QUATERNION ALGEBRAS WITH THE SAME SUBFIELDS

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ABSTRACT. G. Prasad and A. Rapinchuk asked if two quaternion division F-algebras that have the same subfields are necessarily isomorphic. The answer is known to be "no" for some very large fields. We prove that the answer is "yes" if F is an extension of a global field K so that F/K is unirational and has zero unramified Brauer group. We also prove a similar result for Pfister forms and give an application to tractable fields.

1. INTRODUCTION

Gopal Prasad and Andrei Rapinchuk asked the following question in Remark 5.4 of their paper [PR]:

(1.1)   *If two quaternion division algebras over a field F have the same maximal subfields, are the algebras necessarily isomorphic?*

The answer is "no" for some fields F, see §2 below. The answer is "yes" if F is a global field by the Albert-Brauer-Hasse-Minkowski Theorem [NSW, 8.1.17]. Prasad and Rapinchuk note that the answer is unknown even for fields like $\mathbb{Q}(x)$. We prove that the answer is "yes" for this field:

Theorem 1.2. *Let F be a field of characteristic $\neq 2$ that is transparent. If D_1 and D_2 are quaternion division algebras over F that have the same maximal subfields, then D_1 and D_2 are isomorphic.*

The term "transparent" is defined in §6 below. Every retract rational extension of a local, global, real-closed, or algebraically closed field is transparent; in particular $K(x_1, \ldots, x_n)$ is transparent for every global field K of characteristic $\neq 2$ and every n. There are many other examples.

The theorem can be viewed as a statement about symbols in the Galois cohomology group $H^2(F, \mathbb{Z}/2\mathbb{Z})$. We also state and prove an analogue for symbols in Galois cohomology of $H^d(F, \mathbb{Z}/2\mathbb{Z})$—i.e., d-Pfister quadratic forms—in Theorem 9.1 below.

Notation. A *global field* is a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_p(t)$ for some prime p. A *local field* is a completion of a global field with respect to a discrete valuation.

2. An example

Example 2.1. Several people (in no particular order: Markus Rost, Kelly McKinnie, Adrian Wadsworth, Murray Schacher, Daniel Goldstein...) noted that some hypothesis on the field F is necessary for the conclusion of the theorem to hold. Here is an example to illustrate this.

Let F_0 be a field of characteristic $\neq 2$. We follow [Lam 05] for notation on quadratic forms, except that we define the symbol $\langle\!\langle a_1, \ldots, a_d \rangle\!\rangle$ to be the d-Pfister form $\otimes_{i=1}^d \langle 1, -a_i \rangle$. For a Pfister form ϕ, we put ϕ' for the unique form such that $\phi \cong \langle 1 \rangle \oplus \phi'$.

Suppose that F_0 has quaternion division algebras Q_1, Q_2 that are not isomorphic. Write $\phi_i := \langle\!\langle a_i, b \rangle\!\rangle$ for the norm form of Q_i. The Albert form $\phi_1 - \phi_2$ has anisotropic part similar to $\gamma = \langle\!\langle a_1 a_2, b \rangle\!\rangle$.

Fix an extension E/F_0 and a proper quadratic extension $E(\sqrt{c})$ contained in Q_i but not Q_{i+1}, with subscripts taken modulo 2. Equivalently, $-c$ is represented by ϕ_i' but not by ϕ_{i+1}'. Put $q_{i,c} := \langle c \rangle \oplus \phi_{i+1}'$; it is anisotropic. Further, its determinant c is a nonsquare, so $q_{i,c}$ is not similar to ϕ_i, ϕ_{i+1}, nor γ. All three of those forms remain anisotropic over the function field $E(q_{i,c})$ of $q_{i,c}$ by [Lam 05, X.4.10(3)]. That is, $Q_1 \otimes E(q_{i,c})$ and $Q_2 \otimes E(q_{i,c})$ are division and contain $E(q_{i,c})(\sqrt{c})$, and the two algebras are still distinct.

We build a field F from F_0 inductively. Suppose we have constructed the field F_j for some $j \geq 0$. Let F_{j+1} be the composita of the extensions $F_j(q_{i,c})$ for c as in the previous paragraph with $E = F_j$ and $i = 1, 2$. Let F be the colimit of the F_j.

By construction, $Q_1 \otimes F$ and $Q_2 \otimes F$ are both division, are not isomorphic to each other, and have the same quadratic subfields. This shows that some hypothesis on the field F is necessary for the conclusion of the theorem to hold.

One cannot simply omit the hypothesis "division" in the theorem, by Example 3.6.

Furthermore, Theorem 1.2 does not generalize to division algebras of degree > 2, even over number fields. One can easily construct examples using the local-global principle for division algebras over a number field, see [PR, Example 6.5] for details.

