Isomorphism criteria for Witt rings of real fields

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Abstract. We prove isomorphism criteria for Witt rings and reduced Witt rings of certain types of real fields. Refined criteria are obtained under the additional assumption that the field be SAP. This leads to a generalization of a result by Koprowski on Witt equivalence of function fields of transcendence degree 1 over a real closed field. Isomorphism criteria are also obtained for Witt rings of hermitian forms over a quadratic extension of a real base field and for Witt groups of hermitian forms over a quaternion algebra with a real field as center. All these criteria are expressed in terms of properties involving topological subspaces of the space of orderings of the base field.

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1 Introduction

Throughout, we will only consider fields of characteristic different from 2. In [5], Harrison gave necessary and sufficient conditions for two fields to have isomorphic Witt rings:

Theorem 1.1 (Harrison)**.** *Let K and L be two fields. Then the following are equivalent:*

(1) There is a ring isomorphism $W(K) \simeq W(L)$.

(2) There is a group isomorphism $t : K^*/K^{*2} \to L^*/L^{*2}$ sending -1 to -1 such that the quadratic form $\langle 1, -x, -y, xy\rangle$ is hyperbolic over K if and only if the quadratic form $\langle 1, -t(x), -t(y), t(x)t(y)\rangle$ is hyperbolic over L for all $x,\ y \in K^*$.

Over global fields, in [10], Perlis, Szymiczek, Conner and Litherland showed that the previous criterion can be expressed in terms of (nontrivial) places of the considered fields:

Theorem 1.2 (Perlis, Szymiczek, Conner, Litherland)**.** *Let K and L be two global fields and denote by Ω_K (resp. Ω_L) the set of nontrivial places over K (resp. L). Then the following are equivalent:*

(1) There is a ring isomorphism $W(K) \simeq W(L)$.

(2) There is a pair of maps (t, T) with a group isomorphism $t : K^*/K^{*2} \to L^*/L^{*2}$ and a bijection $T : \Omega_K \to \Omega_L$ such that we have the following equality of Hilbert symbols

$$(x, y)_P = (tx, ty)_{T(P)}$$

for all $x,\ y \in K^*/K^{*2}$ and for all $P \in \Omega_K$.

If K is a field, denote by X_K the topological space of all orderings in K. The purpose of this paper is to prove isomorphism criteria for several types of Witt rings or Witt groups over real base fields. It turns out that our results can be stated in the same way as Theorem 1.2 by replacing the set of nontrivial places by certain subspaces of X_K and local conditions on Hilbert symbols by local positivity conditions.

Before we state our first main result in the case of Witt rings of real fields, let us recall that the u-invariant $u(K)$ of a field is defined as

$$u(K) = \sup\{\dim q \mid q \text{ is an anisotropic torsion quadratic form over } K\}\ ,$$

where q is said to be torsion if its Witt class is a torsion element in $W(K)$.

Theorem 1.3. *Let K and L be two real fields. Consider the following three statements:*

(1) There is a ring isomorphism $W(K) \simeq W(L)$.

(2) There is a pair of maps (t, T) with a group isomorphism $t : K^*/K^{*2} \to L^*/L^{*2}$ and a homeomorphism $T : X_K \to X_L$ such that x is positive at P if and only if $t(x)$ is positive at $T(P)$, for all $x \in K^*/K^{*2}$ and for all $P \in X_K$.

(3) There is a pair of maps (t, T) with a group isomorphism $t : K^*/K^{*2} \to L^*/L^{*2}$ and a bijection $T : X_K \to X_L$ such that x is positive at P if and only if $t(x)$ is positive at $T(P)$, for all $x \in K^*/K^{*2}$ and for all $P \in X_K$.

Then (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3)*. If, in addition, $u(K),\ u(L) \leq 2$, then these statements are equivalent.*

A crucial ingredient in the proof is another result due to Harrison which asserts that there is a bijection between the ideals of characteristic 0 in the spectrum of the Witt ring of a field and the orderings of that field. This result and several other facts and notations are recalled in the first part of Section 2, the second part being dedicated to the proof of Theorem 1.3.

In Section 3, we further assume that the base fields are SAP fields, i.e. that they have the strong approximation property, see Definition 3.2. Our second main result is the following refinement of Theorem 1.3:

Theorem 1.4. *Let K and L be two SAP fields such that $u(K),\ u(L) \leq 2$. Then the following are equivalent:*

(1) There is a ring isomorphism $W(K) \simeq W(L)$.

(2) There is a homeomorphism $X_K \simeq X_L$ and a group isomorphism $\dot\sigma(K)/K^{*2}$ $\simeq \dot\sigma(L)/L^{*2}$.

Here, and in the sequel, $\sigma(K)$ stands for the set of elements of K that can be expressed as a sum of squares in K, and we write $\dot\sigma(K)$ for $\sigma(K) \setminus \{0\}$ (which is a subgroup of K^*). The proof of this result very much relies on a study of the ring structure of the set of clopen subspaces of X_K. This is done in the first two parts of Section 3, with the last part containing the proof of Theorem 1.4 and a slight generalization of a result of Koprowski about the Witt equivalence of real algebraic function fields over real closed fields, see Corollary 3.6.

In the last two sections, we adapt the ideas used in Section 2 and 3 to other situations. In the case of reduced Witt rings, it turns out that analogs of Theorem 1.3 and 1.4 can be proved: see Theorem 4.3 and Corollary 4.6. Section 4 is dedicated to the proof of these two results after recalling some elementary facts about preorderings and reduced Witt rings.

In Section 5, we show how to generalize Theorem 1.3 in the case of the Witt ring of hermitian forms over a quadratic extension of a real field endowed with its nontrivial automorphism, and the Witt group of hermitian forms over a quaternion division algebra central over a real field and endowed with its canonical involution. The corresponding results are stated as Theorem 5.4 and Theorem 5.9. As for usual Witt rings, an important step in the proof of each of these results is to determine bijections between certain subspaces of the space of orderings and certain subsets of the spectra of the Witt rings adapted to each situation.

2 Witt rings

Before turning to the proof of Theorem 1.3, let us fix some terminology and recall some basic facts about Witt rings and orderings. For further informations, we refer to [9].

2.1 Notations and preliminary results

Let K be a field. The notation $\langle a_1, \cdots, a_n\rangle$ will refer to the diagonal quadratic form $a_1X_1^2 + \cdots + a_nX_n^2$ where the a_i's belong to K^*. We shall denote by $W(K)$ the *Witt ring* of K and by $I(K)$ its fundamental ideal. The nth power of the fundamental ideal shall be denoted by $I^n(K)$. It is additively generated by the *n-fold Pfister forms*

$$\langle\langle a_1, \cdots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle.$$

The *signed discriminant* of a (diagonal) quadratic form $\langle a_1, \cdots, a_n\rangle$ is defined to be the following element of K^*/K^{*2}:

$$(-1)^{\frac{n(n+1)}{2}} a_1 \cdots a_n.K^{*2}.$$

It induces a group isomorphism $d_\pm : I(K)/I^2(K) \simeq K^*/K^{*2}$.

An *ordering* on K is a subset $P \subsetneq K$ such that $P + P \subseteq P$, $P.P \subseteq P$, $P \cup -P = K$. It follows that $-1 \notin P$ and that $P \cap -P = \{0\}$. ¿From the very definition, any ordering P contains $\sigma(K)$. By Artin-Schreier theory, a field K

has an ordering if and only if $-1 \notin \sigma(K)$, in which case K is called *formally real*, or *real* for short. Furthermore, $\sigma(K) = \bigcap P$ where P runs over all orderings of K. If P is an ordering and $x \in K^*$, then x is said to be *positive* (resp. *negative*) at P, denoted by $x \underset{P}{>} 0$ (resp. $x \underset{P}{<} 0$), if $x \in P$ (resp. $-x \in P$). Note that we can talk about the positivity or the negativity of any square class, a square being always positive at any ordering. The set X_K of all orderings on K is a topological space with a subbasis given by the clopen sets

$$H(a) := \{P \in X_K \mid a \in P\},$$

where a ranges over K^*. This topology is known as the *Harrison topology.*

The signature of a quadratic form q at an ordering P shall be denoted by $\operatorname{sgn}_P(q)$. It induces a surjective ring homomorphism $\operatorname{sgn}_P : W(K) \to \mathbb{Z}$.

Recall that by Pfister's local-global principle, the torsion quadratic forms over a real field K are exactly the forms q with total signature zero, i.e. $\operatorname{sgn}_P(q) = 0$ for all $P \in X_K$. Hence, $u(K)$, if finite, is always even for a real field K.

