DIMENSIONS OF ANISOTROPIC INDEFINITE QUADRATIC FORMS II

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Abstract. Let F be a field of characteristic different from 2. The u-invariant and the Hasse number $\widetilde{u}$ of a field F are classical and important field invariants pertaining to quadratic forms. These invariants measure the suprema of dimensions of anisotropic forms over F that satisfy certain additional properties. We prove new relations between these invariants and we give a new characterization of fields with finite Hasse number, the first one of its kind that uses intrinsic properties of quadratic forms and which, conjecturally, allows an 'algebro-geometric' characterization of fields with finite Hasse number. We also construct various examples of fields with infinite Hasse number and prescribed finite values of u that satisfy additional properties pertaining to the space of orderings of the field.

1. Introduction

Throughout this paper, fields are assumed to be of characteristic different from 2 and quadratic forms over a field are always assumed to be finite-dimensional and nondegenerate. The u-invariant of a field F is one of the most important field invariants pertaining to quadratic forms. Originally, it was defined to be the supremum of the dimensions of anisotropic quadratic form over F. As such, it doesn't yield any useful information in the case of formally real fields (or real fields for short) as in that case, the quadratic form given by a sum of an arbitrary finite number of squares is always anisotropic. To provide a meaningful invariant also in the case of real fields, Elman and Lam [EL2] modified the definition as follows:

$$u(F) := \sup\{\dim\varphi \mid \varphi \text{ is an anisotropic torsion form over } F\}\ ,$$

where 'torsion' means torsion when considered as an element in the Witt ring WF. Over a nonreal field, all forms are torsion forms, whereas over a real field, Pfister's Local-Global Principle implies that torsion forms are exactly those forms that have zero signature with respect to every ordering of the field (see, e.g., [L3, Ch. VIII, Th. 3.2]). If F is real and there are no anisotropic torsion forms, one puts $u(F) = 0$ (for instance, for $F = \mathbb{R}$).

2000 *Mathematics Subject Classification.* primary: 11E04; secondary: 11E10, 11E81, 14C25.

Key words and phrases. quadratic form, Pfister form, Pfister neighbor, real field, ordering, strong approximation property, effective diagonalization, u-invariant, Hasse number, Pythagoras number, function field of a quadratic form, Rost correspondence, Rost projector.

There is a vast body of literature on the question of which values can be realized as u-invariant of a field, and on the problem of determining the u-invariant of certain fields such as finitely generated extensions of local or global fields. Despite much progress in recent years, many of these questions remain unanswered. For example, in the case where K is a finitely generated extension of $\mathbb{Q}_p$ of transcendence degree $m \geq 1$, the finiteness of $u(K)$ was not known until only recently, when Leep showed (based on results by Heath-Brown [He]) that in this situation one has $u(K) = 2^{m+2}$. Before that, this was only known in the case $m = 1$ and $p > 2$ due to Parimala-Suresh [PaS].

In the case where K is real and finitely generated over $\mathbb{R}$ of transcendence degree m, it was Witt who showed that in the case $m = 1$, one has $u(K) = 2$. For $m \geq 2$ the best currently known estimates are $2^m \leq u(K) \leq 2^{m+2} - 2m - 6$ due to Becher [B2] (for $m = 2$, this is due to Elman-Lam [EL2]).

As for possible values of the u-invariant, it is not too hard to show that u cannot take the values $3, 5, 7$ (see, e.g., [L3, Ch. XI, Prop. 6.8]). It was Merkurjev [M2] who showed that every even $n \in \mathbb{N}$ can be realized as u-invariant of a field, thus also producing the first values different from 2-powers (see also [T]). Izhboldin [I] produced the first field with $u = 9$, and examples of fields with $u = 2^r + 1$ for all $r \geq 3$ were constructed by Vishik [V]. It is still not known whether other values than the above ones are possible or not.

If F is a real field, for a form φ over F to be isotropic, it is clearly necessary for φ to be indefinite at each ordering of F, i.e., for φ to be *totally indefinite* or *t.i.* for short. This leads to another field invariant, the *Hasse number* $\widetilde{u}$ defined as

$$\widetilde{u}(F) := \sup\{\dim\varphi \mid \varphi \text{ is an anisotropic t.i. form}\} .$$

One puts $\widetilde{u}(F) = 0$ if there are no anisotropic t.i. forms over F. Clearly, $u(F) \leq \widetilde{u}(F)$, with equality in the case of nonreal fields since being totally indefinite is then an empty condition.

In [H6], $u(F)$ and $\widetilde{u}(F)$ were related to another field invariant, the so-called symbol length $\lambda(F)$, which is defined to be the smallest integer n such that each element in the exponent-2-part ${}_2\mathrm{Br}(F)$ of the Brauer group is Brauer equivalent to a product of $\leq n$ quaternion algebras. (By Merkurjev's theorem [M1], each element in ${}_2\mathrm{Br}(F)$ is equivalent to a product of quaternion algebras.)

In the present paper, we focus on finiteness criteria for u and $\widetilde{u}$. In particular, we derive a criterion for the finiteness of $\widetilde{u}$ that is of a purely quadratic form intrinsic nature, the first one of its kind, and which conjecturally can be translated into a 'algebro-geometric' criterion for the finiteness of $\widetilde{u}$.

To formulate the result, we need to introduce two properties. The first one is for a field to have *effective diagonalization*, ED for short, see §2 for the definition. The second property is the *Pfister neighbor property* $PN(n)$ for $n \in \mathbb{N}$. A field F having property $PN(n)$ means that each form of dimension $2^n + 1$ over F is a Pfister neighbor (see §2 for the definition of Pfister neighbor).

The main result of this paper is the following.

Theorem 1.1. *Let F be field. Then the following statements are equivalent:*
(i) $\widetilde{u}(F) < \infty$.
(ii) $u(F) < \infty$ *and F is ED.*
(iii) *F has property $PN(n)$ for some $n \geq 2$.*

The equivalence (i)$\Longleftrightarrow$(ii) is originally due to Elman-Prestel [EP], but we will give a new and simple proof that at the same time allows to improve considerably all previously known estimates for $\widetilde{u}$ in terms of u for ED-fields.

The paper is structured as follows. In §2, we recall some basic definitions and notations. In §3, we relate the property ED to the *strong approximation property* SAP and a certain property S_1 concerning torsion binary forms. We give a new proof of the fact that ED is equivalent to SAP plus S_1, originally due to Prestel-Ware [PW]. In §4, we prove the equivalence (i)$\Longleftrightarrow$(ii) in the above theorem, using a method that allows us to derive various estimates for $\widetilde{u}$ in terms of u. In §5, we construct examples of fields with prescribed values for u and the Pythagoras number p and which are SAP but not S_1, and which are S_1 but not SAP, respectively (such fields will thus have infinite $\widetilde{u}$). In §6, we prove the equivalence (i)$\Longleftrightarrow$(iii), giving in fact estimates on $\widetilde{u}$ in terms of n if the field has property $PN(n)$, $n \geq 2$. Since property $PN(2)$ is equivalent to F being linked (see Proposition 6.3), we will thus also recover as corollary a famous result on the u-invariant and the Hasse number of linked fields due to Elman-Lam [EL3], [E].

2. Definitions and notations

For all undefined terminology and basic facts about quadratic forms we refer to [L3]. Let φ and ψ be quadratic forms over a field F. We denote isometry, orthogonal sum and tensor product of φ and ψ by $\varphi \cong \psi$, $\varphi \perp \psi$ and $\varphi \otimes \psi$, respectively. Since all fields are assumend to be of characteristic not 2 and all quadratic forms are assumed to be nondegenerate, any quadratic form φ over F can be diagonalized: $\varphi \cong \langle a_1, \ldots, a_n \rangle$ for some $a_i \in F^*$.

φ is called a subform of ψ, $\varphi \subset \psi$, if there exists a form τ such that $\psi \cong \varphi \perp \tau$. A hyperbolic plane $\mathbb{H}$ is a form isometric to $\langle 1, -1 \rangle$. A form φ is called hyperbolic if it is an orthogonal sum of hyperbolic planes. φ is isotropic if it represents 0 nontrivially (i.e., there exists $x \in V \setminus \{0\}$ such that $\varphi(x) = 0$, where V denotes the underlying vector space of φ). Equivalently, φ is isotropic if $\mathbb{H} \subset \varphi$. Otherwise, φ is called anisotropic. By Witt decomposition, any form φ has a unique decomposition (up to isometry) of the shape $\varphi \cong \varphi_{\mathrm{an}} \perp \varphi_h$ with φ_{an} anisotropic and φ_h hyperbolic. The Witt index of φ is then defined to be $i_W(\varphi) = \frac{1}{2} \dim \varphi_h$.

We denote by $D_F(\varphi)$ the set of nonzero elements represented by φ, i.e. $D_F(\varphi) = \{q(x) \mid x \in V\} \cap F^*$, where V denotes the underlying vector space of φ. For simplicity, we write $D_F(n)$ for $D_F(\langle \underbrace{1, \ldots, 1}_{n} \rangle)$. $D_F(\infty)$ denotes the set of all nonzero sums of squares in F: $D_F(\infty)=\bigcup_{n=1}^{\infty}D_F(n)$. By Artin-Schreier theory, for nonreal fields, i.e. fields that do not possess any ordering, one has $D_F(\infty)=F^*$. Whereas for real fields F (fields that do have orderings), $D_F(\infty)$ consists of all elements that are totally positive, i.e. positive with respect to each ordering (see below).

WF will denote the Witt ring of F, and Witt equivalence of two forms φ and ψ (i.e. equality as elements in the Witt ring) will be denoted by $\varphi=\psi$. IF is the fundamental ideal of classes of forms of even dimension in WF, and $I^nF=(IF)^n$ is its n-th power.

A quadratic form of type $\langle 1,-a_1\rangle\otimes\ldots\otimes\langle 1,-a_n\rangle$ $(a_i\in F^*)$ is called an n-fold Pfister form, and we write $\langle\langle a_1,\ldots,a_n\rangle\rangle$ for short. P_nF (resp. GP_nF) denotes the set of all isometry classes of n-fold Pfister forms (resp. of forms similar to n-fold Pfister forms). A form φ is a Pfister neighbor if there exists a Pfister form π and $a\in F^*$ such that $\varphi\subset a\pi$ and $\dim\varphi>\frac{1}{2}\dim\pi$ (in which case φ is called a Pfister neighbor of π). An important property of Pfister forms is that they are either hyperbolic or anisotropic. Furthermore, if φ is a Pfister neighbor of Pfister forms π and ρ, then $\pi\cong\rho$, and φ is anisotropic iff π is anisotropic.

I^nF is additively generated by n-fold Pfister forms. The Arason-Pfister Hauptsatz (APH for short) states that if φ is an anisotropic form in I^nF, then $\dim\varphi\geq 2^n$, and if $\dim\varphi=2^n$, then in fact $\varphi\in GP_nF$ ([AP, Hauptsatz, Kor. 3], [L3, Ch. X, Hauptsatz 5.1, Th. 5.6]).

Let F be a real field and let X_F denote its space of orderings. X_F is a compact totally disconnected Hausdorff space with a subbasis of the topology given by the clopen sets $H(a)=\{P\in X_F\,|\,a>_P 0\}$, $a\in F^*$. F is said to satisfy the *strong approximation property* SAP if given any disjoint closed subsets U,V of X_F there exists $a\in F^*$ such that $U\subset H(a)$ and $V\subset H(-a)$. Since nonreal fields have no orderings, such fields are SAP by default.

If φ is a form over F, say, $\varphi\cong\langle a_1,\ldots,a_n\rangle$, then the signature of φ at an ordering $P\in X_F$ is defined by

$$\operatorname{sgn}_P(\varphi)=\#\{i\,|\,a_i>_p 0\}-\#\{i\,|\,a_i<_p 0\}$$

(note that $\operatorname{sgn}_P(\varphi)$ as defined above is independent of the chosen diagonalization by Sylvester's Law of Inertia). By Pfister's Local-Global Principle, φ being torsion is equivalent to $\operatorname{sgn}_P(\varphi)=0$ for all $P\in X_F$. Note also that any torsion elements in WF are always 2-primary torsion (see, e.g., [L3, Theorem 3.2]). The torsion part of WF is denoted by W_tF, and we define $I_t^nF=I^nF\cap W_tF$. For nonreal fields, we have $WF=W_tF$.

φ is called positive (resp. negative) definite at $P\in X_F$ if $\operatorname{sgn}_P(\varphi)=\dim\varphi$ (resp. $\operatorname{sgn}_P(\varphi)=-\dim\varphi$), and it is called indefinite at P if it is not definite at P. A totally positive definite (t.p.d.) form is a form that is totally definite at each $P\in X_F$, similarly one defines totally negative definite (t.n.d.) and totally indefinite (t.i.).

