ALGEBRAIC CONES

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Abstract. A characterization of algebraic cones in terms of actions of the one-dimensional multiplicative algebraic monoid $\mathbf{M}_{\mathrm{m}}$ and the algebraic group $\mathbf{G}_{\mathrm{m}}$ is given.

This note answers a question of K. Adjamagbo asked in [A].

Below all algebraic varieties are taken over an algebraically closed field k.

An irreducible algebraic variety X is called *a cone* if X is affine and its coordinate algebra $k[X]$ admits a connected $\mathbf{N}$-grading:

$$k[X] = \bigoplus_{d\in\mathbf{N}} k[X]_d, \qquad k[X]_0 = k. \tag{1}$$

Let $\mathbf{M}_{\mathrm{m}}$ be the multiplicative algebraic monoid whose underlying variety is the affine line $\mathbf{A}^1$ and the multiplication $\mu\colon \mathbf{A}^1\times\mathbf{A}^1\to\mathbf{A}^1$ is defined by that in k, i.e., $\mu^*(T) = T\otimes T$ where T is the standard coordinate function on $\mathbf{A}^1$. The group of units of $\mathbf{M}_{\mathrm{m}}$ is $\mathbf{M}_{\mathrm{m}}\setminus\{0\} = \mathbf{G}_{\mathrm{m}}$. We put

$$\chi_d\colon \mathbf{G}_{\mathrm{m}} \longrightarrow \mathbf{G}_{\mathrm{m}}, \qquad t\mapsto t^d, \tag{2}$$

One says that $\mathbf{M}_{\mathrm{m}}$ *acts* on a variety Y if a morphism $\alpha\colon \mathbf{M}_{\mathrm{m}}\times Y\to Y$ is given, such that $\alpha(g,\alpha(h,y)) = \alpha(gh,y)$ and $\alpha(1,y) = y$ for all $g,h,\in\mathbf{M}_{\mathrm{m}}$, $y\in Y$. We write $g(y) := \alpha(g,y)$. The restriction of α to $\mathbf{G}_{\mathrm{m}}\times Y$ is the usual group action of $\mathbf{G}_{\mathrm{m}}$ on Y. The set $\mathbf{M}_{\mathrm{m}}(y) := \{g(y)\mid g\in\mathbf{M}_{\mathrm{m}}\}$ is called an $\mathbf{M}_m$-*orbit* of y (warning: different $\mathbf{M}_m$-orbits may have a nonempty intersection). If $\mathbf{M}_{\mathrm{m}}(y) = y$, then y is called a *fixed point* of the action.

Theorem. *Let X be an irreducible algebraic variety. Consider the properties:*

(i) *X is a cone;*
(ii) *there is an action of $\mathbf{M}_{\mathrm{m}}$ on X with a unique fixed point;*
(iii) *there is an action of $\mathbf{G}_{\mathrm{m}}$ on X with a fixed point and without other closed orbits.*

Then (i)⇒(ii)⇒(iii) *and, if X is normal,* (iii)⇒(i).

Proof. We may assume that $\dim X > 0$.

(i)⇒(ii) Let X be a cone. Consider a grading (1). Then formula $t\cdot f := t^d f$ for $t\in\mathbf{G}_m$, $f\in k[X]_d$, defines an action of $\mathbf{G}_m$ on $k[X]$ by k-algebra automorphisms. In turn, it defines an (algebraic) action of $\mathbf{G}_m$ on X. As grading (1) is connected,

$$k[X]^{\mathbf{G}_{\mathrm{m}}} = k. \tag{3}$$

Since $k[X]^{\mathbf{G}_{\mathrm{m}}}$ separates closed $\mathbf{G}_{\mathrm{m}}$-orbits (see [MF, Cor. A1.3]), from (3) and [Bor, Cor. I.1.8] we deduce that there is a unique such orbit O. The coordinate algebra $k[O]$

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of O does not contain $\mathbf{G}_{\mathrm{m}}$-stable proper ideals as the zero set of such an ideal would be a proper $\mathbf{G}_{\mathrm{m}}$-stable subset of O. Hence, as closedness of O implies surjectivity of the map $k[X] \to k[O]$, $f \mapsto f|_O$, the $\mathbf{G}_{\mathrm{m}}$-stable ideal $\bigoplus_{d>0} k[X]_d$ vanishes on O. Therefore, O is a single point.