3. Discrete valuations: good residue characteristic

Fix a field F with a discrete valuation v. We write $\bar{F}$ for the residue field and $\hat{F}$ for the completion. (Throughout this note, for fields with a discrete valuation and elements of such a field, we use bars and hats to indicate the corresponding residue field/completion and residues/image over the completion.)

For a central simple F-algebra A, we write $\hat{A}$ for $A \otimes \hat{F}$. It is Brauer-equivalent to a central division algebra B over $\hat{F}$. As v is complete on $\hat{F}$, it extends uniquely to a valuation on B by setting $v(b) := \frac{1}{\deg B} v(\mathrm{Nrd}_B(b))$ for

$b \in B^{\times}$, see, e.g., [R, 12.6]. We say that A is *unramified* if $v(B^{\times}) = v(F^{\times})$—e.g., if $\hat{F}$ splits A—and *ramified* if $v(B^{\times})$ is strictly larger. That is, A is ramified or unramified if v is so on B. Note that these meanings for ramified and unramified are quite a bit weaker than the the usual definitions (which usually require that the residue division algebra of B be separable over $\bar{F}$).

The valuation ring in $\hat{F}$ is contained in a noncommutative discrete valuation ring S in B, and the residue algebra of S is a (possibly commutative) division algebra $\bar{B}$ whose center contains $\bar{F}$. One has the formula

$$\dim_{\bar{F}} \bar{B} \cdot [v(B^{\times}) : v(F^{\times})] = \dim_{\hat{F}} B$$

(see [W, p. 393] for references). That is, A is unramified if and only if $\dim_{\bar{F}} \bar{B} = \dim_{\hat{F}} B$.

We use a few warm-up lemmas.

Lemma 3.1. *Let A_1, A_2 be central simple algebras over a field F that has a discrete valuation. If there is a separable maximal commutative subalgebra that embeds in one of the $\hat{A}_i$ and not the other, then the same is true for the A_i.*

Proof. Let $\hat{L}$ be a separable maximal commutative subalgebra of $\hat{A}_1$ that is not a maximal subalgebra of $\hat{A}_2$. Choose $\hat{\alpha} \in \hat{L}$ so that $\hat{L} = \hat{F}(\hat{\alpha})$. Since A_1 is dense in $\hat{A}_1$, we can choose $\alpha \in A_1$ as close as we want to $\hat{\alpha}$. Thus we can choose the Cayley-Hamilton polynomial of α to be as close as we need to that of $\hat{\alpha}$. In particular, by Krasner's Lemma (applied as in [R, 33.8]), we can assume that $\hat{F}(\alpha) \cong \hat{F}(\hat{\alpha})$. In particular, $\hat{F}(\alpha)$ is not a maximal subalgebra of $\hat{A}_2$, which implies that $F(\alpha)$ is not a maximal subalgebra of A_2 and is by construction a maximal subfield of A_1. Since $\hat{F}(\alpha) \cong \hat{F}(\hat{\alpha})$, the minimal polynomial of α over F has nonzero discriminant and hence is separable. $\square$

Lemma 3.2. *Let A_1, A_2 be central division algebras of prime degree over a field F with a discrete valuation. If*

    (1) *$\hat{A}_1$ is split and $\hat{A}_2$ is not; or*

    (2) *both $\hat{A}_1$ and $\hat{A}_2$ are division, but only one is ramified,*

then there is a separable maximal subfield of A_1 or A_2 that does not embed in the other.

Proof. The statement is obvious from Lemma 3.1. In case (1), we take take the diagonal subalgebra in $\hat{A}_1$. In case (2), we take a maximal separable subfield that is ramified. $\square$

For the rest of this section, we fix a prime p and let F be a field with a primitive p-th root of unity ζ; we suppose that F has a discrete valuation v with residue field of characteristic different from p.

A *degree p symbol algebra* is an (associative) F-algebra generated by elements α, β subject to the relations $\alpha^p = a$, $\beta^p = b$, and $\alpha\beta = \zeta\beta\alpha$. It is central simple over F [D, p. 78] and we denote it by $(a, b)_F$ or simply (a, b).

