NOTE ON WITT GROUP AND KO-THEORY OF COMPLEX GRASSMANNIANS

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ABSTRACT. For a complex Grassmannian X, there is the isomorphism between the Balmer's Witt group and the quotient of topological K-theories so that $W^*(X) \cong KO^{2*}(X)/KU^{2*}(X)$.

1. INTRODUCTION

Let X be a smooth variety over a field k with $1/2$. The Witt group $W(X)$ is the quotient of the Grothendieck group of vector bundles with quadratic forms over X, by subbundles V with quadratic forms which admit Lagragian subbundles E (i.e., E is its own orthogonal complement in V).

Hence when $k = \mathbb{C}$, there is the natural map

$$W(X) \to KO^0(X(\mathbb{C}))/KU^0(X(\mathbb{C})).$$

Here $KO^0(-)$ and $KU^0(-)$ is the usual (topological) real and complex K-theories. One purpose of this paper is to show that this map is isomorphic when $X = M_{m,n}$ the complex Grassmannian of m-planes in an $m+n$-plane. Moreover, we have the isomorphism

$$W^*(X) \cong KO^{2*}(X(\mathbb{C}))/KU^{2*}(X(\mathbb{C}))$$

where $W^*(X)$ is the Balmer's Witt group with $W^0(X) = W(X)$.

The right hand side of the above isomorphism is computed explicitly by Hara and Hara-Kono [Ha],[Ha-Ko] by using Atiyah-Hirzebruch spectral sequence, here the computation of Sq^2 is most important.

On the other hand, $W^*(X)$ is given recently by Balmer-Calmes [Ba-Ca2] in complete general forms, using $m \times n$-framed *even* Young diagrams. However we can also get the results (for $k = \mathbb{C}$) by using Pardon and Balmer-Walter spectral sequences by using the computation of Sq^2 by Hara-Kono. The another purpose of this paper is to explain the relation between the results by Balmer-Calmes and Hara-Kono.

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2. KO-theory

We explain the KO-theory of Garassmannian according to Hara [Ha] and Hara-Kono [Ha-Ko]. Let $M_{m,n}$ be the complex Grassmannian $GL_m(\mathbb{C}^{m+n})$ of m-planes in $\mathbb{C}^{m+n}$. Then there is the homeomorphism

$$M_{m+n} \cong U(m+n)/(U(m)\times U(n)).$$

By using the Serre spectral sequence induced from the fiber sequence

$$U(m+n)/U(m)\times U(n) \to BU(m)\times BU(n) \to BU(m+n),$$

we get the cohomology for any field K

$$(2.1) \quad H^*(M_{m,n};K) \cong K[a_1,...,a_m,b_1,...,b_n]/(c_1,...,c_{m+n})$$

where a_i, b_j, c_k are Chern classes induced from maps in the above fibering, and $c_i = \sum a_{i-j}b_j$. (See also [Fu], [La] and the arguments in §4 bellow.)

Recall the coefficient ring of the (topological) KO^*-theory is

$$KO^* \cong \mathbb{Z}[\mu,\mu^{-1},\eta,w]/(2\eta,\eta^3,w^2-4\mu)$$

with $|\mu| = -8, |w| = -2, |\eta| = -1$. To compute $KO^*(M_{m,n})$, we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(M_{m,n};KO^*) \Longrightarrow KO^*(M_{m,n}).$$

It is well known that the first differential is ([Ha])

$$d_2(x) = \eta\otimes Sq^2(\bar{x})$$

where $\bar{x}\in H^*(M_{m,n};\mathbb{Z}/2)$ is the mod 2 reduction of x. The Squaring operation is well known (from Wu formula) [Ha]

$$Sq^2(a_1) = a_1^2, \quad Sq^2(a_{2i}) = a_{2i+1} + a_{2i}a_1 \quad for\ i\geq 1.$$

Let $H(m,n)$ be the homology $H(H^*(M_{m,n};\mathbb{Z}/2);Sq^2)$ with the differential Sq^2. Hara and Kono get this homology