A form φ is called universal if $D_F(\varphi) = F^*$. Isotropic forms are always universal. If F is real, then $x \in D_F(\varphi)$ clearly implies that $x >_P 0$ (resp. $x <_P 0$) if φ is positive (resp. negative) definite at P. If the converse also holds, i.e. if

$$D_F(\varphi) \;=\; \{x \in F^* \mid \; x >_P 0 \text{ (resp. } x <_P 0\text{) if } \varphi \text{ is positive (resp. negative) definite at } P\}\,.$$

then φ is called *signature-universal* (*sgn-universal* for short). Over a real field we thus have that a form is universal if and only if it is t.i. and sgn-universal. One readily sees that if $\widetilde{u}(F) < \infty$ then any form φ with $\dim \varphi \geq \widetilde{u}(F)$ is sgn-universal.

A form φ over a real field F is said to have *effective diagonalization* ED if it has a diagonalization $\langle a_1, \ldots, a_n\rangle$ such that $H(a_i) \subset H(a_{i+1})$ for $1 \leq i \leq n-1$. F is said to be ED if each form over F has ED. Since nonreal fields have no orderings, such fields are ED by default.

The property S_1 for a real field is defined as follows:

(S_1) *Every binary torsion form represents a totally positive element (i.e. an element in $D_F(\infty)$).*

In addition to $u(F)$ and $\widetilde{u}(F)$, we will also need the *Pythagoras number* $p(F)$ (resp. *level* $s(F)$) of a field, the smallest $n \in \mathbb{N}$ such that each $D_F(n) = D_F(\infty)$ (resp. $-1 \in D_F(n)$), provided such an n exists. If no such n exists, one puts $p(F) = \infty$ (resp. $s(F) = \infty$). In particular, F is nonreal iff $s(F) < \infty$ by Artin-Schreier, in which case there exists $k \in \mathbb{N}_0$ such that $s(F) = 2^k$, and one has $p(F) \in \{2^k, 2^k + 1\}$ (see, e.g., [L3, Ch. XI, Th. 2.2, Th. 5.6]). In contrast, it was shown in [H4] that to any $n \in \mathbb{N}$ there exists a real field F such that $p(F) = n$.

3. ED equals SAP plus S_1

The following theorem is due to Prestel-Ware [PW]. We give a new proof based mainly on the study of binary forms.

Theorem 3.1. *F has ED if and only if F has SAP and S_1.*

To prove this, we use alternative descriptions of the properties involved.

Lemma 3.2. *Let F be a real field.*

(i) *F is SAP if and only if for all $a, b \in F^*$ there exists $c \in F^*$ such that $H(c) = H(a) \cap H(b)$ (or equivalently, there exists $d \in F^*$ such that $H(d) = H(a) \cup H(b)$).*

(ii) *F is ED if and only if for all $a, b \in F^*$, there exists $c, d \in F^*$ such that $\langle a, b\rangle \cong \langle c, d\rangle$ and $H(c) = H(a) \cap H(b)$ (or equivalently, $H(d) = H(a) \cup H(b)$).*

(iii) *F has property S_1 if and only if, for all $a \in F^*$, $s \in D_F(\infty)$, and $x \in D_F(\langle 1, as\rangle)$, there exists $t \in D_F(\infty)$ such that $tx \in D_F(\langle 1, a\rangle)$.*

Proof. (i) This is well known, see, e.g., [L1, Prop. 17.2].

(ii) The 'only if' is nothing else but ED for binary forms. As for the converse, we use induction on the dimension n of forms. Forms of dimension ≤ 2 have ED by assumption. So let φ be a form of dimension $n \geq 3$. Then we can write $\varphi = \langle a_1, \ldots, a_n\rangle$ and we may assume that $\langle a_2, \ldots, a_n\rangle$ is already an ED. Write $\langle a_1, a_2\rangle \cong \langle b_1, b_2\rangle$ with $H(b_1) = H(a_1) \cap H(a_2)$ (so $\langle b_1, b_2\rangle$ is an ED of $\langle a_1, a_2\rangle$). Then $\varphi \cong \langle b_1, b_2, a_3, \ldots, a_n\rangle$. Now let $\langle c_2, \ldots, c_n\rangle$ be an ED of $\langle b_2, a_3, \ldots, a_n\rangle$. Then one readily checks that $\langle b_1, c_2, \ldots, c_n\rangle$ is an ED of φ.

(iii) 'if': Let $\langle u, v\rangle \cong u\langle 1, uv\rangle$ be torsion. Then $uv = -s$ with $s \in D_F(\infty)$. Put $a = -s$. Then $\langle 1, -1\rangle \cong \langle 1, as\rangle$ which is hyperbolic and hence represents u. But then, by assumption, there exists $t \in D_F(\infty)$ such that tu is represented by $\langle 1, a\rangle \cong \langle 1, -s\rangle$ and hence t is represented by $u\langle 1, -s\rangle \cong \langle u, v\rangle$.

'only if': $x \in D_F(\langle 1, sa\rangle)$ implies that there exists $y \in F^*$ such that $\langle 1, sa\rangle \cong \langle x, y\rangle$. Now the torsion form $xa\langle s, -1\rangle$ represents some $u \in D_F(\infty)$ by S_1. Hence $\langle sa, -a\rangle \cong \langle xu, -xus\rangle$ and hence

$$\langle 1, sa, -a\rangle \cong \langle 1, xu, -xus\rangle \cong \langle -a, x, y\rangle$$

Thus, $\langle 1, a\rangle = \langle x, xus, -xu, y\rangle$ in WF, so $x\langle 1, us, -u, xy\rangle$ is isotropic and there exists $v \in D_F(\langle 1, us\rangle) \cap D_F(\langle u, -xy\rangle)$. Note that $us \in D_F(\infty)$, so $v \in D_F(\infty)$. Hence, $\langle 1, us\rangle \cong \langle v, vus\rangle$ and $\langle -u, xy\rangle \cong \langle -v, vuxy\rangle$, and we get $\langle 1, a\rangle \cong x\langle vus, vuxy\rangle \cong \langle xvus, vuy\rangle$, thus $xt \in D_F(\langle 1, a\rangle)$ with $t := vus \in D_F(\infty)$. $\square$

Proof of Theorem 3.1. 'only if': Clearly, ED implies SAP. Now let $\langle a, b\rangle$ be any binary torsion form. Then $\mathrm{sgn}_P(\langle a, b\rangle) = 0$, so $H(a) \cap H(b) = \emptyset$, and by ED, there exists $c \in -D_F(\infty)$ and $d \in D_F(\infty)$ such that $\langle a, b\rangle \cong \langle c, d\rangle$, in particular, d is a totally positive element represented by $\langle a, b\rangle$ and we have established S_1.

'if': Let F be SAP and S_1. We will verify the alternative description of ED from Lemma 3.2(ii). Let $\langle a, b\rangle$ be any binary form. By SAP, there exists $d' \in F^*$ such that $H(a) \cup H(b) = H(d')$. Then $\langle a, b, -d'\rangle$ is t.i., thus the form $\varphi \cong \langle a, b, -d', -d'ab\rangle \cong -d'\langle\langle ad', bd'\rangle\rangle$ has total signature zero and is therefore torsion. Hence, there exists some $n \in F$ such that for $\sigma_n \cong \langle\langle -1\rangle\rangle^{\otimes n} \cong \langle 1, 1\rangle^{\otimes n}$, we have that $\sigma_n \otimes \langle a, b, -d', -d'ab\rangle \in GP_{n+2}F$ is hyperbolic. But then its Pfister neighbor $\sigma_n \otimes \langle a, b\rangle \perp \langle -d'\rangle$ is isotropic. It follows that there exist $u, v \in D_F(\sigma_n) \subset D_F(\infty)$ such that $d' \in D_F(\langle ua, vb\rangle)$, and hence $ad'u \in D_F(\langle 1, abuv\rangle)$. Now $uv \in D_F(\infty)$, and by Lemma 3.2(iii), there exists $w \in D_F(\infty)$ such that $ad'uw \in D_F(\langle 1, ab\rangle)$, i.e. $d := d'uw \in D_F(\langle a, b\rangle)$. In particular, there exists $c \in F^*$ such that $\langle a, b\rangle \cong \langle c, d\rangle$. Since $uw \in D_F(\infty)$, we have $H(d) = H(d') = H(a) \cup H(b)$ as required. $\square$

4. Relations between the Hasse number and the u-invariant

In this section, we will only consider real fields since for nonreal field, $u(F) = \widetilde{u}(F)$, and most of the statements below are trivially true. It is quite possible for a real field F that $u(F)$ is finite but $\widetilde{u}(F)$ is infinite. Elman-Prestel [EP, Th. 2.5] gave the following necessary and sufficient criterion for the finiteness of $\widetilde{u}(F)$:

Theorem 4.1. *$\widetilde{u}(F) < \infty$ if and only if $u(F) < \infty$ and F has ED.*

Furthermore, in the case $\widetilde{u}(F) < \infty$, the value of $\widetilde{u}(F)$ was related to that of $u(F)$. In fact, it was proved that if $\widetilde{u}(F)$ is finite, then $\widetilde{u}(F) \leq (h(F)-1)(u(F)+2)$ (cf. [EP, Prop. 2.7]). Here, $h(F)$ denotes the *height* of the field F, i.e. the exponent of the torsion subgroup W_tF. If finite, $h(F)$ is the smallest 2-power such that $2^d \cdot W_tF = 0$. If F is not real, then $h(F) = 2D_F(\infty)$, and if F is real, then $h(F)$ is the smallest 2-power $2^d \geq p(F)$ (cf. [L3, Ch. XI, Def. 5.4, Th. 5.6]).

Hornix [Hor1, Th. 3.9] showed that in Elman and Prestel's bound above one can replace the 2-power $h(F)$ by 2^b for any integer $b \geq 0$ such that there exists a t.p.d. b-fold Pfister form over F which represents all totally positive elements in F (i.e., which is sgn-universal). Note that the smallest 2^d such that the d-fold Pfister form $\langle\langle -1, \ldots, -1\rangle\rangle$ represents all totally positive elements is nothing but $h(F)$, so that Hornix' result implies the one by Elman and Prestel.

The main purpose of this section is to improve these upper bounds for $\widetilde{u}(F)$ and at the same time give a new and simplified proof that $u(F) < \infty$ plus ED is equivalent to $\widetilde{u}(F) < \infty$. Let us start with a simple and well known lemma.

Lemma 4.2. *For any field F, if $p(F) > 2^n$ then $\widetilde{u}(F) \geq u(F) \geq 2^{n+1}$. In particular, $p(F) \leq u(F) \leq \widetilde{u}(F)$.*

Proof. If $p(F) > 2^n$, then there exists $x \in D_F(\infty) \setminus D_F(2^n)$, so $2^n \times \langle 1\rangle \perp \langle -x\rangle$ is anisotropic. But this is a Pfister neighbor of the form $\langle\langle -1, \ldots, -1, x\rangle\rangle \in P_{n+1}F$ which is therefore also anisotropic and which is furthermore torsion. Hence $u(F) \geq 2^{n+1}$. It follows immediately that $p(F) \leq u(F) \leq \widetilde{u}(F)$. $\square$

Proposition 4.3. *Suppose that F has ED and that there exists an n-dimensional t.p.d. sgn-universal form ρ. Then*

$$\widetilde{u}(F) \leq \frac{n}{2}(u(F) + 2)\ .$$

Proof. We may clearly assume that $u(F)$ (and hence $p(F)$) is finite. The form $p(F) \times \langle 1\rangle$ is t.p.d. and sgn-universal, so we may assume that $n \leq p(F)$. If $n = 1$ then F is obviously pythagorean and $u(F) = 0$. Since F has ED, any t.i. form φ over F contains a binary torsion form β as a subform. But then β is isotropic as $u(F) = 0$, hence φ is isotropic. It follows that $\widetilde{u}(F) = 0$ and the above inequality is clearly satisfied. So we may assume that $2 \leq n \leq p(F) = p$ and we have $\widetilde{u}(F) \geq u(F) \geq p \geq n$ by Lemma 4.2.

If $\widetilde{u}(F) = u(F)$ there is nothing to show. So we may assume that $\widetilde{u}(F) > u(F)$. Then there exist anisotropic t.i. forms of dimension $> u(F) \geq n$. Let φ_0 be any such form, and write $\dim \varphi_0 = m = rn+k+1$ with $r \geq 1$ and $0 \leq k \leq n-1$. Since F is ED and thus SAP, we may assume after scaling that $0 \leq \operatorname{sgn}_P \varphi_0 \leq \dim \varphi_0 - 2 = rn+k-1$ for all orderings P on F.

Let $\varphi_1 = a_0(\varphi_0 \perp -\rho)_{\mathrm{an}}$, where a_0 is chosen such that $0 \leq \operatorname{sgn}_P \varphi_1$ for all orderings P.

