CANONICAL DIMENSION OF PROJECTIVE $\mathrm{PGL}_1(A)$-HOMOGENEOUS VARIETIES

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ABSTRACT. Let A be a central division algebra over a field F with $\operatorname{ind} A = n$. For integers $1 \leq d_1 < d_2 < \cdots < d_k \leq n-1$, let $X_{d_1,d_2,\ldots,d_k}(A)$ be the variety of flags of right ideals $I_1 \subset I_2 \subset \cdots \subset I_k$ of A with I_i of reduced dimension d_i. In computing canonical p-dimension of such varieties, for p prime, we can reduce to the case of generalized Severi-Brauer varieties $X_e(A)$ with $\operatorname{ind} A$ a power of p divisible by e. We prove that canonical 2-dimension (and hence canonical dimension) equals dimension for all $X_e(A)$ with $\operatorname{ind} A = 2e$ a power of 2.

1. CANONICAL p-DIMENSION

We begin by recalling the definitions of canonical p-dimension, p-incompressibility, and equivalence.

Let X be a scheme over a field F, and let p be a prime or zero. A field extension K of F is called a *splitting field of* X (or is said to *split* X) if $X(K) \neq \emptyset$. A splitting field K is called *p-generic* if, for any splitting field L of X, there is an F-place $K \rightharpoonup L'$ for some finite extension L'/L of degree prime to p. In particular, K is 0-generic if for any splitting field L there is an F-place $K \rightharpoonup L$.

The canonical p-dimension of a scheme X over F was originally defined [1, 7] as the minimal transcendence degree of a p-generic splitting field K of X. When X is a smooth complete variety, the original algebraic definition is equivalent to the following geometric one [7, 9].

Definition 1.1. Let X be a smooth complete variety over F. The *canonical p-dimension* $\operatorname{cdim}_p(X)$ *of* X is the minimal dimension of the image of a morphism $X' \to X$, where X' is a variety over F admitting a dominant morphism $X' \to X$ with $F(X')/F(X)$ finite of degree prime to p. The canonical 0-dimension of X is thus the minimal dimension of the image of a rational morphism $X \dashrightarrow X$.

In the case $p = 0$, we will drop the p and speak simply of *generic* splitting fields and canonical *dimension* $\operatorname{cdim}(X)$.

For a third definition of canonical p-dimension as the essential p-dimension of the detection functor of a scheme X, we refer the reader to Merkurjev's comprehensive exposition [9] of essential dimension.

For a smooth complete variety X, the inequalities

$$\operatorname{cdim}_p(X) \leq \operatorname{cdim}(X) \leq \dim(X)$$

are clear from Definition 1.1. Note also that if X has a rational point, then $\operatorname{cdim}(X) = 0$ (though the converse is not true).

Definition 1.2. When a smooth complete variety X has canonical p-dimension as large as possible, namely $\operatorname{cdim}_p(X) = \dim(X)$, we say that X is *p-incompressible*.

It follows immediately that if X is p-incompressible, it is also *incompressible* (i.e. 0-incompressible).

When two schemes X and Y over a field F have the same class of splitting fields, we call them equivalent and write $X \sim Y$. In this case

$$\operatorname{cdim}_p(X) = \operatorname{cdim}_p(Y)$$

for all p. If X and Y are smooth complete varieties, then they are equivalent if and only if there exist rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$.

2. Reductions

Let A be a central division algebra over a field F with $\operatorname{ind} A = n$. We consider the problem of computing the canonical p-dimension of the following varieties.

Definition 2.1. For integers $1 \leq d_1 < d_2 < \cdots < d_k \leq n-1$, define $X_{d_1,d_2,\ldots,d_k}(A)$ to be the variety of flags of right ideals $I_1 \subset I_2 \subset \cdots \subset I_k$ of A with I_i of reduced dimension d_i. When the algebra A is understood, we write simply $X_{d_1,d_2,\ldots,d_k}$.

When $k = 1$ we get the generalized Severi-Brauer varieties $X_d(A)$ of A. In particular, $X_1(A)$ is the Severi-Brauer variety of A.

It is known [8, Th. 1.17] that the generalized Severi-Brauer variety $X_{d_1}(A)$ has a rational point over an extension field K/F if and only if the index $\operatorname{ind} A_K$ divides d_1. As a consequence, $X_{d_1}(A) \sim X_d(A)$, where $d := \gcd(\operatorname{ind} A, d_1)$. We record the easy generalization of this fact to varieties $X_{d_1,d_2,\ldots,d_k}(A)$.