Lemma 3.3. *If A is a degree p symbol algebra over F, then A is isomorphic to $(a, b)_F$ for some $a, b \in F^\times$ with $v(a) = 0$.*

Proof. Write A as (a, b) where $a = x\pi^{n_a}$ and $b = y\pi^{n_b}$ where x, y have value 0 and π is a uniformizer for F. We may assume that p divides neither n_a nor n_b, so $n_a = sn_b$ modulo p. But A is isomorphic to $(a(-b)^{-s}, b)$ because $(-b, b)$ is split [D, p. 82, Cor. 5], and $v(a(-b)^{-s})$ is divisible by p. $\square$

3.4. Let (a, b) be a symbol algebra as in Lemma 3.3 and write $\hat{F}$ for the completion of F at v. If $(a, b) \otimes \hat{F}$ is not division, certainly (a, b) is *unramified*. Otherwise, there are two cases:

- If $v(b)$ is divisible by p, then we can assume $v(b) = 0$, (a, b) is unramified, and $\overline{(a, b)}$ is the symbol algebra $(\bar{a}, \bar{b})_{\bar{F}}$. (The residue algebra— which is division—contains $(\bar{a}, \bar{b})_{\bar{F}}$, hence they must be equal.)
- If $v(b)$ is not divisible by p, then (a, b) is ramified and $\overline{(a, b)}$ is the field extension $\bar{F}(\sqrt[p]{\bar{a}})$. (By hypothesis, a is not a p-th power in $\hat{F}$, so by Hensel's Lemma $\bar{a}$ is not a p-th power in $\bar{F}$.)

Proposition 3.5. *Let A_1 and A_2 be degree p symbol algebras over F that are division algebras. If*

(1) *A_1 ramifies and A_2 does not; or*
(2) *A_1 and A_2 both ramify but their ramification defines different extensions of the residue field $\bar{F}$,*

then there is a maximal subfield L of one of the A_i that does not split the other.

Proof. Write A_i as (a_i, b_i). We can assume $v(a_1) = v(a_2) = 0$. If both the A_i ramify, then the $v(b_i)$ must be prime to p and the field extensions of the ramification of the A_i must be $\bar{F}(\sqrt[p]{\bar{a}_i})$. If these are different fields, then the field $L_2 := F(\sqrt[p]{a_2})$ cannot split A_1 because it does not split the ramification of A_1.

Thus assume A_2 is unramified and A_1 ramifies. In particular, $A_1 \otimes \hat{F}$ is division, and we are done by Lemma 3.2. $\square$

Example 3.6. In the proposition, the hypothesis that A_1 and A_2 are both division is necessary. Indeed, for $\ell \neq 2$, the field $\mathbb{Q}_\ell$ of ℓ-adic numbers has two quaternion algebras: one split and one division. Only one is ramified, but both are split by every quadratic extension of $\mathbb{Q}_\ell$.

4. DISCRETE VALUATIONS: BAD RESIDUE CHARACTERISTIC

In this section, we consider quaternion algebras over a field with a dyadic discrete valuation. The purpose of this section is to prove:

Proposition 4.1. *Let A_1, A_2 be quaternion division algebras over a field F of characteristic 0 that has a dyadic discrete valuation. If $A_1 \otimes A_2$ is Brauer-equivalent to a ramified quaternion division algebra with residue algebra a*

separable quadratic extension of $\bar{F}$, then there is a quadratic extension of F that embeds in A_1 or A_2 but not both.

What really helps nail down the structure of $A_1 \otimes A_2$ is the hypothesis that the residue algebra is a separable quadratic extension. In particular, the division algebra underlying $A_1 \otimes A_2$ is "inertially split" in the language of [JW].

Example 4.2. Let B be a quaternion division algebra over a field F of characteristic 0 that is complete with respect to a dyadic discrete valuation, and suppose that the residue algebra $\bar{B}$ contains a separable quadratic extension $\bar{L}$ of F. One can lift $\bar{L}/\bar{F}$ to a *unique* quadratic extension L/F and write $B = (L, b)$ for some $b \in F^\times$ that we may assume has value 0 or 1. One finds two possibilities:

- If $v(b) = 0$, then $\bar{B}$ is a quaternion algebra over $\bar{F}$ generated by elements i, j such that
$$\bar{F}(i) = \bar{L}, \ i^2 + i \in \bar{F}, \ j^2 = \bar{b}, \quad \text{and} \quad ij = j(i+1).$$
  In this case B is unramified.
- If $v(b) = 1$, then $\bar{B} = \bar{L}$ and B is ramified.

For detailed explanation of these claims, see, e.g., [GP]. (Unlike the situation in the preceding section, there are more than just these two possibilities for general quaternion algebras over F, but in those other cases the residue algebra $\bar{B}$ is a purely inseparable extension of $\bar{F}$ of dimension 2 or 4 with $\bar{B}^2 \subseteq \bar{F}$. Note that if L/F is unramified and $\bar{L}/\bar{F}$ is purely inseparable, then L/F is *not* uniquely determined by $\bar{L}/\bar{F}$.)

Proof of Proposition 4.1. Put B for the quaternion division F-algebra underlying $A_1 \otimes A_2$. By the hypothesis on its residue algebra, B remains division when we extend scalars to the completion $\hat{F}$. Hence at least one of $\hat{A}_1$, $\hat{A}_2$ is division; by Lemma 3.2 (using that A_1 and A_2 are division) we may assume that both are division and that the valuation has the same ramification index on both algebras. By Lemma 3.1, we may assume that F is complete.