A prime ideal I in a ring R is said to be of characteristic p where p is a prime number (resp. 0) if the quotient ring R/I has characteristic p (resp. 0) and we will say that char $I = p$ (resp. 0). In the case where $R = W(K)$, we easily see that a prime ideal I has characteristic p (resp. 0) if and only if the quotient $W(K)/I$ is isomorphic (as a ring) to $\mathbb{Z}/p\mathbb{Z}$ (resp. $\mathbb{Z}$), see [9, Chapter VIII]. Following [9, Chapter VIII], let

$$Y_K := \{I \in \operatorname{Spec}\bigl(W(K)\bigr) \mid \operatorname{char} I = 0\}.$$

Theorem 2.1 (Harrison)**.** *There is a bijection between X_K and Y_K.*

Proof. The bijection is defined as follows. To any $P \in X_K$, we can associate the ideal $I_P := \ker \operatorname{sgn}_P \in Y_K$. Conversely, from $I \in Y_K$, one can construct an ordering by setting $P_I := \{0\} \cup \{a \in K^* \mid \langle a \rangle \equiv 1 \mod I\}$. See [9, Chapter VIII, Proposition 7.4] for more details. □

2.2 Proof of Theorem 1.3

During the proof of Theorem 1.3, we will need the following Lemma:

Lemma 2.2. *Let K and L be real fields. Suppose that there is a pair of maps (t, T) with a group isomorphism $t : K^*/K^{*2} \to L^*/L^{*2}$ and a bijection $T : X_K \to X_L$ such that x is positive at P if and only if $t(x)$ is positive at $T(P)$ for all $x \in K^*/K^{*2}$ and for all $P \in X_K$. Then there exists a pair (t, T) with the above properties and in addition $t(-1) = -1$.*

Proof. By assumption, $-t(-1)$ is positive at all orderings of L, so $-t(-1) \in \dot\sigma(L)$, say $t(-1) = -\sum_{i=1}^n {u_i}^2$. Now if $\sum_{i=1}^n {u_i}^2 \neq 1 \in L^*/L^{*2}$, then let $\{-1, -\sum_{i=1}^n {u_i}^2, \cdots\}$ be a $\mathbb{F}_2$-basis of the $\mathbb{F}_2$-vector space L^*/L^{*2}. Let $\tilde{t} : L^*/L^{*2} \to L^*/L^{*2}$ be the $\mathbb{F}_2$-vector space automorphism which exchanges -1 and $-\sum_{i=1}^n {u_i}^2$ but fixes all other basis vectors. Let $t_1 = \tilde{t} \circ t$. Now, t_1 is a group isomorphism between K^*/K^{*2} and L^*/L^{*2} and $t_1(-1) = -1$. Moreover, by decomposing any element $t(x) \in L^*/L^{*2}$ in the previous $\mathbb{F}_2$-basis, we easily see that $t(x)$ is positive at Q if and only if $t_1(x)$ is positive at Q for any $Q \in X_L$. □

Proof of Theorem 1.3: (1) $\Rightarrow$ (2) We will use the notations of the proof of Theorem 2.1. Let Φ be a ring isomorphism between $W(K)$ and $W(L)$. By Theorem 1.1, there exists a group isomorphism $t : K^*/K^{*2} \to L^*/L^{*2}$ with $t(-1) = -1$. More precisely, if $a \in K^*/K^{*2}$, $t(a)$ is defined to be the signed discriminant of $\Phi(\langle 1, -a\rangle)$: $t(a) = d_{\pm}(\Phi(\langle 1, -a\rangle)) \in L^*/L^{*2}$.

Now, we show how to construct T. As Φ is a ring isomorphism, it is easy to see that $I \in Y_K$ if and only if $\Phi(I) \in Y_L$. Now, consider the following diagram:

$$\begin{tikzcd} X_K \arrow[r, shift left, "\lambda_K"] \arrow[d, dotted, "T"] \arrow[dr, phantom, "\circlearrowleft"] & Y_K \arrow[l, shift left, "\mu_K"] \arrow[d, "\Phi"] \\ X_L \arrow[r, shift left, "\lambda_L"] & Y_L \arrow[l, shift left, "\mu_L"] \end{tikzcd}$$

In this diagram, $\lambda_* : X_* \to Y_* : P \mapsto I_P$ is the bijection in Theorem 2.1, and $\mu_* : Y_* \to X_* : I \mapsto P_I$ is its inverse.

Define $T = \mu_L \circ \Phi \circ \lambda_K : X_K \to X_L$. More precisely, if $P \in X_K$, then $T(P) := \{0\} \cup \{b \in L^* \mid \langle b\rangle \equiv 1 \mod \Phi(I_P)\}$. In particular, $q \in I_P$ if and only if $\Phi(q) \in I_{T(P)}$.

Next, we show the compatibility of (t, T) with respect to the positivity at each ordering. Let $x \in K^*/K^{*2}$ and $P \in X_K$ such that $x \underset{P}{>} 0$. This means that $\langle 1, -x\rangle \in I_P$, thus $\Phi(\langle 1, -x\rangle) \in I_{T(P)}$. By construction, $t(x)$ is the signed discriminant of $\Phi(\langle 1, -x\rangle)$. Let $\Phi(\langle x\rangle) = \langle b_1, \cdots, b_{2n+1}\rangle$ (as a ring homomorphism, Φ sends the units of $W(K)$ onto the units of $W(L)$ which are necessarily Witt classes of forms of odd dimension). We easily verify that

$$t(x) = (-1)^n b_1 \cdots b_{2n+1}.$$

As

$$0 = \mathrm{sgn}_{T(P)}\big(\Phi(\langle 1, -x\rangle)\big) = \mathrm{sgn}_{T(P)}(\langle 1, -b_1, \cdots, -b_{2n+1}\rangle),$$

there are exactly n (resp. $n+1$) b_i's which are negative (resp. positive) at $T(P)$. Therefore, $t(x)$ is positive at $T(P)$. The converse is similar.

Lastly, we show that T is in fact a homeomorphism. To show that T is continuous with respect to the corresponding Harrison topologies, it suffices to show that $T^{-1}\big(H(b)\big)$ is open in X_K for every $b \in L^*$. Let $b \in L^*$ and $a \in K^*$ be such that $t(a) = b$. Now for each $P \in X_K$, we have that a is positive at P if and only if b is positive at $T(P)$. This readily implies that $T^{-1}\big(H(b)\big) = H(a)$ thus proving that the bijection T is continuous. Similarly, T^{-1} is continuous and T is a homeomorphism.

(2) $\Leftrightarrow$ (3) If (3) holds, then clearly $T(H(a)) = H(t(a))$ for all $a \in K^*$, and by the definition of the Harrison topology, T is clearly a homeomorphism which establishes (2). The converse is trivial.

Finally, assume that $u(K), u(L) \leq 2$.

(3) $\Rightarrow$ (1) The signature at an ordering of any n-fold Pfister form is 0 or 2^n, so we have

$$\mathrm{sgn}_P\big(\langle\langle x, y\rangle\rangle\big) \quad = \quad \mathrm{sgn}_{T(P)}\big(\langle\langle t(x), t(y)\rangle\rangle\big), \tag{2.1}$$

for all $x,\ y \in K^*/K^{*2}$ and for all $P \in X_K$. If $x,\ y \in K^*/K^{*2}$, as $u(K) \leq 2$, the 2-fold Pfister form $\langle\langle x, y\rangle\rangle$ is hyperbolic if and only if $\langle\langle x, y\rangle\rangle$ is torsion which in turn is equivalent to the fact that $\operatorname{sgn}_P\big(\langle\langle x, y\rangle\rangle\big) = 0$ for every $P \in X_K$, by Pfister's local-global principle (see [9, Chapter VIII, Theorem 3.2]). Using (2.1), Pfister's local-global principle and the fact that $u(L) \leq 2$, we obtain that the 2-fold Pfister form $\langle\langle x, y\rangle\rangle$ is hyperbolic over K if and only if the 2-fold Pfister form $\langle\langle t(x), t(y)\rangle\rangle$ is hyperbolic over L. Now, we get condition (1) by using Lemma 2.2 and Theorem 1.1. □

Remarks 2.3. We have $u(K) \leq 2$ if and only if $I^2(K)$ is torsionfree (see [9, Chapter XI, Proposition 6.26 (2)]). For example, if K is an extension of transcendence degree 1 over a real closed field k, then $u(K) \leq 2$ (see [3, Theorems E, I]).