We have $i_W(\varphi_0 \perp -\rho) \leq n-1$, for otherwise one could write $\varphi_0 \cong \rho \perp \tau$ for some form τ. Since φ_0 is t.i. and since F has ED, this implies that there exists a totally positive x such that $-x$ is represented by τ. But then the form φ_0 contains the subform $\rho \perp \langle -x\rangle$ which is isotropic as ρ is t.p.d. and sgn-universal, clearly a contradiction. This implies that

$$\dim \varphi_1 \geq \dim \varphi_0 + n - 2(n-1) = (r-1)n + (k+1) + 2\ .$$

Note also that $\operatorname{sgn}_P(\varphi_0 \perp -\rho) = \operatorname{sgn}_P \varphi_0 - n$ for each ordering P. Hence, one obtains

$$\operatorname{sgn}_P \varphi_1 \leq \max\{(r-1)n + k - 1, n\}$$

for each ordering P. Note that if $r \geq 2$, then φ_1 is again t.i. as $0 \leq \operatorname{sgn}_P \varphi_1 < \dim \varphi_1$ for all orderings P. We see that if we apply this procedure altogether $r-1$ times, we get a form φ_{r-1} which is anisotropic, t.i., and such that

$$\dim \varphi_{r-1} \geq n + (k+1) + 2(r-1)\ ,$$

$$0 \leq \operatorname{sgn}_P \varphi_{r-1} \leq \max\{n+k-1, n\} \text{ for all orderings } P.$$

We therefore have

$$\dim \varphi_{r-1} - \operatorname{sgn}_P \varphi_{r-1} \geq \min\{2r, k+2r-1\}\ .$$

Since $\dim \varphi_{r-1} - \operatorname{sgn}_P \varphi_{r-1}$ is even, this yields $\dim \varphi_{r-1} - \operatorname{sgn}_P \varphi_{r-1} \geq 2r$ for all orderings P. By ED, the anisotropic form φ_{r-1} contains a torsion subform φ_t of dimension $\geq 2r$. Hence $u(F) \geq 2r$ and thus $u(F) + 2 \geq 2(r+1)$. On the other hand, by assumption $m = rn + k + 1 \leq n(r+1)$. These two inequalities together imply $m \leq \frac{n}{2}(u(F)+2)$. It follows readily that $\widetilde{u}(F) \leq \frac{n}{2}(u(F)+2)$. □

Proof of Theorem 4.1. The 'only if' part is easy and left to the reader. As for the 'if' part, we have $\infty > u(F) \geq p(F)$ by Lemma 4.2, and if we put $\rho = p(F) \times \langle 1\rangle$, then Proposition 4.3 immediately yields $\widetilde{u}(F) \leq \frac{p(F)}{2}(u(F)+2) < \infty$. □

If we put $\rho = p(F) \times \langle 1\rangle$ as in the proof above, and we let m be such that $2^m + 1 \leq p = p(F) \leq 2^{m+1}$, then the bound of Elman and Prestel yields $\widetilde{u}(F) \leq (2^{m+1}-1)(u(F)+2)$. Thus, in the 'worst' case $p = 2^{m+1}$, our bound $\widetilde{u}(F) \leq 2^m(u(F)+2)$ is almost by a factor 2 better as m gets large. In the 'best' case $p = 2^m + 1$, our bound $\widetilde{u}(F) \leq (2^{m-1} + \frac{1}{2})(u(F)+2)$ is almost by a factor 4 better as m gets large.

A similar improvement of the general bound found by Hornix is also obtained this way: Just let ρ be the b-fold Pfister form in Hornix' result and put $n = 2^b$.

In [GV], an invariant $m(F)$ of the field F has been defined to be the smallest integer $n \geq 1$ such that there exists an anisotropic universal n-dimensional form over F. If there is no anisotropic universal form over F, then one defines $m(F) = \infty$. Note that if F is real, universal forms must necessarily be t.i.. Let us introduce for our purposes another related invariant, $\widetilde{m}(F)$, which is defined to be the smallest integer $n \geq 1$ such that there exists an n-dimensional t.p.d. sgn-universal form.

Again, we put $\widetilde{m}(F) = \infty$ if there are no t.p.d. sgn-universal forms. If $p(F) < \infty$, we have that $p(F) \times \langle 1 \rangle$ is sgn-universal. Hence $\widetilde{m}(F) \leq p(F)$. With this new invariant, Proposition 4.3 immediately implies

Corollary 4.4. *Suppose that $\widetilde{u}(F) < \infty$. Then*

$$\widetilde{u}(F) \leq \frac{\widetilde{m}(F)}{2}(u(F) + 2) \ .$$

Next, we give another bound which will lead to further improvements of the bound by Elman and Prestel.

Proposition 4.5. *Suppose that $u(F) < \infty$ and that F has ED (or, equivalently, that $\widetilde{u}(F) < \infty$). Let $\rho = \langle 1 \rangle \perp \rho'$ be a t.p.d. m-fold Pfister form, $m \geq 1$, such that its pure part ρ' is sgn-universal. Then*

$$\widetilde{u}(F) \leq 2^{m-2}(u(F) + 6) \ .$$

If $m = 2$ then $\widetilde{u}(F) \leq u(F) + 4$.

Proof. If $m = 1$, then $\dim \rho' = 1$ and the assumptions imply that F is pythagorean, hence $\widetilde{u} = u = 0$ and there is nothing to show. So we may assume $m \geq 2$. Furthermore, if d is an integer such that $2^d \leq p(F) = p \leq 2^{d+1} - 1$, then we may assume that $m \leq d + 1$. For we have that $(2^{d+1} - 1) \times \langle 1 \rangle$ is the pure part of $\langle\langle -1, \ldots, -1 \rangle\rangle \in P_{d+1}F$ and it is totally positive definite and sgn-universal. We proceed similarly as before, but this time we put $\widetilde{u} = \widetilde{u}(F) = r2^m + k + 1$ with $r \geq 0$ and $0 \leq k \leq 2^m - 1$.

If $r = 0$ then we have $\widetilde{u} \leq 2^m$. If $2^d + 1 \leq p \leq 2^{d+1} - 1$ then $u \geq 2^{d+1} \geq 2^m$ by Lemma 4.2, and thus necessarily $u = \widetilde{u}$ and there is nothing to show. Suppose that $p = 2^d$ so that in particular $u \geq 2^d$. Our previous bound yields $\widetilde{u} \leq 2^{d-1}(u + 2)$. If $m = d + 1$, then $2^{d-1}(u + 2) < 2^{m-2}(u + 6)$ and there is nothing to show. If $m \leq d$, then we have $\widetilde{u} = k + 1 \leq 2^m \leq 2^d \leq u$ and thus $\widetilde{u} = u$, again there is nothing to show. So we may assume that $r \geq 1$.

Let φ_0 be an anisotropic t.i. form of dimension $\widetilde{u}$. As before, we may this time assume that $\dim \varphi_0 - 2 = r2^m + k - 1 \geq \operatorname{sgn}_P \varphi_0 \geq 0$ for all orderings P.

We claim that $i_W(\varphi_0 \perp -\rho) \leq 2^m - 2$. Indeed, otherwise φ_0 would contain a subform $\tilde{\rho}$ of dimension $2^m - 1$ with $\tilde{\rho} \subset \rho$. Now it is well known that all codimension 1 subforms of a Pfister form are similar to its pure part. Hence, φ_0 would contain a subform similar to ρ', and since φ_0 is t.i. and by ED, φ_0 would contain a subform similar to $\rho' \perp \langle -x \rangle$ for some totally positive x. By assumption, $\rho' \perp \langle -x \rangle$ is isotropic, a contradiction.

Thus, we obtain as in the proof of the previous lemma an anisotropic t.i. form φ_1 such that

$$\dim \varphi_1 \geq (r - 1)2^m + k + 1 + 4 \ ,$$

$$0 \leq \operatorname{sgn}_P \varphi_1 \leq \max\{(r - 1)2^m + k - 1, 2^m\} \ ,$$

and reiterating this construction $r-1$ times, we get an anisotropic t.i. form φ_{r-1} such that

$$\dim \varphi_{r-1} \geq 2^m + k + 1 + 4(r-1) \ ,$$
$$0 \leq \operatorname{sgn}_P \varphi_{r-1} \leq \max\{2^m + k - 1, 2^m\} \text{ for all orderings } P.$$

This yields $\dim \varphi_{r-1} - \operatorname{sgn}_P \varphi_{r-1} \geq 4r - 2$ for all orderings P, and thus, by ED, the existence of an anisotropic torsion subform φ_t of φ_{r-1} with $\dim \varphi_t \geq 4r-2$. In particular, $u + 6 \geq 4(r+1)$. On the other hand, $\widetilde{u} \leq 2^m(r+1)$ and thus $\widetilde{u} \leq 2^{m-2}(u+6)$.

Now if $m = 2$, we have $\dim \varphi_{r-1} \geq 4r + k + 1 = \dim \varphi_0$ and $0 \leq \operatorname{sgn}_P \varphi_{r-1} \leq \max\{4 + k - 1, 4\}$. In particular, since all the forms φ_i are anisotropic and t.i., it follows readily from the construction and the fact that $\widetilde{u} = 4r + k + 1$ that $\dim \varphi_0 = \dim \varphi_1 = \ldots = \varphi_{r-1} = \widetilde{u}$. Note also that $0 \leq k \leq 3$, so that by repeating our construction one more time, we obtain an anisotropic t.i. form φ_r such that $\dim \varphi_r = \widetilde{u}$ and $\operatorname{sgn}_P \varphi_r \leq 4$ for all orderings P. Thus, φ_r contains a torsion subform of dimension $\geq \widetilde{u} - 4$ and therefore $\widetilde{u} \leq u + 4$. $\square$

Let us investigate how we can use this result to give bounds for $\widetilde{u}$ in terms of p and u which in certain cases will further improve our previous bound $\widetilde{u} \leq \frac{p}{2}(u+2)$ (and thus the bound found by Elman and Prestel).

Suppose first that $p = 2^d$. Then our first bound yields $\widetilde{u} \leq 2^{d-1}(u+2)$. A priori, we do not know whether there exist t.p.d. m-fold Pfister forms, $m \leq d$, whose pure parts are sgn-universal. However, we do know by the definition of p that $\langle\langle -1, \ldots, -1 \rangle\rangle \in P_{d+1}F$ is such a Pfister form of fold $d+1$. But then our second bound only yields $\widetilde{u} \leq 2^{d-1}(u+6)$, which is worse.

Suppose now that d is an integer with $2^d + 1 \leq p \leq 2^{d+1} - 1$. Again, the Pfister form $\langle\langle -1, \ldots, -1 \rangle\rangle \in P_{d+1}F$ is t.p.d. and its pure part is sgn-universal, so we can use our second bound for $m = d+1$. For $p = 2^d + 1$, $d \geq 1$, we get $2^{d-1}(u+6) - \frac{p}{2}(u+2) = 2^{d+1} - \frac{1}{2}u - 1$. In this case, the first bound is better when $u \leq 2^{d+2} - 4$ (note that we will have $u \geq 2^{d+1}$), they are the same for $u = 2^{d+2} - 2$, and for $u \geq 2^{d+2}$ the second bound is sharper.

Let us look a little closer at the case $p = 2$. Here, our first bound yields $\widetilde{u} \leq u + 2$ if $\widetilde{u}$ is finite. Under the additional assumption that $I_t^3 F = 0$, we can show a little more.

Proposition 4.6. *Suppose that $I_t^3 F = 0$, and that $u(F) < \infty$ and F has ED (or, equivalently, that $\widetilde{u}(F) < \infty$). If there exists a t.p.d. sgn-universal binary form ρ over F, then $u(F) = \widetilde{u}(F)$.*

Proof. By [ELP, Th. H], $I_t^3 F = 0$ implies that $\widetilde{u} = \widetilde{u}(F)$ is even. By Proposition 4.3, $\widetilde{u} \leq u + 2$. So let us assume that $\widetilde{u} \neq u$, i.e. $\widetilde{u} = u + 2$. The proof of Proposition 4.3 then shows that there exists an anisotropic t.i. form φ (which is nothing but the form φ_{r-1} in the proof) with $\dim \varphi = \widetilde{u}$ and which contains a torsion subform φ_t, $\dim \varphi_t = \dim \varphi - 2 = u$. After scaling, we may assume that $\varphi_t \perp \langle 1 \rangle \subset \varphi$. Let $d =$

$d_\pm\varphi_t$. Then $\varphi_t \perp \langle 1,-d\rangle \in I^2F$, and since $\mathrm{sgn}_P\,\varphi_t = 0$ and $\mathrm{sgn}_P\,\varphi_t \perp \langle 1,-d\rangle \in 4\mathbb{Z}$, it follows that $\varphi_t \perp \langle 1,-d\rangle \in I_t^2F$. As $\dim\varphi_t \perp \langle 1,-d\rangle = u+2$, this form must be isotropic. Thus, $\varphi_t \perp \langle 1\rangle \cong \psi \perp \langle d\rangle$. Comparing discriminants and signatures, it follows that $\psi \in I_t^2F$. So for every $x \in F^*$, one $\langle 1,-x\rangle \otimes \psi \in I_t^3F = 0$, thus $\psi \cong x\psi$ which implies that ψ is universal, hence the subform $\psi \perp \langle d\rangle$ of φ is isotropic, a contradiction. □

The following is an immediate consequence.