The residue division algebra $\bar{A}_1$ is distinct from $\bar{F}$ and we can find a quadratic subfield L of A_1 so that the residue field $\bar{L}$ is a quadratic extension of $\bar{F}$. As L is a maximal subfield of A_1, it is also one for A_2 and hence B. Certainly, $\bar{L}$ is a subfield of the residue algebra $\bar{B}$; as both have dimension 2 over $\bar{F}$ they are equal. That is, $\bar{L}$ is separable. We write $A_i = (L, a_i)$ and $B = (L, b)$ for some $a_i, b \in F^\times$. As the valuation does not ramify on L we may assume that $v(b) = 1$. By the previous paragraph, we may write $A_i = (L, a_i)$ for some $a_i \in F^\times$. As $\bar{A}_i$ contains the separable extension $\bar{L}/\bar{F}$, we have $v(a_1) = v(a_2)$ as in Example 4.2. Then $A_1 \otimes A_2 \otimes B$—which is split—is Brauer-equivalent to $(L, a_1 a_2 b)$, which is ramified (because $v(a_1 a_2 b) = 1 + 2v(a_i)$ is odd) and in particular not split. This is a contradiction, which completes the proof of the proposition. $\square$

5. Unramified cohomology

For a field F of characteristic not 2, we write $H^d(F)$ for the Galois cohomology group $H^d(F, \mathbb{Z}/2\mathbb{Z})$. We remind the reader that for $d = 1$, this group is $F^\times/F^{\times 2}$ and for $d = 2$ it is identified with the 2-torsion in the Brauer group of F.

Definition 5.1. Fix an integer $d \geq 2$. We define $H_u^d(F)$ to be the subgroup of $H^d(F)$ consisting of classes that are unramified at every nondyadic discrete valuation of F (in the usual sense, as in [GMS, p. 19]) and are killed by $H^d(F) \to H^d(R)$ for every real closure R of F.

Example 5.2.   (1) A local field F (of characteristic $\neq 2$) has $H_u^2(F) = 0$ if and only if F is non-dyadic. Indeed, F has no orderings, so it suffices to consider the unique discrete valuation on F.
   (2) A global field F (of characteristic $\neq 2$) has $H_u^2(F) = 0$ if and only if F has at most 1 dyadic place. This is Hasse's local-global theorem for central simple algebras [NSW, 8.1.17].
   (3) A real closed field F has $H_u^d(F) = 0$ for all $d \geq 2$; this is trivial.

A local field F (of characteristic $\neq 2$) has $H_u^d(F) = 0$ for $d \geq 3$. This is trivial because such a field has no orderings and cohomological 2-dimension 2.

Lemma 5.3. *If $F(\sqrt{-1})$ has cohomological 2-dimension $< d$, then $H_u^d(F)$ is zero.*

Proof. By [A, Satz 3], an element $x \in H^d(F)$ is zero at every real closure if and only if $x \cdot (-1)^r$ is zero for some $r \geq 0$. On the other hand, there is an exact sequence for every n [KMRT, 30.12(1)]:

$$H^n(F(\sqrt{-1})) \xrightarrow{\text{cor}} H^n(F) \xrightarrow{\cdot(-1)} H^{n+1}(F) \xrightarrow{\text{res}} H^{n+1}(F(\sqrt{-1})$$

For $n \geq d$, the two end terms are zero, so the cup product with $(-1)^r$ defines an isomorphism $H^d(F) \xrightarrow{\sim} H^{d+r}(F)$ for all $r \geq 0$. $\square$

We immediately obtain:

Corollary 5.4. *A global field F of characteristic $\neq 2$ has $H_u^d(F) = 0$ for all $d \geq 3$.* $\square$

Corollary 5.5. *If F has transcendence degree d over a real-closed field, then $H_u^{d+1}(F) = 0$.*

Proof. $F(\sqrt{-1})$ has transcendence degree d over an algebraically closed field, hence has cohomological dimension $\leq d$ [Se, §II.4.3]. $\square$

We define a discrete valuation of an extension F/F_0 to be a discrete valuation on F that vanishes on F_0.

Proposition 5.6. *For every extension F/F_0, the homomorphism* res: $H^d(F_0) \to H^d(F)$ *restricts to a homomorphism $H^d_u(F_0) \to H^d_u(F)$. If additionally F/F_0 is unirational and*

(5.7)   *every class in $H^d(F)$ that is unramified at every discrete valuation of F/F_0 comes from $H^d(F_0)$,*

then the homomorphism $H^d_u(F_0) \to H^d_u(F)$ is an isomorphism.