By the embedding theorem [PV, Theorem 1.5] we may assume that X is a closed $\mathbf{G}_{\mathrm{m}}$-stable subset of a finite dimensional $\mathbf{G}_{\mathrm{m}}$-module V and X is not contained in its proper submodule. But the set of zeros of all linear functions on V that vanish on X is a submodule containing X. Hence we have the embedding $V^* \hookrightarrow k[X]$, $\ell \mapsto \ell|_X$, of $\mathbf{G}_{\mathrm{m}}$-modules. As the weights of V are the inverses of that of V^*, by (3) this yields that the origin 0_V is the unique $\mathbf{G}_{\mathrm{m}}$-fixed point in V; in particular, $O = 0_V$. By (1) this also yields that, for every $\mathbf{G}_{\mathrm{m}}$-weight χ_d of V^*, we have $d \in \mathbf{N}$. Hence by (2) there are $d_1, \ldots, d_n \in \mathbf{N}$ and a basis $e_1, \ldots, e_n$ of V such that $t(e_i) = t^{d_i} e_i$ for every $t \in \mathbf{G}_{\mathrm{m}}$ and i. This shows that the action of $\mathbf{G}_m$ on V extends to the action of $\mathbf{M}_{\mathrm{m}}$ on V by putting $0(v) = 0_V$ for every $v \in V$. As $\mathbf{M}_{\mathrm{m}}$ is the closure of $\mathbf{G}_{\mathrm{m}}$ and X is $\mathbf{G}_{\mathrm{m}}$-stable and closed, X is $\mathbf{M}_{\mathrm{m}}$-stable as well. By the same reason, 0_V is a fixed point of $\mathbf{M}_{\mathrm{m}}$. There are no other such points because $V^{\mathbf{G}_{\mathrm{m}}} = \{0_V\}$.

(ii)$\Rightarrow$(iii) Assume that $\mathbf{M}_m$ acts on an irreducible variety X with a unique fixed point x. Restrict this action to $\mathbf{G}_m$. Take a point $y \in X$. If the $\mathbf{G}_m$-orbit $\mathbf{G}_m(y)$ is closed in X, then $\mathbf{G}_m(y) = \mathbf{M}_m(y)$ since $\mathbf{G}_m$ is dense in $\mathbf{M}_m$.

We claim that then $\mathbf{G}_m(y)$ is a single point, whence $\mathbf{G}_m(y) = x$ because of the uniqueness of x. Indeed, assume the contrary, i.e., $\dim \mathbf{G}_m(y) = 1$. The orbit map $\varphi_y \colon \mathbf{M}_{\mathrm{m}} \to \mathbf{M}_m(y)$, $g \mapsto g(y)$, is a surjective morphism of one-dimensional smooth varieties such that every fiber is finite and, for every point $z \in \mathbf{M}_{\mathrm{m}}(y)$, $z \neq \varphi_y(0)$, the cardinality of $\varphi_y^{-1}(z)$ is equal to the order s of the $\mathbf{G}_{\mathrm{m}}$-stabilizer of y while the cardinality of $\varphi_y^{-1}(\varphi_y(0))$ is $s+1$. By [G, Sect. 2, Cor. 2] this is impossible, a contradiction. Thus, x is a unique closed $\mathbf{G}_m$-orbit.