Corollary 4.7. *Suppose that $p(F) = 2$ and $\widetilde{u}(F) < \infty$. If $I_t^3F = 0$ then $u(F) = \widetilde{u}(F)$. In particular, if $u(F) \leq 6$ or $\widetilde{u}(F) \leq 8$, then $\widetilde{u}(F) = u(F)$.*

Let us summarize our bounds for $\widetilde{u}$ in terms of u and p.

Theorem 4.8. *Let F be a real field. Suppose that $u(F) < \infty$ and that F has ED (or, equivalently, that $\widetilde{u}(F) < \infty$). Let $m \geq 2$ be an integer.*

(i) *$p(F) = 1$ if and only if $\widetilde{u}(F) = u(F) = 0$.*
(ii) *If $p(F) = 2$ then $\widetilde{u}(F) \leq u(F) + 2$. If in addition $I_t^3F = 0$ then $\widetilde{u}(F) = u(F) = 2n$ for some integer $n \geq 1$.*
(iii) *If $p(F) = 3$ then $\widetilde{u}(F) \leq u(F) + 4$.*
(iv) *If $p(F) = 2^m$ then $\widetilde{u}(F) \leq 2^{m-1}(u(F) + 2)$.*
(v) *If $p(F) = 2^m + 1$ then $\widetilde{u}(F) \leq (2^{m-1} + \frac{1}{2})(u(F) + 2)$ if $u(F) \leq 2^{m+2} - 2$, and $\widetilde{u}(F) \leq 2^{m-1}(u(F) + 6)$ if $u(F) \geq 2^{m+2} - 2$.*
(vi) *If $2^m + 2 \leq p(F) \leq 2^{m+1} - 1$, then $\widetilde{u}(F) \leq 2^{m-1}(u(F) + 6)$.*

It is, however, difficult to say at this point how good our bounds really are. In fact, we know extremely little about fields with $u(F) < \widetilde{u}(F) < \infty$. The only values which could be realized so far are fields where $u(F) = 2n$ and $\widetilde{u}(F) = 2n + 2$ for any $n \geq 2$ (see [L2], [Hor2], [H6]), and fields with $u(F) = 8$ and $\widetilde{u}(F) = 12$, see [H5, Cor. 6.4].

5. Fields with finite u-invariant and infinite Hasse number

In [EP, §5], one finds examples of non-SAP fields F with prescribed u-invariant 2^n, $n \geq 1$. These examples were obtained using the method of intersection of henselian fields (cf. [P2]). In this section, we will apply Merkurjev's method of constructing fields with even u-invariant and modify it in a way such that these fields will be real and such that either they will be non-SAP or they will not have the property S_1. In particular, for such fields the Hasse number will be infinite by Theorem 4.1, and it also illustrates the independence of the properties SAP and S_1.

Let us first recall some well known results and some special cases of Merkurjev's index reduction theorem which we will use in the sequel. We refer to [M2], [T] for details. See also [L3, Ch. V.3] for basic results on Clifford invariants $c(q) \in_2 \mathrm{Br}(F)$ for quadratic forms q over F and how to compute them, and [L3, Ch. X] for basic results on function fields $F(q)$ of quadratic forms q over F.

Lemma 5.1. (i) *Let $Q_i = (a_i, b_i)$, $1 \le i \le n$, be quaternion algebras over F with associated norm forms $\langle\langle a_i, b_i\rangle\rangle \in P_2F$. Let $A = \bigotimes_{i=1}^n Q_i$ (over F). Then there exist $r_i \in F^*$, $1 \le i \le n$, and a form $q \in I^2F$, $\dim q = 2n+2$ such that $c(q) = [A] \in \mathrm{Br}_2 F$ and $q = \sum_{i=1}^n x_i\langle\langle a_i, b_i\rangle\rangle$ in WF. (We will call such a form q an* Albert form *associated to A.) Furthermore, if A is not Brauer equivalent to a product of $< n$ quaternion algebras (in particular if A is a division algebra), then every Albert form associated to A is anisotropic.*

(ii) *If q is a form over F with either $\dim q = 2n+2$ and $q \in I^2F$, or $\dim q = 2n+1$, or $\dim q = 2n$ and $d_\pm q \ne 1$, then there exist quaternion algebras $Q_i = (a_i, b_i)$, $1 \le i \le n$, such that for $A = \bigotimes_{i=1}^n Q_i$ we have $c(q) = [A]$, and there exists an Albert form φ associated to A such that $q \subset \varphi$.*

(iii) *If A is a division algebra and if ψ is a form over F of one of the following types:*
(a) $\dim \psi \ge 2n+3$,
(b) $\dim \psi = 2n+2$ *and* $d_\pm\psi \ne 1$,
(c) $\dim \psi = 2n+2$, $d_\pm\psi = 1$ *and* $c(\psi) \ne [A] \in \mathrm{Br}_2 F$,
(d) $\psi \in I^3F$,
then A stays a division algebra over $F(\psi)$.

Let us also recall some basic facts on the property SAP and weakly isotropic forms which we will use and which are essentially well known. Recall that a form q over F is called weakly isotropic if $n \times q$ is isotropic for some $n \ge 1$ (over nonreal F, all forms are clearly weakly isotropic as $WF = W_tF$),

Lemma 5.2. (i) *F is SAP if and only if for every $a, b \in F^*$ the form $\langle 1, a, b, -ab\rangle$ is weakly isotropic.*

(ii) *Suppose that $a, b \in F^*$ are such that $\langle 1, a, b, -ab\rangle$ is not weakly isotropic. Let $t \in D_F(\infty)$. Then $\langle 1, a, b, -ab\rangle_{F(\sqrt{t})}$ is not weakly isotropic.*

Proof. (i) See [P1, Satz 3.1], [ELP, Th. C].

(ii) Suppose $\langle 1, a, b, -ab\rangle_{F(\sqrt{t})}$ is weakly isotropic. Then there exists an integer $n \ge 1$ such that $n \times \langle 1, a, b, -ab\rangle_{F(\sqrt{t})}$ is isotropic. The isotropy over $F(\sqrt{t})$ implies that $n \times \langle 1, a, b, -ab\rangle$ contains a subform similar to $\langle 1, -t\rangle$ (see, e.g., [L3, Ch. VII, Th. 3.1]). Since t is totally positive, it can be written as a sum of, say, m squares in F. But then $m \times \langle 1, -t\rangle$ is isotropic. Hence $mn \times \langle 1, a, b, -ab\rangle$ is isotropic and thus $\langle 1, a, b, -ab\rangle$ is weakly isotropic. □

Corollary 5.3. *Suppose that $a, b \in F^*$ are such that $\langle 1, a, b, -ab\rangle$ is not weakly isotropic.*

(i) *Let F_{pyth} be the pythagorean closure of F (inside some algebraic closure of F). Then $\langle 1, a, b, -ab\rangle_{F_{pyth}}$ is not weakly isotropic. In particular, if F is not SAP, then F_{pyth} is not SAP.*

(ii) *Let ψ be a form over F such that ψ is isotropic over F_{pyth}. Then $\langle 1, a, b, -ab\rangle_{F(\psi)}$ is not weakly isotropic. In particular, if F is not SAP, then $F(\psi)$ is not SAP. This is always the case if ψ contains a subform τ, $\dim \tau \geq 2$, such that $|\operatorname{sgn}_P(\tau)| \leq 1$ for all orderings P of F.*

Proof. (i) follows immediately from the previous lemma and the fact that F_{pyth} can be obtained as the compositum of all extensions K/F (inside an algebraic closure of F) which are of the form $F = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = K$ for some n, where $F_{i+1} = F_i(\sqrt{1 + a_i^2})$ for some $a_i \in F_i$.

(ii) Since ψ is isotropic over F_{pyth}, the extension $F_{pyth}(\psi)/F_{pyth}$ is purely transcendental. Then $\langle 1, a, b, -ab\rangle_{F_{pyth}(\psi)}$ is not weakly isotropic because $\langle 1, a, b, -ab\rangle_{F_{pyth}}$ is not weakly isotropic and because anisotropic forms (here, $n \times \langle 1, a, b, -ab\rangle_{F_{pyth}}$) stay anisotropic over purely transcendental extensions.

Now suppose ψ has a subform τ with $\dim \tau \geq 2$ and $|\operatorname{sgn}_P(\tau)| \leq 1$ for all orderings P of F. Since $\dim \tau \equiv \operatorname{sgn}_P(\tau) \pmod 2$, we have two cases. If $\operatorname{sgn}_P(\tau) = 0$ for all P, then $\tau \in W_tF$. Hence $\tau_{F_{pyth}}$ is hyperbolic and $\psi_{F_{pyth}}$ is isotropic.

If $|\operatorname{sgn}_P(\tau)| = 1$ for all P (which implies that $\dim \tau$ is odd and ≥ 3), then let $d \in F^*$ such that $q = \tau \perp \langle d\rangle \in I^2F$. It follows readily that in fact $q = \tau \perp \langle d\rangle \in I_t^2F$. Thus, $q_{F_{pyth}}$ is hyperbolic and the codimension 1 subform $\tau_{F_{pyth}}$ is isotropic. Again, $\psi_{F_{pyth}}$ is isotropic. □

Theorem 5.4. *Let $\mathcal{N}'$ be the set of pairs of integers (p, u) such that either $p = 1$ and $u = 0$ or $u = 2n \geq 2^m \geq p \geq 2$ for some integers m and n. Let $\mathcal{N} = \mathcal{N}' \cup \{(p, \infty); p \geq 2\}$.*

(i) *If F is a real field, then $(p(F), u(F)) \in \mathcal{N}$. If in addition $I_t^kF = 0$ then $p(F) \leq 2^{k-1}$.*

(ii) *Let E be a real field and let $(p, u) \in \mathcal{N}$. Then there exists a real field extension F/E such that F is non-SAP, F has property S_1 and $(p(F), u(F)) = (p, u)$. In particular, $\widetilde{u}(F) = \infty$.*

(iii) *Let E be a real field and let $(p, u) \in \mathcal{N}$ such that $p \leq 2^{k-1}$, $k \geq 1$. Then there exists a real field extension F/E such that F is non-SAP, F has property S_1, $I_t^kF = 0$ and $(p(F), u(F)) = (p, u)$. In particular, $\widetilde{u}(F) = \infty$.*

Proof. (i) Clearly, $u(F)$ is either even or infinite. It is also obvious that $p(F) = 1$ implies $u(F) = 0$. The fact that $p(F) \geq 2$ implies the existence of an integer m such that $u(F) \geq 2^m \geq p(F)$ has been mentioned in the proof of Proposition 4.3. If $p(F) > 2^{n-1}$ then there exists an $x \in D_F(\infty)$ such that $2^{n-1} \times \langle 1\rangle \perp \langle -x\rangle$ is anisotropic. This form is t.i. and a Pfister neighbor of $\langle\langle -1, \ldots, -1, x\rangle\rangle \in P_nF$ which is therefore torsion and anisotropic. Hence $I_t^nF \neq 0$. This yields the claim.

(ii) First, let us remark that if $u(F) \leq 2$, then F automatically has property S_1. In fact, S_1 means that to each torsion binary form β over F there exists an integer $n \geq 1$ such that $(n \times \langle 1\rangle) \perp \beta$ is isotropic. But if $u(F) \leq 2$, then $\langle 1\rangle \perp \beta$ is isotropic as it is a Pfister neighbor of some torsion 2-fold Pfister form which itself is hyperbolic as $I_t^2F = 0$.

To realize the value $(p,u) = (1,0)$, let F_0 be the pythagorean closure of E. Consider the iterated power series field $F = F_0((x))((y))$. By Springer's theorem (cf. [L3, Ch. VI, §1]), $u(F) = 2^2u(F_0) = 0$ and $p(F) = p(F_0) = 1$. Note that we have $W_tF = I_tF = 0$. Furthermore, F is not SAP as $\langle 1, x, y, -xy\rangle$ is not weakly isotropic.

To get the non-SAP field F with $p(F) = u(F) = 2$, let $F_1 = F_0(x, y)$ be the rational function field in two variables. Note that again F_1 is not SAP as $\langle 1, x, y, -xy\rangle$ is not weakly isotropic. Let $\varphi = \langle 1, -(1 + x^2)\rangle$, which is anisotropic and torsion as $1 + x^2 \in D_{F_1}(\infty) \setminus F_1^2$. We now construct an infinite tower $F_1 \subset F_2 \subset \ldots$ such that over each F_i, φ stays anisotropic and $\langle 1, x, y, -xy\rangle$ will not be weakly isotropic.