Theorem 2.1. *(Hara-Kono [Ha-Ko]) Let $B(k,l)$ be the graded algebra*

$$B(k,l) = \mathbb{Z}/2[a_2^2,...,a_{2k}^2,b_2^2,...,b_{2l}^2)/(c_2^2,...,c_{2k+2k}^2).$$

Then we have the isomorphism

$$H(m,n) \cong \begin{cases} B(k,l) \quad if\ (m,l) = (2k,2k),\ (2k+1,2l),\ (2k,2l+1) \\ B(k,l)\oplus B(k,l)\{a_{2k+1}b_{2l}\} \quad if\ (m,l) = (2k+1,2l+1). \end{cases}$$

They also proved

Theorem 2.2. *([Ha-Ko]) The Atiyah-Hirebruch spectral sequence collapses from $E_3^{*,*'}$-term.*

For the proof of this theorem, Hara and Kono used the natural maps $U(n) \to Sp(n)$ and $Sp(n) \to U(2n)$. Let us write $N_{m,n} = Sp(m+n)/Sp(m) \times Sp(n)$ and consider the maps

$$M_{m,n} \xrightarrow{q} N_{m,n} \xrightarrow{c'} M_{2m,2n}.$$

The cohomology is also computed as the case $U(n)$,

$$H^*(N_{n,m}; \mathbb{Z}/2) \cong \mathbb{Z}/2[q_1, ..., q_m, r_1, ..., r_n]/(s_1, ..., s_{m+n}),$$

$s_i = \sum q_{i-j} r_j$ with $q^* q_i = a_i^2$ and $c'^* a_{2i} = q_i$ ($c'^*(a_{2i-1}) = 0$).

Note $KO^{odd} = KO^{8*-1}$ and $H^*(N_{m,n}) = H^{4*}(N_{m,n})$. By the dimensional reason for differntial $deg(d_r) = (r, -r+1)$, we know the Atiyah-Hirzebruch spectral sequence for $KO^*(N_{m,n})$ collapses from $E_2^{*,*'}$-term, that means

$$grKO^*(N_{m,n}) \cong KO^* \otimes H^*(N_{m,n}; \mathbb{Z}).$$

Proof of Theorem 2.2. By the naturality of the spectral sequence, the maps q^*, c'^* are defined as maps of spectral sequences. The fact $q^* q_i = a_i^2$ implies that $d_r(a_i^2) = 0$ for all $r > 1$. The fact $c'^* a_{2i} = q_i$ implies that $a_{2i}^2 \neq 0$ in $E_\infty^{*,*'}$ and moreover each nonzero element in $B(k,l)$ is also nonzero in $E_\infty^{*,*'}$. If $d_r(a_{2k+1} b_{2l}) \neq 0$, then it is contained in $B(k,l)$ by dimensional reason ; this is a contradiction to the preceding result. □

There is the well known exact sequence for topological space X

$$(1) \quad \to KO^*(X) \xrightarrow{\eta} KO^*(X) \xrightarrow{c} KU^*(X) \xrightarrow{r} KO^*(X) \to$$

where c is the complexification and r is the restriction maps. Therefore

$$KO^*(X)/KU^*(X) \cong \eta KO^*(X).$$

From the above theorem, we see

$$E_\infty^{odd,*} \cong E_\infty^{8*-1,*} \cong \mathbb{Z}/2\{\eta\}[\mu, \mu^{-1}] \otimes H(m,k).$$

Hence we have

Corollary 2.3. $grKO^{2*}(M_{m,n})/KU^{2*}(M_{m,n}) \cong H(m,n)$.

3. Balmer's Witt group.

For a smooth X over a field k with $1/2 \in k$, let $W(X)$ denote the Witt group of X. Balmer defined the periodic Witt group $W^i(X) \cong W^{i+4}(X),\ (i \in \mathbb{Z})$ with $W^0(X) = W(X)$.