Proposition 2.2. *If* $d := \gcd(\operatorname{ind} A, d_1, d_2, \ldots, d_k)$*, then*

$$X_{d_1,d_2,\ldots,d_k}(A) \sim X_d(A)$$

and thus, for any p*,*

$$\operatorname{cdim}_p(X_{d_1,d_2,\ldots,d_k}(A)) = \operatorname{cdim}_p(X_d(A)).$$

Proof. If $X_{d_1,d_2,\ldots,d_k}(A)$ has a rational point over an extension field K/F, then by definition A_K has right ideals of reduced dimensions $d_1, d_2, \ldots, d_k$. This is the case if and only if $\operatorname{ind} A_K$ divides each of the d_i, or equivalently, $\operatorname{ind} A_K$ divides d (since $\operatorname{ind} A_K$ always divides $\operatorname{ind} A$).

Reading the argument backwards, $\operatorname{ind} A_K$ dividing d implies the existence of right ideals $I_1, I_2, \ldots, I_k$ in A_K with reduced dimensions $d_1, d_2, \ldots, d_k$. In fact, the $I_1, \ldots, I_k$ can be chosen to form a flag. Suppose $d_i = m_i \operatorname{ind} A_K$ and $A_K \simeq M_t(D)$ for some division algebra D. Then we take I_i to be the set of matrices in $M_t(D)$ whose $t - m_i$ last rows are zero. □

Hence it is enough to compute $\operatorname{cdim}_p(X_d(A))$ *for* d *dividing* $\operatorname{ind} A$.

If the index of A factors as $\operatorname{ind} A = q_1 q_2 \cdots q_r$ with the q_j powers of distinct primes p_j, then there exist central division algebras A_j of index q_j for $j = 1, \ldots, r$ such that

$$A \simeq A_1 \otimes A_2 \otimes \cdots \otimes A_r.$$

Proposition 2.3. *Given a positive integer* $1 \leq d \leq \operatorname{ind} A - 1$*, with* q_j *as above, define* $e_j := \gcd(d, q_j)$ *for* $j = 1, \ldots, r$*. Then*

$$X_d(A) \sim X_{e_1}(A_1) \times X_{e_2}(A_2) \times \cdots \times X_{e_r}(A_r)$$

and thus, for any p,

$$\mathrm{cdim}_p(X_d(A)) = \mathrm{cdim}_p(X_{e_1}(A_1) \times X_{e_2}(A_2) \times \cdots \times X_{e_r}(A_r)).$$

Proof. The variety $X_d(A)$ has a rational point over an extension field K/F if and only if $\operatorname{ind} A_K$ divides d. Because

$$\operatorname{ind} A_K = (\operatorname{ind}(A_1)_K) \cdots (\operatorname{ind}(A_r)_K),$$

this condition is equivalent to $\operatorname{ind}(A_j)_K$ dividing d for all j, or to $\operatorname{ind}(A_j)_K$ dividing e_j for all j (since $\operatorname{ind}(A_j)_K$ always divides $\operatorname{ind} A_j = q_j$). This holds if and only if each $X_{e_j}(A_j)$ has a rational point over K, which is equivalent to the product of the $X_{e_j}(A_j)$ having a rational point over K. □

The proposition gives the following upper bound on canonical p-dimension:

$$\mathrm{cdim}_p(X_d(A)) \leq \dim \prod_{j=1}^{r} X_{e_j}(A_j) = \sum_{j=1}^{r} \dim X_{e_j}(A_j) = \sum_{j=1}^{r} e_j(q_j - e_j). \tag{1}$$

If p is prime, then there exists a finite, p-coprime extension K of F which splits the algebras A_j for all j with $p_j \neq p$. Since canonical p-dimension does not change under such an extension [9, Prop. 1.5 (2)], $\mathrm{cdim}_p(X_d(A)) = 0$ unless some $p_s = p$, in which case

$$\mathrm{cdim}_p(X_d(A)) = \mathrm{cdim}_p(X_{e_s}(A_s)).$$

We see that it is enough, when p is prime, to compute the canonical p-dimension of varieties of the form $X_e(A)$ with $\operatorname{ind} A$ *a prime power divisible by e. When $p = 0$, it is enough to compute the canonical dimension of products of such varieties.*

3. Known results for Severi-Brauer varieties

We now recall what is already known about the canonical p-dimension of Severi-Brauer varieties $X_1(A)$, the $d = 1$ case.