Proof. Fix $y \in H^d_u(F_0)$. Every ordering of F restricts to an ordering on F_0; write R and R_0 for the corresponding real closures. By hypothesis, the image of y in $H^d(R_0)$ is zero, hence it also has zero image under the composition $H^d(F_0) \to H^d(R_0) \to H^d(R)$. It is easy to see that res(y) is unramified at every nondyadic discrete valuation of F, and the first claim follows.

We now prove the second sentence. The natural map $H^d(F_0) \to H^d(F)$ is injective by [GMS, p. 28]. Fix $x \in H^d_u(F)$. As it is unramified at every discrete valuation of F/F_0, it is the restriction of some element $x_0 \in H^d(F_0)$ by (5.7). We will show that x_0 belongs to $H^d_u(F_0)$.

Fix an extension $E := F_0(x_1, x_2, \ldots, x_n)$ containing F. Every ordering on F_0 extends to an ordering v on E (and hence also on F) by a recipe as in [Lam 80, pp. 9–11]. As the map $H^d((F_0)_v) \to H^d(F_v)$ is an injection by Lemma 5.8 below, we deduce that x_0 is killed by every real-closure of F_0.

One extends a nondyadic discrete valuation v on F_0 to E by setting $v(x_i) = 0$ and $\bar{x}_i$ to be transcendental over $\bar{F}_0$ as in [B, §VI.10.1, Prop. 2]; then the residue field of $\bar{E}$ is $\bar{F}_0(x_1, x_2, \ldots, x_n)$ and the natural map $H^{d-1}(\bar{F}_0) \to H^{d-1}(\bar{E})$ is injective, hence so is $H^{d-1}(\bar{F}_0) \to H^{d-1}(\bar{F})$. It follows that the image of x_0 in $H^{d-1}(\bar{F}_0)$ is zero, which completes the proof. $\square$

Here is the promised lemma:

Lemma 5.8. *If $R_1 \subseteq R_2$ are real-closed fields, then the natural map $H^d(R_1) \to H^d(R_2)$ is an isomorphism for every $d \geq 0$.*

Proof. $H^d(R_i)$ is the usual group cohomology $H^d(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$. Obviously $H^0(R_i) = H^1(R_i) = \mathbb{Z}/2\mathbb{Z}$, and 2-periodicity for the cohomology of finite cyclic groups shows that $H^d(R_i) = \mathbb{Z}/2\mathbb{Z}$ for all $d \geq 0$, with nonzero element $(-1)^d$. $\square$

Example 5.9. Recall that an extension F of F_0 is *retract rational* if it is the field of fractions of an F_0-algebra domain S and there are F_0-algebra homomorphisms $S \to F_0[x_1, \ldots, x_n](1/s) \to S$ whose composition is the identity on S, where $F_0[x_1, \ldots, x_n]$ is a polynomial ring and $s \in F_0[x_1, \ldots, x_n]$ is a nonzero element. Obviously such an extension is unirational, and (5.7) holds by [Sa, 11.8] or [M, Prop. 2.15], so $H^d_u(F_0) \cong H^d_u(F)$.

6. Transparent fields

Definition 6.1. Fix a field F of characteristic $\neq 2$. For each dyadic valuation v, fix a completion $\hat{F}$ of F. Write K for the limit of the finite

unramified extensions of $\hat{F}$ with separable residue field. This extension is called the *maximal unramified extension of $\hat{F}$* (which highlights the fact that usually the word "unramified" is quite a bit stronger than how we are using it). There is a well-defined ramification map

$$H^2(K/\hat{F}, \mu_2) \to H^1(\bar{F}, \mathbb{Z}/2\mathbb{Z}) \quad \text{s.t.} \quad (a, b) \mapsto \begin{cases} 0 & \text{if } v(a) = v(b) = 0 \\ (\bar{a}) & \text{if } v(a) = 0 \ \& \ v(b) = 1 \end{cases}$$

We say that F is *transparent* if for every division algebra $D \neq F$ of exponent 2 such that $[D]$ belongs to $H_u^2(F)$, there is some dyadic place v of F and completion $\hat{F}$ such that

(1) $D \otimes \hat{F}$ is split by the maximal unramified extension of $\hat{F}$, and
(2) the image of $[D]$ in $H^1(\bar{F}, \mathbb{Z}/2\mathbb{Z})$ is not zero.

Another way to phrase these hypotheses is: $D \otimes \hat{F}$ *is division and the residue division algebra $\bar{D}$ is a separable quadratic extension of $\bar{F}$.*

Roughly speaking, transparent fields are those for which nonzero 2-torsion elements of the Brauer group can be detected using ramification.