Balmer and Walter [Ba-Wa] define the Gersten-Witt complex

$$0 \to W(k(X)) \to \oplus_{x\in X^{(1)}} W(k(x)) \to ... \oplus_{x\in X^{(n)}} W(k(x)) \to 0.$$

Let $H^*(W(X))$ denote the cohomology group of the above cochain complex, with $W(k(X))$ places in degree 0. From the above complex, we have the (Balmer-Walter) spectral sequence

$$E(BW)_2^{r,s} \cong \begin{cases} H^r(W(X))\ (s=4s') & \Longrightarrow W^{r+s}(X) \\ 0\ \ (s \neq 0(mod(4))) & \end{cases} .$$

By the affirmative answer of the Milnor conjecture of quadratic forms by Orlov-Vishik-Voevodsky [Or-Vi-Vo], we have the isomorphism of graded rings $H^*(k(x);\mathbb{Z}/2) \cong grW^*(k(x))$. Using this fact, Pardon ([Pa],[To]) defined the spectral sequence

$$E(P)_2^{r,s} \cong H^r_{Zar}(X; H^s_{\mathbb{Z}/2}) \Longrightarrow H^r(W(X)) \cong E(BW)_2^{r,4s}$$

so that the differential d_r has degree $(1, r-1)$ for $r \geq 2$. Here $H^n_{\mathbb{Z}/2}$ the Zarisky sheaf induced from the presheaf $H^n_{et}(V;\mathbb{Z}/2)$ for open subset V of X.

The above sheaf cohomology $H^r_{Zar}(X; H^s_{\mathbb{Z}/2})$ relates the motivic cohomology $H^{*,*'}(X;\mathbb{Z}/2)$ (see [Vo1-3]). Let $\tau \in H^{0,1}(Speck(k);\mathbb{Z}/2) \cong \mathbb{Z}/2$ be a generator. (Hence $H^{*,*'}(Speck(\mathbb{C});\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau]$.) Then we get the long exact sequence from the solution of Beilinson-Lichtenbaum conjecture by Voevodsky [Or-Vi-Vo]

$$\to H^{m,n-1}(X;\mathbb{Z}/2) \xrightarrow{\times\tau} H^{m,n}(X;\mathbb{Z}/2)$$

$$\to H^{m-n}_{Zar}(X; H^n_{\mathbb{Z}/2}) \to H^{m+1,n-1}(X;\mathbb{Z}/2) \xrightarrow{\times\tau} .$$

Therefore, we have

Lemma 3.1. $E(P)_2^{m-n,n} \cong H^{m-n}_{Zar}(X; H^n_{\mathbb{Z}/2}) \cong$

$$H^{m,n}(X;\mathbb{Z}/2)/(\tau) \oplus Ker(\tau)|H^{m+1,n-1}(X;\mathbb{Z}/2).$$

In particular, $E(P)_2^{m,m} \cong H^{2m,m}(X;\mathbb{Z}/2) \cong CH^m(X)/2$. Moreover Totaro proved

Lemma 3.2. *(Totaro [To]) If $x \in E(P)_2^{m,m} \cong CH^m(X)/2$, then $d_2(x) = Sq^2(x)$.*

Now we consider the case $X = M_{m,n}$ and $k = \mathbb{C}$. Since $M_{m,n}$ is cellular, we see

$$H^{2*,*'}(M_{m,n};\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau]\otimes CH^*(M_{m,n})/2 \cong \mathbb{Z}/2[\tau]\otimes H^{2*}(M_{m,n};\mathbb{Z}/2).$$

Hence we have

$$E(P)_2^{*,*'} \cong E(P)_2^{*,*} \cong H^{2*}(M_{m,n};\mathbb{Z}/2).$$

From the result of Totaro, we have

$$E(P)_3^{*,*'} \cong H(H^{2*}(M_{m,n};\mathbb{Z}/2); Sq^2) = H(m,n).$$

By dimensional reason of differential $deg(d_r) = (1, r-1)$, it is immediate that the spectral sequence collapses from $E_3^{*,*'}$-term, i.e., $E(P)_3^{*,*'} \cong E_\infty^{*,*'}$.