For any p, if $d = 1$ in (1) above, then all of the $e_j = 1$, and the upper bound becomes

$$\mathrm{cdim}_p(X_1(A)) \leq \sum_{j=1}^{r} (q_j - 1). \tag{2}$$

In the special case r=1 and $p = p_1$, it is shown in [1, Th. 11.4], based on Karpenko's [6, Th. 2.1], that the inequality (2) is actually an equality. Thus, for general A, we have

$$\mathrm{cdim}_{p_j}(X_1(A)) = \mathrm{cdim}_{p_j}(X_1(A_j)) = q_j - 1$$

for $j = 1, 2, \ldots, r$, while $\mathrm{cdim}_p(X_1(A)) = 0$ for all other *primes* p [7, Ex. 5.10].

Now let $p = 0$, $d = 1$. When $r = 1$, we again have equality in (2), since canonical dimension is bounded below by canonical p-dimension for every prime p. In [4, Th. 1.3], (2) is proven also to be an equality in the case $\operatorname{ind} A = 6$ (i.e. $r = 2$, $q_1 = 2$, $q_2 = 3$) provided that $\operatorname{char} F = 0$. The authors of [4] suggest that equality may indeed hold for any A when $p = 0$, $d = 1$.

4. 2-Incompressibility of $X_e(A)$ for $\operatorname{ind} A = 2e$ a power of 2

If A is a central division algebra with $\operatorname{ind} A = 4$, the variety $X_2(A)$ is known to be 2-incompressible. Indeed, if the exponent of A is 2, then $X_2(A)$ is isomorphic to a 4-dimensional projective quadric hypersurface called the *Albert quadric* of A [10, §5.2]. Such a quadric has first Witt index 1 [13, p. 93], hence is 2-incompressible by [5, Th. 90.2]. If the exponent of A is 4, we can reduce to the exponent 2 case by extending to the function field of the Severi-Brauer variety of $A \otimes A$.

In what follows, we show 2-incompressibility for an infinite family of varieties which includes the varieties of the form $X_2(A)$ (with $\operatorname{ind} A = 4$) mentioned above.

Theorem 4.1. *Let $e = 2^a$, $a \geq 1$. For a central division algebra A with $\operatorname{ind} A = 2e$, the variety $X_e := X_e(A)$ is 2-incompressible. Thus*

$$\operatorname{cdim}_2(X_e) = \operatorname{cdim}(X_e) = \dim(X_e) = e(2e - e) = e^2 = 4^a.$$

We briefly recall some terminology from [5, §62 and §75]. Let X and Y be schemes with $\dim X = e$. A *correspondence of degree zero $\delta : X \rightsquigarrow Y$ from X to Y* is just a cycle $\delta \in \mathrm{CH}_e(X \times Y)$. The *multiplicity* $\operatorname{mult}(\delta)$ of such a δ is the integer satisfying $\operatorname{mult}(\delta) \cdot [X] = p_*(\delta)$, where p_* is the push-forward homomorphism

$$p_* : \mathrm{CH}_e(X \times Y) \to \mathrm{CH}_e(X) = \mathbb{Z} \cdot [X].$$

The exchange isomorphism $X \times Y \to Y \times X$ induces an isomorphism

$$\mathrm{CH}_e(X \times Y) \to \mathrm{CH}_e(Y \times X)$$

sending a cycle δ to its *transpose* δ^t.

To prove that a variety X is 2-incompressible, it suffices to show that for any correspondence $\delta : X \rightsquigarrow X$ of degree zero,

$$\operatorname{mult}(\delta) \equiv \operatorname{mult}(\delta^t) \pmod 2. \tag{3}$$

Indeed, suppose we have $f : X' \to X$ and a dominant $g : X' \to X$ with $F(X')/F(X)$ finite of odd degree. Let $\delta \in \mathrm{CH}(X \times X)$ be the pushforward of the class $[X']$ along the induced morphism $(g, f) : X' \to X \times X$. By assumption, $\operatorname{mult}(\delta)$ is odd, so by (3) we have that $\operatorname{mult}(\delta^t)$ is odd. It follows that $f_*([X'])$ is an odd multiple of $[X]$ and in particular is nonzero, so f is dominant.

We will check that the condition (3) holds for the variety X_e. A correspondence of degree zero $\delta : X_e \rightsquigarrow X_e$ is just an element of $\mathrm{CH}_{e^2}(X_e \times X_e)$. Using the method of Chernousov and Merkurjev described in [2], we can decompose the Chow motive of $X_e \times X_e$ as follows. See also [3] for examples of similar computations.

