CROSS-SECTIONS, QUOTIENTS, AND REPRESENTATION RINGS OF SEMISIMPLE ALGEBRAIC GROUPS

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"*Is Steinberg's theorem* [...] *only true for simply connected groups* [...]*? What happens for* GP(1)*, for instance? Is there a rational section of G over $I(G)$* ("*invariants*") *in this case?* [...] *Is it true that $I(G)$ is a rational variety* [...]?"

A. Grothendieck*, Letter to J.-P. Serre,* January 15, 1969*,* [GS, pp. 240–241].

Abstract. Let G be a connected semisimple algebraic group over an algebraically closed field k. In 1965 Steinberg proved that if G is simply connected, then in G there exists a closed irreducible cross-section of the set of closures of regular conjugacy classes. We prove that in arbitrary G such a cross-section exists if and only if the universal covering isogeny $\tau\colon \widehat{G} \to G$ is bijective. In particular, for $\operatorname{char} k = 0$, the converse to Steinberg's theorem holds. The existence of a cross-section in G implies, at least for $\operatorname{char} k = 0$, that the algebra $k[G]^G$ of class functions on G is generated by $\operatorname{rk} G$ elements. We describe, for arbitrary G, a minimal generating set of $k[G]^G$ and that of the representation ring of G and answer two Grothendieck's questions on constructing the generating sets of $k[G]^G$. We prove the existence of a rational cross-section in any G (for $\operatorname{char} k = 0$, this has been proved earlier in [CTKPR]). We also prove that the existence of a rational section of the quotient morphism for G is equivalent to the existence of a rational W-equivariant map $T \dashrightarrow G/T$ where T is a maximal torus of G and W the Weyl group. We show that both properties hold if the isogeny τ is central.

1. Introduction

Below all algebraic varieties are taken over an algebraically closed field k. We use the standard notation and conventions of [Bor] and [Sp].

Let G be a connected semisimple algebraic group, $G \neq \{e\}$. Let $(G/\!\!/G, \pi_G)$ be a categoricalRMR quotient for the conjugating action of G on itself, i.e., $G/\!\!/G$ is an affine variety and

$$\pi_G\colon G \longrightarrow G/\!\!/G$$

a surjective morphism such that $\pi_G^*\big(k[G/\!\!/G]\big)$ is the algebra $k[G]^G$ of class functions on G. Every fiber of π_G is then the closure of a regular conjugacy class (i.e., that of the maximal dimension) and such classes in general position are closed [Ste$_1$, Theorem 6.11, Cor. 6.13, and Sect. 2.14].

Definition 1.1. A closed irreducible subvariety S of G is called a *cross-section* (of the collection of fibers of π_G) in G if S intersects at a single point every fiber of π_G.

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The elements of S are the "canonical forms" of the elements of a dense constructible subset of G with respect to conjugation. The image of any *section* of π_G (i.e., a morphism $\sigma\colon G/\!\!/G \to G$ such that $\pi_G \circ \sigma = \mathrm{id}_{G/\!\!/G}$) is the example of such S and, for $\operatorname{char} k = 0$, every cross-section in G is obtained in this manner (see Remark 1 in Section 6).

In 1965 Steinberg gave an explicit construction of a section of π_G for every simply connected semisimple group G (see his celebrated paper [Ste$_1$]). Its image is the cross-section that intersects every regular conjugacy class and does not intersect other conjugacy classes.

In this paper we explore what happens in the general case, i.e., when G is not necessarily simply connected. In this case the following two facts about cross-sections in G for $\operatorname{char} k = 0$ are known.

First, by [CTKPR, Theorem 0.3] in every connected semisimple algebraic group G there is a *rational section* of π_G, i.e., a section over a dense open subset of $G/\!\!/G$ (local section).

Second, by Kostant's theorem [K, Theorem 0.10] there is an infinitesimal counterpart of Steinberg's cross-section: for the adjoint action of G on its Lie algebra $\operatorname{Lie} G$, there is a closed irreducible subvariety in $\operatorname{Lie} G$ that intersects every regular G-orbit at a single point.

In order to formulate our result consider the universal covering of G, i.e., an isogeny

$$\tau\colon \widehat{G} \longrightarrow G$$

such that $\widehat{G}$ is a simply connected semisimple algebraic group and the composition of τ with every projective rational representation of G lifts to a linear one of $\widehat{G}$.

We prove the following

Theorem 1.2. *Let G be a connected semisimple algebraic group.*

(i) *The following properties are equivalent:*
   (a) *there is a cross-section in G;*
   (b) *the isogeny τ is bijective.*

(ii) *If $\sigma\colon G/\!\!/G \to G$ is a section of π_G, then the cross-section $\sigma(G/\!\!/G)$ in G intersects every regular conjugacy class and does not intersect other conjugacy classes.*

Remark 1.3. The isogeny τ is bijective if and only if it is either an isomorphism or purely inseparable (radical). The latter holds if and only if $\operatorname{char} k = p > 0$ and p divides the order of the fundamental group of G.

Statement (ii) of the next corollary answers a question posed in [CTKPR, p. 4].

Corollary 1.4. *Let G be a connected semisimple algebraic group.*

(i) *If a section of π_G exists, then τ is bijective.*

(ii) *For $\operatorname{char} k = 0$, the following properties are equivalent:*
   (a) *there is a section of π_G;*
   (b) *there is a cross-section in G;*
   (c) *G is simply connected.*

Theorem 1.2 is proved in Section 2.

One can show (see below Lemma 3.1) that if a cross-section in G exists, then, at least for $\operatorname{char} k = 0$, the variety $G/\!\!/G$ is smooth (the converse is not true). The known criterion of smoothness of $G/\!\!/G$ (Theorem 3.2) may be interpreted as that of the existence of $\operatorname{rk} G$ generators of $k[G]^G$. In Section 3 we consider the general case and describe a minimal generating set of $k[G]^G$ and singularities of $G/\!\!/G$ for any G. This is based on the property that actually $G/\!\!/G$ is a toric variety of a maximal torus T

of G. In particular, it also implies the affirmative answer to Grothendieck's question cited in the epigraph:

Corollary 3.7. *$G/\!\!/G$ and T/W are the rational varieties.*

Here W is the Weyl group of G, i.e., the quotient of T in its normalizer $N_G(T)$, acting on T via conjugation.

Parallel to this we describe a minimal generating set of the representation ring $R(G)$ of G. Note that finding generators of $R(G)$ attracted people's attention during long time, in particular, because of the bearing on the K-theory (cf., e.g., [Hus, Chap. 13] where the generators of $R(G)$ are found for some classical G's utilizing the ad hoc bulky arguments; see also [A]). Singularities of $G/\!\!/G$ attracted the attention as well (see [Sl, Sects. 3.15, 4.5]).

The precise formulations of these results are given below in Theorems 3.5, 3.9 and Lemma 3.10.

Constructing the generating sets of $k[G]^G$ is the topic of yet two GROTHENDIECK's questions asked in [GS, p. 241]. In Section 4 we answer the first question in the negative and the second in the positive.

In Section 5 we consider rational sections of π_G and *rational cross-sections* in G, i.e., irreducible closed subsets S of G that intersect at a single point every fiber of π_G over a point of a dense open subset of $G/\!\!/G$. The closure of the image of a rational section of π_G is the example of such S and, for $\operatorname{char} k = 0$, every rational cross-section in G is obtained in this way.

First we show that the existence of a rational section of π_G is equivalent to another property. Namely, W also acts on G/T as follows:

$$w \cdot gT := g\dot{w}^{-1}T, \tag{1}$$

where $\dot{w} \in N_G(T)$ is a representative of w. We prove

Theorem 1.5. *Let G be a connected semisimple algebraic group. The following properties are equivalent:*

(i) *there is a rational section of π_G;*
(ii) *there is a W-equivariant rational map $T \dashrightarrow G/T$.*

Then we consider the existence problem and prove the following.

Recall (see [Bor, 22.3]) that the isogeny τ is called *central* if $\ker \tau$ lies in the center of G and $\ker d\tau_e$ lies in the center of $\operatorname{Lie} G$.

The next theorem answers the other Grothendieck's question cited in the epigraph.

Theorem 1.6. *Let G be a connected semisimple algebraic group.*

(i) *There is a rational cross-section in G.*
(ii) *If the isogeny τ is central, then there is a rational section of π_G.*

For $\operatorname{char} k = 0$, this theorem has been proved earlier in [CTKPR, Theorem 0.3]. The strategy and the essential part of our proof are the same: we use the relevant characteristic free results from [CTKPR], but bypass Theorem 2.12 from this paper (whose proof is based on the assumption $\operatorname{char} k = 0$) by exploring properties of π_G and proving that versality of G holds in any characteristic; this permits us to use STEINBERG's section of $\pi_{\widehat{G}}$ in place of KOSTANT's cross-section in $\operatorname{Lie} G$ used in [CTKPR].

Theorems 1.5 and 1.6 yield the following

Corollary 1.7. *Let G be a connected semisimple algebraic group. If the isogeny τ is central, then there is a W-equivariant rational map $T \dashrightarrow G/T$.*

Section 6 contains some remarks, questions, and an example of a cross-section S in G such that $\pi_G|_S$ is not separable (hence S is not the image of a section of π_G).

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2. Cross-sections in G

Fix a choice of Borel subgroup $\widehat{B}$ of $\widehat{G}$ and maximal torus $\widehat{T} \subset \widehat{B}$. Denote by $X(\widehat{T})$ the character lattice of $\widehat{T}$ in additive nonation. For $\lambda \in X(\widehat{T})$, denote by t^λ the value of $\lambda\colon \widehat{T} \to \mathbf{G}_m$ at $t \in \widehat{T}$. Let $\varpi_1, \ldots, \varpi_r \in X(\widehat{T})$ be the system of fundamental weights of $\widehat{T}$ with respect to $\widehat{B}$.

Let $\varrho_i\colon \widehat{G} \to \mathbf{GL}(V_i)$ be an irreducible representation of $\widehat{G}$ with ϖ_i as the highest weight. Let $\chi_{\varpi_i} \in k[\widehat{G}]^{\widehat{G}}$ be the character of ϱ_i.

Let $\widehat{C}$ be the center of $\widehat{G}$; it is a finite subgroup of $\widehat{T}$. The conjugating action of $\widehat{G}$ on itself commutes with the action of $\widehat{C}$ on $\widehat{G}$ by left translations. Therefore the latter action descends to $\widehat{G}/\!\!/\widehat{G}$ and
$$\pi_{\widehat{G}}\colon \widehat{G} \longrightarrow \widehat{G}/\!\!/\widehat{G}$$
becomes a $\widehat{C}$-equivariant morphism.

