THERE IS NO "THEORY OF EVERYTHING" INSIDE E_8

JACQUES DISTLER AND SKIP GARIBALDI

ABSTRACT. We analyze certain subgroups of real and complex forms of the Lie group E_8, and deduce that any "Theory of Everything" obtained by embedding the gauge groups of gravity and the Standard Model into a real or complex form of E_8 lacks certain representation-theoretic properties required by physical reality. The arguments themselves amount to representation theory of Lie algebras along the lines of Dynkin's classic papers and are written for mathematicians.

1. INTRODUCTION

Recently, the preprint [1] by Garrett Lisi has generated a lot of popular interest. It boldly claims to be a sketch of a "Theory of Everything", based on the idea of combining the local Lorentz group and the gauge group of the Standard Model in a real form of E_8 (necessarily not the compact form, because it contains a group isogenous to $\mathrm{SL}(2,\mathbb{C})$). The purpose of this paper is to explain some reasons why an entire class of such models—which include the model in [1]—cannot work, using mostly mathematics with relatively little input from physics.

The mathematical set up is as follows. Fix a real Lie group E. We are interested in subgroups $\mathrm{SL}(2,\mathbb{C})$ and G of E so that:

(ToE1) G is connected, reductive, compact, and centralizes $\mathrm{SL}(2,\mathbb{C})$

We complexify and decompose $\mathrm{Lie}(\mathrm{E}) \otimes \mathbb{C}$ as a direct sum of representations of $\mathrm{SL}(2,\mathbb{C})$ and G. We identify $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}$ with $\mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}}$ and write

$$\mathrm{Lie}(\mathrm{E}) = \bigoplus_{m,n\geq 1} m \otimes n \otimes V_{m,n} \tag{1.1}$$

where m and n denote the irreducible representation of $\mathrm{SL}_{2,\mathbb{C}}$ of that dimension and $V_{m,n}$ is a complex representation of $G \times \mathbb{C}$. (Physicists would usually write 2 and $\bar{2}$ instead of $2 \otimes 1$ and $1 \otimes 2$.) Of course,

$$\overline{m \otimes n \otimes V_{m,n}} \simeq n \otimes m \otimes \overline{V_{m,n}}$$

and since the action of $\mathrm{SL}(2,\mathbb{C}).G$ on $\mathrm{Lie}(\mathrm{E})$ is defined over $\mathbb{R}$, we deduce that $\overline{V_{m,n}} \simeq V_{n,m}$. We further demand that

(ToE2) $V_{2,1}$ is a complex representation of G, and

(ToE3) $V_{m,n} = 0$ if $m + n > 4$.

We recall the definition of complex representation and explain the physical motivation for these hypotheses in the next section. Roughly speaking, (ToE1) is a trivial requirement based on trying to construct a Theory of Everything along the lines suggested by Lisi, (ToE2) is the statement that the gauge theory (with gauge group G) is *chiral*, as required by

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the Standard Model, and (ToE3) is the requirement that the model not contain any "exotic" higher-spin particles. In fact, physics requires slightly stronger hypotheses on $V_{m,n}$, for $m + n = 4$. We will not impose the stronger version of (ToE3).

Definition 1.2. A *ToE subgroup* of a real Lie group E is a subgroup generated by a copy of $\mathrm{SL}(2,\mathbb{C})$ and a subgroup G such that (ToE1), (ToE2), and (ToE3) hold.

Our main result is:

Theorem 1.3. *There are no ToE subgroups in (the transfer of) the complex* E_8 *nor in any real form of* E_8.

Notation. Unadorned Lie algebras and Lie groups mean ones over the real numbers. We use a subscript $\mathbb{C}$ to denote complex Lie groups—e.g., $\mathrm{SL}_{2,\mathbb{C}}$ is the (complex) group of 2-by-2 complex matrices with determinant 1. We can view a d-dimensional complex Lie group $G_{\mathbb{C}}$ as a $2d$-dimensional real Lie group, which we denote by $R(G_{\mathbb{C}})$. (Algebraists call this operation the "transfer" or "Weil restriction of scalars".) We use the popular notation of $\mathrm{SL}(2,\mathbb{C})$ for the transfer $R(\mathrm{SL}_{2,\mathbb{C}})$ of $\mathrm{SL}_{2,\mathbb{C}}$; it is a double covering of the "restricted Lorentz group", i.e., of the identity component $\mathrm{SO}(3,1)_0$ of $\mathrm{SO}(3,1)$.

1.4. Strategy and main results. Our strategy for proving Theorem 1.3 will be as follows. We will first catalogue, up to conjugation, all possible embedding of $\mathrm{SL}(2,\mathbb{C})$ satifying the hypotheses of (ToE3). The list is remarkably short. Specifically, every ToE subgroup of E is contained in $\mathrm{SL}(2,\mathbb{C}) \cdot G_{\max}$, where $G_{\max}$ is the maximal compact, connected, reductive subgroup of the centralizer of $\mathrm{SL}(2,\mathbb{C})$ in E. The proof of Theorem 1.3 shows that the only possibilities are:

E	$G_{\max}$
$\mathrm{E}_{8(8)}$	$\mathrm{Spin}(5) \times \mathrm{Spin}(7)$
$\mathrm{E}_{8(-24)}$	$\mathrm{Spin}(11)$ or $\mathrm{Spin}(9) \times \mathrm{Spin}(3)$
$R(\mathrm{E}_{8,\mathbb{C}})$	E_7, $\mathrm{Spin}(13)$ or $\mathrm{Spin}(12)$.

We then note that the representation, $V_{2,1}$, of $G_{\max}$ (and hence, of any $G \subseteq G_{\max}$) has a self-conjugate structure. In other words, (ToE2) fails.

2. Physics background

One of the central features of modern particle physics is that the world is described by a *chiral gauge theory*.

2.1. Let M a four dimensional pseudo-Riemannian manifold, of signature $(3,1)$, which we will take to be oriented, time-oriented and spin. Let G be a compact Lie group. The data of a *gauge theory on* M*, with gauge group* G consists of a connection, A, on a principal G-bundle, $P \to M$, and some "matter fields" transforming as sections of vector bundle(s) associated to unitary representations of G.

Of particular interest are the *fermions* of the theory. The orthonormal frame bundle of M is a principal $\mathrm{SO}(3,1)_0$ bundle. A choice of spin structure defines a lift to a principal $\mathrm{Spin}(3,1)_0 = \mathrm{SL}(2,\mathbb{C})$ bundle. Let $S_\pm \to M$ be the irreducible spinor bundles, associated, via the defining two-dimensional representation and its complex conjugate, to this $\mathrm{SL}(2,\mathbb{C})$ principal bundle.

The *fermions of our gauge theory* are denoted

$$\psi \in \Gamma(S_+ \otimes V), \qquad \overline{\psi} \in \Gamma(S_- \otimes \overline{V})$$

where $V \to M$ is a vector bundle associated to a (typically reducible) representation, R, of G.

Definition 2.2. A *real structure* on a representation V (over $\mathbb{C}$) is an antilinear map, $J : V \to V$, satisfying $J^2 = 1$. Physicists call a representation possessing a real structure *real*.