3 SAP fields

The purpose of this section is to prove isomorphism criteria for Witt rings over SAP fields. We first study the space of orderings of such a field.

3.1 Homeomorphisms of spaces of orderings of real fields

Let $\mathcal{X}$ be a topological space that is boolean (= compact, totally disconnected and Hausdorff). Then the set of clopen subsets $\operatorname{Clop}(\mathcal{X})$ is a boolean ring with addition given by the symmetric difference $\triangle$ and multiplication given by the intersection $\cap$. It is clear that homeomorphic boolean spaces $\mathcal{X}$ and $\mathcal{X}'$ give rise to isomorphic boolean rings $\operatorname{Clop}(\mathcal{X})$ and $\operatorname{Clop}(\mathcal{X}')$.

Furthermore, if B is a boolean ring then $\operatorname{Spec}(B)$ becomes a boolean topological space under the Zariski topology, and again it is clear that isomorphic boolean rings B and B' give rise to homeomorphic boolean spaces $\operatorname{Spec}(B)$ and $\operatorname{Spec}(B')$.

It is a consequence of Stone duality that any boolean space $\mathcal{X}$ is homeomorphic to $\operatorname{Spec}(\operatorname{Clop}(\mathcal{X}))$ (see [14]).

Since spaces of orderings of fields are boolean spaces under the Harrison topology, we can apply the above to get

Corollary 3.1. *Let K and L be real fields. Then the following are equivalent:*

(1) There is a homeomorphism $\Phi : X_K \to X_L$.

(2) There is a ring isomorphism $\Psi : (\operatorname{Clop}(X_K), \triangle, \cap) \to (\operatorname{Clop}(X_L), \triangle, \cap)$.

3.2 The group structure of $(K^*/\dot{\sigma}(K), \times)$ for SAP fields

Definition 3.2. A field K satisfies the Strong Approximation Property (we also say that K is SAP for short) if one of the following equivalent conditions holds:

(1) For any two disjoint closed subsets $A,\ B$ of X_K, there exists an $a \in K^*$ such that a is positive at every $P \in A$ (i.e. $A \subset H(a)$) and negative at every $P \in B$ (i.e. $B \subset H(-a)$).

(2) If A is a clopen subset of X_K, then $A = H(a)$ for some $a \in K^*$.

Note that a nonreal field may be called SAP as both conditions are empty. It is known that algebraic number fields, fields of transcendence degree at most 1 over a real closed field or the field of Laurent series in one variable over any uniquely ordered real field are SAP fields.

In the sequel, for any $a \in K^*$, we put

$$H^-(a) := \{P \in X_K \mid -a \in P\} .$$

Then $H^-(a) = H(-a) \in \mathrm{Clop}(X_K)$, and we have $H^-(a) \triangle H^-(b) = H^-(ab)$.

The purpose of this subsection is to give two descriptions of the multiplicative group $K^*/\dot{\sigma}(K)$ in the case where K is an SAP field. For us, the more important one is the following.

Lemma 3.3. *Let K be real and SAP. Then the map*

$$t : \; K^*/\dot{\sigma}(K) \to (\mathrm{Clop}(X_K), \triangle) \; : \; a \mapsto H^-(a)$$

is a well-defined group isomorphism.

Proof. Clearly, t is well-defined, and it is surjective as K is SAP. Moreover $H^-(a) = H^-(b)$ if and only if ab is positive at all $P \in X_K$ if and only if $ab \in \dot{\sigma}(K)$ by Artin's Theorem. This proves that t is bijective. Also, $H^-(a) \triangle H^-(b) = H^-(ab)$, thus proving that t is a group isomorphism. □

Remark 3.4. In general, one always has a group isomorphism $K^*/\dot{\sigma}(K) \simeq (\mathcal{H}, \triangle)$ as above (where $\mathcal{H}$ denotes the set of all $H^-(a)$) as was first noticed by Craven in [2]. In the case where K is SAP, we obviously have $\mathcal{H} = \mathrm{Clop}(X_K)$.

For $n \in \mathbb{N}$, denote by $I_t^n(K)$ the ideal of torsion forms in $I^n(K)$, i.e. $I_t^n(K) = I^n(K) \cap W_t(K)$. The following result (which won't be used in the sequel) is another way to describe the group $K^*/\dot{\sigma}(K)$ under a more restictive hypothesis.

Proposition 3.5. *Let K be a field with with $I_t^n(K) = 0$ for some $n \in \mathbb{N}$.*

(1) The map

$$\Phi : K^*/\dot{\sigma}(K) \to H^n(K, \mu_2) \; : \; a \bmod \dot{\sigma}(K) \to (a) \cup \underbrace{(-1) \cup \cdots \cup (-1)}_{n-1}$$

is well defined and an injective group homomorphism.

(2) Φ is an isomorphism if and only if K is SAP.

Proof. (1) Let us for short write $(a_1, \cdots, a_n)$ for the symbol $(a_1) \cup \cdots \cup (a_n) \in H^n(K, \mu_2)$. Using the injectivity of the Milnor map $e_n : I^nK/I^{n+1}K \to H^n(K, \mu_2)$ on n-fold Pfister forms, we get that

$$\begin{aligned}
&(a, -1, \cdots, -1) = (b, -1, \cdots, -1) \\
&\iff \langle\langle a, -1, \cdots, -1\rangle\rangle \cong \langle\langle b, -1, \cdots, -1\rangle\rangle \\
&\iff \mathrm{sgn}_P(\langle\langle a, -1, \cdots, -1\rangle\rangle - \langle\langle b, -1, \cdots, -1\rangle\rangle) = 0 \quad (\forall P \in X_K) \\
&\iff ab \in \dot{\sigma}(K)
\end{aligned}$$

(in the second equivalence, we use that $\langle\langle a,-1,\cdots,-1\rangle\rangle - \langle\langle b,-1,\cdots,-1\rangle\rangle \in I^nK$ together with Pfister's local-global principle and the fact that $I_t^nK = 0$). This shows well-definition and injectivity of Φ. It is clear that Φ is then also a group homomorphism.

(2) Suppose in addition that K is SAP. Since e_n is also surjective (by the Milnor conjecture), to show surjectivity of Φ, it suffices to show that each n-fold $\langle\langle a_1,\cdots,a_n\rangle\rangle$ is isometric to $\langle\langle a,-1,\cdots,-1\rangle\rangle$ for some $a \in K^*$. Since $I_t^nK = 0$ and by Pfister's local-global principle, it suffices to show that there exists $a \in K^*$ such that $H^-(a_1)\cap\cdots\cap H^-(a_n) = H^-(a)$, but this clearly holds for SAP fields.

Conversely, suppose K is not SAP. Then there must exist $a_1, a_2 \in K^*$ such that for any $a \in K^*$, we have $H^-(a_1)\cap H^-(a_2) \neq H^-(a)$. In particular, for any $a \in K^*$ there exists $P \in X_K$ with $\mathrm{sgn}_P(\langle\langle a_1,a_2,-1,\cdots,-1\rangle\rangle - \langle\langle a,-1,\cdots,-1\rangle\rangle) \neq 0$. By similar arguments as before, we conclude that $(a_1,a_2,-1,\cdots,-1)$ is not in the image of Φ. □

3.3 Isomorphism criteria for the Witt ring of an SAP field

The purpose of this subsection is to prove two isomorphism criteria for the Witt rings of certain SAP fields. We first begin by the proof of Theorem 1.4.

Proof of Theorem 1.4: (1) $\Rightarrow$ (2) By Theorem 1.3, there is a pair of maps (t,T), where t is a group isomorphism $t : K^*/K^{*2} \to L^*/L^{*2}$ and T is a homeomorphism $T : X_K \to X_L$ such that x is positive at P if and only if $t(x)$ is positive at $T(P)$, for all $x \in K^*/K^{*2}$ and for all $P \in X_K$. By Artin's Theorem, it easily follows that $x \in \dot\sigma(K)$ if and only if $t(x) \in \dot\sigma(L)$, hence t induces a group isomorphism $\dot\sigma(K)/K^{*2} \simeq \dot\sigma(L)/L^{*2}$.