(iii)$\Rightarrow$(i) Now assume that $\mathbf{G}_{\mathrm{m}}$ acts on X with a unique fixed point x and without other closed orbits. Then, for every point $y \in X$, $y \neq x$, the closure $\overline{\mathbf{G}_m(y)}$ of the orbit $\mathbf{G}_m(y)$ is one-dimensional and

$$x \in \overline{\mathbf{G}_m(y)}. \tag{4}$$

Assume further that X is normal. Then by Sumihiro's theorem [S, Cor. 2 of Lemma 8] there is a $\mathbf{G}_{\mathrm{m}}$-stable affine open neighborhood U of x. We claim that $X = U$. Indeed, if not, then $X \setminus U$ is a nonempty $\mathbf{G}_{\mathrm{m}}$-stable closed subset in X and (4) is impossible for $y \in X \setminus U$, a contradiction.

Thus, $X = U$, hence X is affine.

As elements of $k[X]^{\mathbf{G}_{\mathrm{m}}}$ are constant on $\mathbf{G}_{\mathrm{m}}$-orbits, (4) implies that $f(y) = f(x)$ for every $f \in k[X]^{\mathbf{G}_{\mathrm{m}}}$ and $y \in X$; whence (3) holds.

Now let $k[X]_d$ be the χ_d-isotypic component of the $\mathbf{G}_{\mathrm{m}}$-module $k[V]$. Then

$$k[X] = \bigoplus_{d \in \mathbf{Z}} k[X]_d \tag{5}$$

is a $\mathbf{Z}$-grading of the k-algebra $k[X]$. By (3) it is connected, i.e., $k[X]_0 = k$.

We claim that there are no integers $d_1 > 0$ and $d_2 < 0$ such that $k[X]_{d_i} \neq 0$ for $i = 1, 2$. Indeed, assume the contrary. Then there is a point $y \in X$ such that $k[X]_{d_i}$, $i = 1, 2$, is not in the kernel of $k[X] \to k[\overline{\mathbf{G}_m(y)}]$, $f \mapsto f|_{\overline{\mathbf{G}_m(y)}}$. Hence, for every $i = 1, 2$, the χ_{d_i}-isotypic component of the $\mathbf{G}_m$-module $k[\overline{\mathbf{G}_m(y)}]$ is nonzero. This implies that there is an integer $d \neq 0$ such that the $\mathbf{G}_m$-stable maximal ideal $\{f \in k[\overline{\mathbf{G}_m(y)}] \mid f(x) = 0\}$ of $k[\overline{\mathbf{G}_m(y)}]$ has the nonzero χ_d- and χ_d^{-1}-isotypic components. Let p and q be the nonzero elements of resp. the first and second of them. Then pq

is constant on $\mathbf{G}_m(y)$ and hence on $\overline{\mathbf{G}_m(y)}$. As $pq(x) = 0$, this means that $pq = 0$ contrary to the irreducibility of $\overline{\mathbf{G}_m(y)}$, a contradiction.

Thus, $k[X]_d = 0$ in (5) either for all negative or for all positive d's. Replacing, if necessary, the action of $\mathbf{G}_m$ on X by $g \cdot y := g^{-1}(y)$, we may assume that the first possibility is realized, i.e., (1) holds. This completes the proof. $\square$

Remark. The following example shows that, in general, without the assumption of normality of X the implications (iii)⇒(i) and (ii)⇒(i) do not hold.

Example. Let X be the image of the morphism

$$\nu\colon \mathbf{P}^1 \longrightarrow \mathbf{P}^2,\ \ (a_0 : a_1) \mapsto (p^3 : q^2t - p^3 : q^3 - qp^2), \quad p = a_1 - a_0,\ q = a_1 + a_0.$$

X is the projective plane cubic with an ordinary double point $O = (1 : 0 : 0)$ and $\mathbf{P}^1 \to X$, $x \mapsto \nu(x)$, is the normalization map. Formula $\alpha(t, (a_0 : a_1)) = (a_0 : ta_1)$ defines an action of $\mathbf{G}_{\rm m}$ on $\mathbf{P}^1$ that descends to X by means of ν, see [P]. For this action, O is a unique fixed point and $X \setminus O$ is an orbit. This action extends to the one of $\mathbf{M}_{\rm m}$ by putting $0(x) = O$ for every $x \in X$.

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