A *quaternionic structure* on a representation V (over $\mathbb{C}$) is an antilinear map, $J : V \to V$, satisfying $J^2 = -1$. Physicists call a representation possessing a quaternionic structure *pseudoreal*.

Subsuming these two subcases, we will say that a representation V (over $\mathbb{C}$) has a *self-conjugate structure* if there is an antilinear map $J : V \to V$, satisfying $J^4 = 1$. Physicists call a representation V, which does not possess a self-conjugate structure, *complex*.

For an alternative view, we suppose that V is an irreducible representation (over $\mathbb{C}$) of a real reductive Lie group G and exploit [2, §7]. If V has a real structure J, then the subset V' of elements of V fixed by J is a real vector space that is a representation of G such that $\mathrm{End}_G(V') = \mathbb{R}$ and $V' \otimes \mathbb{C}$ is canonically identified with V. A quaternionic structure on V defines a real structure on $V \oplus V$ via $(v_1, v_2) \mapsto (Jv_2, -Jv_1)$ such that $(V \oplus V)'$ is irreducible with $\mathrm{End}_G((V \oplus V)') \cong \mathbb{H}$. If V is complex, then $V \oplus \overline{V}$ has an essentially unique real structure and $\mathrm{End}_G((V \oplus \overline{V})') \cong \mathbb{C}$.

Definition 2.3. A gauge theory, with gauge group G, is said to be *chiral* if the representation, R by which the fermions (2.1) are defined, is complex in the above sense. By contrast, a gauge theory is said to be *nonchiral* if the representation R in 2.1 has a self-conjugate structure.

Note that whether a gauge theory is chiral depends crucially on the choice of G. A gauge theory might be chiral for gauge group G, but *nonchiral* for a subgroup $H \subset G$, because there exists a self-conjugate structure on R, compatible with H, even though no such structure exists, compatible with the full group G.

Conversely, suppose that a gauge theory is nonchiral for the gauge group G. It is also necessarily nonchiral for any gauge group $H \subset G$.

2.4. GUTs. The Standard Model is a chiral gauge theory with gauge group

$$G_{\mathrm{SM}} := (\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/(\mathbb{Z}/6\mathbb{Z})$$

Various grand unified theories (GUTs) proceed by embedding G_{SM} is some (usually simple) group, G_{GUT}. Popular choices for G_{GUT} are $\mathrm{SU}(5)$ [3], $\mathrm{Spin}(10)$, E_6, and the Pati-Salam group, $(\mathrm{Spin}(6) \times \mathrm{Spin}(4))/(\mathbb{Z}/2\mathbb{Z})$ [4].

It is easiest to explain what the fermion representation of G_{SM} is after embedding G_{SM} in $G_{\mathrm{GUT}} := \mathrm{SU}(5)$. Let W be the five-dimensional defining representation of $\mathrm{SU}(5)$. The representation R from 2.1 is the direct sum of three copies of

$$R_0 = \wedge^2 W \oplus \overline{W}$$

Each such copy is called a “generation” and is 15-dimensional. One identifies each of the 15 weights of R_0 with left-handed fermions: 6 quarks (two in a doublet, each in three colors), two leptons (e.g., electron and its neutrino), 6 antiquarks, and a positron. With three generations, R is 45-dimensional.

For the other choices of GUT group, the analogue of a generation (R_0) is higher-dimensional, containing additional fermion which are not seen at low energies. When decomposed under $G_{\mathrm{SM}} \subset G_{\mathrm{GUT}}$, the representation decomposes as $R_0 + R'$, where R' is a real representation of G_{SM}. In $\mathrm{Spin}(10)$, a generation is the 16-dimensional half-spinor representation. In E_6, it is the 27, and for the Pati-Salam group, it is the

$(4,1,2) \oplus (\overline{4},2,1)$ representation. In each case, these representations are complex representations (in the above sense) of G_{GUT}, and the complex-conjugate representation is called an "anti-generation."

So far, we have described a chiral gauge theory in a fixed (pseudo) Riemannian structure on M. Lisi's proposal [1] is to try to combine the spin connection on M, and the gauge connection on P into a single dynamical framework. This motivates Definition 1.2 of a ToE subgroup.

Fix a ToE subgroup—say, with $G = G_{\mathrm{SM}}$—in some real Lie group E. The action of central element $-1 \in \mathrm{SL}(2,\mathbb{C})$ provides a $\mathbb{Z}/2\mathbb{Z}$ grading on the Lie algebra of E. This $\mathbb{Z}/2\mathbb{Z}$ grading allows one to define a sort of superconnection associated to E (precisely what sort of superconnection is explained in a blog post by the first author [5].) In the proposal of [1], we are supposed to identify each of the generators of $\mathrm{Lie}(\mathrm{E})$ as either a boson or a fermion. The Spin-Statistics Theorem [6] says that fermions transform as spinorial representations of $\mathrm{Spin}(3,1)$; bosons transform as "tensorial" representations (representation which lift to the double cover, $\mathrm{SO}(3,1)$). To be consistent with the Spin-Statistics Theorem, we must, therefore, require that the fermions belong to the -1-eigenspace of the aforementioned $\mathbb{Z}/2\mathbb{Z}$ action, and the bosons to the $+1$-eigenspace.

In fact, to agree with 2.1, we should require that the -1 eigenspace (when tensored with $\mathbb{C}$) decompose as a direct sum of two-dimensional representations (over $\mathbb{C}$) of $\mathrm{SL}(2,\mathbb{C})$, corresponding to "left-handed" and "right-handed" fermions, in the sense of 2.1.

Remark 2.5. In the language of (ToE3), $m+n =$ odd are fermions and $m+n =$ even are bosons. In Lisi's setup, the bosons are 1-forms on M, with values in a vector bundle associated to the aforementioned $\mathrm{Spin}(3,1)_0$ principal bundle via the $m \otimes n$ representation (with $m+n$ even). The case $m+n=4$ is special; these correspond to the gravitational degrees of freedom in Lisi's theory. $(3\otimes 1)\oplus(1\otimes 3)$ is the adjoint representation; these correspond to the spin connection. The 1-form with values in the $2\otimes 2$ representation is the vierbein. It is a serious result from physics (see sections 13.1, 25.4 of [7]) that a unitary interacting theory is incompatible with massless particles in higher representation ($m+n \geq 6$). But, in light of the difficulties in making physical sense of the bosonic sector of Lisi's theory, it would be cleaner — meaning demanding less input from physics— to focus on the fermionic sector and forbid the presence of gravitinos ($m+n=5$) or yet-higher spin fermionic fields. (ToE3), as stated, forbids both. In §9, we will revisit the possibility of admitting gravitinos.

2.6. Dimension considerations. Elaborating on the discussion above, in a Theory of Everything one wishes to identify weight vectors in $V_{2,1}$ and $V_{1,2}$ with left- and right-handed fermions. As there are $3\times 15 = 45$ known fermions of each chirality, we find that the -1-eigenspace must have dimension at least $2\times 2\times 45 = 180$.

In case E is a real form of E_8, the -1-eigenspace has dimension 112 or 128 (this is implicit in Elie Cartan's classification of real forms of E_8 as in [8, p. 518, Table V]),[1] so no identification of the fermions as distinct weight vectors in $\mathrm{Lie}(\mathrm{E})$ (as in Table 9 in [1]) can be compatible with the Spin-Statistics Theorem and the existence of three generations.