(2) $\Rightarrow$ (1) Suppose that $\Phi : X_K \simeq X_L$ is a homeomorphism and that $\beta : \dot\sigma(K)/K^{*2} \simeq \dot\sigma(L)/L^{*2}$ is a group isomorphism. As K and L are SAP fields, Φ induces a ring isomorphism $\Psi : \mathrm{Clop}(X_K) \simeq \mathrm{Clop}(X_L)$ by Corollary 3.1. Combining this with Lemma 3.3, we deduce that there is a group isomorphism $\gamma : K^*/\dot\sigma(K) \simeq L^*/\dot\sigma(L)$. We have the following diagram with exact rows (where π_K and π_L are the canonical surjections):

$$\begin{tikzcd}
0 \arrow[r] & \dot\sigma(K)/K^{*2} \arrow[r] \arrow[d, "\beta"] & K^*/K^{*2} \arrow[r, "\pi_K"] \arrow[d, dotted, "\alpha"] & K^*/\dot\sigma(K) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \dot\sigma(L)/L^{*2} \arrow[r] & L^*/L^{*2} \arrow[r, "\pi_L"] & L^*/\dot\sigma(L) \arrow[r] & 0
\end{tikzcd}$$

By the five-lemma, as β and γ are group isomorphisms, there exists a group isomorphism $\alpha : K^*/K^{*2} \simeq L^*/L^{*2}$ such that the above diagram commutes.

Finally, we prove that (α,Φ) respects the positivity. Let $x \in K^*/K^{*2}$ be such that $x \underset{P}{>} 0$ for a $P \in X_K$. Then $\pi_K(x) \underset{P}{>} 0$ so $P \notin H^-\big(\pi_K(x)\big)$. As $\Psi\big(H^-(\pi_K(x))\big) = H^-\big(\gamma(\pi_K(x))\big)$ we have $\gamma\big(\pi_K(x)\big) \underset{\Phi(P)}{>} 0$ hence $\alpha(x) \underset{\Phi(P)}{>} 0$. The converse is similar. Now, we apply Theorem 1.3 to (α,Φ) to conclude. □

As a consequence, one obtains the following result essentially due to Koprowski in [7]:

Corollary 3.6. *Let $K(t)$ be the rational function field in one variable over a real closed field K, and let $L/K(t)$ be a finite extension. Then*

$$\begin{array}{rl} W(L) \cong W(K(t)) & \textit{if } L \textit{ is real;} \\ W(L) \cong W(K(t)(\sqrt{-1}\,)) & \textit{if } L \textit{ is nonreal and } \sqrt{-1} \in L; \\ W(L) \cong W(K(t)(\sqrt{-(t^2+1)}\,)) & \textit{if } L \textit{ is nonreal and } \sqrt{-1} \notin L; \end{array}$$

All these cases are mutually exclusive.

Proof. It is well known that $u(L) = u(K(t)) = u(K(t)(\sqrt{-1}\,)) = 2$. Indeed, $K(t)(\sqrt{-1}\,) = K(\sqrt{-1}\,)(t)$ and hence $L(\sqrt{-1}\,)$ is C_1 since $K(\sqrt{-1}\,)$ is algebraically closed, so $u(L(\sqrt{-1}\,)) \leq 2$ and thus $u(L) \leq 2$ (see, e.g., [11, Ch. 6, Thm. 2.12]). Clearly, $L^{*2} \neq \dot{\sigma}(L)$, so $u(L) = 2$. Also, such fields L are always SAP (see, e.g., [3, Theorem I]).

Now one readily observes that the irreducible polynomials $t^2 + a \in K[t]$, $a \in K^{*2}$, are $\mathbb{F}_2$-independent in $\dot{\sigma}(K(t))/K(t)^{*2}$ (considered as $\mathbb{F}_2$-vector space). Thus, we get for the cardinalities that $\mathrm{card}(K(t)) = \mathrm{card}(K) = \mathrm{card}(K^{*2})$ since K is clearly infinite and Euclidean. It follows that $\mathrm{card}(\dot{\sigma}(K(t))/K(t)^{*2}) = \mathrm{card}(K)$. Now if L is a finite algebraic extension of $K(t)$, then clearly $\mathrm{card}(L) = \mathrm{card}(K(t)) = \mathrm{card}(K)$. By Kummer theory, there exists a subset $S \subset K^{*2}$ (again with cardinality $\mathrm{card}(S) = \mathrm{card}(K^{*2}) = \mathrm{card}(K)$) such that $\{t^2 + a \mid a \in S\}$ still forms a $\mathbb{F}_2$-independent subset in the $\mathbb{F}_2$-vector space $\dot{\sigma}(L)/L^{*2}$, and we immediately get that $\mathrm{card}(\dot{\sigma}(L)/L^{*2}) = \mathrm{card}(\dot{\sigma}(K(t))/K(t)^{*2})$.

Suppose first that L is nonreal. The above together with the fact that both $K(t)(\sqrt{-1}\,)$ and $K(t)(\sqrt{-(t^2+1)}\,)$ are also nonreal shows that

$$\begin{array}{rcl} L^*/L^{*2} & \cong & K(t)(\sqrt{-1}\,)^*/K(t)(\sqrt{-1}\,)^{*2} \\ & \cong & K(t)(\sqrt{-(t^2+1)}\,)^*/K(t)(\sqrt{-(t^2+1)}\,)^{*2} \end{array}$$

(since for nonreal fields F we have $F^* = \dot{\sigma}(F)$).

Since $u(L) = u(K(t)(\sqrt{-1}\,)) = u(K(t)(\sqrt{-(t^2+1)}\,)) = 2$, any 2-fold Pfister form over any of these fields will be hyperbolic. Thus, in order to invoke Harrison's criterion (Theorem 1.1), it suffices to check whether or not there exists such an isomorphism between the square class groups that sends -1 to -1.

If $-1 \in L^{*2}$, then any isomorphism $L^*/L^{*2} \cong K(t)(\sqrt{-1}\,)^*/K(t)(\sqrt{-1}\,)^{*2}$ will do as in both fields, $-1 = 1$ modulo squares. Hence, by Harrison's criterion, $W(L) \cong W(K(t)(\sqrt{-1}))$.

Suppose that $-1 \notin L^{*2}$. One readily checks that $-1 \notin K(t)(\sqrt{-(t^2+1)}\,)^{*2}$. Since the cardinalities of the square class groups are the same, it is now possible to choose an isomorphism between them that sends -1 to -1. Hence, by Harrison's criterion, $W(L) \cong W(K(t)(\sqrt{-(t^2+1)}\,))$.

If L is real, then we have $u(L) = u(K(t)) = 2$ and thus $\dot{\sigma}(K(t))/K(t)^{*2} \cong \dot{\sigma}(L)/L^{*2}$ by the above. We also have $X_{K(t)} \cong X_L$ (see Proposition in the Appendix). Hence, $W(L) \cong W(K(t))$ by Theorem 1.4.

The fact that all these cases are mutually exclusive follows readily from the fact that for the level s, we have $s(K(t)) = \infty$, $s(K(t)(\sqrt{-1})) = 1$ and $s(K(t)(\sqrt{-(t^2+1)}\,)) = 2$, and that for any field F, the characteristic of $W(F)$ is either 0 (if $s(F) = \infty$, i.e. F is real), or $2s(F)$ (if $s(F) < \infty$, i.e. F is nonreal). $\square$

Remark 3.7. Koprowski [7, Prop. 2.2, Cor. 4.2] gives the above classification in the case where the field of constants of L is real closed (i.e. K is the field of constants), which obviously corresponds to the cases where L is real or L is nonreal and $\sqrt{-1} \notin L$. His proof does not explicitly make use of the fact that in the case where L is real, $X_{K(t)} \cong X_L$. Instead, his methods are based on the use of real places and the notion of quaternion-symbol equivalence. In view of Theorem 1.4 which works for all SAP fields of u-invariant ≤ 2, it seemed therefore natural to base our proof on the above homeomorphism of the respective spaces of orderings, and it became desirable to have a self-contained proof of this homeomorphism. Such a proof can be found in the Appendix.

4 Reduced Witt rings

In this section, we adapt Theorem 1.3 and 1.4 to the case of reduced Witt rings. Before stating and proving our results, we recall some notations and basic facts. For further details, see [8].

4.1 Notations and preliminary results

Let K be a real field. Recall that T is a *preordering* over K if $T \subsetneq K$, $K^2 \subset T$, $T+T \subseteq T$ and $T.T \subseteq T$. For a preordering T over K, we thus have $\sigma(K) \subseteq T$. Note that $\sigma(K)$ is the smallest preordering in K which is consequently called the *weak preordering* of K. We write X_K/T for the non-empty subset (see [8, Corollary 1.4]) of X_K consisting of all orderings P containing T. Note that X_K/T is a closed space of X_K and is a topological space with the induced topology. Note also that $T^* = T \setminus \{0\}$ is a subgroup of K^*.