The construction is as follows. Having constructed F_i with the desired properties, $i \geq 1$, let F_{i+1} be the compositum of all function fields of 3-dimensional t.i. forms over F_i. Since anisotropic 2-dimensional forms stay anisotropic over the function fields of forms of dimension ≥ 3 (see, e.g. [H1, Th. 1]), φ will stay anisotropic over F_{i+1}. By Cor. 5.3, $\langle 1, x, y, -xy\rangle$ will not be weakly isotropic over F_{i+1}. Now let $F = \bigcup_{i=1}^{\infty} F_i$. The above shows that φ_F is anisotropic so that in particular $u(F) \geq 2$, and $\langle 1, x, y, -xy\rangle_F$ is not weakly isotropic so that F is not SAP. Let $q \in P_2F \cap W_tF$. Any 3-dimensional subform of q is t.i. and thus isotropic by construction of F. Thus, q is hyperbolic. In particular, $I_t^2F = 0$ as I_t^2F is generated as an ideal by torsion 2-fold Pfister forms (cf. [EL1, Th. 2.8]). By [EL2, Prop. 1.8], this implies $u(F) \leq 2$ and thus $u(F) = p(F) = 2$. Clearly, $I_t^2F = 0$.

To get those values (p,u) of $\mathcal{N}$ with $u \geq 4$, we use a construction quite similar to that in the proofs of [H4, Th. 2, Th. 3].

So let $p \geq 2$, $F_1 = F_0(x_1, x_2, \ldots, y_1, y_2, \ldots)$ be the rational function field in an infinite number of variables x_i, y_j over F_0. Clearly, F_1 is not SAP as, for example, the form $q = \langle 1, x_1, x_2, -x_1x_2\rangle$ is not weakly isotropic. Let $a = 1 + x_1^2 + \ldots + x_{p-1}^2$ and let $\varphi = \langle \underbrace{1, \ldots, 1}_{p-1}, -a\rangle$ which is anisotropic by a well known result of Cassels (cf. [L3, Ch. IX, Cor. 2.4]). Let $n \geq 2$ and consider the multiquaternion algebra

$$A_n = (1 + x_1^2, y_1) \otimes \ldots \otimes (1 + x_{n-1}^2, y_{n-1})$$

over F_1. Then A is a division algebra over F_1 and it will stay a division algebra over $F_1(\sqrt{-1})$ (see, e.g. [H3, Lem. 2]). By Lemma 5.1, there exists a $2n$-dimensional form ψ_n such that in WF_1 we have $\psi_n = \sum_{i=1}^{n-1} c_i\langle\langle 1 + x_{i-1}^2, y_{i-1}\rangle\rangle$ for suitable $c_i \in F_1^*$. Since $1 + x_{i-1}^2 \in D_{F_1}(\infty)$, the forms $\langle\langle 1 + x_{i-1}^2, y_{i-1}\rangle\rangle$ are torsion and thus $\psi_n \in I_t^2F_1$. Furthermore, ψ_n is anisotropic as A_n is division (this stays true over $F_1(\sqrt{-1})$).

Let now $n \geq 2$ and $p \geq 2$ be such that $2n \geq 2^m \geq p$ for some integer m. Suppose that K is any real field extension of F_1 such that q_K is not weakly isotropic, $(A_n)_{K(\sqrt{-1})}$ is division and φ_K is anisotropic. Consider the following three types of quadratic forms over K:

$$\mathcal{C}_1(K) = \{\langle\underbrace{1,\ldots,1}_{p}, -b\rangle \mid b \in D_K(\infty)\}\ ,$$

$$\mathcal{C}_2(K) = \{\underbrace{\langle 1,\ldots,1\rangle}_{2p} \perp \beta \mid \dim\beta = 2,\ \beta \in W_tF\}\ ,$$

$$\mathcal{C}_3(K) = \{\alpha \mid \alpha \in W_tK,\ \dim\alpha \geq 2n+2\}\ .$$

Let $\rho \in \mathcal{C}_1(K) \cup \mathcal{C}_2(K) \cup \mathcal{C}_3(K)$. Then $(A_n)_{K(\rho)(\sqrt{-1})}$ is division so that in particular $(\psi_n)_{K(\rho)}$ is anisotropic. For $\rho \in \mathcal{C}_i(K)$, $i = 1, 2$, this follows as $\rho_{K(\sqrt{-1})}$ is isotropic (recall that in this case $\langle 1, 1\rangle \subset \rho$) and therefore $K(\rho)(\sqrt{-1}) = K(\sqrt{-1})(\rho)$ is purely transcendental over $K(\sqrt{-1})$. In the case $\rho \in \mathcal{C}_3(K)$ this is a consequence of Lemma 5.1(iii).

Also, $\varphi_{K(\rho)}$ is anisotropic. This follows from [H4, Cor.] if $\rho \in \mathcal{C}_1(K)$, and from [H1, Th. 1] by comparing dimensions if $\rho \in \mathcal{C}_i(K)$, $i = 2, 3$.

q will not be weakly isotropic over $K(\rho)$ by Corollary 5.3.

As before, we now construct a tower of fields $F_1 \subset F_2 \subset \ldots$ as follows. Having constructed F_i, we let F_{i+1} be the compositum of all function fields of forms in $\mathcal{C}_1(F_i) \cup \mathcal{C}_2(F_i)$. Let $F = \bigcup_{i=1}^{\infty} F_i$. By the above, $(\psi_n)_F$ is anisotropic (and torsion), so that $u(F) \geq 2n$. On the other hand, torsion forms of dimension $> 2n$ will be isotropic by construction. Thus $u(F) = 2n$.

φ_F is also anisotropic. Hence $p(F) \geq p$. By construction, all forms in $\mathcal{C}_1(F)$ are isotropic and thus $p(F) = p$.

q_F is not weakly isotropic and therefore F is not SAP. In particular $\widetilde{u}(F) = \infty$.

Finally, F has property S_1 as all forms in $\mathcal{C}_2(F)$ are isotropic by construction.

To obtain the values (p, ∞) with $p \geq 2$, we do the same construction as before, but this time only with forms in $\mathcal{C}_i(F)$, $i = 1, 2$. This will again yield a non-SAP field F with property S_1 and with $p(F) = p$. However, this time we have that $(A_n)_F$ will be a division algebra for each $n \geq 2$, so that $(\psi_n)_F$ will be an anisotropic torsion form of dimension $2n$ for each $n \geq 2$. In particular, $u(F) = \infty$.

(iii) If $k \leq 2$ then $I_t^2F = 0$ and thus $u(F) \leq 2$. These cases have already been dealt with in the proof of (ii). So suppose that $k \geq 3$. We repeat the steps in (ii), but when taking composites of function fields, we now include also function fields of forms in

$$\mathcal{C}_4(K) = \{\alpha \mid \alpha \in I_t^kK,\ \dim\alpha \geq 2^k\}$$

in addition to those in $\mathcal{C}_i$, $1 \leq i \leq 3$ (resp. $\mathcal{C}_1$, $\mathcal{C}_2$ in the case $u = \infty$). Since by APH we have that anisotropic forms in I^kF must be of dimension $\geq 2^k$, we immediately see that by construction $I_t^kF = 0$.

$(A_n)_F$ will still be a division algebra by Lemma 5.1(iii) as we only consider in addition function fields of forms in I_t^k with $k \geq 3$. Thus, ψ_n will be anisotropic as above and we get again that $u(F) = u$. Since $\dim\varphi = p \leq 2^{k-1}$, it follows from [H1, Th. 1] that φ_F will still be anisotropic as we only consider in addition function fields of forms which have dimension $\geq 2^k$. We conclude similarly as above that $p(F) = p$.

Using the same reasoning as above, Cororollary 5.3 implies that q_F is not weakly isotropic and therefore F is not SAP, so that in particular $\widetilde{u}(F) = \infty$. Obviously, F will again have the property S_1. $\square$

Remark 5.5. In [EP, § 5], examples of real fields F with $u(F) = 2^n$ have been constructed for each integer $n \geq 1$ with the property that $u(F(\sqrt{a})) = \infty$ and $p(F(\sqrt{a})) = 2$. $u(F(\sqrt{a})) = \infty$ implies that F is non-SAP by [EP, Cor. 2.4]. It is also indicated how to obtain such a field which *does not* satisfy S_1 (resp. certain properties S_n which generalize S_1), see [EP, Rem. 5.3].

We will now construct real SAP fields F such that $\widetilde{u}(F) = \infty$ and $u(F) = 2n$ for a given n. First, we note that it will be impossible to realize such examples for all values in $\mathcal{N}$ (cf. Theorem 5.4).

Proposition 5.6. *Let F be real and SAP. If $u(F) \leq 2$ then $u(F) = \widetilde{u}(F)$.*

Proof. As remarked in the proof of Theorem 5.4, $u(F) \leq 2$ implies that F has property S_1. Since F is SAP by assumption, we thus have $\widetilde{u}(F) < \infty$. Now $p(F) \leq u(F) \leq 2$, and by Theorem 4.8 we have $u(F) = \widetilde{u}(F)$. $\square$

Theorem 5.7. *Let $\mathcal{N}$ be as in Theorem 5.4.*

(i) *If F is a real SAP field with $\widetilde{u}(F) = \infty$, then $u(F) \geq 4$ and $(p(F), u(F)) \in \mathcal{N}$. Furthermore, $I_t^2 F \neq 0$. If in addition $I_t^k F = 0$, $k \geq 3$, then $p(F) \leq 2^{k-1}$.*

(ii) *Let E be a real field and let $(p, u) \in \mathcal{N}$ with $u \geq 4$. Then there exists a real field extension F/E such that F is SAP, F does not have property S_1 and $(p(F), u(F)) = (p, u)$. In particular, $\widetilde{u}(F) = \infty$.*

(iii) *Let E be a real field and let $(p, u) \in \mathcal{N}$ with $u \geq 4$ and such that $p \leq 2^{k-1}$, $k \geq 3$. Then there exists a real field extension F/E such that F is SAP, F does not have property S_1, $I_t^k F = 0$ and $(p(F), u(F)) = (p, u)$. In particular, $\widetilde{u}(F) = \infty$.*

Proof. (i) If $I_t^2 F = 0$, then $u(F) \leq 2$ by [EL2, Prop. 1.8]. The result now follows from Theorem 5.4 and Proposition 5.6.

(ii) We proceed as in the proof of Theorem 5.4(ii) for the case $(p, u) \in \mathcal{N}$ and $2n = u \geq 4$, except for the definition of F_1, which now will be the power series field in one variable t over the field which was denoted by F_1 in the proof of Theorem 5.4(ii): $F_1 = F_0(x_1, x_2, \ldots, y_1, y_2, \ldots)((t))$. We keep the notations for A_n, ψ_n, $\mathcal{C}_1(K)$, $\mathcal{C}_3(K)$. We redefine $\mathcal{C}_2(K)$:

$$\mathcal{C}_2(K) = \{\langle 1, 1\rangle \otimes \langle 1, x, y, -xy\rangle \mid x, y \in K^*\} .$$

We construct a tower of fields $F_1 \subset F_2 \subset \ldots$ as follows. Having constructed F_i, we let F_{i+1} be the compositum of all function fields of forms in $\mathcal{C}_1(F_i) \cup \mathcal{C}_2(F_i) \cup \mathcal{C}_3(F_i)$. Let $F = \bigcup_{i=1}^{\infty} F_i$.

Exactly as in the proof of Theorem 5.4(ii), it follows that $(u(F), p(F)) = (p, u)$. It remains to show that F is SAP and does not have property S_1.

Now by construction, for all $x, y \in F^*$ we have that $\langle 1, 1\rangle \otimes \langle 1, x, y, -xy\rangle$ is isotropic. In particular, each form $\langle 1, x, y, -xy\rangle$ is weakly isotropic, which shows by Lemma 5.2 that F is SAP.

Now let $d = 1 + x_1^2$ and consider the form $\mu_m = m \times \langle 1\rangle \perp t\langle 1, -d\rangle$ which is anisotropic over F_1 by Springer's theorem. Let $L_1 = F_1$ and $L_1' = F_1' = F_0(x_1, x_2, \ldots, y_1, y_2, \ldots)$. We now construct a tower of fields $L_1 \subset L_2 \subset \ldots$ such that L_i will be the power series field in the variable t over some L_i', $L_i = L_i'((t))$, such that $F_i \subset L_i$, and $(\mu_m)_{L_i}$ anisotropic for all $m \geq 0$ and all $i \geq 1$. This then shows that $(\mu_m)_{F_i}$ is anisotropic for all $m \geq 0$, $i \geq 1$, and therefore $(\mu_m)_F$ will be anisotropic for all $m \geq 0$. It follows that the torsion form $(-t\langle 1, -d\rangle)_F$ does not represent any element in $D_F(\infty)$. Thus, F does not have property S_1.