Endow the r-dimensional affine space $\mathbf{A}^r$ with the linear action of $\widehat{T}$ by the formula
$$t \cdot (a_1, \ldots, a_r) := (t^{\varpi_1} a_1, \ldots, t^{\varpi_r} a_r), \qquad t \in \widehat{T},\ \ (a_1, \ldots, a_r) \in \mathbf{A}^r. \tag{2}$$

Lemma 2.1.

(i) *The $\widehat{T}$-stabilizer of the point $(1, \ldots, 1) \in \mathbf{A}^r$ is trivial. In particular, the considered action of $\widehat{T}$ on $\mathbf{A}^r$ is faithful.*

(ii) *There is a $\widehat{C}$-equivariant isomorphism*
$$\lambda\colon \widehat{G}/\!\!/\widehat{G} \xrightarrow{\simeq} \mathbf{A}^r.$$

Proof. Since $\varpi_1, \ldots, \varpi_r$ generate $X(\widehat{T})$, we have
$$\bigcap_{i=1}^{r} \{t \in T \mid t^{\varpi_i} = 1\} = \{e\}. \tag{3}$$
But (2) entails that the $\widehat{T}$-stabilizer of the point $(1, \ldots, 1)$ coincides with the right-hand side of equality (3). This proves (i).

By [Ste$_1$, Theorems 6.1, 6.16] the k-algebra $k[\widehat{G}]^{\widehat{G}}$ is freely generated by $\chi_{\varpi_1}, \ldots, \chi_{\varpi_r}$ and the morphism
$$\theta\colon \widehat{G} \longrightarrow \mathbf{A}^r, \qquad \theta(g) = (\chi_{\varpi_1}(g), \ldots, \chi_{\varpi_r}(g)),$$
is surjective. Hence there is an isomorphism $\lambda\colon \widehat{G}/\!\!/\widehat{G} \longrightarrow \mathbf{A}^r$ such that the following diagram is commutative:
$$\begin{tikzcd} & \widehat{G} \arrow[dl, "\pi_{\widehat{G}}"'] \arrow[dr, "\theta"] & \\ \widehat{G}/\!\!/\widehat{G} \arrow[rr, "\lambda"] & & \mathbf{A}^r \end{tikzcd} \quad . \tag{4}$$

The morphism θ is $\widehat{C}$-equivariant. Indeed, let $c \in \widehat{C}$. Since ϱ_i is irreducible, Schur's lemma entails that $\varrho_i(c) = \mu_{i,c}\, \mathrm{id}_{V_i}$ for some $\mu_{i,c} \in k$. On the other hand, since $c \in \widehat{T}$, any highest vector in V_i with respect to $\widehat{B}$ is an eigenvector of c with the eigenvalue c^{ϖ_i}. Hence $\mu_{i,c} = c^{\varpi_i}$. Therefore, for every $g \in \widehat{G}$, by (2) we have
$$\begin{aligned} \theta(cg) &= \big(\chi_{\varpi_1}(cg), \ldots, \chi_{\varpi_r}(cg)\big) \\ &= \big(\mathrm{trace}\,\big(\varrho_1(cg)\big), \ldots, \mathrm{trace}\,\big(\varrho_r(cg)\big)\big) \end{aligned}$$

$$
\begin{aligned}
&= \big(\operatorname{trace}\big(\varrho_1(c)\varrho_1(g)\big), \ldots, \operatorname{trace}\big(\varrho_1(c)\varrho_r(g)\big)\big) \\
&= \big(\operatorname{trace}\big(c^{\varpi_1}\varrho_1(g)\big), \ldots, \operatorname{trace}\big(c^{\varpi_r}\varrho_r(g)\big)\big) \\
&= \big(c^{\varpi_1}\operatorname{trace}\big(\varrho_1(g)\big), \ldots, c^{\varpi_r}\operatorname{trace}\big(\varrho_r(g)\big)\big) \\
&= \big(c^{\varpi_1}\chi_{\varpi_1}(g), \ldots, c^{\varpi_r}\chi_{\varpi_r}(g)\big) \\
&= c\cdot\theta(g),
\end{aligned}
$$

as claimed.

Since both θ and $\pi_{\widehat{G}}$ are $\widehat{C}$-equivariant and $\pi_{\widehat{G}}$ is surjective, commutativity of diagram (4) entails that λ is $\widehat{C}$-equivariant as well. This proves (ii). $\square$

Corollary 2.2. *Let g be a nonidentity element of $\widehat{C}$. Then there is no g-stable cross-section in $\widehat{G}$.*

Proof. Assume the contrary and let $\widehat{S}$ be a g-stable cross-section in $\widehat{G}$. Since $\pi_{\widehat{G}}$ is $\widehat{C}$-equivariant, $\pi_{\widehat{G}}|_{\widehat{S}}\colon \widehat{S} \to \widehat{G}/\!\!/\widehat{G}$ is a bijective g-equivariant morphism. As, by Lemma 2.1(ii), there is a point of $\widehat{G}/\!\!/\widehat{G}$ fixed by $\widehat{C}$, hence by g, this implies that there is a point of $\widehat{S}$ fixed by g. But for the action of $\widehat{C}$ on $\widehat{G}$ by left translations, the stabilizer of every point is trivial, a contradiction with $g \neq e$. $\square$

Given an element h of an algebraic group H, we shall denote its conjugacy class in H by $H(h)$:

$$H(h) := \{shs^{-1} \mid s \in H\}. \tag{5}$$

Lemma 2.3. *Let H and $\widetilde{H}$ be connected algebraic groups and let $\sigma\colon \widetilde{H} \to H$ be an isogeny. Then the following properties hold:*

(i) *σ is a finite morphism;*

(ii) *$\sigma\big(\widetilde{H}(h)\big) = H\big(\sigma(h)\big)$ and $\dim \widetilde{H}(h) = \dim H\big(\sigma(h)\big)$ for every $h \in \widetilde{H}$;*

(iii) *if $\widetilde{H}(h)$ is a regular conjugacy class in $\widetilde{H}$ (i.e., that of the maximal dimension), then $\sigma\big(\widetilde{H}(h)\big)$ is a regular conjugacy class in H;*

(iv) *if H and $\widetilde{H}$ are semisimple, then for every $h \in \widetilde{H}$,*

$$\sigma\big(\pi_{\widetilde{H}}^{-1}\big(\pi_{\widetilde{H}}(h)\big)\big) = \pi_H^{-1}\big(\pi_H\big(\sigma(h)\big)\big).$$

Proof. The varieties H and $\widetilde{H}$ are normal (even smooth) and the fiber of σ over every point of H is a finite set whose cardinality does not depend on this point. Hence (cf. [G$_1$, Sect. 2, Cor. 3]) $\widetilde{H}$ is the normalization of H in the field of rational functions on $\widetilde{H}$ and σ is the normalization map. This proves (i).

The first equality in (ii) holds as σ is an epimorphism of groups. The second follows the first and theorem on dimension of fibers, cf., e.g., [Bor, AG 10.1]. This proves (ii).

As σ is surjective, (iii) follows from (ii).

Since the fibers of $\pi_{\widetilde{H}}$ and π_H are the closures of regular conjugacy classes and, by (i), the map σ is closed, (iv) follows from (iii). $\square$

Corollary 2.4. *Let $\widetilde{G}$ be a connected semisimple algebraic group and let $\sigma\colon \widetilde{G} \to G$ be a bijective isogeny.*

(i) *If $\widetilde{S}$ is a cross-section in $\widetilde{G}$, then $\sigma(\widetilde{S})$ is a cross-section in G.*

(ii) *If S is a cross-section in G, then $\sigma^{-1}(S)$ is a cross-section in $\widetilde{G}$.*

The same holds if "cross-section" is replaced with "rational cross-section".

Proof. By Lemma 2.3(i) the bijective map σ is closed. Hence it is a homeomorphism. Both claims follow from this, the definitions of cross-section and rational cross-section, and Lemma 2.3(iv). $\square$

Lemma 2.5. *Assume that there is a subgroup Z of $\widehat{C}$ such that $G = \widehat{G}/Z$ and τ is the quotient morphism $\widehat{G} \to \widehat{G}/Z$. Then there is a morphism*

$$\varphi\colon \widehat{G}/\!\!/\widehat{G} \longrightarrow G/\!\!/G \tag{6}$$

such that

(i) *$(G/\!\!/G, \varphi)$ is a categorical quotient for the action of Z on $\widehat{G}/\!\!/\widehat{G}$;*
(ii) *the following diagram is commutative:*

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\tau} & G \\ {\scriptstyle \pi_{\widehat{G}}}\downarrow & & \downarrow{\scriptstyle \pi_G} \\ \widehat{G}/\!\!/\widehat{G} & \xrightarrow{\varphi} & G/\!\!/G \end{array} \quad ; \tag{7}$$

(iii) *for every point $x \in \widehat{G}/\!\!/\widehat{G}$, the following equality holds:*

$$\tau\big(\pi_{\widehat{G}}^{-1}(x)\big) = \pi_G^{-1}\big(\varphi(x)\big). \tag{8}$$

Proof. As τ^*, $\pi^*_{\widehat{G}}$, and π^*_G are injections, there is a unique morphism (6) such that $\tau^* \circ \pi^*_G = \pi^*_{\widehat{G}} \circ \varphi^*$, i.e., diagram (7) is commutative.

Consider the action of $\widehat{G}$ on G via the isogeny τ and the conjugating action of G on itself. The isogeny τ is then $\widehat{G}$-equivariant and $\widehat{G}$-orbits in G are G-conjugacy classes, so we have $k[G]^G = k[G]^{\widehat{G}}$. Since the conjugating action of $\widehat{G}$ on itself commutes with the action of Z by left translations, we have

$$\begin{aligned}\pi^*_{\widehat{G}}\big(\varphi^*(k[G/\!\!/G])\big) &= \tau^*\big(\pi^*_G(k[G/\!\!/G])\big) = \tau^*\big(k[G]^G\big) = \tau^*\big(k[G]^{\widehat{G}}\big) = \big(\tau^*(k[G])\big)^{\widehat{G}} \\ &= \big(k[\widehat{G}]^Z\big)^{\widehat{G}} = \big(k[\widehat{G}]^{\widehat{G}}\big)^Z = \big(\pi^*_{\widehat{G}}(k[\widehat{G}/\!\!/\widehat{G}])\big)^Z = \pi^*_{\widehat{G}}\big(k[\widehat{G}/\!\!/\widehat{G}]^Z\big).\end{aligned}$$

Thus, $\varphi^*(k[G/\!\!/G]) = k[\widehat{G}/\!\!/\widehat{G}]^Z$. This proves (i) and (ii). Lemma 2.3(iv) and commutativity of diagram (7) imply (iii). $\square$

Below, given a variety Z, we denote by $\mathrm{T}_{z,Z}$ the tangent space of Z at a point z.