Fix a preordering T over K. A *T-form* ϕ over K is an expression $\langle a_1, \cdots, a_n\rangle_T$ where $a_1, \cdots, a_n \in K^*$. If $P \in X_K/T$, the P-signature of ϕ is just the sum of the signatures at P of the a_i's. Note that we can talk about the positivity of an element of K^*/T^* at $P \in X_K/T$, an element of T being always positive at such an ordering. Two T-forms ϕ, ϕ' are *T-isometric* and we write $\phi \simeq_T \phi'$ if they have the same dimension and the same signatures at every element of X_K/T. A T-form is *T-hyperbolic* if $\operatorname{sgn}_P \phi = 0$ at every $P \in X_K/T$. As in the absolute theory, the set of the T-isometry classes with the orthogonal sum and the tensor product is a semi-ring. By definition, the reduced Witt ring $W_T(K)$ is the Grothendieck ring of the T-isometry classes of T-forms modulo the ideal generated by T-hyperbolic forms.

By viewing a quadratic form $\langle a_1, \cdots, a_n\rangle$ as a T-form, we obtain a surjective ring homomorphism $W(K) \to W_T(K)$. The image of $I(K)$ by this homomorphism is $I_T(K)$ which is nothing but the ideal of T-isometry classes of even-dimensional T-forms. The nth power of this ideal is denoted by $I_T^n(K)$ and is generated by the T-Pfister forms $\langle\langle a_1, \cdots, a_n\rangle\rangle_T$ (with the obvious notation). The signed discriminant $d_{T,\pm}$ of a T-form $\phi = \langle a_1, \cdots, a_n\rangle_T$ is the following element of K^*/T^*:

$$(-1)^{\frac{n(n+1)}{2}} a_1 \cdots a_n.T^*.$$

Moreover, it induces a group isomorphism $d_{T,\pm} : I_T(K)/I_T^2(K) \simeq K^*/T^*$.

We say that two T-forms are chain T-equivalent if we can change one to another by a finite sequence of transformations of the three following types:

$$(A) \qquad \langle a_1, \cdots, a_n\rangle_T \to \langle t_1 a_1, \cdots, t_n a_n\rangle_T, \quad t_1, \cdots, t_n \in T^*,$$

$$(B) \qquad \langle a_1, \cdots, a_i, \cdots, a_j, \cdots, a_n\rangle_T \to \langle a_1, \cdots, a_i + a_j, \cdots, a_i a_j(a_i + a_j), \cdots, a_n\rangle_T \quad (\text{where } a_i + a_j \neq 0).$$

$$(C) \qquad \langle a_1, \cdots, a_i, \cdots, a_j, \cdots, a_n\rangle_T \to \langle a_1, \cdots, a_j, \cdots, a_i, \cdots, a_n\rangle_T$$

If two T-forms are chain T-equivalent, it is clear that they are T-isometric but the converse also holds by Witt's Chain Equivalence Theorem:

Theorem 4.1 ([8, Theorem 1.28])**.** *If two T-forms are T-isometric then they are chain T-equivalent.*

We denote by Y_K^T the set of prime ideals of characteristic 0 in $W_T(K)$. The following is an analogue of Theorem 2.1.

Proposition 4.2. *There is a bijection between X_K/T and Y_K^T.*

Proof. The definition of the bijection is the same as in the proof of Theorem 2.1. If $P \in X_K/T$, then $I_P := \ker(\mathrm{sgn}_P)$ is an element of Y_K^T. Conversely, if $I \in Y_K^T$, then $P_I := \{0\} \cup \{a \in K^* \mid \langle a\rangle_T \equiv 1 \mod I\}$ is an element of X_K/T, and these two maps are inverse to each other. □

4.2 Statement and proof of the criterion

Theorem 4.3. *Let T be a preordering over the real field K and S be a preordering over the real field L. Then the following are equivalent:*

(1) There is a ring isomorphism $W_T(K) \simeq W_S(L)$.

(2) There is a group isomorphism $t : K^*/T^* \to L^*/S^*$ sending -1 to -1 such that the T-form $\langle 1, -x, -y, xy\rangle_T$ is T-hyperbolic over K if and only if the S-form $\langle 1, -t(x), -t(y), t(x)t(y)\rangle_S$ is S-hyperbolic over L for all $x, y \in K^*$.

(3) There is a pair of maps (t, D) with a group isomorphism $t : K^*/T^* \to L^*/S^*$ and a homeomorphism $D : X_K/T \to X_L/S$ such that x is positive at P if and only if $t(x)$ is positive at $D(P)$, for all $x \in K^*/T^*$ and for all $P \in X_K/T$.

(4) There is a pair of maps (t, D) with a group isomorphism $t : K^*/T^* \to L^*/S^*$ and a bijection $D : X_K/T \to X_L/S$ such that x is positive at P if and only if $t(x)$ is positive at $D(P)$, for all $x \in K^*/T^*$ and for all $P \in X_K/T$.

Proof. The equivalence of (1) and (2) is the "reduced" version of Harrison's Criterion. It can be proved by readily adapting the proofs of Harrison's criterion found, for example, in [9, Ch. XII, Theorem 1.8] or [10] to the reduced setting, using chain T-equivalence (Theorem 4.1) instead of the usual chain equivalence, $d_{T,\pm}$ instead of $d_\pm$, etc. We leave the details to the reader.

(1) ⇒ (3) Let Φ be a ring isomorphism between $W_T(K)$ and $W_S(L)$. The signed discriminant induces a group isomorphism $t : K^*/T^* \simeq L^*/S^*$ defined by $t(a) := d_\pm\big(\Phi(\langle 1, -a\rangle_T)\big)$ for $a \in K^*/T^*$. By Proposition 4.2, we get the desired

bijection D between X_K/T and X_L/S. The implication can now be shown by applying the "reduced" analogs of the arguments used in the proof of (1) $\Rightarrow$ (2) in Theorem 1.3.

(3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (2) First note that (4) implies that $t(-1) = -1$. Indeed, $-t(-1) \underset{P}{>} 0$ for each $P \in X_L/S$ thus

$$-t(-1) \in \bigcap_{Q \in X_L/S} Q \quad = \; S,$$

(see [8, Theorem 1.6]).

Note also that the compatibility between (t, D) and the positivity readily implies that the 2-fold T-Pfister form $\langle\langle a, b\rangle\rangle_T$ is T-hyperbolic if and only if the 2-fold S-Pfister form $\langle\langle t(a), t(b)\rangle\rangle_S$ is S-hyperbolic. □

Remarks 4.4. (1) If T and S are the corresponding weak preorderings and if K, L are pythagorean, Theorem 4.2 is nothing but Theorem 1.1 in this particular case.
(2) If T and S are the corresponding weak preorderings, then we have the following ring isomorphism $W_T(K) \simeq W(K)/W_t(K)$, $W_S(L) \simeq W(L)/W_t(L)$ (see [8, Chapter 1]) where W_t denotes the torsion subgroup of the corresponding Witt ring. Hence Theorem 4.2 can be viewed as Harrison's criterion for W/W_t in this case.

If K and L are two fields such that $W(K) \simeq W(L)$ as rings, then, by the preceding remark, $W_T(K) \simeq W_S(L)$ as rings if T (resp. S) is the weak preordering of K (resp. of L). The following example shows that the converse is false:

Example 4.5. Let K and L be any uniquely ordered fields with weak preorderings T and S, respectively. Since every element of K (resp. L) is a sum of squares or the negative of a sum of squares, one clearly has $W_T(K) \simeq W_S(L) \simeq \mathbb{Z}$. But $W(K)$ and $W(L)$ need not be isomorphic. For example, $\mathbb{R}$ and $\mathbb{Q}$ are uniquely ordered but obviously $W(\mathbb{R}) \not\simeq W(\mathbb{Q})$.

