-2}\mu_{4i} - \lambda_{4i}\mu_{4i-2} \; (i \neq 1),$$
$$and \qquad d_4 = \lambda_1\mu_4 - \lambda_4\mu_1.$$

Lemma 13.1. *Let $\xi_m = x_1x_2x_5x_6....x_{4m-3}x_{4m-2} \in H^*(BG'_m;\mathbb{Z}/p)$. Then $\xi_m \in H^*(BG'_n;\mathbb{Z}/p)/(Ng)$ is not detected by elementary abelian $p-groups$.*

Proof. Suppose that ξ_m is detected by

$$\langle g_1, g_2, ..., g_{2m}\rangle \cong \mathbb{Z}/p \oplus ... \oplus \mathbb{Z}/p$$

such that $\xi_m|\langle g_1, ..., g_{2m}\rangle \neq 0$, and $x_i|\langle g_i\rangle \neq 0$ for some i. Take generators adequately, let $g_i = a^{\Lambda_i}$ such that

$$\Lambda_1 = (1, 0, *, *|0, 0, *, *|....|0, 0, *, *),$$
$$\Lambda_2 = (0, 1, *, *|0, 0, *, *|....|0, 0, *, *),$$
$$\Lambda_5 = (0, 0, *, *|1, 0, *, *|....|0, 0, *, *),$$
$$\Lambda_6 = (0, 0, *, *|0, 1, *, *|....|0, 0, *, *),$$

.....

$$\Lambda_{4m-3} = (0, 0, *, *|0, 0, *, *|....|1, 0, *, *),$$
$$\Lambda_{4m-2} = (0, 0, *, *|0, 0, *, *|....|0, 1, *, *).$$

Let $i > 1$. Consider the commutativity of $L = \Lambda_{4i-3}$ and $M = \Lambda_1$. Since $d_{4i-1} = \lambda_{4i-3}\mu_{4i-1} - \lambda_{4i-1}\mu_{4i-3} = 0$, we see

$$\mu_{4i-1} = 0 \quad from \; \mu_{4i-3} = 0 \; and \; \lambda_{4i-3} = 1.$$

Similarly we have $\mu_{4i} = 0$ from the commutativity of Λ_{4i-2} and Λ_1. Thus we see

$$\Lambda_1 = (1, 0, *, *|0, 0, 0, 0|....|0, 0, 0, 0).$$

We also have $\Lambda_2 = (0, 1, *, *|0, 0, 0, 0|....|0, 0, 0, 0)$. Let $\Lambda = \Lambda_1$ and $M = \Lambda_2$. By the commutativity and facts $d_3 = d_4 = 0$, we see $\mu_3 = \mu_4 = 0$, that is,

$$\Lambda_2 = (0, 1, 0, 0|0, 0, 0, 0|....|0, 0, 0, 0).$$

However this is a contradiction, indeed,

$$d = \sum_i \lambda_{4i-3}\mu_{4i-2} - \lambda_{4i-2}\mu_{4i-3} + \lambda_{4i-1}\mu_{4i} - \lambda_{4i}\mu_{4i-1} = 1 \neq 0.$$

□

Lemma 13.2. *$Q_0...Q_{2m-2}(\xi_m) \neq 0$ in $H^*(BG_m;\mathbb{Z}/p)$.*

Proof. We recall that G'_m is the central product of G and copies of G', that means

$$0 \to \oplus^m \mathbb{Z}/p \to G \times G' \times ... \times G' \stackrel{pr.}{\to} G'_m \to 0.$$

The map induces the map of cohomologies

$$pr.^* : H^*(BG'_m; \mathbb{Z}/p) \to H^*(BG; \mathbb{Z}/p) \otimes H^*(BG'; \mathbb{Z}/p)^{\otimes n}.$$

The operation Q_i is derivative, we have

$$Q_0 ... Q_{2m-2}(\xi_m) =$$

$$Q_0(x_1x_2)Q_1Q_2(x_5x_6)...Q_{2m-3}Q_{2m-2}(x_{4m-3}x_{4m-2}) + ...$$

Here from Lemma 12.1,

$$Q_1(x_1x_2) = y_1x_2 - y_2x_1 \neq 0 \quad in\ H^*(BG; \mathbb{Z}/p).$$

For $i > 1$, we see

$$Q_{2i-3}Q_{2i-2}(x_{4i-3}x_{4i-2}) = y_{4i-3}^{p^{2i-2}}\, y_{4i-2}^{p^{2i-3}} - y_{4i-3}^{p^{2i-3}}\, y_{4i-2}^{p^{2i-2}}.$$

This is nonzero in $H^*(BG'; \mathbb{Z}/p)$ because it is nonzero in the restriction image

$$H^*(BG'; \mathbb{Z}/p) \to H^*(B\langle a_1a_3^{-1}, a_2a_4^{-1}\rangle; \mathbb{Z}/p) \cong \mathbb{Z}/p[y'_1, y'_2] \otimes \Lambda(x'_1, x'_2)$$

where $y_{4i-3}, y_{4i-1} \mapsto y'_1$ and $y_{4i-2}, -y_{4i} \mapsto y'_2$. □

Corollary 13.3. *Elements in $H^{2m}(BG'_m; \mathbb{Z}/p)/(Ng)$ is not detected by abelian subgroups, namely, the restriction map*

$$Res : H^{2m}(BG'_m; \mathbb{Z}/p)/(Ng) \to \Pi_{A;abelian} H^{2m}(BA; \mathbb{Z}/p)/(Ng)$$

is not injective.

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