Proof of Theorem 1.2. First, we shall prove criterion (i).

1. By Steinberg's theorem, $\widehat{G}$ has a cross-section. Hence, by Corollary 2.4, if τ is bijective, then there exists a cross-section in G as well.

So we may assume that τ is not bijective and we then have to prove that there is no cross-section in G. Solving this problem, we may assume that τ is separable. Indeed, if this is not the case, then by [Bor, Prop. 17.9] there exist a connected semisimple algebraic group $\widetilde{G}$ and a commutative diagram of isogenies

$$\begin{array}{ccccc} \widehat{G} & & \xrightarrow{\quad\tau\quad} & & G \\ & {\scriptstyle \mu}\searrow & & \nearrow{\scriptstyle \sigma} & \\ & & \widetilde{G} & & \end{array} \quad , \tag{9}$$

where μ is separable and σ is purely inseparable. As σ is bijective, Corollary 2.4 then reduces the problem to proving that there is no cross-section in $\widetilde{G}$, i.e., we may replace G by $\widetilde{G}$ and τ by μ.

So from now on we may (and shall) assume that τ is a separable isogeny of degree $\geqslant 2$. This means that there is a nontrivial subgroup Z of $\widehat{C}$ such that $G = \widehat{G}/Z$ and τ is the quotient morphism $\widehat{G} \to \widehat{G}/Z$.

2. Now, arguing on the contrary, assume that there is a cross-section S in G.

Claim 1. (i) *For every point $x \in \widehat{G}/\!\!/\widehat{G}$, the intersection*

$$\pi_{\widehat{G}}^{-1}(x) \cap \tau^{-1}(S) \tag{10}$$

is a nonempty subset of a single Z-orbit; in particular, it is finite.

(ii) *There is a nonempty open subset U of $\widehat{G}/\!\!/\widehat{G}$ such that, for every $x \in U$, intersection* (10) *is a single point.*

Proof of Claim 1. Consider diagram (7). Since $S \cap \pi_G^{-1}\big(\varphi(x)\big)$ is a single point g, we deduce from (8) that intersection (10) is contained in $\tau^{-1}(g)$. This proves (i) as the fibers of τ are Z-orbits.

By Lemma 2.1(i) there is a nonempty open subset U in $\widehat{G}/\!\!/\widehat{G}$ such that the $\widehat{C}$-stabilizer of every point of U is trivial. Take a point $x \in U$. Assume that intersection (10) contains two points g_1 and $g_2 \neq g_1$. By (i) there exists an element $z \in Z$ such that $g_2 = zg_1$. As $\pi_{\widehat{G}}$ is $\widehat{C}$-equivariant, $x = \pi_{\widehat{G}}(g_2) = \pi_{\widehat{G}}(zg_1) = z \cdot \pi_{\widehat{G}}(g_1) = z \cdot x$. Thus, z belongs to the $\widehat{C}$-stabilizer of x. The definition of U then implies that $z = e$. Hence $g_1 = g_2$, a contradiction. This proves (ii). $\square$

3. Since all the fibers of τ are finite, every irreducible component of $\tau^{-1}(S)$ has dimension $\leqslant \dim S = r$ and at least one of them has dimension r.

Claim 2. (i) *There is a unique r-dimensional irreducible component $\widehat{S}$ of $\tau^{-1}(S)$.*

(ii) $\tau(\widehat{S}) = S$.

Proof of Claim 2. Let $\widehat{S}$ be an r-dimensional irreducible component of $\tau^{-1}(S)$. Then $\tau(\widehat{S})$ contains an open subset of S. Since τ is closed, this proves (ii).

¿From (ii) we conclude that

$$\pi_G\big(\tau(\widehat{S})\big) = G/\!\!/G. \tag{11}$$

But by Lemma 2.5 the fibers of φ in commutative diagram (7) are finite. This and (11) imply that $\pi_{\widehat{G}}(\widehat{S})$ contains a nonempty open subset of $\widehat{G}/\!\!/\widehat{G}$.

Let now $\widehat{S}'$ be another r-dimensional irreducible components of $\tau^{-1}(S)$. Then, as above, $\pi_{\widehat{G}}(\widehat{S}')$ contains a nonempty open subset of $\widehat{G}/\!\!/\widehat{G}$ as well. Therefore, $\pi_{\widehat{G}}(\widehat{S}) \cap \pi_{\widehat{G}}(\widehat{S}')$ contains a nonempty open subset V of $\widehat{G}/\!\!/\widehat{G}$. We may assume that $V \subseteq U$ for U from Claim 1(ii). The latter then yields that $\pi_{\widehat{G}}^{-1}(V) \cap \widehat{S} = \pi_{\widehat{G}}^{-1}(V) \cap \widehat{S}'$. As both sides of this equality are the open subsets of respectively $\widehat{S}$ and $\widehat{S}'$, we infer that $\widehat{S} = \widehat{S}'$. This proves (i). $\square$

4. As $\widehat{S}$ is a unique r-dimensional irreducible component of the Z-stable variety $\tau^{-1}(S)$, we conclude that $\widehat{S}$ is Z-stable. We shall now show that $\widehat{S}$ is a cross-section in $\widehat{G}$. As this property contradicts Corollary 2.2, the proof of (i) will be then completed.

5. Let x be a point of $\widehat{G}/\!\!/\widehat{G}$. As S is a section of G, the intersection $S \cap \pi_G^{-1}\big(\varphi(x)\big)$ is a single point $g \in G$. By Claim 2(ii) there is a point $\widehat{g} \in \widehat{S}$ such that $\tau(\widehat{g}) = g$. Commutativity of diagram (7) then entails that x and $\widehat{x} := \pi_{\widehat{G}}(\widehat{g})$ are in the same fiber of φ. Since the fibers of φ are Z-orbits, there is an element $z \in Z$ such that $x = z \cdot \widehat{x}$. As $\pi_{\widehat{G}}$ is Z-equivariant, this yields $\pi_{\widehat{G}}(z\widehat{g}) = x$. But $z\widehat{g} \in \widehat{S}$ as $\widehat{S}$ is Z-stable and $\widehat{g} \in \widehat{S}$. Hence $\pi_{\widehat{G}}^{-1}(x) \cap \widehat{S} \neq \varnothing$, i.e.,

$$\pi_{\widehat{G}}(\widehat{S}) = \widehat{G}/\!\!/\widehat{G}. \tag{12}$$

6. It follows from Claim 1(i),(ii) and (12) that $\pi_{\widehat{G}}|_{\widehat{S}}$ is the surjective morphism with finite fibers, bijective over an open subset of $\widehat{G}/\!\!/\widehat{G}$. As $\widehat{G}$ is normal, $\widehat{G}/\!\!/\widehat{G}$ is

normal as well. Let $\nu\colon \widetilde{S} \to \widehat{S}$ be the normalization. Then the surjective morphism $\pi_{\widehat{G}}|_{\widehat{S}} \circ \nu\colon \widetilde{S} \to \widehat{G}/\!\!/\widehat{G}$ of normal varieties has finite fibers and is bijective over an open subset of $\widehat{G}/\!\!/\widehat{G}$. Hence $\pi_{\widehat{G}}|_{\widehat{S}} \circ \nu$ is bijective (see [G_1, Sect. 2, Cor. 2]). Whence $\pi_{\widehat{G}}|_{\widehat{S}}$ is bijective as well, i.e., $\widehat{S}$ is a cross-section in $\widehat{G}$. This completes the proof of (i).

We now turn to the proof of (ii).

Let $S := \sigma(G/\!\!/G)$. Take a point $x \in S$ and put $y := \pi_G(x)$. As $\pi_G|_S\colon S \to G/\!\!/G$ is the isomorphism (σ is its inverse), $d(\pi_G|_S)_x$ is the isomorphism as well. Hence $(d\pi_G)_x$ is surjective. As $\dim \mathrm{T}_{y,G/\!\!/G} \geqslant \dim G/\!\!/G = r$, this implies that there are functions $f_1, \ldots, f_r \in k[G]^G$ such that $(df_1)_x, \ldots, (df_r)_x$ are linearly independent. By [Ste_1, Theorem 8.7] this yields that x is regular. As S intersects every fiber of π_G at a single point and every such fiber contains a unique regular orbit, this proves (ii). Thus, the proof of Theorem 1.2 comes to a close. $\square$

3. Singularities of $G/\!\!/G$ and generators of $k[G]^G$ and $R(G)$

The following lemma shows that there is a link between the existence of a cross-section in G and smoothness of $G/\!\!/G$.

Lemma 3.1. *Let* $\mathrm{char}\, k = 0$. *If a surjective morphism* $\alpha\colon X \to Y$ *of irreducible varieties admits a section* $\sigma\colon Y \to X$, *then smoothness of* X *implies smoothness of* Y.

Proof. Arguing on the contrary, assume that y is a singular point of Y, i.e.,

$$\dim \mathrm{T}_{y,Y} > \dim Y. \tag{13}$$

Put $x = \sigma(y) \in X$. Since $\alpha \circ \sigma = \mathrm{id}_Y$, the composition $d\alpha_x \circ d\sigma_y$ is the identity map of $\mathrm{T}_{y,Y}$. Hence $d\alpha_x$ is surjective, i.e., $\mathrm{rk}\, d\alpha_x = \dim \mathrm{T}_{y,Y}$. By (13) this yields

$$\mathrm{rk}\, d\alpha_x > \dim Y. \tag{14}$$

As $\mathrm{char}\, k = 0$, there is a dense open subset U of X such that $\mathrm{rk}\, d\alpha_z = \dim Y$ for every point $z \in U$, see [H, 14.4]. As $z \mapsto \dim \ker d\alpha_z$ is the upper semi-continuous function [H, 14.6], we conclude that smoothness of X implies that $\mathrm{rk}\, d\alpha_z \leqslant \dim Y$ for every point $z \in X$. This contradicts (14). $\square$

This prompts to explore smoothness of $G/\!\!/G$. The answer is known:

Theorem 3.2 ([Ste_3, §3], [R_1, Prop. 4.1], [R_2, Prop. 13.3])**.** *Let* $\mathrm{char}\, k \neq 2$. *The following properties are equivalent:*

(i) *$G/\!\!/G$ is smooth;*
(ii) *$G/\!\!/G$ is isomorphic to the affine space* $\mathbf{A}^r$*;*
(iii) *$G = G_1 \times \cdots \times G_s$ where every G_i is either a simply connected simple algebraic group or isomorphic to $\mathbf{SO}_{n_i}$ for an odd n_i.*

This criterion of smoothness of $G/\!\!/G$ may be also interpreted as that of the existence of r generators of the algebra of class functions on G. Below we describe a minimal system of generators of this algebra and singularities of $G/\!\!/G$ in the general case. This also yields a minimal system of generators of the representation ring of G.