Example 6.2.   (0) If $H_u^2(F) = 0$—e.g., if F is real closed—then F is (vacuously, trivially) transparent.

(1) *If F is local, then F is transparent.* If F is nondyadic, this is Example 5.2(1). Otherwise, for each nonzero $x \in H_u^2(F)$, the residue division algebra is necessarily a separable quadratic extension of $\bar{F}$, because $\bar{F}$ is perfect with zero Brauer group.

(2) If F is global, then F is transparent. Indeed, $H_u^2(F)$ consists of classes ramified only at dyadic places, and every such place has finite residue field.

The *unramified Brauer group* of an extension F/F_0 is the subgroup of the Brauer group of F consisting of elements that are unramified at every discrete valuation of F/F_0.

Proposition 6.3. *If F_0 is transparent, F/F_0 is unirational, and the unramified Brauer group of F/F_0 is zero, then F is transparent.*

Proof. Suppose there is a nonzero $x \in H_u^2(F)$. By Proposition 5.6, x is the image of some nonzero $x_0 \in H_u^2(F_0)$ and there is a dyadic valuation v of F_0 such that conditions (1) and (2) of Definition 6.1 holds for the division algebra D_0 represented by x_0. Extend v to a valuation on F as in the proof of Proposition 5.6. The maximal unramified extension of $\hat{F}$ (with respect to v) contains the maximal unramified extension of $\hat{F_0}$, so it kills x_0 and hence also x; this verifies condition (6.1.1). For condition (6.1.2), it suffices to note that the map $H^1(\bar{F_0}, \mathbb{Z}/2\mathbb{Z}) \to H^1(\bar{F}, \mathbb{Z}/2\mathbb{Z})$ is injective as in the proof of Proposition 5.6. $\square$

Example 6.4. A retract rational extension of a transparent field is transparent, cf. Example 5.9.

7. Proof of the main theorem

We now prove Theorem 1.2. We assume that D_1 and D_2 are not isomorphic and that there is some quadratic extension of F that is contained in both of them, hence that $D_1 \otimes D_2$ has index 2. We will produce a quadratic extension of F that is contained in one and not the other.

Suppose first that there is a real closure R of F that does not split $D_1 \otimes D_2$. Then one of the D_i is split by R and the other is not; say D_1 is split. We can write $D_1 = (a_1, b_1)$ where a_1 is positive in R. The field $F(\sqrt{a_1})$ is contained in D_1, but not contained in D_2 because $D_2 \otimes R$ is division.

If there is a nondyadic discrete valuation of F where $D_1 \otimes D_2$ is ramified, then Proposition 3.5 provides the desired quadratic extension.

Finally, suppose that $D_1 \otimes D_2$ is split by every real closure and is unramified at every nondyadic place, so $D_1 \otimes D_2$ belongs to $H_u^2(F)$. By the original hypotheses on D_1 and D_2, the class $[D_1 \otimes D_2]$ is represented by a quaternion division algebra B. But F is transparent, so Proposition 4.1 provides a quadratic extension of F that embeds in D_1 or D_2 but not the other. $\qquad\square$

8. Pfister forms and nondyadic valuations

With the theorem from the introduction proved, we now set to proving a version of it for Pfister forms. The goal of this section is the following proposition, which is an analogue of Proposition 3.5.

Proposition 8.1. *Let F be a field with a nondyadic discrete valuation. Let ϕ_1, ϕ_2 be d-Pfister forms over F for some $d \geq 2$. If*

(1) *ϕ_1 is ramified and ϕ_2 is not; or*
(2) *ϕ_1 and ϕ_2 both ramify, but with different ramification,*

then there is a $(d-1)$-Pfister form γ over F such that γ divides ϕ_1 or ϕ_2 but not both.

The proof given below for case (1) of the proposition was suggested to us by Adrian Wadsworth, and is much simpler than our original proof for that case. It makes use of the following lemma:

Lemma 8.2. *Let $\psi = \langle a_1, \dots, a_n \rangle$ be a nondegenerate quadratic form over a field F with a nondyadic discrete valuation v. If $v(a_i) = 0$ for all i and ψ is isotropic over the completion $\hat{F}$ of F, then there is a 2-dimensional nondegenerate subform q of ψ such that $q \otimes \hat{F}$ is isotropic.*

Proof. As $\phi \otimes \hat{F}$ is isotropic, there exist $x_i \in \hat{F}$ not all zero so that $\sum a_i x_i^2 = 0$. Scaling by a uniformizer, we may assume that $v(x_i) \geq 0$ for all i and that at least one x_i has value 0. Let j be the smallest index such that $v(x_j) = 0$; there must be at least two such indices, so $j < n$.