Suppose we have constructed $L_i = L_i'((t))$. Note that necessarily L_i is real as $(\mu_m)_{L_i}$ is anisotropic for all $m \geq 0$. Let $P_i \in X_{L_i'}$ be any ordering and M_i' be the compositum over L_i' of the function fields of all forms (defined over L_i') in

$$\mathcal{C}'(L_i') = \{\alpha \mid \alpha \text{ indefinite at } P_i,\, \dim \alpha \geq 3\}\ .$$

Let $M_i = M_i'((t))$.

Now let $\rho \in \mathcal{C}_1(F_i) \cup \mathcal{C}_2(F_i) \cup \mathcal{C}_3(F_i)$ and consider $L_i(\alpha)$. By Springer's theorem, $\rho_{L_i} \cong \beta \perp t\gamma$ where β, γ are defined over L_i'. Suppose $\rho \in \mathcal{C}_1(F_i)$. Then $\rho \cong p \times \langle 1\rangle \perp \langle -b\rangle$ with $b \in D_{L_i}(\infty)$. But then, up to a square, $b \in D_{L_i'}(\infty)$ and thus $\rho_{L_i} \in \mathcal{C}'(L_i')$. Hence, ρ_{M_i} is isotropic and therefore $M_i(\rho)/M_i$ is purely transcendental.

Suppose $\rho = \langle 1, 1\rangle \otimes \langle 1, x, y, -xy\rangle \in \mathcal{C}_2(F_i)$. Then either ρ_{L_i} is already defined over L_i', in which case it is a t.i. form of dimension 8 and thus in $\mathcal{C}'(L_i')$. Or there exist $a, b \in L_i'^*$ such that $\rho \cong \langle 1, 1\rangle \otimes \langle 1, a\rangle \perp bt\langle 1, 1\rangle \otimes \langle 1, -a\rangle$. then either $\langle 1, 1\rangle \otimes \langle 1, a\rangle$ is indefinite at P_i and thus in $\mathcal{C}'(L_i')$, or $\langle 1, 1\rangle \otimes \langle 1, -a\rangle$ is indefinite at P_i and thus in $\mathcal{C}'(L_i')$. In any case, we see that ρ_{M_i} is isotropic, and again $M_i(\rho)/M_i$ is purely transcendental.

Finally, suppose that $\rho \in \mathcal{C}_3(F_i)$. Then $\rho_{L_i} \in W_t L_i$, and if we write $\rho \cong \beta \perp t\gamma$ with β and γ defined over L_i', then $\beta \in W_t L_i'$ and $\gamma \in W_t L_i'$. Now $\dim \rho \geq 6$, and hence $\dim \beta \geq 4$ or $\dim \gamma \geq 4$. Hence $\beta \in \mathcal{C}'(L_i')$ or $\gamma \in \mathcal{C}'(L_i')$. As above, we conclude that ρ_{M_i} is isotropic and that $M_i(\rho)/M_i$ is purely transcendental.

Now let N_i be the compositum of the function fields of all forms α_{M_i} with $\alpha \in \mathcal{C}_1(F_i) \cup \mathcal{C}_2(F_i) \cup \mathcal{C}_3(F_i)$. By the above, N_i/M_i is purely transcendental. Let B be a transcendence basis so that $N_i = M_i(B) = M_i'((t))(B)$. We now put $L_{i+1}' = M_i'(B)$ and $L_{i+1} = L_{i+1}'((t)) = M_i'(B)((t))$. There are obvious inclusions $F_{i+1} \subset N_i = M_i'((t))(B) \subset M_i'(B)((t)) = L_{i+1}$. Since M_i' is obtained from L_i' by taking function fields of forms indefinite at P_i, we see that P_i extends to an ordering on M_i' and thus clearly also to orderings on L_{i+1}.

It remains to show that μ_m stays anisotropic over L_{i+1}. Now $m \times \langle 1\rangle$ is clearly anisotropic over the real field L_{i+1}'. Also, $\langle 1, -d\rangle$, which is anisotropic over L_i' by assumption, stays anisotropic over L'_{i+1} as L'_{i+1} is obtained by taking function fields of forms of of dimension ≥ 3 over L'_i followed by a purely transcendental extension. By Springer's theorem, $(\mu_m)_{L_{i+1}} = (m \times \langle 1\rangle \perp t\langle 1, -d\rangle)_{L_{i+1}}$ is anisotropic.

To get the values of type (p, ∞), we adjust the above arguments as in the proof of Theorem 5.4(iii).

(iii) This follows easily by combinig the proof of part (ii) above with that of Theorem 5.4(iii). We leave the details to the reader. $\square$

Remark 5.8. Let K be any real field over E with $u(K) = 2n$ and such that K is uniquely ordered. For $n \geq 2$, such fields have been constructed in [H4, Th. 2]. The construction there can also readily be used to get such a K for $n = 1$.

Now consider $F = K((t))$, the power series field in one variable t over K. By Springer's theorem, $u(F) = 4n = 2u(K)$. Since K is uniquely ordered, we have that F is SAP (cf. [ELP, Prop. 1]). Since $u(K) > 0$, K is not pythagorean. So let $d \in D_K(\infty) \setminus K^{*2}$. Then the form $(m \times \langle 1\rangle) \perp t\langle 1, -d\rangle$ is anisotropic for all m (again by Springer's theorem), and since $t\langle 1, -d\rangle$ is torsion, we see that F does not have property S_1. Hence $\widetilde{u}(F) = \infty$.

This rather simple construction yields SAP fields with $u(F) = 4n$ and $\widetilde{u}(F) = \infty$ for all $n \geq 1$, but it does not provide examples where $u(F) = 4n + 2$, $n \geq 1$. Furthermore, one checks easily that it will not yield examples of SAP fields with $\widetilde{u}(F) = \infty$, $u(F) > 4$ and $I^3_t F = 0$, which do exist by the above theorem.

6. Linkage of fields and the Pfister neighbor property

Let us recall the notion of linkage of fields.

Definition 6.1. Let $n \geq 1$ be an integer. A field F is called *n-linked* if to any n-fold Pfister forms π_1 and π_2 over F there exist an $a \in F^*$ and $(n-1)$-fold Pfister forms σ_1 and σ_2 such that $\pi_i \cong \langle\langle a\rangle\rangle \otimes \sigma_i$, $i = 1, 2$. F is called *linked* if F is 2-linked.

Obviously, every field is 1-linked, so the property of n-linkage is only interesting for $n \geq 2$ The following properties are well known, [EL3, § 2], [H2].

Proposition 6.2. *Let $n \geq 2$. The following are equivalent.*

(i) *F is n-linked.*
(ii) *F is m-linked for all $m \geq n$.*
(iii) *The classes of n-fold Pfister forms in $I^nF/I^{n+1}F$ form an additive subgroup of $I^nF/I^{n+1}F$.*
(iv) *To each form $\varphi \in I^nF$ there exists a form $\pi \in P_nF$ such that $\varphi \equiv \pi \bmod I^{n+1}F$.*
(v) *Each anisotropic form $\varphi \in I^nF$ can be written as $\varphi \cong \pi_1 \perp \ldots \perp \pi_r$ with $\pi_i \in GP_nF$, $1 \leq i \leq r$ (φ is then said to have simple decomposition).*
(vi) *To each anisotropic form $\varphi \in I^nF$ there exists a form $\sigma \in P_{n-1}F$ and an even-dimensional form τ over F such that $\varphi \cong \sigma \otimes \tau$.*

Furthermore, if F is n-linked, $n \geq 2$, then $I_t^{n+2}F = 0$.

We can extend this list of equivalences further for linked (i.e. 2-linked) fields. These equivalent formulations for the linkage property are essentially well known, so we will only sketch the proofs for some of them and give the relevant references for the others.

Proposition 6.3. *The following properties of a field F are equivalent :*

(i) *F is linked.*
(ii) *If Q_1 and Q_2 are quaternion algebras over F then there exist $a, b_1, b_2 \in F^*$ such that $Q_i \cong (a, b_i)_F$, $i = 1, 2$, i.e. Q_1 and Q_2 have a common slot.*
(iii) *The classes of quaternion algebras over F form a subgroup of the exponent-2-part ${}_2\mathrm{Br}(F)$ of the Brauer group of F.*
(iv) *To every element $[A] \in {}_2\mathrm{Br}(F)$ there exists a quaternion algebra $(a, b)_F$, $a, b \in F^*$, such that $[A] = [(a, b)_F]$.*
(v) *Every 6-dimensional form in I^2F is isotropic.*
(vi) *Every 5-dimensional form over F is a Pfister neighbor.*
(vii) *To every anisotropic form φ in I^2F there exists a quadratic extension L/F such that φ_L is hyperbolic.*

Proof. (i)$\Leftrightarrow$(ii). Let $Q_i = (u_i, v_i)_F$, $u_i, v_i \in F^*$, $i = 1, 2$, and let $\pi_i = \langle\langle u_i, v_i\rangle\rangle \in P_2F$ be the norm form associated to Q_i. The equivalence of (i) and (ii) follows readily from the fact that two quaternion algebras are isomorphic if and only if their associated norm forms are isometric, cf. [L3, Ch. III, Th. 2.5], [S, Ch. 2, Th. 11.9].

(ii)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(iv) follows readily from Merkurjev's theorem [M1] which implies that every element in ${}_2\mathrm{Br}(F)$ is Brauer equivalent to a product of quaternion algebras.

(iv)$\Rightarrow$(ii). Let Q_1 and Q_2 be quaternion algebras over F. By assumption, there exists a quaternion algebra Q_3 such that $[Q_1 \otimes Q_2 \otimes Q_3] = 1 \in {}_2\mathrm{Br}(F)$. A well known theorem of Albert then states that there exist $a, b_1, b_2, b_3 \in F^*$ such that $Q_i \cong (a, b_i)_F$, $1 \leq i \leq 3$ (cf. [S, Ch. 2, Th. 13.6]).

(i)$\Leftrightarrow$(v). Up to a scalar multiple, a 6-dimensional I^2-form φ can be written as $\varphi \cong \langle -x, -y, xy, u, v, -uv\rangle$, $x, y, u, v \in F^*$. The equivalence now follows readily from the fact that φ is isotropic iff $\langle -x, -y, xy\rangle$ and $\langle -u, -v, uv\rangle$ represent a common element iff there exist $a, b_1, b_2 \in F^*$ such that $\langle\langle x, y\rangle\rangle \cong \langle\langle a, b_1\rangle\rangle$ and $\langle\langle u, v\rangle\rangle \cong \langle\langle a, b_2\rangle\rangle$.

(v)$\Leftrightarrow$(vi) Let φ be a 5-dimensional form over F and let $d \in F^*$. Then $\varphi \perp \langle -d\rangle \in I^2F$ iff $d = \det(\varphi) \in F^*/F^{*2}$. The equivalence is now an easy consequence of the well known fact that φ is a Pfister neighbor iff φ represents d iff $\varphi \perp \langle -d\rangle$ is isotropic (see, e.g., [L3, Ch. X, Prop. 4.19]).

(i)$\Leftrightarrow$(vii). Let $L = F(\sqrt{a})$ be a quadratic extension, and let φ be an anisotropic form over F. Then it is well known that φ_L is hyperbolic iff there exists a form τ over F such that $\varphi \cong \langle\langle a\rangle\rangle \otimes \tau$ (see, e.g., [L3, Ch. VII, Th. 3.2], [S, Ch. 2, Th. 5.2]).

The equivalence corresponds therefore to the equivalence (i)⇔(vi) in the previous proposition for $n = 2$. □

Linked fields have a very well understood theory of quadratic forms. In particular, we have a complete knowledge of the possible values of the u-invariant and the Hasse number of linked fields. We will now state these results which are due to Elman and Lam [EL3] and Elman [E, Th. 4.7].

Theorem 6.4. *Let F be a linked field. Then $u(F) = \widetilde{u}(F) \in \{0, 1, 2, 4, 8\}$. In particular, $I_t^4 F = 0$. Furthermore, let $n \in \{0, 1, 2\}$. Then $\widetilde{u}(F) \leq 2^n$ iff $I_t^{n+1} F = 0$.*

Note that $u(F) = \widetilde{u}(F) = 0$ can only occur when F is real, whereas $u(F) = \widetilde{u}(F) = 1$ implies that F is nonreal.