Let $B := \tau(\widehat{B})$ and $T := \tau(\widehat{T})$. This is respectively a Borel subgroup and a maximal torus of G. We consider the lattice $\mathrm{X}(T)$ of characters of T as the sublattice of $\mathrm{X}(\widehat{T})$ identifying $\mu \in \mathrm{X}(T)$ with $\tau^*(\mu) \in \mathrm{X}(\widehat{T})$. Then $\mathrm{X}(\widehat{T})$ is the weight lattice of $\mathrm{X}(T)$. The monoid of highest weights of simple $\widehat{G}$-modules (with respect to $\widehat{B}$ and $\widehat{T}$) is

$$\widehat{\mathcal{D}} := \mathbf{N}\varpi_1 + \cdots + \mathbf{N}\varpi_r, \qquad \mathbf{N} = \{0, 1, 2, \ldots\}. \tag{15}$$

and that of simple G-modules (with respect to B and T) is

$$\mathcal{D} := \widehat{\mathcal{D}} \cap \mathrm{X}(T). \tag{16}$$

Let W be the Weyl group of $\widehat{G}$, i.e., the quotient of $\widehat{T}$ in its normalizer, acting on $\widehat{T}$ via conjugation. The Weyl group of T is naturally identified with W (see [Bor, Prop. 11.20 and Cor. 2(d) in 13.17]).

If $\varpi \in \mathcal{D}$ and $E(\varpi)$ is a simple G-module with ϖ as the highest weight, we denote by $\chi_\varpi \in k[G]^G$ the character of $E(\varpi)$.

Given a nonzero commutative ring A with identity element and a commutative monoid M, we denote by $A[M]$ the semigroup ring of M over A. We identify $A[M]$ with $A \otimes_{\mathbf{Z}} \mathbf{Z}[M]$ in the natural way. If S is a submonoid of the multiplicative monoid of $A[M]$ whose elements are linearly independent over A, then the subring of $A[M]$ generated by S is naturally identified with $A[S]$. In particular, we consider $A[\mathrm{X}(T)]$ and $A[\mathcal{D}]$ as the subrings of $A[\mathrm{X}(\widehat{T})]$. The former is stable with respect to the natural action of W on $\mathbf{Z}[\mathrm{X}(\widehat{T})]$. Using the notation and terminology of Bourbaki [Bou$_2$, VI.3], we denote by e^μ the element of $\mathbf{Z}[\mathrm{X}(\widehat{T})]$ corresponding to $\mu \in \mathrm{X}(\widehat{T})$ and put

$$S(e^\mu) := \sum_{\nu \in W\cdot\mu} e^\nu \in \mathbf{Z}[\mathrm{X}(\widehat{T})]^W. \tag{17}$$

Given an algebraic group H, we denote by $R(H)$ the *representation ring* of H: its additive group is the Grothendieck group of the category of finite dimensional algebraic H-modules with respect to exact sequences and the multiplication is induced by tensor product of modules. Using τ, we identify $R(G)$ in the natural way with the subring of $R(\widehat{G})$.

If E is a finite dimensional algebraic G-module and E_μ is its weight space of a weight $\mu \in \mathrm{X}(T)$, then the formal character of E,

$$\mathrm{ch}_G[E] := \sum_{\mu \in \mathrm{X}(T)} (\dim E_\mu) e^\mu, \tag{18}$$

is an element of $\mathbf{Z}[\mathrm{X}(T)]^W$ depending only on the class $[E]$ of E in $R(G)$. Clearly,

$$\mathrm{ch}_G[E \otimes E'] = \mathrm{ch}_G[E]\,\mathrm{ch}_G[E']. \tag{19}$$

According to [Se, 3.6], the homomorphism of $\mathbf{Z}$-modules

$$\mathrm{ch}_G\colon R(G) \longrightarrow \mathbf{Z}[\mathrm{X}(T)]^W, \qquad [E] \mapsto \mathrm{ch}_G[E], \tag{20}$$

is an isomorphism. By (19) it is an isomorphism of rings.

Definition 3.3. Let $\varpi \in \widehat{\mathcal{D}}$. We say that an element $x \in \mathbf{Z}[\mathrm{X}(\widehat{T})]^W$ is *ϖ-sharp* if the following property (M) holds:

(M) e^ϖ is the unique maximal term of x.

Example 3.4. The elements $S(e^\varpi)$ and $\mathrm{ch}_{\widehat{G}}[E(\varpi)]$ are ϖ-sharp (this follows, e.g., from [Bou$_2$, VI.1.6, Prop. 18] and [Hum$_1$, 31.3, Theorem]). $\square$

Property (M) implies that the support of a ϖ-sharp element x lies in $\varpi + \mathrm{X}(T)$. This and [Bou$_2$, VI.3.4, formula (6)] yield

$$x = S(e^\varpi) + \text{sum of some } S(e^{\varpi'})\text{'s with } \varpi' \in \widehat{\mathcal{D}},\ \varpi' < \varpi. \tag{21}$$

By [Bou$_2$, VI.3.2, Lemma 2] if an element x' is a ϖ'-sharp, then xx' is $(\varpi+\varpi')$-sharp.

Now fix a ϖ_i-sharp element $x_{\varpi_i} \in \mathbf{Z}[\mathrm{X}(\widehat{T})]^W$, $i = 1, \ldots, r$, and put

$$x_\varpi := x_{\varpi_1}^{m_1} \cdots x_{\varpi_r}^{m_r} \qquad \text{for} \qquad \varpi = m_1\varpi_1 + \cdots + m_r\varpi_r \in \widehat{\mathcal{D}}.$$

By [Bou$_2$, VI.3.4, Theorem 1] the set $\{x_\varpi \mid \varpi \in \widehat{\mathcal{D}}\}$ is then a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(\widehat{T})]^W$. As $\{e^\mu \mid \mu \in \mathrm{X}(T)\}$ is a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(T)]$ and the support of

x_ϖ lies in $\varpi + \mathrm{X}(T)$, we deduce from this and (16) that $\{x_\varpi \mid \varpi \in \mathcal{D}\}$ is a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(T)]^W$. Hence the homomorphism of the $\mathbf{Z}$-modules

$$\vartheta\colon \mathbf{Z}[\mathrm{X}(T)]^W \to \mathbf{Z}[\mathcal{D}], \qquad \vartheta(x_\varpi) = e^\varpi \quad \text{for} \quad \varpi \in \mathcal{D}, \tag{22}$$

is an isomorphism. Since $x_{\varpi+\varpi'} = x_\varpi x_{\varpi'}$, it is, in fact, an isomorphism of rings.

As by Dedekind's theorem $\{f_\mu\colon T \to k, t \mapsto t^\mu \mid \mu \in \mathrm{X}(\widehat{T})\}$ is a basis of the vector space $k[\widehat{T}]$ over k, the k-linear map $k[\widehat{T}] \to k[\mathrm{X}(\widehat{T})]$, $f_\mu \mapsto e^\mu$, is the isomorphism of k-algebras. We identify them by means of this isomorphism. So we have $k[T] = k[\mathrm{X}(T)]$ and

$$k[T/W] = k[T]^W = k[\mathrm{X}(T)]^W. \tag{23}$$

Finally, take into account that by [Ste$_1$, 6.4] the restriction map

$$\mathrm{res}\colon k[G/\!\!/G] = k[G]^G \longrightarrow k[T]^W, \qquad \mathrm{res}(f) = f|_T, \tag{24}$$

is an isomorphism of k-algebras. Summing up, we obtain

Theorem 3.5.

(i) *$G/\!\!/G$ and T/W are the affine toric varieties of T whose algebras of regular functions are isomorphic to $k[\mathcal{D}]$.*

(ii) *In the diagram*

$$k[G/\!\!/G] \xrightarrow{\mathrm{res}} k[T/W] \xrightarrow{\mathrm{id}\otimes\vartheta} k[\mathcal{D}]$$

(*see* (24), (23), (22)) *both maps are the isomorphisms of k-algebras.*

(iii) *Let F be the simple subring of k. Then the image of $F \otimes_{\mathbf{Z}} R(G)$ in $k[G]^G$ under the composition of the ring isomorphisms*

$$k \otimes_{\mathbf{Z}} R(G) \xrightarrow{\mathrm{id}\otimes\mathrm{ch}_G} k[\mathrm{X}(T)]^W = k[T]^W \xrightarrow{\mathrm{res}^{-1}} k[G]^G \tag{25}$$

is an F-form of $k[G/\!\!/G]$ isomorphic to $F\otimes_{\mathbf{Z}}R(G)$. In particular, if $\mathrm{char}\, k = 0$, *it is a $\mathbf{Z}$-form of $k[G/\!\!/G]$ isomorphic to $R(G)$.*

Remark 3.6. The fact that "multiplicative invariants" of finite reflection groups are semigroup algebras is already in the literature, first implicitly, then explicitly, see the historical account in [L$_1$, Introduction]. Essentially, the main ingredients date back to [Ste$_1$, §6] and [Bou$_2$,VI §3].

Since toric varieties are rational, Theorem 3.5(i) yields

Corollary 3.7. *$G/\!\!/G$ and T/W are rational varieties.*

In the next statement Theorem 3.5 is applied to finding a minimal system of generators of the algebra $k[G]^G$ and that of the ring $R(G)$.

Let $\mathcal{H}$ be the Hilbert basis of the monoid $\mathcal{D}$, i.e., the set of all its indecomposable elements:

$$\mathcal{H} = \mathcal{D}_+ \setminus 2\mathcal{D}_+ \quad \text{where} \quad \mathcal{D}_+ := \mathcal{D} \setminus \{0\}, \quad 2\mathcal{D}_+ := \mathcal{D}_+ + \mathcal{D}_+. \tag{26}$$

The set $\mathcal{H}$ is finite, generates $\mathcal{D}$, and every generating set of $\mathcal{D}$ contains $\mathcal{H}$ (see, e.g., [L$_2$, 3.4]).

Remark 3.8. There is an algorithm for computing $\mathcal{H}$, see [Stu, 13.2] (cf. also Example 3.11 below).

Theorem 3.9.

(i) *The cardinality of every generating set of the algebra $k[G]^G$ of class functions on G is not less than the cardinality of $\mathcal{H}$. The same holds for every generating set of the representation ring $R(G)$ of G.*

(ii) $\{[E(\varpi)] \mid \varpi \in \mathcal{H}\}$ *is a generating set of the ring $R(G)$.*

(iii) $\{\chi_\varpi \mid \varpi \in \mathcal{H}\}$ *is a generating set of the algebra* $k[G]^G$.