[1] Alternatively, the marvelous bound on the trace from [9, Th. 3] implies that for every element x of order 2 in a reductive complex Lie group G, the -1-eigenspace of $\mathrm{Ad}(x)$ has dimension $\leq (\dim G + \operatorname{rank} G)/2$. In particular, when G is a real form of E_8, the -1-eigenspace has dimension ≤ 128.

3. $\mathfrak{sl}_2$ SUBALGEBRAS AND THE DYNKIN INDEX

3.1. In [10, §2], Dynkin defined the *index* of an inclusion $f : \mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2$ of simple complex Lie algebras as follows. Fix a Chevalley basis of the two algebras, so that the Cartan $\mathfrak{h}_1$ of $\mathfrak{g}_1$ is contained in the Cartan $\mathfrak{h}_2$ of $\mathfrak{g}_2$. The Chevalley basis identifies $\mathfrak{h}_i$ with the complexification $Q_i^\vee \otimes \mathbb{C}$ of the coroot lattice $Q_i^\vee$ of $\mathfrak{g}_i$, and the inclusion f gives an inclusion $Q_1^\vee \otimes \mathbb{C} \hookrightarrow Q_2^\vee \otimes \mathbb{C}$. Fix the Weyl-invariant inner product $(\ ,\ )_i$ on $Q_i^\vee$ so that $(\alpha^\vee, \alpha^\vee)_i = 2$ for short coroots $\alpha^\vee$. Then the *Dynkin index* of the inclusion is the ratio $(f(\alpha^\vee), f(\alpha^\vee))_2/(\alpha^\vee, \alpha^\vee)_1$ where $\alpha^\vee$ is a short coroot of $\mathfrak{g}_1$. For example, the irreducible representation $\mathfrak{sl}_2 \to \mathfrak{sl}_n$ has index $\binom{n+1}{3}$ by [10, Eq. (2.32)].

3.2. We now consider the case where $\mathfrak{g}_1 = \mathfrak{sl}_2$ and write simply $\mathfrak{g}$ and $Q^\vee$ for $\mathfrak{g}_2$ and $Q_2^\vee$. In §8 of that same paper (or see [11, §VIII.11]), Dynkin proved that after conjugating by an element of the automorphism group of $\mathfrak{g}$, one can assume that the Cartan subalgebra of $\mathfrak{sl}_2$ is contained in the given Cartan subalgebra of $\mathfrak{g}$ and that the image h of a simple root of $\mathfrak{sl}_2$ in $Q^\vee \otimes \mathbb{C}$ satisfies the strong restrictions:

$$h = \sum_{\delta \in \Delta} p_\delta \delta^\vee \quad \text{for } p_\delta \text{ real and non-negative [10, Lemma 8.3],}$$

where Δ denotes the set of simple roots of $\mathfrak{g}$ and further that

$$\delta(h) \in \{0, 1, 2\} \quad \text{for all } \delta \in \Delta.$$

But note that for each simple root δ, the fundamental irreducible representation of $\mathfrak{g}$ with highest weight dual to δ has weight p_δ (as a representation of $\mathfrak{sl}_2$), hence *p_δ is an integer.*

As a consequence of this and specifically [10, Lemma 8.2], one can identify an $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ up to conjugacy by writing the Dynkin diagram of $\mathfrak{g}$ and putting the number $\delta(h)$ at each vertex; this is the *marked Dynkin diagram* of the $\mathfrak{sl}_2$ subalgebra.

Here is an alternative formula for computing the index of an $\mathfrak{sl}_2$ subalgebra from its marked Dynkin diagram. Write $\kappa_\mathfrak{g}$ and $m^\vee$ for the Killing form and dual Coxeter number of $\mathfrak{g}$. We have:

$$\text{(Dynkin index)} = \frac{1}{2}(h,h) = \frac{1}{4m^\vee}\kappa_\mathfrak{g}(h,h) = \frac{1}{2m^\vee} \sum_{\text{positive roots } \alpha \text{ of } \mathfrak{g}} \alpha(h)^2, \tag{3.3}$$

where the second equality is by, e.g., [12, §5], and the third is by the definition of $\kappa_\mathfrak{g}$. One can calculate the number $\alpha(h)$ by writing α as a sum of positive roots and applying the marked Dynkin diagram for h.

Lemma 3.4. *For every simple complex Lie algebra $\mathfrak{g}$, there is a unique copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$ of index 1, up to conjugacy.*

Proof. The index of an $\mathfrak{sl}_2$-subalgebra is $(h,h)/2$, where the defining vector h belongs to the coroot lattice $Q^\vee$. If $\mathfrak{g}$ is not of type B, then the coroot lattice is not of type C, and the claim amounts to the statement that the vectors of minimal length in the coroot lattice are actually roots. This follows from the constructions of the root lattices in [13, §12.1].

Otherwise $\mathfrak{g}$ has type B and is $\mathfrak{so}_n$ for some odd $n \geq 5$. The conjugacy class of an $\mathfrak{sl}_2$-subalgebra is determined by the restriction of the natural n-dimensional representation; they are parameterized by partitions of n (i.e., $\sum n_i = n$) so that $n_1 \geq \cdots \geq n_s > 0$, $n_1 > 1$, and there are an even number of even n_i's [14, §6.2.2]. The index of the composition $\mathfrak{sl}_2 \to \mathfrak{so}_n \to \mathfrak{sl}_n$ is then $\sum \binom{n_i+1}{3}$; we must classify those partitions such that this sum equals the Dynkin index of $\mathfrak{so}_n \to \mathfrak{sl}_n$, which is 2. The unique such partition is $2 \geq 2 \geq 1 \geq \cdots \geq 1 > 0$. $\square$

If $\mathfrak{g}$ has type C, F_4, or G_2, then the same argument shows that there is up to conjugacy a unique copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$ with index 2, 2, or 3 respectively. We will see below that for $\mathfrak{g}$ of type B_6, there are two conjugacy classes of $\mathfrak{sl}_2$-subalgebras of index 2. This amounts to the fact that there are vectors in the C_6 root lattice that are not roots but have the same length as a root, cf. Exercise 5 in §12 of [13].

4. Copies of $\mathfrak{sl}_{2,\mathbb{C}}$ in the complex E_8

We first prove some facts about copies of $\mathfrak{sl}_{2,\mathbb{C}}$ in the complex Lie algebra $\mathfrak{e}_8$ of type E_8. The 69 conjugacy classes of such are listed in Table 20 on p. 182 of [10]; we are interested in just a few of these, without using Dynkin's table.

Fix a pinning for $\mathfrak{e}_8$; this includes a maximal toral subalgebra $\mathfrak{h}$, a set of simple roots $\mathrm{D} := \{\alpha_i \mid 1 \le i \le 8\}$ (numbered

$$\begin{matrix} 1 & 3 & 4 & 5 & 6 & 7 & 8 \\ & & 2 & & & & \end{matrix} \tag{4.1}$$

as in [15]), and fundamental weights ω_i dual to α_i. As all roots of the E_8 root system have the same length, we can and do identify the root system with its inverse root system.