4.3 An isomorphism criterion for the reduced Witt ring of an SAP field

Let K be a real field and T be a preordering on K. If we denote by $C(X_K/T, \mathbb{Z})$ the ring of continuous functions from X_K to $\mathbb{Z}$ (we endow $\mathbb{Z}$ with the discrete topology), then the signatures induce a ring monomorphism

$$\operatorname{sgn} : W_T(F) \hookrightarrow C(X_K/T, \mathbb{Z}) : q \mapsto \big(P \mapsto \operatorname{sgn}_P(q)\big),$$

see [8, §1]. In fact, we can be more precise. Denote by $\mathbf{1}$ the element of $C(X_K/T, \mathbb{Z})$ which sends every ordering to 1. The fact that $\operatorname{sgn}_P(q) \equiv \dim q$ mod 2 for every $P \in X_K$ and every $q \in W(K)$, implies that the previous map induces a ring monomorphism

$$\Phi_T : W_T(K) \hookrightarrow \mathbb{Z}.\mathbf{1} + C(X_K/T, 2\mathbb{Z}).$$

In the case of SAP fields, we have the following criterion for reduced Witt rings:

Corollary 4.6. *Let K (resp L) be a SAP field and T (resp. S) be a preordering over K (resp. L). Then the following are equivalent:*

(1) There is a ring isomorphism $W_T(K) \simeq W_S(L)$.

(2) There is a homeomorphism $X_K/T \simeq X_L/S$.

(3) There is a ring isomorphism $\mathbb{Z}.\mathbf{1} + C(X_K/T, 2\mathbb{Z}) \simeq \mathbb{Z}.\mathbf{1} + C(X_L/S, 2\mathbb{Z})$.

Proof. (1) $\Rightarrow$ (2) This is obvious by Theorem 4.3.
(2) $\Rightarrow$ (1) Let $D : X_K/T \simeq X_L/S$ be a homeomorphism. Then it induces a ring isomorphism $\widetilde{D} : \mathrm{Clop}(X_K/T) \simeq \mathrm{Clop}(X_L/S)$. Define

$$\theta_K : \begin{cases} K^*/T^* & \rightarrow & \mathrm{Clop}(X_K/T) \\ a & \mapsto & H_T^-(a) := \{P \in X_K/T \mid a \notin P\} \end{cases}.$$

It is easy to see that θ_K is well-defined, surjective as K is a SAP field and injective as $H_T^-(a) = H_T^-(b)$ if and only if $ab \in T^*$. Now we define the group isomorphism $t := {\theta_L}^{-1} \circ \widetilde{D} \circ \theta_K : K^*/T^* \simeq L^*/S^*$ (where θ_L is defined in an analogous way as θ_K). Finally, let $x \in K^*/T^*$. We have $t(x) = {\theta_L}^{-1}\big(\widetilde{D}(H_T^-(x))\big) = y$ where $\widetilde{D}(H_T^-(x)) = H_S^-(y)$. Let $P \in X_K/T$. Then

$$\begin{array}{rcl} x \text{ is positive at } P & \Longleftrightarrow & P \notin H_T^-(x) \\ & \Longleftrightarrow & D(P) \notin H_S^-(y) \\ & \Longleftrightarrow & y = t(x) \text{ is positive at } D(P). \end{array}$$

Now we apply Theorem 4.3 to conclude.

(1) $\Longleftrightarrow$ (3) This is an immediate consequence of the fact that for any SAP field K and any preordering T on K, we have that the above map $\Phi_T : W_T(K) \rightarrow \mathbb{Z}.\mathbf{1} + C(X_K/T, 2\mathbb{Z})$ is an isomorphism (see [8, Theorem 16.2]). $\square$

5 Isomorphism criteria for Witt groups of hermitian forms

The ideas of the proof of Theorem 1.3 can be adapted to two particular situations involving Witt groups of hermitian forms. First, we are interested in finding such a criterion for the Witt ring of hermitian forms over a quadratic field extension with its nontrivial automorphism. In the second situation, we deal with the Witt group of hermitian forms over a quaternion division algebra endowed with its canonical involution. In each of these two cases, we have to recall some notations and notions and we refer to [13, Chapter VII, Chapter X] for further details.

5.1 Witt rings of hermitian forms over quadratic field extensions

Let k be a field, K/k be a quadratic field extension and σ_K be the nontrivial k-automorphism of K. Every nondegenerate hermitian form over (K, σ_K) can be diagonalized and we will use the usual notation $\langle a_1, \cdots, a_n\rangle$, $a_i \in k$, for such a diagonalization. We denote by $W(K, \sigma_K)$ the Witt ring of nondegenerate hermitian forms over (K, σ_K) and by $I(K, \sigma_K)$ its fundamental ideal. The *signed discriminant* of a hermitian form $\langle a_1, \cdots, a_n\rangle$ is defined to be the following element of $K^*/N_{K/k}(K^*)$ (where $N_{K/k}(K^*)$ denotes the usual norm group of K/k):

$$(-1)^{\frac{n(n+1)}{2}} a_1 \cdots a_n N_{K/k}(K^*).$$

It induces a group isomorphism $d_\pm : I(K, \sigma_K)/\big(I(K, \sigma_K)\big)^2 \simeq K^*/N_{K/k}(K^*)$.

In [4, Theorem 1.3], the first author has proved the following analogue of Theorem 1.1 for Witt rings of hermitian forms over quadratic field extensions:

Theorem 5.1 (Grenier-Boley)**.** *Let k and l be two fields. Let $K = k(\sqrt{a})$ (resp. $L = l(\sqrt{b})$) be a quadratic field extension of k (resp. l). Then, the following are equivalent:*

(1) There is a ring isomorphism $W(K, \sigma_K) \simeq W(L, \sigma_L)$.

(2) There is a group isomorphism $t : k^*/N_{K/k}(K^*) \to l^*/N_{L/l}(L^*)$ sending -1 to -1 such that the quadratic form $\langle\langle a, x, y\rangle\rangle$ is hyperbolic over k if and only if the quadratic form $\langle\langle b, t(x), t(y)\rangle\rangle$ is hyperbolic over l for all $x,\ y \in k^*$.

Suppose now that k is a real field and that $K = k(\sqrt{a})$ is a quadratic field extension of k. Let

$$X_k^a := \{P \in X_k \mid a \underset{P}{<} 0\} = H(-a).$$

Note that, if $P \in X_k^a$ and if $x \in N_{K/k}(K^*)$ then x is positive at P. Consequently, we can talk about the positivity or the negativity of any norm class at an ordering in X_k^a.

To each hermitian form (V, h) over (K, σ_K) one can associate a k-quadratic form defined by $q_h(x) := h(x, x)$ for $x \in V$, where V is considered as a k-vector space. Recall that if, over K, $h \simeq \langle a_1, \ldots, a_n\rangle$ (where $a_i \in k$), then, over k, $q_h \simeq \langle\langle a\rangle\rangle \otimes \langle a_1, \ldots, a_n\rangle$. The signature of the hermitian form h at $P \in X_k^a$ is then defined by $\operatorname{sgn}_P(h) := \frac{1}{2}\operatorname{sgn}_P(q_h)$ which is an integer. For each $P \in X_k^a$, we thus get a surjective ring homomorphism $\operatorname{sgn}_P : W(K, \sigma_K) \to \mathbb{Z}$.

We will also need the following notation:

$$Z_k^a := \{I \in \operatorname{Spec}\big(W(K, \sigma_K)\big) \mid \operatorname{char} I = 0\}.$$

As a first step toward Theorem 5.4, we can prove the following analogue of Theorem 2.1 and Proposition 4.2:

Theorem 5.2. *There is a bijection between X_k^a and Z_k^a.*

Proof. The bijection is defined in the same way as in the proof of Theorem 2.1. We will sketch some of the details in this hermitian setting for the reader's convenience.

First of all, a prime ideal I of $W(K, \sigma_K)$ has characteristic 0 if and only if $W(K, \sigma_K)/I$ is isomorphic (as a ring) to $\mathbb{Z}$. If $P \in X_k^a$, the ring homomorphism $\operatorname{sgn}_P : W(K, \sigma_K) \to \mathbb{Z}$ is onto, hence $I_P := \ker(\operatorname{sgn}_P)$ is an element of Z_k^a.

If $I \in Z_k^a$, we put

$$P_I := \{0\} \cup \{c \in k^* \mid \langle c \rangle \equiv 1 \mod I\},$$

(here, the notation $\langle c \rangle$ refers to a one-dimensional hermitian form). This is in fact an element of X_k^a. If $c \in k^*$ such that $\langle c \rangle \not\equiv 1 \mod I$ then $\langle c \rangle \equiv -1 \mod I$ and hence $P_I \cup -P_I = k$. We clearly have $P_I.P_I \subseteq P_I$ and $P_I \subsetneq k$. Let $c,\ d \in P_I$. Then the hermitian forms $\langle c, d \rangle$ and $\langle c+d, (c+d)cd \rangle$ are isometric. Now $e := c + d \neq 0$ and $2 \equiv 2\langle e \rangle \mod I$. As $\operatorname{char} I \neq 2$, this implies that $e \in P_I$. Now the hermitian form $\langle 1, a \rangle$ is hyperbolic since the quadratic form $\langle\langle a \rangle\rangle \otimes \langle 1, a \rangle \simeq \langle 1, -a \rangle \otimes \langle 1, a \rangle$ is clearly hyperbolic. This implies that a is negative at P_I, i.e. $P_I \in X_k^a$.