Proof. (i) Let Y be the affine toric variety of T with $k[Y] = k[\mathcal{D}]$. The linear span I of $\{e^\varpi \mid \varpi \in \mathcal{D}_+\}$ over k is a maximal T-invariant ideal in $k[Y]$. Hence I/I^2 is the cotangent space of Y at the T-fixed point v where I vanishes. As I^2 is the linear span of $\{e^\varpi \mid \varpi \in 2\mathcal{D}_+\}$ over k, this and (26) yield

$$\dim \mathrm{T}_{v,Y} = \dim I/I^2 = |\mathcal{H}|. \tag{27}$$

Now take into account that, given an affine algebraic variety X, the algebra $k[X]$ can be generated by d elements if and only if X admits a closed embedding in $\mathbf{A}^d$. Hence $d \geqslant \dim \mathrm{T}_{x,X}$ for every point $x \in X$. This, Theorem 3.5(i),(iii), and (27) prove (i).

(ii) Let $\mu \in \mathcal{D}$. As $\mathcal{H}$ generates $\mathcal{D}$, we deduce from Example 3.4 that there is a μ-sharp monomial M^μ in the elements of the set $\{\mathrm{ch}_G[E(\varpi)] \mid \varpi \in \mathcal{D}\}$. By (21) we have

$$M^\mu = S(e^\mu) + \text{sum of some } S(e^{\mu'})\text{'s with } \mu' \in \mathcal{D},\ \mu' < \mu. \tag{28}$$

But $\{S(e^\mu) \mid \mu \in \mathcal{D}\}$ is a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(T)]^W$ (see [Bou$_2$, VI.3.4, Lemma 3]). By [Bou$_2$, VI.3.4, Lemma 4] and (28) we then conclude that the set $\{M^\mu \mid \mu \in \mathcal{D}\}$ generates the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(T)]^W$. This means that the ring $\mathbf{Z}[\mathrm{X}(T)]^W$ is generated by the set $\{\mathrm{ch}_G[E(\varpi)] \mid \varpi \in \mathcal{H}\}$. As (20) is an isomorphism of rings, this proves (ii).

(iii) It follows from (ii) that the set $\{1 \otimes [E(\varpi)] \mid \varpi \in \mathcal{H}\}$ generates the ring $k \otimes_{\mathbf{Z}} R(G)$. But formula (18) shows that χ_ϖ is the image of $1 \otimes E[(\varpi)]$ under the composition of the ring isomorphisms in diagram (25). This proves (iii). □

Since the Weyl chambers are simplicial cones, Theorem 3.5(i) implies, at least for $\mathrm{char}\, k = 0$, that $G/\!\!/G$ and T/W are isomorphic to the quotient of $\mathbf{A}^r$ by a linear action of a certain finite abelian group (see, e.g., [O, Prop. 1.25]). In particular, $G/\!\!/G$ and T/W may have only finite quotient singularities. Below this finite group and its action on $\mathbf{A}^r$ are explicitly described assuming that τ is separable.

The latter assumption means that there is a subgroup Z of $\widehat{C}$ such that $G = \widehat{G}/Z$ and τ is the quotient morphism $\widehat{G} \to \widehat{G}/Z$. In this situation we have

$$\mathrm{X}(T) = \{\mu \in \mathrm{X}(\widehat{T}) \mid c^\mu = 1 \text{ for every } c \in Z\}. \tag{29}$$

Consider the $\widehat{T}$-orbit map of the point $(1, \ldots, 1) \in \mathbf{A}^r$:

$$\iota\colon \widehat{T} \longrightarrow \mathbf{A}^r, \qquad \iota(t) = t \cdot (1, \ldots, 1). \tag{30}$$

The map $\iota^*\colon k[\mathbf{A}^r] \to k[\widehat{T}] = k[\mathrm{X}(\widehat{T})]$ is an embedding as ι is dominant by Lemma 2.1(i). Let $y_1, \ldots, y_r$ be the standard coordinate functions on $\mathbf{A}^r$. Then (2) and (30) yield

$$\iota^*(y_i) = e^{\varpi_i}. \tag{31}$$

From (2) we deduce that $k[\mathbf{A}^r]^Z$ is the linear span over k of all monomials $y^{m_1} \cdots y^{m_r}$ with $m_1, \ldots, m_r \in \mathbf{N}$ such that $c^{m_1\varpi_i + \cdots + m_r\varpi_r} = 1$ for every $c \in Z$. By (29) the latter condition is equivalent to the inclusion $m_1\varpi_i + \cdots + m_r\varpi_r \in \mathrm{X}(T)$. This, (31), (15), and (16) imply that $\iota^*(k[\mathbf{A}^r]^Z) = k[\mathcal{D}]$. Thus, taking into account Theorem 3.5, we obtain the isomorphisms of k-algebras

$$k[\mathbf{A}^r]^Z \xrightarrow{\iota^*} k[\mathcal{D}] \xrightarrow{(\mathrm{id}\otimes\vartheta)^{-1}} k[T/W] \xrightarrow{(\mathrm{res})^{-1}} k[G/\!\!/G].$$

that, in turn, induce the isomorphisms of varieties $G/\!\!/G \to T/W \to \mathbf{A}^r/Z$.

By means of a special parametrization of $\widehat{T}$ one obtains an explicit description of the elements of $\widehat{C}$ well adapted for computing $k[\mathbf{A}^r]^Z$. Since $\widehat{G} = \widehat{G}_1 \times \cdots \times \widehat{G}_s$ and $\widehat{C} = \widehat{C}_1 \times \cdots \times \widehat{C}_s$ where every $\widehat{G}_i$ is a nontrivial normal simply connected simple subgroup of $\widehat{G}$ and $\widehat{C}_i$ is the center of $\widehat{G}_i$, it suffices to describe this parametrization for simple groups $\widehat{G}$. The answer is given below in Lemma 3.10.

Namely, let $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_r \in X(\widehat{T})$ be the system of simple roots of $\widehat{T}$ with respect to $\widehat{B}$ and let $\widehat{\alpha}_i^\vee \colon \mathbf{G}_m \to \widehat{T}$ be the coroot corresponding to $\widehat{\alpha}_i$. Then, for every $s \in \mathbf{G}_m$,

$$\big(\widehat{\alpha}_i^\vee(s)\big)^{\varpi_j} = \begin{cases} s & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases} \tag{32}$$

If $\langle\ ,\ \rangle$ is the natural pairing between the lattices of characters and cocharacters of $\widehat{T}$, we put $n_{ij} := \langle \widehat{\alpha}_i, \widehat{\alpha}_j^\vee \rangle$. So $(n_{ij})_{i,j=1}^r$ is the Cartan matrix of $\widehat{G}$.

By [Ste$_2$, Lemma 28(b),(d) and its Cor. (a)] the map

$$\nu \colon \mathbf{G}_m^r \longrightarrow \widehat{T}, \qquad \nu(s_1, \ldots, s_r) = \widehat{\alpha}_1^\vee(s_1) \cdots \widehat{\alpha}_r^\vee(s_r), \tag{33}$$

is an isomorphism of groups and

$$\widehat{C} = \{\widehat{\alpha}_1^\vee(s_1) \cdots \widehat{\alpha}_r^\vee(s_r) \mid s_1^{n_{i1}} \cdots s_r^{n_{ir}} = 1 \text{ for every } i = 1, \ldots, r\}. \tag{34}$$

By (32) and (33) we have

$$\big(\nu(s_1, \ldots, s_r)\big)^{\varpi_i} = \big(\widehat{\alpha}_1^\vee(s_1)\big)^{\varpi_i} \cdots \big(\widehat{\alpha}_r^\vee(s_r)\big)^{\varpi_i} = s_i.$$

This and (2) imply that, for every $s = (s_1, \ldots, s_r) \in \mathbf{G}_m^r$ and $(a_1, \ldots, a_r) \in \mathbf{A}^r$, we have

$$\nu(s) \cdot (a_1, \ldots, a_r) = (s_1 a_1, \ldots, s_r a_r).$$

Lemma 3.10. *For every simple simply connected group $\widehat{G}$, the subgroup $\nu^{-1}(\widehat{C})$ of the torus $\mathbf{G}_m^r$ is described in the following Table* 1 (*simple roots in* (33) *are numerated as in* [Bou$_2$]):

Table 1.

type of $\widehat{G}$	$\nu^{-1}(\widehat{C})$
A_r	$\{(s, s^2, s^3, \ldots, s^r) \mid s^{r+1} = 1\}$
B_r	$\{(1, \ldots, 1, s^2) \mid s^2 = 1\}$
C_r	$\{(s, 1, s, 1, \ldots, s^{r \bmod 2}) \mid s^2 = 1\}$
D_r, r odd	$\{(s^2, 1, s^2, 1, \ldots, s^2, s, s^{-1}) \mid s^4 = 1\}$
D_r, r even	$\{(s, 1, s, 1, \ldots, s, 1, st, t) \mid s^2 = t^2 = 1\}$
E_6	$\{(s, 1, s^{-1}, 1, s, s^{-1}) \mid s^3 = 1\}$
E_7	$\{(1, s, 1, 1, s, 1, s) \mid s^2 = 1\}$
E_8	$\{(1, 1, 1, 1, 1, 1, 1, 1)\}$
F_4	$\{(1, 1, 1, 1)\}$
G_2	$\{(1, 1)\}$

Proof. By (34) an element $(s_1, \ldots, s_r) \in \mathbf{G}_m^r$ lies in $\nu^{-1}(\widehat{C})$ if and only if $(s_1, \ldots, s_r)$ is a solution of the system of equations

$$\begin{aligned} x_1^{n_{11}} \cdots x_r^{n_{1r}} &= 1, \\ \cdots&\cdots\cdots \\ x_1^{n_{r1}} \cdots x_r^{n_{rr}} &= 1, \end{aligned} \tag{35}$$

where $(n_{ij})_{i,j=1}^r$ is the Cartan matrix of $\widehat{G}$.