Example 4.2. Taking any root of E_8, one can generate a copy of $\mathfrak{sl}_{2,\mathbb{C}}$ in $\mathfrak{e}_8$ with index 1. Doing this with the highest root gives an $\mathfrak{sl}_{2,\mathbb{C}}$ with marked Dynkin diagram

$$\text{index 1:} \quad \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & 0 & & & & \end{matrix}$$

Every index 1 copy of $\mathfrak{sl}_2$ in $\mathfrak{e}_8$ is conjugate to this one by Lemma 3.4.

Example 4.3. One can find a copy of $\mathfrak{sl}_{2,\mathbb{C}} \times \mathfrak{sl}_{2,\mathbb{C}}$ in $\mathfrak{e}_8$ by taking the first copy to be generated by the highest root of E_8 and the second copy to be generated by the highest root of the obvious E_7 subsystem. If you embed $\mathfrak{sl}_{2,\mathbb{C}}$ diagonally in this algebra, you find a copy of $\mathfrak{sl}_{2,\mathbb{C}}$ with index 2 and marked Dynkin diagram

$$\text{index 2:} \quad \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & & \end{matrix}$$

Proposition 4.4. *The following collections of copies of $\mathfrak{sl}_{2,\mathbb{C}}$ in $\mathfrak{e}_8$ are the same:*

(1) *copies such that ± 1 are weights of $\mathfrak{e}_8$ (as a representation of $\mathfrak{sl}_{2,\mathbb{C}}$) and no other odd weights occur.*
(2) *copies such that every weight of $\mathfrak{e}_8$ is in $\{0, \pm 1, \pm 2\}$.*
(3) *copies such that the inclusion $\mathfrak{sl}_{2,\mathbb{C}} \subset \mathfrak{e}_8$ has Dynkin index 1 or 2.*
(4) *copies of $\mathfrak{sl}_{2,\mathbb{C}}$ conjugate to one of those defined in Examples 4.2 or 4.3.*

The equality of (3) and (4) says that $\mathfrak{sl}_{2,\mathbb{C}}$'s of index 1 or 2 are determined (up to conjugacy) by their index. This is a small part of the data contained in Dynkin's Table 20, which is not used in our proof.

Proof of Proposition 4.4. One easily checks that (4) is contained in (1)–(3); we prove the opposite inclusion.

For (3), we identify $\mathfrak{h}$ with the complexification $Q \otimes \mathbb{C}$ of the (co)root lattice Q, hence h with $\sum \alpha_i(h)\omega_i$. By (3.3), the index of h satisfies:

$$\frac{1}{60}\sum_{\alpha} \alpha(h)^2 = \frac{1}{60}\sum_{\alpha}\left(\sum_i \alpha_i(h)\langle\omega_i,\alpha\rangle\right)^2 \ge \sum_i \left(\alpha_i(h)^2 \sum_{\alpha} \frac{\langle\omega_i,\alpha\rangle^2}{60}\right)$$

where the sums vary over the positive roots. We calculate for each fundamental weight ω_i the number $\sum_\alpha \langle \omega_i, \alpha \rangle^2 / 60$:

$$\begin{matrix} 2 & 7 & 15 & 10 & 6 & 3 & 1 \\ & & 4 & & & & \end{matrix} \tag{4.5}$$

As the numbers $\alpha_i(h)$ are all 0, 1, or 2, the numbers (4.5) show that h for an $\mathfrak{sl}_{2,\mathbb{C}}$ with Dynkin index 1 or 2 must be ω_1 (index 2) or ω_8 (index 1).

For (2), the highest root $\tilde{\alpha}$ of E_8 is $\tilde{\alpha} = \sum_i c_i \alpha_i$, where $c_1 = c_8 = 2$ and the other c_i's are all at least 3. As $\tilde{\alpha}(h)$ is a weight of $\mathfrak{e}_8$ relative to a given copy of $\mathfrak{sl}_{2,\mathbb{C}}$, we deduce that an $\mathfrak{sl}_{2,\mathbb{C}}$ as in (2) must have $h = \omega_1$ or ω_8, as claimed.

Suppose now that we are given an h for a copy of $\mathfrak{sl}_{2,\mathbb{C}}$ as in (1). As ± 1 occur as weights, there is at least one 1 in the marked Dynkin diagram.

But note that there cannot be three or more 1's in the marked Dynkin diagram for h. Indeed, for every connected subset S of vertices of the Dynkin diagram of E_8, $\sum_{i \in S} \alpha_i$ is a root [15, §VI.1.6, Cor. 3b]. If the number of 1's in the marked diagram of h is at least three, then one can pick S so that it meets exactly three of the α_i's with $\alpha_i(h) = 1$, in which case $\sum_{i \in S} \alpha_i(h)$ is odd and at least 3, violating the hypothesis of (1).

For sake of contradiction, suppose that there are two 1's in the marked diagram for h. If $\alpha_8(h)$ and $\alpha_i(h)$ are the two 1's, then one can find a root α of E_8 where $\langle \omega_i + \omega_8, \alpha \rangle$ is odd and at least 3, i.e., the coefficients of α_i and α_8 in α have opposite parities and sum at least 3. This is a contradiction, so we may assume that both 1's in the marked Dynkin diagram lie in the E_7 subdiagram. Repeating this argument with smaller and smaller subsystems shows the contradiction.

We are left with the case the there is exactly one 1 in the marked diagram for h. If $\alpha_i(h) = 1$ for some $i \neq 1, 8$, then we find a contradiction because there is a root α of E_8 with α_i-coordinate 3. Therefore $\alpha_i(h) = 1$ only for $i = 1$ or 8 and not for both. By the fact used two paragraphs above, $\beta := \sum_i \alpha_i$ is a root of E_8, so $\beta(h) = \sum \alpha_i(h)$ is odd and must be 1. It follows that $h = \omega_1$ or ω_8. $\square$

4.6. Centralizer for index 1. The $\mathfrak{sl}_{2,\mathbb{C}}$ of index 1 in $\mathfrak{e}_8$ has centralizer the regular subalgebra $\mathfrak{e}_7$ of type E_7. Indeed, it is clear that $\mathfrak{e}_7$ centralizes this $\mathfrak{sl}_{2,\mathbb{C}}$, and conversely $\mathfrak{e}_7 \times \mathfrak{sl}_{2,\mathbb{C}}$ is a maximal proper subalgebra of $\mathfrak{e}_8$, so the centralizer cannot be any larger.

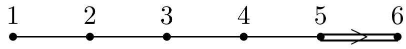
5. Index 2 copies of $\mathfrak{sl}_{2,\mathbb{C}}$ in the complex E_8

Lemma 5.1. *The centralizer of the index 2 $\mathfrak{sl}_{2,\mathbb{C}}$ in $\mathfrak{e}_8$ from Example 4.3 is a copy of $\mathfrak{so}_{13}$ contained in the regular subalgebra $\mathfrak{so}_{14}$ of $\mathfrak{e}_8$.*

Proof. The centralizer of the $\mathfrak{sl}_{2,\mathbb{C}}$ of index 2 in $\mathfrak{e}_8$ is contained in the centralizer of the defining vector h; this centralizer is reductive with semisimple part the regular subalgebra $\mathfrak{so}_{14}$ of type D_7. The centralizer of $\mathfrak{sl}_{2,\mathbb{C}}$ contains the centralizer of the $\mathfrak{sl}_{2,\mathbb{C}} \times \mathfrak{sl}_{2,\mathbb{C}}$ from Example 4.3, which is the regular subalgebra $\mathfrak{so}_{12}$ of type D_6. Furthermore, using the standard representation theory of $\mathfrak{sl}_{2,\mathbb{C}}$, it is easy to see that that the the centralizer of $\mathfrak{sl}_{2,\mathbb{C}}$ has dimension 78 (as is implicitly claimed in the statement of the lemma).