We first realize X_e as a projective homogeneous variety. Let $n := \operatorname{ind} A = 2e = 2^{a+1}$. Let G denote the group $\mathbf{PGL}_1(A)$, and let Π be a set of simple roots for the root system Σ of G. If $\varepsilon_1, \ldots, \varepsilon_n$ are the standard basis vectors of $\mathbb{R}^n$, we may take

$$\Pi = \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n\}.$$

Then X_e is a projective G-homogeneous variety, namely the variety of all parabolic subgroups of G of type S, for the subset $S = \Pi \backslash \{\alpha_e\}$ of the set of simple roots.

Let W denote the Weyl group of the root system Σ. There are $e+1$ double cosets $D \in W_P \backslash W / W_P$ with representatives w as follows, where w_{α_k} denotes the reflection induced by the root α_k.

$$\begin{gathered} 1 \\ w_{\alpha_e} \\ (w_{\alpha_e} w_{\alpha_{e-1}})(w_{\alpha_{e+1}} w_{\alpha_e}) \\ (w_{\alpha_e} w_{\alpha_{e-1}} w_{\alpha_{e-2}})(w_{\alpha_{e+1}} w_{\alpha_e} w_{\alpha_{e-1}})(w_{\alpha_{e+2}} w_{\alpha_{e+1}} w_{\alpha_e}) \\ \vdots \\ (w_{\alpha_e} \cdots w_{\alpha_1}) \cdots (w_{\alpha_{2e-1}} \cdots w_{\alpha_e}) \end{gathered}$$

The subset of Π associated to $w = 1$ is of course $S = \Pi \backslash \{\alpha_e\}$. The general nontrivial representative

$$w = w^{-1} = (w_{\alpha_e} \cdots w_{\alpha_{e-i}}) \cdots (w_{\alpha_{e+i}} \cdots w_{\alpha_e}),$$

for $i \in \{0, \ldots, e-1\}$, has the effect on $\mathbb{R}^n$ of switching the tuple of standard basis vectors $(\varepsilon_{e-i}, \ldots, \varepsilon_e)$ with the tuple $(\varepsilon_{e+1}, \ldots, \varepsilon_{e+1+i})$. The resulting subset associated to w is therefore

$$\Pi \backslash \{\alpha_{e-(i+1)}, \alpha_e, \alpha_{e+(i+1)}\}$$

for $i = 0, \ldots, e-2$ and $\Pi \backslash \{\alpha_e\}$ for $i = e-1$.

From Theorem 6.3 of [2], we deduce the following decomposition of the Chow motive of $X_e \times X_e$, where the relation between the indices i above and l below is $l = i + 1$.

$$M(X_e \times X_e) \simeq M(X_e) \oplus \bigoplus_{l=1}^{e-1} M(X_{e-l,e,e+l})(l^2) \oplus M(X_e)(e^2)$$

This in turn yields a decomposition of the middle-dimensional component of the Chow group of $X_e \times X_e$.

$$\mathrm{CH}_{e^2}(X_e \times X_e) \simeq \mathrm{CH}_{e^2}(X_e) \oplus \bigoplus_{l=1}^{e-1} \mathrm{CH}_{(e-l)(e+l)}(X_{e-l,e,e+l}) \oplus \mathrm{CH}_0(X_e)$$

It now suffices to check the congruence $\mathrm{mult}(\delta) \equiv \mathrm{mult}(\delta^t) \pmod 2$ for δ in the image of any of these summands. We treat the first and last summands separately from the rest.

The embedding of the first summand $\mathrm{CH}_{e^2}(X_e)$ is induced by the diagonal morphism $X_e \to X_e \times X_e$, so the multiplicities of δ and δ^t are equal by symmetry.

For the last summand $\mathrm{CH}_0(X_e)$ we need the following fact.

Proposition 4.2. *Any element of* $\mathrm{CH}_0(X_e)$ *has even degree.*

Proof. If $\mathrm{CH}_0(X_e)$ has an element of odd degree, then there exists a field extension K/F of odd degree over which X_e has a rational point. By [8, Prop. 1.17], $\operatorname{ind} A_K$ divides e. Since the degree of K over F is relatively prime to $\operatorname{ind} A = 2e = 2^{a+1}$, extension by K does not reduce the index of A [11, Th. 3.15a]. Thus $\operatorname{ind} A = \operatorname{ind} A_K$ divides e, a contradiction. □

Let the element $\gamma \in \mathrm{CH}_0(X_e)$ have image $\delta \in \mathrm{CH}_{e^2}(X_e \times X_e)$. By the proposition, $\deg(\gamma)$ is even. For some field E/F over which X_e has a rational point, we set $\bar{X}_e := (X_e)_E$. Since $\mathrm{CH}_0(\bar{X}_e)$ is generated by a single element of degree 1, the image of γ in $\mathrm{CH}_0(\bar{X}_e)$ is divisible by 2. It follows that $\delta \in \mathrm{CH}_{e^2}(\bar{X}_e \times \bar{X}_e)$ is also divisible by 2 and, since multiplicity does not change under field extension, $\mathrm{mult}(\delta)$ is even. The same argument can be applied to δ^t, so $\mathrm{mult}(\delta) \equiv 0 \equiv \mathrm{mult}(\delta^t) \pmod 2$.