Let, for instance, $\widehat{G}$ be of type D_r for even r. Using the explicit form of the Cartan matrix [Bou₂, Planche IV], one immediately verified that every element of $C' := \{(s,1,s,1,\ldots,s,1,st,t) \mid s^2=t^2=1\}$ is a solution of (35). Hence, $C' \subseteq \nu^{-1}(\widehat{C})$. On the other hand, the fundamental group of the root system of type D_r is isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ (see [Sp, 8.1.11] and [Bou₂, Planche IV]). Hence, the SMITH normal form of $(n_{ij})_{i,j=1}^r$ is $\operatorname{diag}(1,\ldots,1,2,2)$. Therefore, there is a basis $\beta_1,\ldots,\beta_r$ of the coroot lattice of $\widehat{T}$ such that, for $(s_1,\ldots,s_r)\in \mathbf{G}_m^r$, we have $\beta_1(s_1)\cdots\beta_r(s_r)\in\widehat{C}$ if and only if $(s_1,\ldots,s_r)$ is a solution of the system

$$x_1=1,\ldots,x_{r-2}=1, x_{r-1}^2=1, x_r^2=1.$$

This yields $|C'|=|\widehat{C}|$; whence $C'=\nu^{-1}(\widehat{C})$.

For the groups of the other types the proofs are similar. □

The following examples illustrate how this can be applied to exploring singularities of $G/\!\!/G$ and finding the minimal generating sets $\{\chi_\varpi \mid \varpi\in\mathcal{H}\}$ and $\{[E(\varpi)] \mid \varpi\in\mathcal{H}\}$ of, respectively, the algebra of class functions on G and the representation ring of G.

Examples 3.11.

(1) Let $\widehat{G}$ be of type B_r (where $\mathsf{B}_1:=\mathsf{A}_1$) and let $\operatorname{char} k\neq 2$. Table 1 implies that $\nu^{-1}(\widehat{C})$ is generated by $(1,\ldots,1,-1)$. Whence $k[\mathbf{A}^r]^{\widehat{C}}=k[y_1,\ldots,y_{r-1},y_r^2]$. Therefore, for the adjoint G, i.e., for $G=\mathbf{SO}_{2r+1}$, the variety $G/\!\!/G$ is isomorphic to $\mathbf{A}^r$ (this agrees with Theorem 3.2) and

$$\mathcal{H}=\{\varpi_1,\ldots,\varpi_{r-1},2\varpi_r\}.$$

(2) Let $\widehat{G}$ be of type D_r, let $\operatorname{char} k\neq 2$, and let $Z:=\{t\in\widehat{C}\mid t^{\varpi_1}=1\}$. Then $G:=\widehat{G}/Z=\mathbf{SO}_{2r}$. Table 1 implies that $\nu^{-1}(Z)$ is generated by $(1,\ldots,1,-1,-1)$. Whence $k[\mathbf{A}^r]^Z=k[y_1,\ldots,y_{r-2},y_{r-1}^2,y_r^2,y_{r-1}y_r]$. Therefore, $G/\!\!/G$ is isomorphic to $\mathbf{A}^{r-2}\times X$ where X is a nondegenerate quadratic cone in $\mathbf{A}^3$ and

$$\mathcal{H}=\{\varpi_1,\ldots,\varpi_{r-2},2\varpi_{r-1},2\varpi_r,\varpi_{r-1}+\varpi_r\}.$$

(3) Let $\widehat{G}$ be of type E_7 and let $\operatorname{char} k\neq 2$. Table 1 implies that $k[\mathbf{A}^7]^{\widehat{C}}$ is minimally generated by y_1,y_3,y_4,y_6 and all the monomials of order 2 in y_2,y_5,y_7. Therefore, if G is adjoint, then $G/\!\!/G$ is isomorphic to $\mathbf{A}^4\times Y$ where Y is the affine cone over the Veronese variety $\nu_2(\mathbf{P^2})$ in $\mathbf{P}^5$ (in particular, the tangent space of the 7-dimensional variety $G/\!\!/G$ at the unique fixed point of T, see Theorem 3.5(i) and (27), is 10-dimensional) and

$$\mathcal{H}=\{\varpi_1,\varpi_3,\varpi_4,\varpi_6,2\varpi_2,2\varpi_5,2\varpi_7,\varpi_2+\varpi_5,\varpi_2+\varpi_7,\varpi_5+\varpi_7.\}. \quad \square$$

4. Yet two Grothendieck's questions

Theorem 3.9 describes a minimal generating set of the algebra $k[G]^G$ of class functions on G. Constructing the generating sets of $k[G]^G$ is the topic of yet two GROTHENDIECK's questions in [GS, p. 241]:

> "[...] When G is an adjoint group, is it possible to generate the affine ring of $I(G)$ with coefficients of the Killing polynomial? In the general case, is it enough to take the coefficients of analogous polynomials for certain linear representations (perhaps arbitrary faithful representations)? [...]"

Below we answer these questions.

Let $\varrho\colon G\to\mathbf{GL}(V)$ be a finite dimensional linear representation of G. Define the set

$$C_\varrho := \{c_{\varrho,i}\in k[G] \mid i=1,\dots,\dim V\}$$

by the equality

$$\det(xI-\varrho(g)) = \sum_{i=0}^{\dim V} x^{\dim V-i}c_{\varrho,i}(g) \quad \text{for every}\ \ g\in G, \tag{36}$$

where x is a variable. If $V=E(\varpi)$ and ϱ determines the G-module structure of $E(\varpi)$, we put $C_\varpi := C_\varrho$.

Clearly, $c_{\varrho,i}\in k[G]^G$ and $c_{\varrho,1}$ is the character of ϱ. Hence by Theorem 3.9(iii)

$$\bigcup_{\varpi\in\mathcal{H}} C_\varpi$$

is a generating set of the algebra $k[G]^G$. This answers the second GROTHENDIECK's question in the affirmative.

In order to answer the first one in the negative it is sufficient to find an adjoint G and two elements $z_1,z_2\in T$ such that

(i) z_1 and z_2 are not in the same W-orbit;
(ii) the spectra of the linear operators $\mathrm{Ad}_G\, z_1$ and $\mathrm{Ad}_G\, z_2$ on the vector space $\mathrm{Lie}\, G$ coincide.

Indeed, property (i) implies that there is a function $f\in k[T]^W$ such that $f(z_1)\neq f(z_2)$. Given isomorphism (24), this means that there is a function $\widetilde{f}\in k[G]^G$ such that $\widetilde{f}(z_1)\neq\widetilde{f}(z_2)$. On the other hand, (36) and property (ii) imply that

$$c_{\mathrm{Ad}_G,i}(z_1)=c_{\mathrm{Ad}_G,i}(z_2) \qquad \text{for every}\ \ i.$$

Therefore, $\widetilde{f}$ is not in the subalgebra of $k[G]^G$ generated by $C_{\mathrm{Ad_G}}$, i.e., the latter is not a generating set of $k[G]^G$.

The following two examples show that one indeed can find G, z_1, and z_2 sharing properties (i) and (ii).

Examples 4.1.

(1) Let $G=H\times H$ where H is a connected adjoint semisimple algebraic group. Let $T=S\times S$ where S is a maximal torus of H. Let W_S be the Weyl group of H naturally acting on S. Take any two elements $a,b\in S$ that are not in the same W_S-orbit and put $z_1:=(a,b)$, $z_2:=(b,a)\in T$. As $W=W_S\times W_S$, property (i) holds. On the other hand, clearly, for every $i=1,2$, the spectrum of $\mathrm{Ad}_G\, z_i$ is the union of the spectra of $\mathrm{Ad}_H\, a$ and $\mathrm{Ad}_H\, b$; whence property (ii) holds.

(2) In this example G is simple, namely, $G=\mathbf{PGL}_3$. Let $\alpha_1,\alpha_2\in X(T)$ be the simple roots of T with respect to B. As the map $T\to\mathbf{G}_{\mathrm{m}}^2$, $t\mapsto(t^{\alpha_1},t^{\alpha_2})$, is surjective (in fact, an isomorphism), for every $u,v\in k$, $uv\neq 0$, there are $z_1,z_2\in T$ such that $z_1^{\alpha_1}=u$, $z_1^{\alpha_2}=v$ and $z_2^{\alpha_1}=v$, $z_2^{\alpha_2}=u$. For these z_1,z_2, property (ii) holds as the set of roots of G with respect to T is $\{\pm\alpha_1,\pm\alpha_2,\pm(\alpha_1+\alpha_2)\}$. Now take u and v such that all elements u, u^{-1}, v, v^{-1}, uv, $u^{-1}v^{-1}$ are pairwise different. Then property (i) holds as there are no $w\in W$ such that $w(\alpha_1)=\alpha_2$ and $w(\alpha_2)=\alpha_1$.

5. Rational cross-sections

Recall from [Ste$_1$, 2.14, 2.15] that an element $x\in G$ is called *strongly regular* if its centralizer G_x is a maximal torus. Such x is regular and semisimple. Strongly regular elements form a dense open subset G_0 of G stable with respect to the conjugating action of G. Every G-orbit in G_0 is regular and closed in G. We put

$$(G/\!\!/G)_0 := \pi_G(G_0) \qquad \text{and} \qquad T_0 := T \cap G_0.$$

Abusing the notation, we denote $\pi_G|_{G_0}$ still by π_G:

$$\pi_G\colon G_0 \longrightarrow (G/\!\!/G)_0. \tag{37}$$

Lemma 5.1.

(i) $(G/\!\!/G)_0$ *is an open smooth subset of* $G/\!\!/G$.
(ii) $\pi_G|_{T_0}\colon T_0 \to (G/\!\!/G)_0$ *is a surjective étale map.*
(iii) $((G/\!\!/G)_0, \pi_G)$ *is the geometric quotient for the action of* G *on* G_0.

Proof. Since $G/\!\!/G$ is normal and all fibers of π_G are of constant dimension and irreducible, π_G is an open map (see [Bor, AG.18.4]). Hence $(G/\!\!/G)_0$ is open in $G/\!\!/G$.

As every element of G_0 is semisimple, it is conjugate to an element of T_0; whence $\pi_G|_{T_0}$ is surjective.

The set T_0 is open in T and W-stable. For every point $t \in T_0$, we have $G_t = T$, hence the W-stabilizer of t is trivial. Thus, the action of W on T_0 is set theoretically free. Since T is smooth, $G/\!\!/G$ is normal, and $(G/\!\!/G, \pi_G|_T)$ is the quotient for the action of W on T (see [Ste$_1$, 6.4]), we deduce from [G$_3$, Exp. I, Théorème 9.5(ii)] and [Bou$_1$, V.2.3, Cor. 4] that $\pi_G|_{T_0}$ is étale and hence $(G/\!\!/G)_0$ is smooth. This proves (i) and (ii).

By (ii) the map $\pi_G\colon G_0 \to (G/\!\!/G)_0$ is separable and surjective. As its fibers are G-orbits and $(G/\!\!/G)_0$ is normal, (iii) follows from [Bor, 6.6]. $\square$

The group W acts on $G/T \times T_0$ diagonally with the action on the first factor defined by formula (1). The group G acts on $G/T \times T_0$ via left translations of the first factor. These two actions commute with each other.