The regular $\mathfrak{so}_{14}$ does not centralize this $\mathfrak{sl}_{2,\mathbb{C}}$. On the other hand, the centralizer is strictly bigger than $\mathfrak{so}_{12}$ because it includes a copy of $\mathfrak{sl}_{2,\mathbb{C}}$ obtained by embedding diagonally in the $\mathfrak{sl}_{2,\mathbb{C}} \times \mathfrak{sl}_{2,\mathbb{C}}$ generated by nilpotents corresponding to the highest root of the D_7 subsystem and $-\alpha_8$. (To check that this $\mathfrak{sl}_{2,\mathbb{C}}$ centralizes the index 2 copy we started with, it is helpful to have a table of structure constants as in [16].) Together with $\mathfrak{so}_{12}$, this generates an $\mathfrak{so}_{13}$ subalgebra of $\mathfrak{so}_{14}$. Dimension count gives the claim. $\square$

5.2. Implicit in the proof above is an inclusion of $\mathfrak{so}_{13}$ in $\mathfrak{e}_8$ and a comparison of the pinnings of the two algebras, and in particular an inclusion of coroot lattices in terms of those pinnings. Number the simple roots of $\mathfrak{so}_{13}$ according to the diagram



We write $\beta_i^\vee$ for the simple coroot corresponding to the simple root i. Here is a translation table between the coroots of $\mathfrak{so}_{13}$ and the (co)roots of $\mathfrak{e}_8$:

$\mathfrak{so}_{13}$	$\beta_1^\vee$	$\beta_2^\vee$	$\beta_3^\vee$	$\beta_4^\vee$	$\beta_5^\vee$	$\beta_6^\vee$
$\mathfrak{e}_8$	α_3	α_4	α_5	α_6	α_7	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7$

The index 2 $\mathfrak{sl}_{2,\mathbb{C}}$ and the copy of $\mathfrak{so}_{13,\mathbb{C}}$ give an $\mathfrak{sl}_{2,\mathbb{C}} \times \mathfrak{so}_{13,\mathbb{C}}$ subalgebra of $\mathfrak{e}_{8,\mathbb{C}}$, and as a representation, $\mathfrak{e}_{8,\mathbb{C}}$ decomposes as a direct sum of irreducibles:

$$1 \otimes \mathfrak{so}_{13,\mathbb{C}} \quad \oplus \quad 2 \otimes (\text{spin}) \quad \oplus \quad 3 \otimes 1 \quad \oplus \quad 3 \otimes (\text{vector})$$

Example 5.3. By Dynkin's game of adding the highest root to the Dynkin diagram and deleting a vertex as in [10, §5], $\mathfrak{so}_{13,\mathbb{C}}$ contains a maximal subalgebra $\mathfrak{so}_{8,\mathbb{C}} \times \mathfrak{sp}_{4,\mathbb{C}}$. In turn, $\mathfrak{so}_8$ contains a maximal subalgebra $\mathfrak{sl}_2 \times \mathfrak{sp}_4$ [17, Th. 1.4]. This gives an $\mathfrak{sl}_2 \times \mathfrak{sp}_4 \times \mathfrak{sp}_4$ subalgebra of $\mathfrak{so}_{13}$.

We remark that this $\mathfrak{sl}_2$ has index 2 in $\mathfrak{e}_8$. As the inclusions $\mathfrak{so}_8 \subset \mathfrak{so}_{13} \subset \mathfrak{e}_8$ have index 1, it suffices to check that $\mathfrak{sl}_2$ has index 2 in $\mathfrak{so}_8$. This follows from the fact that the adjoint representation of $\mathfrak{so}_8$ has index 12, whereas its restriction to $\mathfrak{sl}_2$ decomposes as six copies of the 3-dimensional irreducible representation and a 10-dimensional trivial representation [18, p. 260], so has index $6 \cdot 4 + 10 \cdot 0 = 24$.

The main result of this section is the following:

Proposition 5.4. *Up to conjugacy, there is a unique copy of* $\mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}}$ *in* $\mathrm{E}_{8,\mathbb{C}}$ *so that each inclusion of* $\mathrm{SL}_{2,\mathbb{C}}$ *in* $\mathrm{E}_{8,\mathbb{C}}$ *has index 2. The Lie algebra of the centralizer of this* $\mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}}$ *is an* $\mathfrak{sp}_{4,\mathbb{C}} \times \mathfrak{sp}_{4,\mathbb{C}}$ *subalgebra of* $\mathfrak{so}_{13,\mathbb{C}}$ *in* $\mathfrak{e}_8$.

Proof. An analysis similar to the one in the proof of Proposition 4.4—but more complicated because there is more than one root length and Spin_{13} has a nontrivial center—shows that there are two conjugacy classes of copies of $\mathfrak{sl}_2$ in $\mathfrak{so}_{13}$ of index 2, corresponding to marked Dynkin diagrams

$$\text{(a)} \quad 2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0 \qquad \text{and} \quad \text{(b)} \quad 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0$$

We can pair each of (a) and (b) with the copy of $\mathfrak{sl}_2$ from Example 4.3 to get an $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ subalgebra of $\mathfrak{e}_8$ where both $\mathfrak{sl}_2$'s have index 2. Clearly, these represent the only two E_8-conjugacy classes of such subalgebras. With the marked Dynkin diagram in hand, it is not difficult to calculate the decomposition of $\mathfrak{e}_8$ into a direct sum of irreducible representations of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$.

In case (a), every irreducible summand $m \otimes n$ has $m + n$ even. Therefore, this copy of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ is the Lie algebra of a subgroup of E_8 isomorphic to $(\mathrm{SL}_2 \times \mathrm{SL}_2)/(-1,-1)$.