Fix t_i in the valuation ring of F so that $\bar{t}_i = \bar{x}_i$, and put

$$r := \sum_{i=1}^{j} a_i t_i^2 \quad \text{and} \quad s := \sum_{i=j+1}^{n} a_i t_i^2.$$

By construction of j, $v(r) = 0$. Since $r + s = 0$, also $v(s) = 0$. Now $\langle r, s \rangle$ is a subform of ψ (over F), and

$$\bar{r} + \bar{s} = \overline{r + s} = \overline{\sum a_i t_i^2} = \overline{\sum a_i x_i^2} = 0.$$

So $\langle r, s \rangle$ is isotropic over $\hat{F}$ [Lam 05]. $\qquad\square$

Proof of Proposition 8.1. Obviously, $\langle\langle x\pi, y\pi \rangle\rangle \cong \langle\langle -xy, y\pi \rangle\rangle$, so we may assume that

$$\phi_i = \langle\langle a_{i2}, a_{i3}, \ldots, a_{id}, b_i \rangle\rangle$$

where a_{ij} has value 0 and b_i has value 0 or 1.

In case (2), b_1 and b_2 have value 1 and the residue forms

$$r(\phi_i) = \langle\langle \bar{a}_{i2}, \bar{a}_{i3}, \ldots, \bar{a}_{id} \rangle\rangle$$

are not isomorphic. We take γ to be $\langle\langle a_{12}, a_{13}, \ldots, a_{1d} \rangle\rangle$; it divides ϕ_1. The projective quadric X defined by $\gamma = 0$ is defined over the discrete valuation ring, and $\bar{F}(X)$ does not split $r(\phi_2)$ by [Lam 05, X.4.10]. Hence $F(X)$ does not split ϕ_2 and γ does not divide ϕ_2.

Now suppose we are in case (1); obviously $v(b_1) = 1$. If $\phi_2 \otimes \hat{F}$ is anisotropic, then as ϕ_2 does not ramify, $v(b_2) = 0$ and the second residue form of ϕ_2 is zero. Certainly $\gamma := \langle\langle a_{13}, \ldots, a_{1d}, b_1 \rangle\rangle$ divides ϕ_1. On the other hand, every anisotropic multiple of $\gamma \otimes \hat{F}$ over $\hat{F}$ will have a nonzero second residue form, so ϕ_2 cannot contain γ.

Otherwise, $\phi_2 \otimes \hat{F}$ is hyperbolic, and there exists some 2-dimensional subform q of ϕ_2 that becomes hyperbolic over $\hat{F}$ by Lemma 8.2. It follows that there is a $(d-1)$-Pfister ψ that is contained in ϕ_2 and that becomes hyperbolic over $\hat{F}$. But $\phi_1 \otimes \hat{F}$ is ramified, hence anisotropic, so ϕ_1 cannot contain ψ. $\qquad\square$

9. Theorem for quadratic forms

Theorem 9.1. *Let F be a field of characteristic $\neq 2$ such that $H_u^d(F) = 0$ for some $d \geq 2$. Let ϕ_1 and ϕ_2 be anisotropic d-Pfister forms over F such that, for every $(d-1)$-Pfister form γ, we have: γ divides ϕ_1 if and only if γ divides ϕ_2. Then ϕ_1 is isomorphic to ϕ_2.*

We remark that for $d = 2$, Theorem 9.1 does not include the case where F is a retract rational extension of a global field with more than one dyadic place, a case included in Theorem 1.2.

Proof of Theorem 9.1. Suppose that ϕ_1 and ϕ_2 are d-Pfister forms over F for some $d \geq 2$, $H_u^d(F)$ is zero, and ϕ_1 is not isomorphic to ϕ_2. Then $\phi_1 + \phi_2$ is in $H^d(F) \setminus H_u^d(F)$.

If there is a nondyadic discrete valuation where $\phi_1 + \phi_2$ ramifies, then Proposition 8.1 gives a $(d-1)$-Pfister form that divides one of the ϕ_i and not the other. Otherwise, there is an ordering v of F such that ϕ_1 and ϕ_2 are not isomorphic over the real closure F_v, i.e., one is locally hyperbolic and the other is not. Write $\phi_i = \langle\!\langle a_{i1}, a_{i2}, \ldots, a_{id} \rangle\!\rangle$ and suppose that ϕ_1 is locally hyperbolic, so some a_{1j} is positive; renumbering if necessary, we may assume that it is a_{11}. Then the form $\langle\!\langle a_{12}, a_{13}, \ldots, a_{1d} \rangle\!\rangle$ divides ϕ_1, but it is hyperbolic over F_v and so does not divide ϕ_2. $\qquad\square$