It is a natural question to ask whether one can obtain also good information (for example, upper bounds) on u and $\widetilde{u}$ after weakening the linkage assumption on F. One possible way of generalizing is to assume F to be n-linked for some $n \geq 2$. For $n = 2$ we have linkage and the above result. Suppose F is nonreal and $I^3 F = 0$. Then obviously F is 3-linked as there are no anisotropic 3-fold Pfister forms in the first place. However, Merkurjev constructed to each positive even integer n a field F with $I^3 F = 0$ and $u(F) = n$ (resp. a field with $I^3 F = 0$ and $u(F) = \infty$), [M2]. This shows that if we weaken the linkage assumption by only assuming n-linkage for some $n \geq 3$, the u-invariant can become arbitrarily large and even infinite.

One of the main purposes of this paper is to show that item (vi) in Proposition 6.3 *can* be generalized suitably to yield meaningful bounds on the Hasse number resp. the u-invariant.

Definition 6.5. Let $n \geq 0$ be an integer. A field F is said to have the *Pfister neighbor property of order n*, $PN(n)$ for short, if every form over F of dimension $2^n + 1$ is a Pfister neighbor.

Remark 6.6.
- (i) Every field has property $PN(0)$ and $PN(1)$. F has property $PN(2)$ iff F is linked, see Proposition 6.3.
- (ii) Let $n \geq 2$. Every isotropic form of dimension $2^n + 1$ is a Pfister neighbor. In fact, if $\dim \varphi = 2^n + 1$ and φ is isotropic, then $\varphi \cong \mathbb{H} \perp \psi$ with $\dim \psi = 2^n - 1$. Then $\varphi \perp -\psi \cong \pi \in P_{n+1} F$, where π denotes the hyperbolic $(n + 1)$-fold Pfister form.

Lemma 6.7. *Let $n \geq 2$.*
- (i) *F has property $PN(n)$ if and only if there exists to every form φ over F a form ψ such that $\dim \psi \leq 2^n$ if $\dim \varphi$ even (resp. $\dim \psi \leq 2^n - 1$ if $\dim \varphi$ odd) such that $\varphi \equiv \psi \bmod I^{n+1} F$.*
- (ii) *If F has property $PN(n)$, $n \geq 2$, then F is n-linked. In particular, $I_t^{n+2} F = 0$. Furthermore, F is ED.*

Proof. (i) 'only if': If $\dim \varphi \leq 2^n$, then put $\psi \cong \varphi$. So suppose $\dim \varphi \geq 2^n + 1$. Write $\varphi \cong \psi \perp \tau$ with $\dim \psi = 2^n + 1$. By $PN(n)$, ψ is a Pfister neighbor and there exists ψ', $\dim\psi' = 2^n - 1$ such that $\psi \perp -\psi' \cong \pi \in GP_{n+1}F$. Then, in WF, we have

$$\varphi \equiv \varphi - \pi \equiv \psi' \perp \tau \bmod I^{n+1}F\ .$$

Now $\dim\psi' \perp \tau = \dim\varphi - 2$ and the result follows by an easy induction on the dimension.

'if': Let $\dim\varphi = 2^n + 1$. By assumption, there exists a form ψ, $\dim\psi \le 2^n - 1$ such that $\varphi \perp -\psi \in I^{n+1}F$. If $\dim\psi < 2^n - 1$, then by APH, $\varphi \perp -\psi$ is hyperbolic, i.e. $\varphi = \psi \in WF$, and φ is isotropic by comparing dimensions, hence a Pfister neighbor by Remark 6.6(ii). If $\dim\psi = 2^n - 1$ then $\dim(\varphi \perp -\psi) = 2^{n+1}$ and thus $\varphi \perp -\psi \in GP_{n+1}F$, again by APH, which implies that φ is a Pfister neighbor.

(ii) To show that F is n-linked, let $\varphi \in I^nF$. By (i), there exists ψ such that $\dim\psi \le 2^n$ and $\varphi \equiv \psi \bmod I^{n+1}F$. But clearly $\psi \in I^nF$, and by APH we have that either ψ is hyperbolic (in which case we may replace it by the n-fold hyperbolic Pfister form), or ψ is anisotropic and of dimension 2^n, in which case $\psi \in GP_nF$. Let $x \in F^*$ such that $x\psi \in P_nF$. We then have $\psi \equiv x\psi \bmod I^{n+1}F$, and n-linkage together with $I_t^{n+2}F = 0$ follows from Proposition 6.2.

Now n-linked fields, $n \ge 2$, are easily seen to be SAP. So to establish ED, it suffices to establish property S_1 by Theorem 3.1. Let $\langle a, b\rangle$ be any torsion form. Let $\gamma \cong \langle\underbrace{1,\dots,1}_{2^n-1}\rangle$. Then by $PN(n)$, the form $\gamma \perp \langle -a, -b\rangle$ is a t.i. Pfister neighbor of a Pfister form $\pi \in P_{n+1}F$. Since π contains γ which is a Pfister neighbor (and in fact subform) of $\sigma_n \cong \langle 1, 1\rangle^{\otimes n}$, one necessarily has that σ_n divides π, so there exists $c \in F^*$ such that $\pi \cong \sigma_n \otimes \langle 1, c\rangle$. Now π contains a t.i. Pfister neighbor and is therefore also t.i. and hence torsion. But then $\rho \cong \langle 1, 1\rangle \otimes \sigma_n \otimes \langle 1, c\rangle \in P_{n+2}F$ is torsion as well and therefore hyperbolic by Proposition 6.2. Now $\sigma_n \perp \gamma \perp \langle -a, -b\rangle$ is a Pfister neighbor of ρ. Since ρ is hyperbolic, its neighbor $\sigma_n \perp \gamma \perp \langle -a, -b\rangle$ is isotropic. Hence there exists $x \in D_F(\langle a, b\rangle) \cap D_F(\sigma_n \perp \gamma)$. But clearly, $D_F(\sigma_n \perp \gamma) \subset D_F(\infty)$ which shows that the binary torsion form $\langle a, b\rangle$ represents the totally positive element x. □

The following observation is essentially due Fitzgerald [F, Lemma 4.5(ii)].

Lemma 6.8. *Suppose that $\widetilde{u}(F) \le 2^n$. Let φ be a form over F of dimension $2^n + 1$. Then φ is a Pfister neighbor. In particular, F has $PN(n)$.*

Proof. Since $\widetilde{u}(F) < \infty$ implies that F is SAP, we may assume that after scaling, $\mathrm{sgn}_P(\varphi) \ge 0$ for all $P \in X_F$, and that there exists $c \in F^*$ such that $H(c) = \{P \in X_F \mid \mathrm{sgn}_P(\varphi) = \dim\varphi\}$. In particular, the Pfister form $\langle\langle\underbrace{-1,\dots,-1}_{n}, -c\rangle\rangle \in P_{n+1}F$ is positive definite at all those $P \in X_F$ at which φ is positive definite, and it has signature zero at all those $P \in X_F$ at which φ is indefinite. Let $\psi \cong (\pi \perp -\varphi)_{\mathrm{an}}$. It follows that $|\,\mathrm{sgn}_P(\psi)| \le 2^n - 1$ for all $P \in X_F$. But since $\widetilde{u}(F) \le 2^n$, the anisotropic form ψ must therefore have $\dim\psi \leq 2^n$, so in particular,

$$i_W(\pi \perp -\varphi) = \tfrac{1}{2}(\dim(\pi \perp -\varphi) - \dim\psi) \geq \tfrac{1}{2}(2^{n+1}+1)\ ,$$

and therefore $i_W(\pi \perp -\varphi) \geq 2^n + 1 = \dim\varphi$, which implies that $\varphi \subset \pi$. In particular, φ is a Pfister neighbor of π. □

Theorem 6.9. *If a field F has property $PN(n)$, $n \geq 2$, then either $u(F) \leq \widetilde{u}(F) \leq 2^n$, or $2^{n+1} \leq u(F) \leq \widetilde{u}(F) \leq 2^{n+1} + 2^n - 2$.*

Proof. Let F be a field with property $PN(n)$ for some $n \geq 2$. Suppose that $\widetilde{u}(F) > 2^n$, i.e. there exists an anisotropic t.i. φ with $\dim\varphi = m > 2^n$. By Lemma 6.7, F has ED and so φ can be diagonalized as $\varphi \cong \langle a_1, \ldots, a_m\rangle$ with $-a_1, a_m \in D_F(\infty)$. By removing some of the a_i, $2 \leq i \leq m-1$ if necessary, we will retain a t.i. form, so we may assume that φ is t.i. and $\dim\varphi = 2^n + 1$. But then, by $PN(n)$, φ is a Pfister neighbor of some $\pi \in P_{n+1}F$ which in turn is torsion and anisotropic as its Pfister neighbor φ is t.i. and anisotropic. This shows that $2^{n+1} \leq u(F) \leq \widetilde{u}(F)$.

Now suppose that $\widetilde{u}(F) > 2^{n+1} + 2^n - 2$. By a similar argument as above, we conclude that there exists an anisotropic t.i. form φ with $\dim\varphi = 2^{n+1} + 2^n - 1$. By Lemma 6.7, there exists an anisotropic form ψ of dimension $\leq 2^n - 1$ such that $\varphi \equiv \psi \bmod I^{n+1}F$. Let $\pi \cong (\varphi \perp -\psi)_{\mathrm{an}} \in I^{n+1}F$. Then by dimension count and since φ is anisotropic, we have $2^{n+1} \leq \dim\pi \leq 2^{n+2} - 2$. But since F is $(n+1)$-linked by Lemma 6.7(ii) and Proposition 6.2, anisotropic forms in $I^{n+1}F$ have dimension divisible by 2^{n+1}, so we have $\dim\pi = 2^{n+1}$, which, by APH, implies that $\pi \in GP_{n+1}F$. Also, $\varphi = \pi \perp \psi$ in WF, and by dimension count we have in fact $\varphi \cong \pi \perp \psi$.

After scaling, we may assume that $\pi \in P_{n+1}F$, so that $\mathrm{sgn}_P(\pi) \in \{0, 2^{n+1}\}$. Now φ is t.i., and since F has ED by Lemma 6.7(ii), we can write $\psi \cong \langle a, \ldots\rangle$ with $a <_P 0$ whenever $\mathrm{sgn}_P(\pi) = 2^{n+1}$. But then $\pi \perp \langle a\rangle$ is a t.i. subform of φ. On the other hand, $\pi \perp \langle a\rangle$ is also a Pfister neighbor of $\pi \otimes \langle 1, a\rangle \in P_{n+2}F$. Since $\pi \perp \langle a\rangle$ is t.i., this implies that $\pi \otimes \langle 1, a\rangle$ is torsion and therefore hyperbolic since $I_t^{n+2}F = 0$ by Proposition 6.2 and Lemma 6.7(ii). But then the Pfister neighbor $\pi \perp \langle a\rangle$ is isotropic and therefore also φ, a contradiction. □

We now obtain the equivalence of (i) and (iii) in Theorem 1.1

Corollary 6.10. $\widetilde{u}(F) < \infty$ *if and only if F has $PN(n)$ for some $n \geq 2$.*

Proof. The 'if'-part follows from Theorem 6.9, the converse from Lemma 6.8. □

Remark 6.11. If F is real, then we still get a sufficient criterion for the finiteness of $u(F)$ even if $\widetilde{u}(F) = \infty$. Indeed, for real F, one has that if $u(F(\sqrt{-1}))$ is finite then $u(F)$ is finite, more precisely, one has $u(F) < 4u(F(\sqrt{-1}))$ (see [EKM, Th. 37.4]). Thus, we get the following: If $F(\sqrt{-1})$ has property $PN(n)$ for some $n \geq 2$, then $u(F) < 2^{n+3} + 2^{n+2} - 8$.

Conjecture 6.12. If a field F has property $PN(n)$, $n \geq 2$, then $u(F) \leq \widetilde{u}(F) \leq 2^n$, or $u(F) = \widetilde{u}(F) = 2^{n+1}$.

Corollary 6.13. *For $n \geq 2$, $PN(n)$ implies $PN(m)$ for all $m \geq n+2$. Furthermore, the following are equivalent:*

(i) *Conjecture 6.12 holds.*
(ii) *For $n \geq 2$, $PN(n)$ implies $PN(n+1)$.*

Proof. If $n \geq 2$, then $PN(n)$ implies that $\widetilde{u}(F) \leq 2^{n+2}$, and $PN(m)$ for $m \geq n+2$ follows from Lemma 6.8.

Now suppose that F has $PN(n)$ and that Conjecture 6.12 holds. Then $PN(n+1)$ follows from Lemma 6.8. Conversely, suppose that $n \geq 2$ and that $PN(n)$ implies $PN(n+1)$. Then we have $u(F) \leq \widetilde{u}(F) \leq 2^n$ or $2^{n+1} \leq u(F) \leq \widetilde{u}(F) \leq 2^{n+1}+2^n-2$ because of $PN(n)$, and also $u(F) \leq \widetilde{u}(F) \leq 2^{n+1}$ or $2^{n+2} \leq u(F) \leq \widetilde{u}(F) \leq 2^{n+2}+2^{n+1}-2$ because of $PN(n+1)$. Putting the two together, we obtain $u(F) \leq \widetilde{u}(F) \leq 2^n$ or $u(F) = \widetilde{u}(F) = 2^{n+1}$. □

The only evidence we have as to the veracity of Conjecture 6.12 is the following.