In case (b), we have the following table of multiplicities for $m \otimes n$:

(5.5)

		1	2	3	m
	1	20	20	6	
n	2	20	16	4	
	3	6	4	0	

In particular, it is the Lie algebra of a copy of $\mathrm{SL}_2 \times \mathrm{SL}_2$ in E_8. Combining the $\mathfrak{sl}_2$'s from Examples 4.3 and 5.3 gives an $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ subalgebra with both $\mathfrak{sl}_2$'s of index 2 and the same decomposition of $\mathfrak{e}_8$ into irreducible representations, so it is in the same conjugacy class. Clearly, its centralizer contains the copy of $\mathfrak{sp}_4 \times \mathfrak{sp}_4$ from Example 5.3, and the number 20 in the upper-left corner of (5.5) shows that $\mathfrak{sp}_4 \times \mathfrak{sp}_4$ is the whole centralizer. $\square$

5.6. We can decompose $\mathfrak{e}_8$ into a direct sum of irreducible representations of the $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sp}_4 \times \mathfrak{sp}_4$ subalgebra from the proposition by combining the decomposition of $\mathfrak{e}_8$ into irreducible representations of $\mathfrak{sl}_2 \times \mathfrak{so}_{13}$ from 5.2 with the tables in [18]. Recall that $\mathfrak{sp}_4$ has two fundamental irreducible representations: one that is 4-dimensional symplectic and another that is 5-dimensional orthogonal; we denote them by their dimensions. With this notation and 1.1, we find:

$$V_{2,1} = 5 \otimes 4, \quad V_{1,2} = 4 \otimes 5, \quad V_{2,3} = 1 \otimes 4, \quad \text{and} \quad V_{3,2} = 4 \otimes 1.$$

6. Copies of $\mathrm{SL}(2,\mathbb{C})$ in a real form of E_8

Suppose now that we have a copy of $\mathrm{SL}(2,\mathbb{C})$ inside a real Lie group E of type E_8. Over the complex numbers, we decompose $\mathrm{Lie}(\mathrm{E}) \otimes \mathbb{C}$ into a direct sum of irreducible representations of $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C} \cong \mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}}$; each irreducible representation can be written as $m \otimes n$ where m and n denote the dimension of an irreducible representation of the first or second $\mathrm{SL}_{2,\mathbb{C}}$ respectively. The goal of this section is to prove:

Proposition 6.1. *Maintain the notation of the previous paragraph. If* $\mathrm{Lie}(\mathrm{E})$ *contains no irreducible summands* $m \otimes n$ *with* $m+n > 4$*, then the identity component of the centralizer of* $\mathrm{SL}(2,\mathbb{C})$ *in* E *is a*

(1) *a regular subgroup* $\mathrm{Spin}(7,5)$ *if* E *is split; or*
(2) *a regular subgroup* $\mathrm{Spin}(9,3)$ *or* $\mathrm{Spin}(11,1)$ *if the Killing form of* $\mathrm{Lie}(\mathrm{E})$ *has signature* -24*.*

Proof. Complexifying the inclusion of $\mathrm{SL}(2,\mathbb{C})$ in E and going to Lie algebras gives an inclusion of $\mathfrak{sl}_{2,\mathbb{C}} \times \mathfrak{sl}_{2,\mathbb{C}}$ in the complex Lie algebra $\mathfrak{e}_8$ from §4. The hypothesis on the irreducible summands $m \otimes n$ amounts to the statement that each of the two $\mathfrak{sl}_{2,\mathbb{C}}$'s has index 1 or 2 by Proposition 4.4. As complex conjugation interchanges the two components, they must have the same index.

Suppose first that both $\mathfrak{sl}_2$'s have index 1. Lemma 3.4 (twice) gives that this $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ is conjugate to the one generated by the highest root of E_8 from Example 4.2 (so the second $\mathfrak{sl}_2$ belongs to the centralizer of type E_7) and by the highest root of the E_7 subsystem and makes up the first two summands of an $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{so}_{12}$ subalgebra as in [10, pp. 147, 148]. As this subalgebra has rank 8, it follows that the Lie algebra of the centralizer $\mathfrak{z}$ of $\mathrm{SL}(2,\mathbb{C})$ in E is a real form of $\mathfrak{spin}_{12}$.

We can decompose $\mathrm{Lie}(\mathrm{E}) \otimes \mathbb{C}$ into irreducible representations of $(\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{spin}_{12}) \otimes \mathbb{C}$ using the tables in [18] to find the adjoint representation plus

$$2 \otimes 1 \otimes S_+, \quad 1 \otimes 2 \otimes S_-, \quad \text{and} \quad 2 \otimes 2 \otimes V,$$

where $S_\pm$ denotes the half-spin representations of $\mathfrak{spin}_{12}$ and V is the vector representation. But $\mathrm{Lie}(\mathrm{E})$ is a real representation of $\mathfrak{z}$, so we deduce that V is also a real representation of $\mathfrak{z}$ but S_+ and S_- are not; they are interchanged by the Galois action. The first observation shows that $\mathfrak{z}$ is $\mathfrak{spin}(12-a, a)$ for some $0 \le a \le 6$. The second shows that a must be 1, 3, or 5, as claimed.

It remains to prove the correspondence between a and the real forms of E_8. For $a = 5$, this is clear: the subgroup generated by $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{Spin}(7,5)$ has real rank 6, so it can only be contained in the split real form. Now suppose that $a = 3$ or 1 and that $\mathrm{SL}(2,\mathbb{C})$

is in the split E_8; we will obtain a contradiction. Over $\mathbb{C}$, $\mathrm{SL}(2,\mathbb{C})$ is conjugate to the copy of $\mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}}$ in $\mathrm{E}_{8,\mathbb{C}}$ generated by the highest root of E_8 and the highest root of the natural subsystem of type E_7. One calculates using the tables in [15], e.g., that the element $-1 \in \mathrm{SL}(2,\mathbb{C})$, equivalently, $(-1,-1) \in \mathrm{SL}_2 \times \mathrm{SL}_2$ is $h_{\alpha_2}(-1)\, h_{\alpha_3}(-1)$ where h_{α_i} is the cocharacter corresponding to the coroot $\alpha_i^\vee$ as in [19]. That is, the subgroup fixed by conjugation by this -1 has root system consisting of roots α such that $\langle \omega_1, \alpha \rangle$ is even. These roots form the natural D_8 subsystem of E_8 and our $\mathrm{SL}(2,\mathbb{C}) \cdot \mathrm{Spin}(12-a,a)$ is a standard subgroup. The vector representation of this D_8 restricts to a sum of the vector representation of $\mathrm{Spin}(12-a,a)$ and the 4-dimensional vector representation of $\mathrm{SL}(2,\mathbb{C})$ (which factors through $\mathrm{SO}(3,1)$). In particular, this representation is real, so this D_8 subgroup is, according to [20, p. 161], isogenous to $\mathrm{SO}(8,8)$. But then the invariant symmetric bilinear form on the vector representation has signature 0 but restricts to have signature $\pm(12-2a) \in \{\pm 6, \pm 10\}$ and ± 2 on each summand, which cannot add up to get 0. This is a contradiction, so for $a = 3$ or 1, the real form of E_8 is neither split nor compact.

Now suppose that both $\mathfrak{sl}_2$'s have index 2. When we decompose $\mathfrak{e}_8$ as in 1.1, we find the representation $2 \otimes 3$ with positive multiplicity 4 by (5.5), which violates our hypothesis on the $\mathrm{SL}(2,\mathbb{C})$ subgroup of E. $\square$

7. No Theory of Everything in a real form of E_8

We now prove the second claim in Theorem 1.3, namely that each real form E of E_8 contains no ToE subgroups. Suppose E contains a copy of $\mathrm{SL}(2,\mathbb{C})$ (so E is non-compact) and a subgroup G satisfying (ToE1) and (ToE3). We will show that (ToE2) fails.