The remaining summands are dealt with by the following proposition.

Proposition 4.3. *Let $Fl := X_{d_1,d_2,\ldots,d_k}(A)$ with $d := \gcd(e, d_1, d_2, \ldots, d_k) < e$, and let the correspondence $\alpha : Fl \rightsquigarrow X_e \times X_e$ induce an embedding*

$$\alpha_* : \mathrm{CH}_r(Fl) \hookrightarrow \mathrm{CH}_{e^2}(X_e \times X_e).$$

Then for any δ in the image of α_*, $\mathrm{mult}(\delta) \equiv 0 \equiv \mathrm{mult}(\delta^t) \pmod 2$.

Proof. Consider the diagram below of fiber products, where we select either of the projections p_i and choose the other morphisms accordingly.

$$\begin{tikzcd}
 & (Fl)_{F(X_e)} \arrow[dr] & \\
(Fl \times X_e)_{F(X_e)} \arrow[r] \arrow[ur] \arrow[d] & (X_e)_{F(X_e)} \arrow[r] \arrow[d] & \operatorname{Spec} F(X_e) \arrow[d] \\
Fl \times X_e \times X_e \arrow[r] \arrow[d] & X_e \times X_e \arrow[r, "p_1", "p_2"'] & X_e \\
Fl & &
\end{tikzcd}$$

Taking push-forwards and pull-backs, we get the following diagram which commutes except for the triangle at the bottom. The push-forward by p_i takes a cycle $\delta \in \mathrm{CH}_{e^2}(X_e \times X_e)$ to $\mathrm{mult}(\delta)$ if we chose the first projection p_1 or to $\mathrm{mult}(\delta^t)$ if we chose the second projection p_2.

$$\begin{tikzcd}
 & \mathrm{CH}_0\left((Fl)_{F(X_e)}\right) \arrow[dr, "\deg"] & \\
\mathrm{CH}_0\left((Fl \times X_e)_{F(X_e)}\right) \arrow[r] \arrow[ur] & \mathrm{CH}_0\left((X_e)_{F(X_e)}\right) \arrow[r, "\deg"'] & \mathbb{Z} \arrow[d, equal] \\
\mathrm{CH}_{e^2}\left(Fl \times X_e \times X_e\right) \arrow[r] \arrow[u] & \mathrm{CH}_{e^2}\left(X_e \times X_e\right) \arrow[r, "\mathrm{mult}", "(\mathrm{mult})\circ(\mathrm{transpose})"'] \arrow[u] & \mathbb{Z} \\
\mathrm{CH}_r(Fl) \arrow[u] \arrow[ur, dotted, "\alpha_*"'] & &
\end{tikzcd}$$

Any $\delta \in \mathrm{im}(\alpha_*)$ also lies in the image of $\mathrm{CH}_{e^2}\left(Fl \times X_e \times X_e\right)$, by the definition of the push-forward. Chasing through the diagram, one sees that $\mathrm{mult}(\delta)$ (and similarly $\mathrm{mult}(\delta^t)$) must lie in $\deg \mathrm{CH}_0\left((Fl)_{F(X_e)}\right)$. We will be done if we can show that no element of $\mathrm{CH}_0\left((Fl)_{F(X_e)}\right)$ has odd degree.

Note that

$$Fl_{F(X_e)} = X_{d_1,d_2,\ldots,d_k}(A)_{F(X_e)} \simeq X_{d_1,d_2,\ldots,d_k}\left(A_{F(X_e)}\right),$$

where $A_{F(X_e)}$ has index equal to $\gcd(2e, e) = e$ [12, Th. 2.5]. If some element of $\mathrm{CH}_0\left((Fl)_{F(X_e)}\right)$ has odd degree, then there exists a field extension $K/F(X_e)$ of odd degree over which $(Fl)_{F(X_e)}$ has a rational point. By Proposition 2.2, $X_d(A_{F(X_e)})$ also has a rational point over K. Thus $\operatorname{ind} A_K$ divides $d < e$, which contradicts $\operatorname{ind} A_{F(X_e)} = e$, since an odd degree extension cannot reduce the index of $A_{F(X_e)}$ [11, Th. 3.15a]. □

This completes the proof of the theorem.

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