Finally, one readily checks that the map $P \mapsto I_P$ gives the desired bijection whose inverse is $I \mapsto P_I$. $\square$

Remark 5.3. As a consequence of the previous result, one can determine the spectrum of the ring $W(K, \sigma_K)$. It consists of the fundamental ideal of $W(K, \sigma_K)$ (which is the unique maximal ideal of characteristic 2 in $W(K, \sigma_K)$), $\ker(\operatorname{sgn}_P)$ where $P \in X_k^a$ (which are exactly the prime ideals of characteristic 0 in $W(K, \sigma_K)$) and $\ker(\operatorname{sgn}_P) \mod p$ where p is a prime number and $P \in X_k^a$. The proof of this fact can easily be adapted from the determination of the spectrum of the usual Witt ring, see, e.g., [9, Ch.VIII, Theorem 7.5].

The analogue of Theorem 1.3 in the hermitian context is the following:

Theorem 5.4. *We keep the notations of Theorem 5.1 and we suppose further that k and l are two real fields such that $u(k),\ u(l) \leq 6$. The following are equivalent:*

(1) There is a ring isomorphism $W(K, \sigma_K) \simeq W(L, \sigma_L)$.

(2) There is a pair of maps (t, T) with a group isomorphism $t : k^*/N_{K/k}(K^*) \to l^*/N_{L/l}(L^*)$ such that $t(-1) = -1$, and a homeomorphism $T : X_k^a \to X_l^b$ such that, for all $x \in k^*/N_{K/k}(K^*)$ and for all $P \in X_k^a$, x is positive at P if and only if $t(x)$ is positive at $T(P)$.

(3) There is a pair of maps (t, T) with a group isomorphism $t : k^*/N_{K/k}(K^*) \to l^*/N_{L/l}(L^*)$ such that $t(-1) = -1$, and a bijection $T : X_k^a \to X_l^b$ such that, for all $x \in k^*/N_{K/k}(K^*)$ and for all $P \in X_k^a$, x is positive at P if and only if $t(x)$ is positive at $T(P)$.

Proof. (1) $\Rightarrow$ (2) Let Φ be a ring isomorphism between $W(K, \sigma_K)$ and $W(L, \sigma_L)$. By Theorem 5.1, the map t of condition (2) exists and satisfies $t(-1) = -1$ (more precisely, if $a \in k^*/N_{K/k}(K^*)$, $t(a)$ is defined to be the (hermitian) signed discriminant of $\Phi(\langle 1, -a \rangle)$).

As in the proof of Theorem 1.3, using Theorem 5.2, we construct a bijection T between X_k^a and X_l^b. We also get the compatibility of (t, T) with respect to the positivity at each ordering in the same manner.

Lastly, X_k^a is a topological space with its topology induced by Harrison's topology over X_k. An open subbasis of X_k^a is given by

$$H^a(c) := \{P \in X_k^a \mid c \in P\} = H(c) \cap H(-a),$$

where c ranges over k^*. T is then readily seen to be a homeomorphism for the induced topologies.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Condition (3) and the fact that $u(k),\ u(l) \leq 6$ implies that the quadratic form $\langle\langle a, x, y \rangle\rangle$ is hyperbolic over k if and only if the quadratic form $\langle\langle b, t(x), t(y) \rangle\rangle$ is hyperbolic over l. Now we apply Theorem 5.1 to deduce condition (1). ☐

Remark 5.5. Fields whose u-invariant is at most 6 include global fields and fields of transcendence degree at most 2 over a real closed field (see, e.g., [11, Ch.8, Theorem 2.12]).

5.2 The Witt group of a quaternion division algebra with its canonical involution

Let Q be a quaternion division algebra over a field K with canonical involution γ. We denote by $W(Q,\gamma)$ the Witt group of nondegenerate hermitian forms over (Q,γ). It is easy to see that $W(Q,\gamma)$ is in fact a $W(K)$-module. The goal of this Subsection is to find an isomorphism criterion for Witt modules of quaternion division algebras over the same real base field K. For this purpose, we have to find the object corresponding to the spectrum of the Witt ring in the case of this particular Witt module. This object is in fact nothing but a subset of the support of this module.

Let A be a commutative ring and let M be an A-module. If $I \in \mathrm{Spec}(A)$, we denote by M_I the localized module of M at I. The support of M is a subset of $\mathrm{Spec}(A)$ defined as follows:

$$\mathrm{Supp}(M) = \{I \in \mathrm{Spec}(A) \mid M_I \neq 0\}.$$

Suppose now that $A = W(K)$, and that $M = W(Q,\gamma)$ as above. Then M is an A-module and we set:

$$\mathrm{Supp}^0(M) = \{I \in \mathrm{Supp}(M) \mid \mathrm{char}\, I = 0\}.$$

When K is a real field and $a,\ b \in K^*$, we define

$$X_K^{a,b} = \{P \in X_K \mid a \underset{P}{<} 0 \text{ and } b \underset{P}{<} 0\} = H(-a) \cap H(-b).$$

We have the following description which is similar to the description obtained in Theorem 2.1 and 5.2:

Proposition 5.6. *Let $Q = (a,b)_K$ be a quaternion division algebra endowed with its canonical involution γ over a real field K and M be the Witt group of (Q,γ). Then there is a bijection between* $\mathrm{Supp}^0(M)$ *and* $X_K^{a,b}$.

Proof. Recall that any nondegenerate hermitian form h over (Q,γ) can be diagonalized as $h \simeq \langle a_1, \ldots, a_n \rangle$ with $a_i \in K^*$. Similarly as in the case of hermitian forms over quadratic extensions, we can define the signature of h at P for every $P \in X_K^{a,b}$ using the fact that for any $P \in X_K^{a,b}$ and any $x \in \mathrm{Nrd}_{Q/K}(Q^*)$ we have $x \in P$. For each $P \in X_K^{a,b}$, we thus get a surjective ring homomorphism $\mathrm{sgn}_P : W(Q,\gamma) \to \mathbb{Z}$.

To each $P \in X_K^{a,b}$ we associate $I_P = \ker(\mathrm{sgn}_P)$. We have to show that $I_P \in \mathrm{Supp}^0(M)$, that is $M_{I_P} \neq 0$. Now $M_{I_P} \neq 0$ if and only if there exists $h \in M$ such that for every $s \in W(K) \setminus I_P$, $s.h \neq 0$. We choose $h = \langle 1 \rangle$ and we let $s \in W(K) \setminus I_P$. If $s.h$ is hyperbolic as a hermitian form over (Q,γ) then the quadratic form $\langle\langle a,b \rangle\rangle \otimes s$ is hyperbolic as a quadratic form over K. Now the hypotheses imply that $\mathrm{sgn}_P\left(\langle\langle a,b \rangle\rangle \otimes s\right) \neq 0$ which is a contradiction. This means that $I_P \in \mathrm{Supp}^0(M)$.

Conversely, to each $I \in \mathrm{Supp}^0(M)$ we associate

$$P_I := \{0\} \cup \{x \in K^* \mid \langle x \rangle \equiv 1 \mod I\} .$$

We have to show that $P_I \in X_K^{a,b}$. Suppose otherwise, i.e. $P_I \notin X_K^{a,b}$. Without loss of generality, we may assume that a is positive at P_I. This means that the quadratic form $\langle 1, a \rangle$ is in $W(K) \setminus I$. Let $h \in M$ and $s = \langle 1, a \rangle$. We are going to show that $s.h = 0$. We may assume that $\mathrm{rank}(h) = 1$ and therefore it suffices to show that the quadratic form $\langle\langle a,b \rangle\rangle \otimes s \simeq \langle 1, -a \rangle \otimes \langle 1, -b \rangle \otimes \langle 1, a \rangle$ is hyperbolic which is obvious. This implies that $M_I = 0$ which is a contradiction. Thus, $P_I \in X_K^{a,b}$.