The -1-eigenspace in $\mathrm{Lie}(\mathrm{E})$ is a real representation of $\mathrm{SL}(2,\mathbb{C}) \cdot G$. By Proposition 6.1, G is contained in a copy of $\mathrm{Spin}(12-a,a)$ for $a = 1$, 3, or 5. As in the proof of Proposition 6.1, there is a representation W of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{Spin}(12-a,a)$ defined over $\mathbb{R}$ that is isomorphic to

$$(2 \otimes 1 \otimes S_+) \quad \oplus \quad (1 \otimes 2 \otimes S_-)$$

over $\mathbb{C}$. Now G is contained in the maximal compact subgroup of $\mathrm{Spin}(12-a,a)$, i.e., $\mathrm{Lie}(G)$ is a subalgebra of $\mathfrak{so}(11)$, $\mathfrak{so}(9) \times \mathfrak{so}(3)$, or $\mathfrak{so}(7) \times \mathfrak{so}(5)$. The restriction of the two half-spin representations of $\mathrm{Spin}(12-a,a)$ to the compact subalgebra are equivalent [18, p. 264], and we see that in each case the restriction is *quaternionic*. (To see this, one uses the standard fact that the spin representation of $\mathfrak{so}(2\ell+1)$ is real for $\ell \equiv 0, 3 \pmod 4$ and quaternionic for $\ell \equiv 1, 2 \pmod 4$.) That is, the restrictions of S_+, S_-, and their complex conjugates to the maximal compact subgroup are all equivalent (over $\mathbb{C}$), hence the same is true for their further restrictions to G, and (ToE2) fails. $\square$

Remark 7.1. It is worthwhile noting that, in each of the three cases, it is possible to embed G_{SM} in the centralizer, thus showing that (ToE1) is satisfied. Given such an embedding, a simple computation verifies explicitly that S_+ has a self-conjugate structure as a representation of G_{SM}.

First consider $\mathrm{Spin}(11,1)$. There is an obvious embedding of $G_{\mathrm{GUT}} := \mathrm{Spin}(10)$. Under this embedding, S_+ decomposes as the direct sum of the two half-spinor representations, *i.e.* as a generation and an anti-generation.

For $\mathrm{Spin}(7,5)$, there is an obvious embedding of the Pati-Salam group, $G_{\mathrm{GUT}} := (\mathrm{Spin}(6) \times \mathrm{Spin}(4))/(\mathbb{Z}/2\mathbb{Z})$. Again, S_+ decomposes as the direct sum of a generation and an anti-generation.

Finally, $\mathrm{Spin}(3,9)$ contains $(\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1))/(\mathbb{Z}/6\mathbb{Z})$ as a subgroup. Under this subgroup,

$$S_+ = (3,2,2)_{1/6}\oplus(\overline{3},2,2)_{-1/6}+(1,2,2)_{-1/2}+(1,2,2)_{1/2}$$

where the subscript indicates the $\mathrm{U}(1)$ weights, and the overall normalization is chosen to agree with the physicists' convention for the weights of the Standard Model's $\mathrm{U}(1)_Y$. Embedding the $\mathrm{SU}(2)$ of the Standard model in one of the two $\mathrm{SU}(2)$s, we obtain an embedding of $G_{\mathrm{SM}}\subset\mathrm{Spin}(3,9)$ where, again S_+ has a self-conjugate structure as a representation of G_{SM}.

8. No Theory of Everything in complex E_8

We now complete the proof of Theorem 1.3 by showing that there are no ToE subgroups in the transfer $\mathrm{E} := R(\mathrm{E}_{8,\mathbb{C}})$ of the complex Lie group of type E_8.

8.1. First, recall the transfer $R(G_{\mathbb{C}})$ of a complex group $G_{\mathbb{C}}$ as described, e.g., in [21, §2.1.2]. Its complexification can be viewed as $G_{\mathbb{C}}\times G_{\mathbb{C}}$, where complex conjugation acts via

$$\overline{(g,g')} = (\overline{g'},\overline{g}). \tag{8.2}$$

One can view $R(G_{\mathbb{C}})$ as the subgroup of the complexification consisting of elements fixed by (8.2).

Now consider an inclusion $\phi\colon \mathrm{SL}(2,\mathbb{C}) = R(\mathrm{SL}_{2,\mathbb{C}})\hookrightarrow R(\mathrm{E}_{8,\mathbb{C}})$. Complexifying, we identify $R(\mathrm{SL}_{2,\mathbb{C}})\times\mathbb{C}$ with $\mathrm{SL}_{2,\mathbb{C}}\times\mathrm{SL}_{2,\mathbb{C}}$ and similarly for $R(E_{8,\mathbb{C}})$ and write out ϕ as:

$$\phi(g_1,g_2) = (\phi_1(g_1)\phi_2(g_2),\psi_1(g_1)\psi_2(g_2)) \tag{8.3}$$

for some homomorphisms $\phi_1,\phi_2,\psi_1,\psi_2 : \mathrm{SL}_{2,\mathbb{C}}\to\mathrm{E}_{8,\mathbb{C}}$. As ϕ is defined over $\mathbb{R}$, we have:

$$\phi(g_1,g_2) = \overline{\phi(\overline{g_2},\overline{g_1})} = (\overline{\psi_1(\overline{g_2})\psi_2(\overline{g_1})},\overline{\phi_1(\overline{g_2})\phi_2(\overline{g_1})}),$$

and it follows that $\psi_1(g_1) = \overline{\phi_2(\overline{g_1})}$ and $\psi_2(g_2) = \overline{\phi_1(\overline{g_2})}$. Conversely, given any two homomorphisms $\phi_1,\phi_2\colon\mathrm{SL}_{2,\mathbb{C}}\to\mathrm{E}_{8,\mathbb{C}}$ (over $\mathbb{C}$) with commuting images, the same equations define a homomorphism $\phi\colon\mathrm{SL}(2,\mathbb{C})\to R(\mathrm{E}_{8,\mathbb{C}})$ defined over $\mathbb{R}$.

8.4. Plan of the proof. Now suppose that we have a subgroup $\mathrm{SL}(2,\mathbb{C})\cdot G$ of $R(\mathrm{E}_{8,\mathbb{C}})$ satisfying (ToE1) and (ToE3). Write C for the identity component of the centralizer of the image of the map $(\phi_1,\phi_2)\colon\mathrm{SL}_{2,\mathbb{C}}\times\mathrm{SL}_{2,\mathbb{C}}\to\mathrm{E}_{8,\mathbb{C}}$ from (8.3). Clearly, G is contained in the transfer $R(C)$ of C. In each of the cases below, we verify that

(8.5) C is semisimple,

so the maximal compact subgroup of $R(C)$ is the compact real form $C_{\mathbb{R}}$ of C. Furthermore, in each of the cases below, we will observe that

(8.6) -1 is in the Weyl group of C.

It follows that $C_{\mathbb{R}}$ is an inner form, hence every irreducible representation of $C_{\mathbb{R}}$ is real or quaternionic, hence every representation of $C_{\mathbb{R}}$ is self-conjugate. That is, (ToE2) fails, which is the desired contradiction.

8.7. Case 1: ϕ_1 or ϕ_2 is trivial. Consider the easiest-to-understand case where ϕ_1 or ϕ_2 is the zero map, say ϕ_2. That is, ϕ is the transfer of a homomorphism $\phi_1 : \mathrm{SL}_{2,\mathbb{C}}\to\mathrm{E}_{8,\mathbb{C}}$, which by Proposition 4.4 has index 1 or 2.