The Balmer-Walter spectral sequence also collapses, that is, we will prove

$$E(P)_\infty^{*,*} \cong E(BW)_2^{*,r} \cong E(BW)_\infty^{*,r}, \quad r = 0 \ mod(4).$$

For this we consider the $Sp(n)$-version of above arguments. We consider spectral sequences for $N_{m,n}$. By dimensional reason, the Pardon and Balmer-Walter spectral sequences collapse from E_2-terms. (Note $H^*(N_{m,n};\mathbb{Z}/2) = 0$ for $* \neq 0 \ mod(4)$.) That is

$$grW^*(M_{m,n}) \cong E(BW)_\infty^{*,r} \cong H^{2*}(N_{m,n};\mathbb{Z}/2).$$

Then the arguments of the proof of Theorem 2 also work. Thus we see the collapseness of the Balmer-Walter spectral sequence for $M_{m,n}$. Therefore we have isomorphisms

Theorem 3.3.

$$grW^*(M_{m,n}) \cong H(m,n) \cong grKO^{2*}(M_{m,n})/KU^{2*}(M_{m,n}).$$

4. Young diagram and Witt group

In this section we recall the result of Balmer-Calmes [Ba-Ca1,2], and consider relation to the result of Hara-Kono.

The cohomology $H^*(M_{m,n})$ (or $CH^*(M_{m,n})$) is also computed by induction on n, m. In fact the following exact sequence

$$(4.1) \qquad \to H^{*-|a_n|}(M_{m,n-1}) \xrightarrow{g_*} H^*(M_{m,n}) \xrightarrow{f^*} H^*(M_{m-1,n}) \xrightarrow{\partial}$$

becomes split since $\partial = 0$. Here g_* is the Gysin map for the embedding $M_{m,n-1} \subset M_{m,n}$. The map f^* is induced from

$$M_{m-1,n} \overset{proj.}{\leftarrow} M_{m,n} - M_{m,n-1} \subset M_{m,n}.$$

(See [Ba-Ca1,2] or Laksov [La], [Fu]). Here note $g_*(x) = a_m \cdot x$.

It is well known that the cohomology of $M_{m,n}$ is stated also by using Young diagram. The $m\times n$-framed partition $\lambda=(\lambda_1,...,\lambda_d)$ means

$$n\ge \lambda_1\ge ...\ge \lambda_d\ge 1 \quad and \quad m\ge d.$$

The partition λ corresponds a Young diagram, consisting of λ_i boxes in the i-th row from the top, lined up on the left. Then $m\times n$-framed Young diagrams with $d=|\lambda|=\lambda_1+...+\lambda_d$ form the basis of $H^d(M_{m,n})$, namely, $H^*(M_{m,n})\cong \oplus_{*=|\lambda|}\mathbb{Z}$. This fact is shown as follows.

The Young diagram for the conjugate partition $\tilde{\lambda}$ of λ is obtained by interchanging rows and columns in the diagram. For a Young diagram λ, we can define the Schur polynomial (e.g. see [Fu]) by

$$\Delta_{\lambda}(b)=det(b_{\lambda_i+j-i})\in H^*(M_{m,n}).$$

It is known (Lemma 14.5.1 in [Fu]) $\Delta_{\lambda}(b)=\Delta_{\tilde{\lambda}}(a)$. Hence we have $\Delta_{(k)}(b)=b_k$ and $\Delta_{(\tilde{k})=(1,...,1)}(b)=a_k$. Moreover we see by the above definition of Δ_{λ},

$$\Delta_{\lambda}(b)=b_{\lambda_1}...b_{\lambda_d} \quad mod(F_{>\lambda}).$$

Here $F_{>\lambda}$ is the filtration of elements $b_{\lambda'_1}....b_{\lambda'_{d'}}$ with $\lambda'>\lambda$ by the lexicographical order.

We still know the above $\Delta_{\lambda}(b)$ make a basis $[\lambda]$ of $H^*(M_{m,n})$ from (2.1). However we can also get it by induction by using the short exact sequence (4.1) such that

$$g_*([\lambda])=[(1,...,1)+\lambda]=[(1+\lambda_1,...,1+\lambda_d,1,...,1)],$$

(Indeed, $\Delta_{g_*(\lambda)}(b)=a_m\cdot\Delta_{\lambda}(b)\ mod(F_{>g^*(\lambda)})$.) The induced map f^* is given $f^*(\lambda)=\lambda$ for $d<m$, and $=0$ for $d=m$.