Finally, one readily verifies that the above defined maps $P \mapsto I_P$ and $I \mapsto P_I$ yield bijections between $X_K^{a,b}$ and $\mathrm{Supp}^0(M)$ that are inverse to each other. □

Remark 5.7. With the above notations, the support of $W(Q,\gamma)$ consists of $I(K)$, $\ker(\mathrm{sgn}_P)$ where $P \in X_K^{a,b}$, $\ker(\mathrm{sgn}_P) \mod p$ where p is a prime number and $P \in X_K^{a,b}$. Again, the proof is similar to the determination of the spectrum of $W(K)$ and is omitted.

For quaternion division algebras over the same base field, the first author has proved the following criterion in [4, Corollary 1.4]:

Proposition 5.8 (Grenier-Boley)**.** *Let $Q_1 = (a,b)_K$ (resp. $Q_2 = (c,d)_K$) be a quaternion division algebra over K endowed with its canonical involution γ_1 (resp. γ_2). Then, the following are equivalent:*

(1) There is a $W(K)$-module isomorphism $W(Q_1,\gamma_1) \simeq W(Q_2,\gamma_2)$.

(2) There is a group isomorphism $t : K^*/\mathrm{Nrd}_{Q_1/K}({Q_1}^*) \simeq K^*/\mathrm{Nrd}_{Q_2/K}({Q_2}^*)$ with $t(-1) = -1$ such that the quadratic form $\langle\langle a,b,u,v \rangle\rangle$ is hyperbolic over K if and only if the quadratic form $\langle\langle c,d,t(u),t(v) \rangle\rangle$ is hyperbolic over K for all $u,\ v \in K^*$, where $\mathrm{Nrd}_{Q_i/K}({Q_i}^*)$ denotes the group of reduced norms from the quaternion algebra Q_i for $i = 1,\ 2$.

We are now able to state:

Theorem 5.9. *Let K be a real field with $u(K) \leq 14$. Let $Q_1 = (a,b)_K$, $Q_2 = (c,d)_K$ be quaternion division algebras over K endowed with their respective canonical involutions γ_1 and γ_2. Then the following are equivalent:*

(1) There is a $W(K)$-module isomorphism $W(Q_1,\gamma_1) \simeq W(Q_2,\gamma_2)$.

(2) There is a pair of maps (t,T) with a group isomorphism $t : K^*/\mathrm{Nrd}_{Q_1}(Q_1^*) \to K^*/\mathrm{Nrd}_{Q_2}(Q_2^*)$ such that $t(-1) = -1$, and a homeomorphism $T : X_K^{a,b} \to X_K^{c,d}$ such that, for all $x \in K^*/\mathrm{Nrd}_{Q_1}(Q_1^*)$ and for all $P \in X_K^{a,b}$, x is positive at P if and only if $t(x)$ is positive at $T(P)$.

(3) There is a pair of maps (t, T) with a group isomorphism $t : K^*/\operatorname{Nrd}_{Q_1}(Q_1^*) \to K^*/\operatorname{Nrd}_{Q_2}(Q_2^*)$ such that $t(-1) = -1$, and a bijection $T : X_K^{a,b} \to X_K^{c,d}$ such that, for all $x \in K^*/\operatorname{Nrd}_{Q_1}(Q_1^*)$ and for all $P \in X_K^{a,b}$, x is positive at P if and only if $t(x)$ is positive at $T(P)$.

Proof. The proof is similar to the proofs of Theorem 1.3 and 5.4 by using Proposition 5.6 and Proposition 5.8. We leave the details to the reader. □

Appendix:

The space of orderings of a one-dimensional function field over R

Claus Scheiderer

Let R be a real closed field. We are going to show here that the space of orderings of a real one-dimensional function field K over R depends (up to homeomorphism) only on R, but not on K. This is an easy application of general and well-known techniques from real algebra.

As in the preceding text we denote the topological space of all orderings of a field K by X_K. Let $R(t)$ be the field of rational functions in one variable over R. Let $S \subset R$ be a semi-algebraic set, i.e., a finite union of open intervals and singletons. By $\widehat{S}$ we denote the set of all orderings P of $R(t)$ for which there exist $a, b \in R \cup \{\pm\infty\}$ with $a <_P t <_P b$ and $]a, b[\subset S$. The sets $\widehat{S}$, for S running through the semi-algebraic subsets of R, are precisely the open-closed subsets of the boolean topological space $X_{R(t)}$. The operator $S \mapsto \widehat{S}$ commutes with the finite boolean set-theoretic operations, and $\widehat{S}_1 = \widehat{S}_2$ holds if and only if the set-theoretic difference of S_1 and S_2 is finite. In particular, $\widehat{S} = \widehat{\widehat{S}}$.

It is convenient to consider the affine real line $\mathbb{A}^1(R) = R$ as contained in the projective real line $\mathbb{P}^1(R) = R \cup \{\infty\}$. The semi-algebraic subsets of $\mathbb{P}^1(R)$ are the subsets which intersect R in a semi-algebraic set. For any such set S we define $\widehat{S}$ to be $(S - \{\infty\})\widehat{\ }$.

We remark that the above description of the constructible sets in $X_{R(t)}$ via semi-algebraic sets extends naturally to more general situations. In particular, given any integral algebraic variety V over R and any semi-algebraic subset S of $V(R)$, one defines an open-closed subset $\widehat{S}$ of $X_{R(V)}$ in a similar and natural way. (See any of the standard references like [1], [6] or [12].) The open-closed subsets of $X_{R(V)}$ are precisely these sets $\widehat{S}$, and $\widehat{S}_1 = \widehat{S}_2$ holds if and only if the set-theoretic difference of S_1 and S_2 is not Zariski dense in V.

For the purpose of this appendix, only the case $V = \mathbb{P}^1$ is needed, as explained above. The notation $\widehat{S}$ is not standard and has been chosen *ad hoc* here.

We denote the topological (disjoint) sum of two topological spaces X and Y by $X \amalg Y$, and write $X \approx Y$ to indicate that X and Y are homeomorphic.

Lemma. *Let $I_1, \ldots, I_r$ ($r \geq 1$) be any finite number of non-degenerate intervals in R. Then $X_{R(t)} \approx \widehat{I}_1 \amalg \cdots \amalg \widehat{I}_r$.*

Here, an interval I in R is called non-degenerate if it contains more than one point.

Proof. It is convenient to extend the concept of intervals from R to $\mathbb{P}^1(R)$ in the obvious way. For the proof of the lemma we may restrict to closed intervals. Any two nonempty closed intervals in $\mathbb{P}^1(R)$, different from $\mathbb{P}^1(R)$, are conjugate under the action of $\mathrm{PGL}_2(R)$. From this it follows that $\widehat{I_1} \approx \widehat{I_2}$ holds for any two such intervals I_1, I_2, and also that $\widehat{I} \approx X_{R(t)} - \widehat{I}$ for any such interval I. Since clearly $[-1,1]^\frown \approx [-1,0]^\frown \amalg [0,1]^\frown$, and since $X_{R(t)} \approx [0,1]^\frown \amalg \left(X_{R(t)} - [0,1]^\frown\right)$, the assertion of the lemma follows. □

Proposition. *Let $R \subset K$ be a finitely generated field extension of transcendence degree one. If the field K is real then X_K is homeomorphic to $X_{R(t)}$.*

Proof. Choose t, $x \in K$ with t transcendental over R and $K = R(t,x)$, and let $r\colon X_K \to X_{R(t)}$ be the restriction map. The discriminant of the minimal polynomial of x over $R(t)$ is a nonzero element of $R(t)$. Its real poles and zeros, together with ∞, divide the real t-line into finitely many open intervals $I_1, \ldots, I_r$. For each index $i = 1, \ldots, r$, the restriction $r^{-1}(\widehat{I_i}) \to \widehat{I_i}$ of r is topologically a trivial d_i-sheeted covering, where $0 \leq d_i \leq [K : R(t)]$ is some integer. Therefore, X_K is homeomorphic to a topological sum $\coprod_{i=1}^{r} \coprod_{j=1}^{d_i} \widehat{I_i}$, and so the above Lemma shows that X_K is homeomorphic to $X_{R(t)}$. □

Corollary. *Let $R \subset K$ be a finitely generated field extension of transcendence degree one. Given any finite number of nonempty open-closed subsets $U_1, \ldots, U_r$ of X_K ($r \geq 1$), there is a homeomorphism $X_{R(t)} \approx U_1 \amalg \cdots \amalg U_r$.*

Proof. This follows from a combination of the two previous results. □

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