10. Appendix: tractable fields

We now use the notion of unramified cohomology from §5 to prove some new results and give new proofs of some known results regarding tractable fields. Recall from [CTW] that a field F of characteristic $\neq 2$ is called *tractable* if for every a_1, a_2, a_3, b_1, b_2, $b_3 \in F^\times$ such that the six quaternion algebras (a_i, b_j) are split for all $i \neq j$ and the three quaternion algebras (a_i, b_i) are pairwise isomorphic, the algebras (a_i, b_i) are necessarily split. (This condition was motivated by studying conditions for decomposability of central simple algebras of degree 8 and exponent 2.) We prove:

Proposition 10.1. *If $H_u^2(F) = 0$, then F is tractable.*

Proof. For sake of contradiction, suppose we are given a_i, b_i as in the definition of tractable where the quaternion algebras (a_i, b_i) are not split. If the (a_i, b_i) do not split at some real closure of F, then the a_i and b_i are all negative there. Hence (a_i, b_j) for $i \neq j$ is also not split, a contradiction.

So the (a_i, b_i) ramify at some nondyadic discrete valuation v of F and in particular are division over a the completion of F at v. We replace F with its completion and derive a contradiction. Modifying each a_i or b_i by a square if necessary, we may assume that each one has value 0 or 1. Further, as each (a_i, b_i) is ramified, at most one of the two slots has value 0.

Suppose first that one of the a_i or b_i—say, a_1—has value 0. Then b_1 has value 1 and a_1 is not a square in F. As (a_1, b_j) is split for $j \neq 1$, we deduce that $v(b_j) = 0$. Hence $v(a_2) = v(a_3) = 1$. As (a_3, b_3) is division, b_3 is not a square in F. Since $v(a_2) = 1$ and $v(b_3) = 0$, it follows that (a_2, b_3) is division, a contradiction.

It remains to consider the case where the a_i's and b_i's all have value 1. We can write $a_i = \pi \alpha_i$ and $b_i = \pi \beta_i$, where π is a prime element—say, a_1 so $\alpha_1 = 1$—and α_i, β_i have value 0. In the Brauer group of F, we have for $i \neq j$:

$$0 = (\pi \alpha_i, \pi \beta_j) = (\pi \alpha_i, -\alpha_i \beta_j) = (\pi, -\alpha_i \beta_j) + (\alpha_i, \beta_j).$$

It follows that $-\alpha_i \beta_j$ is a square in F for $i \neq j$. In particular, $-\beta_2$, $-\beta_3$, α_2, and α_3 are all squares. Then

$$(a_2, b_2) = (\pi \alpha_2, \pi \beta_2) = (\pi, -\pi) = 0,$$

a contradiction. $\qquad\square$

Example 10.2. In [CTW], the authors proved that a local field is tractable if and only if it is nondyadic (ibid., Cor. 2.3) and a global field is tractable if and only if it has at most 1 dyadic place (ibid., Th. 2.10). Combining the preceding proposition and Example 5.2 gives a proof of the "if" direction for both of these statements.

Corollary 10.3 ([CTW, Th. 3.17]). *Every field of transcendence degree 1 over a real-closed is tractable.*

Proof. Combine the proposition and Corollary 5.5. □

Example 10.4. The converse of Proposition 10.1 is false. For every odd prime p, $\mathbb{Q}_p((x))$ is tractable by [CTW, Prop. 2.5]. On the other hand, the usual discrete valuation on the formal power series ring is essentially the unique one [EP, 4.4.1], so $H_u^2(\mathbb{Q}_p((x))) = H^2(\mathbb{Q}_p) = \mathbb{Z}/2\mathbb{Z}$.

Corollary 10.5. *Let F_0 be a field of characteristic $\neq 2$ that*
  - *is tractable and*
  - *is global, local, real-closed, or has no 2-torsion in its Brauer group.*

Then every unirational extension F/F_0 satisfying (5.7)—*e.g., every retract rational extension F/F_0—is also tractable.*

Proof. We note that F_0 has $H_u^2(F_0) = 0$. In case F_0 is global or local, then F_0 has one dyadic place or is non-dyadic respectively by [CTW], and the claim follows, see Example 5.2. Then $H_u^2(F)$ is also zero by Proposition 5.6, hence F is tractable. □

Of course, this corollary includes the statement that rational extensions of tractable global fields are tractable, which was a nontrivial result (Example 3.14) in [CTW]. However, a stronger result has already been proved in [Han, Cor. 2.14]: every rational extension of a tractable field is tractable.

We do get a large family of new examples of tractable fields:

Corollary 10.6. *Let F be a global field with at most one dyadic place. If G is an isotropic, simply connected, absolutely almost simple linear algebraic group over F, then the function field $F(G)$ is tractable.*

Proof. The extension $F(G)/F$ is retract rational by Theorems 5.9 and 8.1 in [Gi]. □

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