Consider the G-equivariant morphism

$$\psi\colon G/T \times T_0 \longrightarrow G_0, \qquad (gT, t) \mapsto gtg^{-1}. \tag{38}$$

The proofs of Lemma 5.2 and Corollary 5.4 reproduce that from my letter [P].

Lemma 5.2. ψ *is a surjective étale map.*

Proof. As every G-orbit in G_0 intersects T_0, surjectivity of ψ follows from (38).

Take a point $z \in G/T \times T_0$. We shall prove that $d\psi_z$ is an isomorphism. As $G/T \times T_0$ and G_0 are smooth, this is equivalent to proving that ψ is étale at z. Using that ψ is G-equivariant, we may assume that $z = (eT, s)$, $s \in T_0$.

Let U_α be the one-dimensional unipotent root subgroup of G corresponding to a root α with respect to T and let $\theta_\alpha\colon \mathbf{G}_a \to U_\alpha$ be the isomorphism of groups such that

$$t\theta_\alpha(x)t^{-1} = \theta_\alpha(t^\alpha x) \qquad \text{for all } \ t \in T,\ x \in \mathbf{G}_a,$$

see [Bor, IV.13.18]. Put

$$C_\alpha := \{(\theta_\alpha(x)T, s) \in G/T \times T_0 \mid x \in \mathbf{G}_a\},$$
$$D := \{(eT, t) \in G/T \times T_0 \mid t \in T_0\}.$$

The linear span of all T_{z,C_α}'s and $\mathrm{T}_{z,D}$ is $\mathrm{T}_{z,G/T\times T_0}$. We have

$$\begin{aligned}\psi(\theta_\alpha(x)T, s) &= \theta_\alpha(x)s\theta_\alpha(x)^{-1} = \theta_\alpha(x)s\theta_\alpha(-x)\\ &= \theta_\alpha(x)\theta_\alpha(-s^\alpha x)s = \theta_\alpha((1-s^\alpha)x)s.\end{aligned} \tag{39}$$

Since s is regular, $s^\alpha \neq 1$. Hence (39) shows that ψ maps the curve C_α isomorphically onto the curve

$$\psi(C_\alpha) = \{\theta_\alpha((1-s^\alpha)x)s \mid x \in \mathbf{G}_\alpha\}.$$

Clearly, $\psi(D) = T_0$ and $\psi|_D\colon D \to T_0$ is the isomorphism. But $\mathrm{T}_{e,G}$ is the linear span of $\mathrm{T}_{e,T}$ and the tangent spaces of the curves $\{\theta_\alpha(x) \mid x \in \mathbf{G}_\alpha\}$ at e. Hence $\mathrm{T}_{s,G}$ is the linear span of $\mathrm{T}_{s,T}$ and the tangent spaces at s of the right translations of these curves by s. This implies the claim of the lemma. □

Corollary 5.3. *ψ is separable.*

Corollary 5.4. *(G_0, ψ) is the quotient for the action of W on $G/T \times T_0$.*

Proof. By [Bor, Prop. II.6.6], as G_0 is normal and ψ is surjective and separable, it suffices to prove that the fibers of ψ are W-orbits.

Using (1) and (38) one immediately verifies that the fibers of ψ are W-stable. On the other hand, let $\psi(g_1T, t_1) = \psi(g_2T, t_2)$. By (38) this equality is equivalent to $(g_1^{-1}g_2)t_2(g_1^{-1}g_2)^{-1} = t_1$. By [Ste$_1$, 6.1] the latter, in turn, implies that there is an element $w \in W$ such that

$$\dot{w}t_2\dot{w}^{-1} = (g_1^{-1}g_2)t_2(g_1^{-1}g_2)^{-1}.$$

Hence $g_1^{-1}g_2 = \dot{w}z$ for $z \in G_{t_2}$. As $t_2 \in T$ is strongly regular, this yields that $z \in T$. Therefore,

$$(g_2T, t_2) = (g_1\dot{w}T, \dot{w}^{-1}t_1\dot{w}) = w^{-1} \cdot (g_1T, t_1).$$

Thus, (g_1T, t_1) and (g_2T, t_2) are in the same W-orbit. This completes the proof. □

Let $\pi_2\colon G/T \times T_0 \to T_0$ be the second projection. Clearly, (T_0, π_2) is the geometric quotient for the action of G on $G/T \times T_0$. As ψ is G-equivariant, this implies that there is a morphism $\phi\colon T_0 \to G/\!\!/G$ such that the following diagram is commutative:

$$\begin{tikzcd} G/T \times T_0 \arrow[r, "\psi"] \arrow[d, "\pi_2"'] & G_0 \arrow[d, "\pi_G"] \\ T_0 \arrow[r, "\phi"] & (G/\!\!/G)_0 \end{tikzcd} \quad . \tag{40}$$

Lemma 5.5.

(i) $\phi = \pi_G|_{T_0}$.

(ii) *For every point $t \in T_0$, the restriction of ψ to $\pi_2^{-1}(t)$ is a G-equivariant isomorphism $\pi_2^{-1}(t) \to \pi_G^{-1}(\phi(t))$.*

Proof. Take a point $t \in T_0$. Commutativity of diagram (40) and formula (38) yield that $\pi_G(t) = \pi_G(\psi(eT, t)) = \phi(\pi_2(eT, t)) = \phi(t)$. This proves (i).

Commutativity of diagram (40) implies that the restriction of ψ to $\pi_2^{-1}(t)$ is a G-equivariant morphism $\pi_2^{-1}(t) \to \pi_G^{-1}(\phi(t))$. As both $\pi_2^{-1}(t)$ and $\pi_G^{-1}(\phi(t))$ are the G-orbits and the stabilizers of their points are conjugate to T, this morphism is bijective. By Lemma 5.2 it is separable. Then, as $\pi_G^{-1}(\phi(t))$ is normal, it is an isomorphism. This proves (ii). □

Proof of Theorem 1.5. Assume that (i) holds. Let $\sigma\colon G/\!\!/G \dashrightarrow G$ be a rational section of π_G, i.e., a section of π_G over a dense open subset U of $(G/\!\!/G)_0$. Let S be the closure of $\sigma(U)$. Put $\rho := \pi_G|_S\colon S \to (G/\!\!/G)_0$. Since $\pi_G \circ \sigma = \mathrm{id}$, shrinking U if necessary, we may assume that, for every point $x \in U$, the following properties hold:

(a) $S \cap \pi_G^{-1}(x)$ is a single point s;

(b) $d\rho_s$ is an isomorphism.

Since ψ is an isomorphism on the fibers of π_2, property (a) implies that, for every point $t \in \phi^{-1}(U)$, the W-stable closed set $\psi^{-1}(S)$ intersects $\pi_2^{-1}(t)$ at a single point. From this we infer that $\psi^{-1}(S)$ has a unique irreducible component $\widetilde{S}$ whose image under π_2 is dense in T_0 — the urgument is the same as that in the proof of Claim 2(i) in Section 2. Due to the uniqueness, this $\widetilde{S}$ is W-stable.

Let $V \subseteq \pi_2(\widetilde{S}) \cap \phi^{-1}(U)$ be an open subset of T_0. Replacing it, if necessary, by $\bigcap_{w\in W} w(V)$, we may assume that V is W-stable. Let $\widetilde{\rho}\colon \pi_2^{-1}(V) \cap \widetilde{S} \to V$ be the restriction of π_2 to $\pi_2^{-1}(V) \cap \widetilde{S}$. Then $\widetilde{\rho}$ is a bijective W-equivariant morphism. We claim that it is separable and hence, by ZARISKI's Main Theorem, an isomorphism (as V is normal). Indeed, take a point $\widetilde{s} \in \pi_2^{-1}(V) \cap \widetilde{S}$ and put $\pi_2(\widetilde{s}) = t$. Then property (b), Lemma 5.2, and commutativity of diagram (40) imply that $d\widetilde{\rho}_s\colon \mathrm{T}_{\widetilde{s},\widetilde{S}} \to \mathrm{T}_{t,V}$ is an isomorphism; whence the claim by [Bor, AG.17.3].

Thus, $\widetilde{\rho}^{-1}\colon V \to \pi_2^{-1}(V) \cap \widetilde{S}$ is a rational W-equivariant section of π_2. Its composition with the first projection $G/T \times T_0 \to G/T$ is then a W-equivariant rational map $T \dashrightarrow G/T$. This proves (i)$\Rightarrow$(ii).

Conversely, assume that (ii) holds. Let $\gamma\colon T \dashrightarrow G/T$ be a W-equivariant rational map. Then $\varsigma := (\gamma, \mathrm{id})\colon T_0 \dashrightarrow G/T \times T_0$ is a W-equivariant rational section of π_2, i.e., a section of π_2 over a dense open subset V of T_0. We may assume that $\varsigma(V)$ and $S := \psi(\varsigma(V))$ are open in their closures, $\varsigma\colon V \to \varsigma(V)$ is an isomorphism, and the subsets $\phi(V)$, $\pi_G(S)$ of $G/\!\!/G$ are open and coincide. As above, we may also assume that V is W-stable.

Taking into account that ς is W-equivariant, $\varsigma(V) \cap \pi_2^{-1}(t)$ is a single point for every $t \in V$, and ψ is an isomorphism on the fibers of π_2, we conclude that property (a) holds for every $x \in \varsigma(V)$. Thus, $\rho := \pi_G|_S\colon S \to \phi(V)$ is a bijection.

We claim that ρ is separable, hence an isomorphism as $\phi(V)$ is normal by Lemma 5.1(i). Indeed, $d\phi_t$ is an isomorphism by Lemma 5.5(i) and Lemma 5.1(ii). Let $s = \psi(\varsigma(t)) \in S$. Since the restriction of $(d\pi_2)_{\varsigma(t)}$ to $\mathrm{T}_{\varsigma(t),\varsigma(V)}$ is an isomorphism with $\mathrm{T}_{t,V}$, commutativity of diagram (40) and Lemma 5.2 imply that property (b) holds; whence the claim.

Thus, the composition of $\rho^{-1}\colon \phi(V) \to S$ and the identical embedding $S \hookrightarrow G$ is a rational section of π_G. This proves (ii)$\Rightarrow$(i) and completes the proof of the theorem. $\square$

Recall some definitions from [CTKPR, Sects. 2.2, 2.3, and 3].

Let P be a linear algebraic group acting on a variety X and let Q be its closed subgroup. X is called a (P,Q)-*variety* if in X there is a dense open P-stable subset U, called *a friendly subset*, such that a geometric quotient $\pi_U\colon U \to U/P$ exists and π_U becomes the second projection $P/Q \times \widehat{U/P} \to \widehat{U/P}$ after a surjective étale base change $\widehat{U/P} \to U/P$. If there is a rational section of π_U, one says that X *admits a rational section*. X is called a *versal* (P,Q)-*variety* if U/P is irreducible and, for every its dense open subset $(U/P)_0$ and (P,Q)-variety Y, there is a friendly subset V of Y such that π_V is induced from π_U by a base change $V \to (U/H)_0$.