Let us say that framed Young diagram λ is *strongly even* if all its segments have even length, namely all λ_i, $\tilde{\lambda}_i$ are even. Then its Schur polynomial is written

$$\Delta_{\lambda}(b)=(b_{\lambda_1}^2 b_{\lambda_3}^2...b_{\lambda_{d-1}}^2) \quad mod(F_{>\lambda}).$$

Hence if m or n is even, then set of strongly even $m\times n$-framed diagrams make $\mathbb{Z}/2$-base of the ring $B(k,l)$ given in Theorem 2.1 by Hara-Kono.

Balmer and Calmes results generalize above arguments. We can consider the generalized Witt group $W^i(X;L)$ for $i\in\mathbb{Z}/4$ and $L\in Pic(X)/2$ such that the usual Witt group $W^i(X)=W^i(X;O_X)$.

Let us say that framed Young diagram λ is *even* if all its segments have even length, which are strictly inside of the frame, namely all $\lambda_i-\lambda_{i+1}$ for $1\le i\le d-1$, $\tilde{\lambda}_i-\tilde{\lambda}_{i+1}$ for $1\le i\le \tilde{d}-1$ are even. Let $t(\lambda)$ be half of the perimeter of λ.

Theorem 4.1. *(Balmer-Camles [Ba-Ca2]) The total Witt group*

$$W^{tot}(M_{m,n}) = \oplus_{i\in\mathbb{Z}/4,\ L\in\mathbb{Z}/2} W^i(M_{m,n}; L)$$

has $\mathbb{Z}/2$-basis indexed by even Young diagrams λ. The corresponding base $[\lambda]$ is in $W^{|\lambda|}(M_{m,n}, t(\lambda))$.

Remark. In [Ba-Ca2], the theorem is stated in very generalized situation.

Let m or m be even. Then when $t(\lambda) = 0$, it is easily seen that λ is even means strongly even. So the argument before the above theorem explains the relation of the results by Hara-Kono and Balmer-Calmes.

Next we consider the case $(m, n) = (2k+1, 2l+1)$. Each even $m\times n$-framed diagram λ with $t(\lambda) = 0$ is easily seen strongly even $[\lambda^{se}]$ or $[\Gamma\lambda^{se}]$ which is defined as

$$[(2l+1, 1, ..., \overset{m}{1}) + (0, \lambda^{se})] = [(2l+1, \lambda_1^{se}+1, ..., \lambda_{d^{se}}^{se}+1, 1..., \overset{m}{1})].$$

Note $\mu = (2l+1, 1, .., 1)$ is even but not strongly even, and $|\mu| = odd$, $t(\mu) = 0$. We still know $[\lambda^{se}]$ form the $\mathbb{Z}/2$-basis of $B(k, l)$ in Theorem 2.1. Note that

$$\Delta_\mu(b) = b_{2l+1}a_{2k} = a_{2k+1}b_{2l} \quad mod(F_{>\mu}).$$

Hence $[\Gamma\lambda^{se}]$ form a basis of $B(k, l)\{a_{2k+1}b_{2l}\}$. Therefore we see even $m\times n$-framed Young diagrams form the base of

$$B(k, l) \oplus B(k, l)\{a_{2k+1}b_{2l}\} = H(m, n).$$

Thus we can explain the relation between Hara-Kono and Balmer-Calmes.