If ϕ_1 has index 1, then C is simply connected of type E_7, hence (8.5) and (8.6) hold. If ϕ_1 has index 2, then C is isogenous to $\mathrm{Spin}_{13,\mathbb{C}}$ by Lemma 5.1, and again (8.5) and (8.6) hold.

8.8. Case 2: ϕ_1 and ϕ_2 are injections. Finally, we consider the case where ϕ_1 and ϕ_2 are both injections. Again, (ToE3) and Proposition 4.4 implies that ϕ_1 and ϕ_2 have Dynkin index 1 or 2.

If ϕ_1 and ϕ_2 both have index 1, then over $\mathbb{C}$ this is the same embedding of $\mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}}$ in $\mathrm{E}_{8,\mathbb{C}}$ as the one in the proof of Proposition 6.1. The centralizer C is the standard D_6 subgroup of $\mathrm{E}_{8,\mathbb{C}}$.

If ϕ_1 and ϕ_2 both have index 2, then $\phi_1 \times \phi_2$ gives an embedding as in Proposition 5.4, and C has Lie algebra $\mathfrak{sp}_{4,\mathbb{C}} \times \mathfrak{sp}_{4,\mathbb{C}}$ of type $\mathrm{B}_2 \times \mathrm{B}_2$. Note that (ToE3) fails in this case by (5.5).

Suppose finally that ϕ_1 has index 1 and ϕ_2 has index 2. We conjugate so that $\phi_2(\mathfrak{sl}_2)$ is the copy of $\mathfrak{sl}_2$ from Example 4.3, and (by Lemma 3.4 for the centralizer $\mathfrak{so}_{13}$ of $\phi_2(\mathfrak{sl}_2)$) we can take $\phi_1(\mathfrak{sl}_2)$ to be a copy of $\mathfrak{sl}_2$ generated by the highest root of Spin_{13}. Calculating the weights of the representation of this $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ on $\mathfrak{e}_8$ gives the following decomposition into irreducibles:

(8.9)
$$\begin{array}{cc|cccc} & & 1 & 2 & 3 & m \\ \hline & 1 & 39 & 18 & 1 & \\ n & 2 & 32 & 16 & 0 & \\ & 3 & 10 & 2 & 0 & \end{array}$$

with the same notation as (5.5). In particular, the $\mathrm{A}_1 \times \mathrm{B}_4$ subgroup of Spin_{13} that centralizes the image of $\phi_1 \times \phi_2$ is all of the identity component C of the centralizer in E_8. Again (8.5) and (8.6) hold. (Of course, (8.9) shows that (ToE3) fails anyway.)

This completes the proof of Theorem 1.3. $\square$

9. RELAXING (TOE3)

Technically, $(m,n) = (2,3)$ and $(3,2)$ are *possible* in an interacting theory, but only in the presence of local supersymmetry (i.e., in supergravity theories) [22]. Lisi's framework is not compatible with local supersymmetry, so we excluded this possibility above. If we relax (ToE3) by replacing it with

(ToE3') $$V_{1,4} = V_{4,1} = 0 \text{ and } V_{m,n} = 0 \text{ if } m + n > 6$$

then we still don't find anything further. More precisely, we have the following strengthening of Theorem 1.3.

Theorem 9.1. *There are no subgroups $\mathrm{SL}(2,\mathbb{C}) \cdot G$ satisfying (ToE1), (ToE2), and (ToE3') in the (transfer of the) complex E_8 or any real form of E_8.*

Proof. Note that (ToE3') still forces that in the decomposition of $\mathrm{Lie}(\mathrm{E})$, the representation $V_{m,n} = 0$ if $m > 3$ or $n > 3$, so each of the two $\mathfrak{sl}_{2,\mathbb{C}}$ summands in the complexification of $\mathrm{SL}(2,\mathbb{C})$ have index 1 or 2 by Proposition 4.4. Looking back, we see that we already proved the theorem for the transfer of the complex E_8 in §8.

Suppose we have an $\mathrm{SL}(2,\mathbb{C}) \cdot G$ subgroup of a real form E of E_8. Imitating the proof in §7, we appeal to Proposition 6.1. We now have the additional possibility that the complexification $\mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}}$ of the $\mathrm{SL}(2,\mathbb{C})$ subgroup is such that both $\mathrm{SL}_{2,\mathbb{C}}$'s have index 2 as in §5. The centralizer of such an $\mathrm{SL}(2,\mathbb{C})$ is a real form of $\mathfrak{sp}_{4,\mathbb{C}} \times \mathfrak{sp}_{4,\mathbb{C}}$ by Proposition 5.4. When we decompose $\mathfrak{e}_8$ as in 1.1, we find $V_{2,1}$ and $V_{1,2}$ as in 5.6. As complex conjugation interchanges these two representations, it follows that complex conjugation interchanges the two $\mathfrak{sp}_{4,\mathbb{C}}$ factors, i.e., the centralizer of $\mathrm{SL}(2,\mathbb{C})$ has identity component the transfer $R(\mathrm{Sp}_{4,\mathbb{C}})$ of $\mathrm{Sp}_{4,\mathbb{C}}$. Its maximal compact subgroup is the compact form of $\mathrm{Sp}_{4,\mathbb{C}}$ (also known as $\mathrm{Spin}(5)$), all of whose irreducible representations are self-conjugate. Therefore, (ToE2) fails. $\square$

Remark 9.2. As we have already mentioned, weakening (ToE3) to (ToE3') is only consistent in supergravity theories. In the case at hand, with $G_{\text{max}} = \mathrm{Spin}(5)$, we find

$$V_{3,2} \simeq V_{2,3} = 4, \qquad V_{2,1} \simeq V_{1,2} = 4 \oplus 16 \tag{9.3}$$

where we have indicated the irreducible representations of $\mathrm{Spin}(5)$ by their dimensions. Since the gravitinos transform nontrivially under G_{max} and, given their multiplicity, the only consistent possibility would be a gauged $\mathcal{N} = 4$ supergravity theory (for a recent review of such theories, see [23]). Unfortunately, the rest of the matter content (it suffices to look at $V_{2,1}$) is not compatible with $\mathcal{N} = 4$ supersymmetry. Even if it were, $\mathcal{N} = 4$ supersymmetry would, of course, necessitate that the theory be non-chiral, making it unsuitable as a candidate Theory of Everything.

10. Conclusion

In paragraph 2.6 above, we observed by an easy dimension count that no proposed Theory of Everything constructed using subgroups of a real form E of E_8 has a sufficient number of weight vectors in the -1-eigenspace to identify with all known fermions. The proof of our Theorem 1.3 was quite a bit more complicated, but it also gives much more. It shows that you cannot obtain a *chiral* gauge theory for *any ToE subgroup* of E, whether E is a real form or the complex form of E_8. In particular, it is impossible to obtain even the one-generation Standard Model in this fashion.

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(DISTLER) THEORY GROUP, DEPARTMENT OF PHYSICS, AND TEXAS COSMOLOGY CENTER, UNIVERSITY OF TEXAS, AUSTIN, TX 78712
E-mail address: `distler@golem.ph.utexas.edu`

(GARIBALDI) DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE, 400 DOWMAN DR., EMORY UNIVERSITY, ATLANTA, GA 30322
E-mail address: `skip@member.ams.org`