Now we shall give the characteristic free proofs of the following two statements proved in [CTKPR] for char $= 0$.

Lemma 5.6. *Let X be an irreducible variety endowed with a faithful action of a finite algebraic group H. Then*

(i) *X is an $(H,\{e\})$-variety;*
(ii) *X is a versal $(H,\{e\})$-variety in each of the following cases:*
   (a) *X is a free H-module;*
   (b) *X is a linear algebraic torus and H acts by its automorphisms.*

Proof. (i) Replacing X by its smooth locus, we may assume that X is smooth.

As H is finite, for any nonempty open affine subset U of X, the set $\bigcap_{h\in H} h(U)$ is H-stable, affine, and open in X. So, replacing X by it, we may assume that X is affine. Then, as is well known, for the action of H on X there is a geometric quotient $\pi\colon X \to X/H$ (see, e.g., [Bor, Prop. 6.15]). As X is normal, X/H is normal as well.

Since H is finite and the action is faithful, the points with trivial stabilizer form an open subset of X. Replacing X by it, we may also assume that the action is set-theoretically free, i.e., the H-stabilizer of every point of X is trivial. As X and X/G are normal, by [G_3, Exp. I, Théorème 9.5(ii)] and [Bou_1, V.2.3, Cor. 4] this property implies that π is étale and hence X/H is smooth.

For every base change $\beta\colon Y \to X/H$ of π, the group H acts on $X \times_{X/H} Y$ via X. As the action of H on X is set-theoretically free, taking $Y = X$ and $\beta = \pi$, we obtain

$$X \times_{X/H} X = \bigsqcup_{h\in H} h(D) \qquad \text{where} \quad D := \{(x,x) \mid x \in X\}.$$

¿From this we deduce that in the commutative diagram

$$\begin{tikzcd} H \times X \arrow[rr, "\alpha"] \arrow[dr] & & X \times_{X/H} X \arrow[dl] \\ & X & \end{tikzcd},$$

where $\alpha(h,x) := (h(x), x)$ and two other maps are the second projections, α is an H-equivariant isomorphism. This proves (i).

The proofs of (ii)(a) and (ii)(b) are the same as that of (b) and (d) in [CTKPR, Lemma 3.3] if one replaces in them the references to [CTKPR, Theorem 2.12] (whose proof is based on the assumption $\operatorname{char} k = 0$) by the references to statement (i) of the present lemma. $\square$

Remark 5.7. The proof of (i) shows that, for finite group actions, set-theoretical freeness coincides with that in the sense of GIT, [MF, Def. 0.8].

Lemma 5.8. *G is a versal (G,T)-variety.*

Proof. First we shall give a characteristic free proof of the fact that G is a (G,T)-variety (the proof given in [CTKPR] is based on the assumption $\operatorname{char} k = 0$). By Lemma 5.1(iii) this is equivalent to proving the existence of a dense open subset U of $(G/\!\!/G)_0$ such that after a surjective étale base change $U' \to U$ morphism (37) becomes the second projection $G/T \times U' \to U'$.

Consider the base change of π_G in (40) by means of ϕ. Lemma 5.5(i) implies that

$$F := G_0 \times_{(G/\!\!/G)_0} T_0 = \{(g,t) \in G_0 \times T_0 \mid G(g) = G(t)\} \tag{41}$$

(see (5)). We have the canonical map corresponding to commutative diagram (40):

$$\gamma := \psi \times \mathrm{id}\colon G/T \times T_0 \longrightarrow F, \qquad (gT, t) \mapsto (gtg^{-1}, t). \tag{42}$$

It follows from (41) that γ is surjective; whence F is irreducible. But if for $t \in T_0$ and $g_1, g_2 \in G$ we have $g_1tg_1^{-1} = g_2tg_2^{-1}$, then $g_1T = g_2T$ since $G_t = T$. Therefore, γ is bijective. Lemma 5.2 and (42) show that $d\gamma_x$ is injective for every $x \in G/T \times T_0$. Hence if $\gamma(x)$ lies in the smooth locus F_{sm} of F, then $d\gamma_x$ is the isomorphism. This implies that γ is separable and then, by Zariski'a Main Theorem, that the restriction of γ to $\gamma^{-1}(F_{\mathrm{sm}})$ is an isomorphism $\gamma^{-1}(F_{\mathrm{sm}}) \to F_{\mathrm{sm}}$.

As F_{sm} is G-stable and γ is G-equivariant, $\gamma^{-1}(F_{\mathrm{sm}})$ is a G-stable open subset of $G/T \times T_0$. Hence it is of the form $G/T \times U'$ for an open subset U' of T_0. But Lemmas 5(ii) and 5.5(i) imply that $U := \phi(U')$ is open in $(G/\!\!/G)_0$ and $\phi|_{U'}\colon U' \to U$ is étale. This proves that after the étale base change $\phi|_{U'}\colon U' \to U$ morphism (37) becomes the second projection $G/T \times U' \to U'$. Hence G is a (G,T)-variety.

By Lemma 5.6(b), T is a versal $(W, \{e\})$-variety. The characteristic free arguments from [CTKPR, proof of Prop. 4.3(c)] then show that this fact implies versality of the (G, T)-variety G. This completes the proof of the lemma. $\square$

Proof of Theorem 1.6. Let us first show how to deduce (i) from (ii). Consider commutative diagram (9). As μ is separable, it is central. Therefore, if (ii) holds, there is a rational section of $\pi_{\widetilde{G}}$; whence there is a rational cross-section in $\widetilde{G}$. Then (i) follows from Corollary 2.4.

Now we shall prove (ii). Assume that τ is central. Then the natural morphism $\widehat{G}/\widehat{T} \to G/T$ is an isomorphism by [Bor, Props. 6.13, 22.4].

Using τ, every action of G naturally lifts to an action of $\widehat{G}$ on the same variety. In particular, G is endowed with an action of $\widehat{G}$. But G is a (G, T)-variety by Lemma 5.8(i). As $\widehat{G}/\widehat{T}$ and G/T are isomorphic, this means that G is a $(\widehat{G}, \widehat{T})$-variety. But $\widehat{G}$ is a versal $(\widehat{G}, \widehat{T})$-variety (by Lemma 5.8(i)) that admits a rational section (by Lemma 5.1(iii) and [Ste_1]). Hence by [CTKPR, Theorem 3.6(a)] (the proof of this result is characteristic free) every $(\widehat{G}, \widehat{T})$-variety admits a rational section. In particular, this is so for G. This proves (ii) and completes the proof of the theorem. $\square$

6. Remarks

1. Cross-sections versus sections. If there is a section $\sigma\colon G/\!\!/G \to G$ of π_G, then $\sigma(G/\!\!/G)$ is a cross-section in G. Indeed, as $\mathrm{id}_{k[G/\!\!/G]}$ is the composition of the homomorphisms

$$k[G/\!\!/G] \xrightarrow{\pi_G^*} k[G] \xrightarrow{\sigma^*} k[G/\!\!/G],$$

π_G^* is surjective; by [G_2, Cor. 4.2.3] this means that σ is a closed embedding.

The cross-section $\sigma(G/\!\!/G)$ has the property that the restriction of π_G to $\sigma(G/\!\!/G)$ is an isomorphism $\sigma(G/\!\!/G) \to G/\!\!/G$. Conversely, let S be a cross-section in G. If $\pi_G|_S\colon S \to G/\!\!/G$ is separable, then, since $\pi_G|_S$ is bijective and $G/\!\!/G$ is normal, Zariski's Main Theorem implies that $\pi_G|_S$ is an isomorphism (cf. [Bor, AG 18.2]). So in this case the composition of $(\pi_G|_S)^{-1}$ with the identity embedding $S \hookrightarrow G$ is a section of π_G whose image is S. In particular, if $\operatorname{char} k = 0$, then every cross-section in G is the image of a section of π_G. If $\operatorname{char} k > 0$, then in the general case this is not true.

Example 6.1. Let $G = \mathbf{SL}_3$ and $\operatorname{char} k = p > 0$. Then for every integer $d > 0$,

$$S := \{s(a_1, a_2) \mid a_1, a_2 \in k\}, \qquad \text{where} \quad s(a_1, a_2) := \begin{pmatrix} a_1 & a_2 & 1 \\ 1 & a_1^{p^d} - a_1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

is a cross-section in G such that ρ is not separable. Indeed, as $\chi_{\varpi_i}(g)$ is the sum of principal i-minors of $g \in G$, we have (see Lemma 2.1(ii))

$$(\lambda \circ \rho)\big(s(a_1, a_2)\big) = \big(a_1^{p^d}, a_1\big(a_1^{p^d} - a_1\big) - a_2\big). \qquad \square$$

Similarly, if $\sigma\colon G/\!\!/G \dashrightarrow G$ is a rational section of π_G and S the closure S of its image, then S is a rational cross-section in G such that the restriction of π_G to it is a birational isomorphism with $G/\!\!/G$.

2. Group action on the set of cross-sections. Let $\mathrm{Mor}(G/\!\!/G, G)$ be the group of morphisms $G/\!\!/G \to G$. If S is a cross-section in G and $\gamma \in \mathrm{Mor}(G/\!\!/G, G)$, then

$$\gamma(S) := \{\gamma(s)s\gamma(s)^{-1} \mid s \in S\}$$

is a cross-section in G. This defines an action of $\mathrm{Mor}(G/\!\!/G, G)$ on the set of cross-sections in G. If $\mathrm{char}\, k = 0$, then by [FM] this action is transitive. If $\mathrm{char}\, k > 0$, then in the general case this is not true: in Example 6.1, STEINBERG's section and S are not in the same $\mathrm{Mor}(G/\!\!/G, G)$-orbit since, for the former, the restriction of π_G is separable [Ste_1, Theorem 1.5], but, for the latter, it is not.

3. Lifting *T*-action. By Theorem 3.5 there is an action of T on T/W determining a structure of a toric variety. This action cannot be lifted to T making $\pi_T \colon T \to T/W$ equivariant. This follows from the fact that the automorphism group of the underlying variety of T is $\mathbf{GL}_r(\mathbf{Z}) \ltimes T$.

4. Questions. Given Theorem 1.5 and Corollary 1.7, it would be interesting to construct explicitly an example of a W-equivariant rational map $T \dashrightarrow G/T$ for central τ.

Is there such a map defined on T_0?

Is there a rational section of π_G defined on $(G/\!\!/G)_0$?

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