Balmer and Camles prove their theorem by showing following (long) exact sequence. (They construct the Gysin and the boundary maps as the maps in $W^{total}(X)$.) Let $g : Z \subset X$ be a regular closed immersion of $codim = c \geq 2$, and $U = X - Z$. Let ω_g be the relative canonical bundle (for the definition see [Ba-Ca1]). Then there is the natural exact sequence

$$\to W^{*-c}(Z, \omega_g \otimes L|_Z) \xrightarrow{g_*} W^*(X, L) \xrightarrow{f^*} W^*(Z, L|_Z) \xrightarrow{\partial} .$$

In general $\partial \neq 0$. In fact, when $X = M_{m,n}$, it is proved (Figures 4-6 in [Ba-Ca2]).

$$g_*([\lambda]) = \begin{cases} [(1, .., \overset{m}{1}) + \lambda] & if\ m-d : even \\ 0 & otherwise, \end{cases} \qquad f^*([\lambda]) = \begin{cases} [\lambda] & if\ d < m \\ 0 & otherwise, \end{cases}$$

$$\partial([\lambda]) = \begin{cases} [\lambda - (1, ..., \overset{d}{1})] & if\ \lambda_d : odd \\ 0 & otherwise. \end{cases}$$

In the $\mathbb{A}^1$-homotopy category, Hornbostel [Ho] proved that $W^*(-)$ is represented as a $\mathbb{P}^1$-spectrum. This implies

$$W^{*+1}(\mathbb{P}^1 \wedge X) \cong W^*(X).$$

Therefore we can define the natural map

$$q : W^*(X) \to KO^{2*}(X(\mathbb{C}))/KU^{2*}(X(\mathbb{C})) \quad for\ all\ \ * .$$

We will see that this map induces the isomorphism given preceding and this sections.

First we consider the cases $(m, n) = (2k, 2l)$. Let $\tilde{g} : M_{m,n-2} \subset M_{m,n}$ and $\tilde{f} : M_{m,n} \to M_{m-2,n}$. (Note $\tilde{g}_*(x) = a_m^2 x$.) Then we have the commutative diagram.

$$\begin{CD}
0 @>>> W^0(M_{m,n-2}) @>{\tilde{g}_*}>> W^0(M_{m,n}) @>{\tilde{f}^*}>> W^0(M_{m-2,n}) @>{\tilde{\partial}_*}>> 0\\
@. @VV{q_1}V @V{q_2}VV @V{q_3}VV @.\\
0 @>>> K^0(M_{m,n-2}) @>{\tilde{g}_*}>> K^0(M_{m,n}) @>{\tilde{f}^*}>> K^0(M_{m-2,n}) @>{\tilde{\partial}_*}>> 0
\end{CD}$$

where $K^*(X) = KU^*(X)/KO^*(X)$. The exactness of rows follow from the isomorphism given in the preceding or this sections. (In fact this case $W^*_{m,n} = W^0_{m,n}$.) By the induction and five lemma, we have the isomorphism of q_2. The case m or n even follows from the above result and the naturality.

The case $(m, n) = (2k + 1, 2l + 1)$ is proved as follows. Let $g' : M_{2k,2l} \to M_{2k+1,2l+1}$ and recall ${g'}_*([\lambda]) = [\Gamma\lambda]$. Consider the following diagram

$$\begin{CD}
W^0(M_{2k,2l}) @>{{g'}_*}>{\cong}> W^1(M_{2k+1,2l+1})\\
@VV{q_1}V @V{q_2}VV\\
K^0(M_{2k,2l}) @>{g'_*}>{\cong}> K^2(M_{2k+1,2l+1}).
\end{CD}$$

We see q_2 is isomorphic for this case. Similarly we consider $f' : M_{2k,2l} \to M_{2k+1,2k+1}$. By the isomorphism of preceding or this section, we have the isomorphism $f'^* : W^0(M_{2k+1,2k+1}) \cong W^0(M_{2k,2k})$. This also induces the isomorphism of q for $W^0(M_{m,n}) \to K^0(M_{m,n})$. Thus we can show

Theorem 4.2. *The map $q : W^*(M_{m,n}) \to KO^{2*}(M_{m,n})/KU^{2*}(M_{m,n})$ induces the isomorphism.*

Corollary 4.3. *There is the isomorphism of graded rings*

$$grW^*(M_{m,n}) \cong H(m, n).$$

In particular, we note that $grW^0(M_{2k,2l}) \cong H^*(M_{k,l};\mathbb{Z}/2)$.

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