Lemma 6.14. *$PN(2)$ implies $PN(3)$. In particular, if F has $PN(2)$, then $u(F) \leq \widetilde{u}(F) \leq 4$ or $u(F) = \widetilde{u}(F) = 8$.*

Proof. Suppose F has $PN(2)$ and let φ be any 9-dimensional form over F. Write $\varphi \cong \alpha \perp \beta$ with $\dim \alpha = 5$. Since α is a Pfister neighbor, there exists $\pi \in GP_2F$ such that $\pi \subset \alpha \subset \varphi$ (see, e.g., [L3, Ch. X, Prop. 4.19]). Write $\varphi \cong \pi \perp \gamma$. Then $\dim \gamma = 5$ and γ is also a Pfister neighbor, so there exists $\rho \in GP_2F$ such that $\rho \subset \gamma$. Hence, there exist $a, b, c, d, e, f, g \in F^*$ such that $\varphi \cong a\langle\langle b, c\rangle\rangle \perp d\langle\langle e, f\rangle\rangle \perp \langle g\rangle$.

Since $PN(2)$ implies that F is linked by Proposition 6.3, we may assume that $b = e$, and after scaling (which doesn't change the property of being a Pfister neighbor), we may also assume $a = 1$, so

$$\varphi \cong \langle\langle b, c\rangle\rangle \perp d\langle\langle b, f\rangle\rangle \perp \langle g\rangle \subset \langle\langle b\rangle\rangle \otimes (\langle\langle c\rangle\rangle \perp d\langle\langle f\rangle\rangle \perp \langle g\rangle) .$$

Now $\delta \cong \langle\langle c\rangle\rangle \perp d\langle\langle f\rangle\rangle \perp \langle g\rangle$ has dimension 5 and is therefore again a Pfister neighbor, so as above there exist $h, k, l, m \in F^*$ such that $\delta \cong h\langle\langle k, l\rangle\rangle \perp \langle m\rangle$. We thus get that

$$\varphi \subset \langle\langle b\rangle\rangle \otimes \delta \cong h\langle\langle b, k, l\rangle\rangle \perp m\langle\langle b\rangle\rangle \subset h\langle\langle b, k, l, -hm\rangle\rangle \in GP_4F \ ,$$

which shows that φ is a Pfister neighbor.

The remaining statement now follows from Corollary 6.13. □

Using this, we are now able to give our proof of the Elman-Lam result.

Proof of Theorem 6.4. Let F be linked. By Proposition 6.3, this is equivalent to F having $PN(2)$. By Lemma 6.14, we have that $u(F) \leq \widetilde{u}(F) \leq 4$ or $u(F) = \widetilde{u}(F) = 8$.

All that remains to be shown is that $\widetilde{u}(F) \leq 4$ implies $\widetilde{u}(F) \neq 3$ and $u(F) = \widetilde{u}(F)$.

Now over any ED field, any t.i. form of dimension $n \geq 3$ contains a t.i. subform of dimension $n-1$. Thus, we readily conclude that if $3 \leq \widetilde{u}(F) \leq 4$ then there exists an anisotropic t.i. form φ of dimension 3. But then φ is a Pfister neighbor of some anisotropic $\pi \in P_2F$ that is t.i. and thus torsion, so we readily get $\widetilde{u}(F) \geq u(F) \geq 4$ and thus $\widetilde{u}(F) = u(F) = 4$. Finally, it is clear that if $\widetilde{u}(F) \leq 2$ then $\widetilde{u}(F) = u(F)$. $\square$

Example 6.15. (i) In [B1], Becher studies fields F that possess an anisotropic form φ such that any other anisotropic form over F is a subform of φ. It can be shown that such a form φ is then necessarily an n-fold Pfister form for some $n \in \mathbb{N}_0$ (called *supreme Pfister form*), in which case F is nonreal and $u(F) = \dim \varphi = 2^n$. It is clear that any such field will have property $PN(n-1)$. A well known example of such a field is the iterated power series field $F = \mathbb{C}((X_1))((X_2))\dots((X_n))$, where the supreme Pfister form is given by $\langle\langle X_1, \dots, X_n\rangle\rangle$.

This also shows that for any $n \geq 2$, there exist nonreal fields F with property $PN(n)$ and $u(F) = 2^{n+1}$.

(ii) To get real fields with $PN(n)$ ($n \geq 2$) and $u(F) = \widetilde{u}(F) = 2^{n+1}$, consider the real field $K = \mathbb{Q}(X_1, \cdots, X_n)$. Let $\pi = \langle\langle 2, X_1, \dots, X_n\rangle\rangle$. One readily sees that π is anisotropic and torsion (since $\langle\langle 2\rangle\rangle \cong \langle 1, -2\rangle$ is torsion). Fix an ordering $P \in X_K$. Now consider

$$\mathcal{C} = \{\text{field extensions } L \text{ of } K \text{ s.t. } P \text{ extends to } L \text{ and } \pi_L \text{ anisotropic}\}$$

Clearly, $K \in \mathcal{C}$, $\mathcal{C}$ is closed under direct limits, and if $L \in \mathcal{C}$ and L' is a field with $K \subset L' \subset L$, then $L' \in \mathcal{C}$. Then, by [B1, Theorem 6.1], there exists a field $F \in \mathcal{C}$ such that for any anisotropic form φ over F, $\dim \varphi \geq 2$, one has that $F(\varphi) \notin \mathcal{C}$. We claim that F has a unique ordering (which extends P), that F has $PN(n)$ and that $u(F) = \widetilde{u}(F) = 2^{n+1}$.

Now by construction, F is real with an ordering P' extending P. Suppose there exists $Q \in X_F$ with $Q \neq P'$. Let $a \in F$ such that $a >_{P'} 0$ and $a <_Q 0$, and consider $q \cong (2^{n+1} \times \langle 1\rangle) \perp \langle -a\rangle$. Then q is anisotropic as it is positive definite at Q, and P' (and thus P) extends to $F(q)$ as q is indefinite at P'. However, since $\dim q = 2^{n+1} + 1 > 2^{n+1} = \dim \pi$, π stays anisotropic over $F(q)$. Hence $F(q) \in \mathcal{C}$, a contradiction. Thus, $X_F = \{P'\}$.

In particular, since π_F is torsion and anisotropic, we have $u(F) \geq 2^{n+1}$. Suppose $\widetilde{u}(F) > 2^{n+1}$. Then there exists an anisotropic t.i. form τ with $\dim \tau > 2^{n+1}$. A similar reasoning as above shows that $F(\tau) \in \mathcal{C}$, again a contradiction. Hence $\widetilde{u}(F) \leq 2^{n+1}$ and we have $u(F) = \widetilde{u}(F) = 2^{n+1}$.

Now let ψ be any form of dimension $2^n + 1$ over F. If ψ is isotropic, it is a Pfister neighbor (Remark 6.6). So assume that ψ is anisotropic. Suppose first that ψ is t.i. and consider $\rho = (\pi_F \perp -\psi)_{\mathrm{an}}$. Then $2^n - 1 \leq \dim \rho$. If $\dim \rho > 2^n - 1$ then $\dim \rho \geq 2^n + 1 = \dim \psi$ and $|\operatorname{sgn}_{P'} \rho| = |\operatorname{sgn}_{P'} \psi| \leq 2^n - 1$, so in particular ρ is t.i. and thus P' extends to $F(\rho)$. Since we cannot have $F(\rho) \in \mathcal{C}$, we must therefore have that $\pi_{F(\rho)}$ is isotropic and hence hyperbolic, so ρ is similar to a subform of π_F. Thus, there exists $x \in F^*$ and a form γ, $\dim\gamma \leq 2^n - 1$ with $x\pi_F \cong \rho \perp \gamma$. Thus, in WF, we get $x\pi_F = \pi_F \perp -\psi \perp \gamma$. But $\pi_F \perp -x\pi_F \in P_{n+2}F$ is torsion, therefore isotropic since $u(F) = 2^{n+1}$ and thus hyperbolic (this actually shows that $x\pi_F \cong \pi_F$ for any $x \in F^*$). Hence, we have $\psi = \gamma$ in WF with ψ anisotropic and $\dim\psi > \dim\gamma$, a contradiction. It then follows that $\dim\rho = 2^n - 1$ and therefore $\pi_F \cong \rho \perp \psi$, showing that ψ is a Pfister neighbor of π_F.

Now suppose that ψ is definite at the unique ordering P' of F. After scaling, we may assume that ψ is positive definite. Let $\sigma = 2^{n+1} \times \langle 1\rangle \in P_{n+1}F$. If ψ is a subform of σ then it is a Pfister neighbor and we are done. So suppose that ψ is not a subform of σ and let $\eta \cong (\sigma \perp -\psi)_{\mathrm{an}}$. We then have that $\dim\eta \geq 2^n + 1$ whereas $\mathrm{sgn}_{P'}\eta = 2^n - 1$. In particular, η is t.i., and P' extends to $F(\eta)$. But $F(\eta) \notin \mathcal{C}$, so we must have that $\pi_{F(\eta)}$ is isotropic and hence hyperbolic, and as above we have that $\pi_F \cong \eta \perp \delta$ for some form δ with $\dim\delta \leq 2^n - 1$. In WF, we thus get $\sigma \perp -\pi_F = \psi \perp -\delta \in I^{n+1}F$. Now since $\dim\psi = 2^n + 1 \geq \dim\delta + 2$, we have that $\psi \perp -\delta$ is of dimension $\leq 2^{n+1}$ but not hyperbolic. By APH, we necessarily have that $\dim\delta = 2^n - 1$ and $\psi \perp -\delta \in GP_{n+1}F$, so ψ is a Pfister neighbor, showing that F has property $PN(n)$.

Let us finally remark that in this example, the proof shows that π_F is the unique anisotropic torsion $(n+1)$-fold Pfister form over F, and that there are two anisotropic (positive definite) $(n+1)$-fold Pfister forms, namely σ and $(\sigma \perp -\pi)_{\mathrm{an}}$. This also implies that $I^{n+1}F/I^{n+2}F \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. □

We finish this paper with some remarks on a possible geometric interpretation of the property $PN(n)$ which can be formulated in the language of Chow groups. We refer to [Kar], [EKM, §80].

Let φ be a (nondegenerate) quadratic form of dimension $n + 2 \geq 3$, and let $X = X_\varphi$ be the smooth projective n-dimensional quadric $\{\varphi = 0\}$ over F. We call X (an)isotropic if φ is (an)isotropic. Let $\overline{F}$ denote the algebraic closure of F and let $\overline{X} = X_{\overline{F}}$. Let l_0 be the class of a rational point in $\mathrm{CH}^n(\overline{X})$, the Chow group of 0-dimensional cycles, and let $1 \in \mathrm{CH}^0(X)$ be the class of X. A *Rost correspondence* on X is an element $\rho \in \mathrm{CH}^n(X \times X)$ which, over $\overline{F}$, is equal to $l_0 \times 1 + 1 \times l_0 \in \mathrm{CH}^n(\overline{X} \times \overline{X})$. A *Rost projector* is a Rost correspondence that is also an idempotent in the ring of correspondences on X. It is known that if a quadric has a Rost correspondence, then it has in fact also a Rost projector (see [Kar, Rem. 1.4]). The study of Rost correspondences/projectors has proven to be crucial in the motivic theory of quadrics.

It is known that if X is isotropic, then $l_0 \times 1 + 1 \times l_0$ is actually the unique Rost projector on X (see [Kar, Lem. 5.1]). For anisotropic forms, the situation is much more complicated.

One knows the following (see [Kar, Prop. 6.2, 6.4]):

Theorem 6.16. *Let φ be an anisotropic form over F of dimension ≥ 3, and let $X = X_\varphi$.*

(i) *If X possesses a Rost projector, then* $\dim\varphi = 2^n + 1$ *for some $n \geq 1$ (see Karpenko* [Kar, Prop. 6.2, 6.4]*).*
(ii) *If φ is a Pfister neighbor of dimension $2^n + 1$ then X has a unique Rost projector (considered as element in $CH^r(X \times X)$, $r = 2^n - 1$) (see Izhboldin-Vishik* [IV, Th. 1.12] *for* $\mathrm{char}(F) = 0$*, Elman-Karpenko-Merkurjev* [EKM, Cor. 80.11] *in the general case).*

In view of part (i), it is natural to ask whether or not the converse of part (ii) also holds. This is still an open problem (see also [Kar, Conj. 1.6]):

Conjecture 6.17. If an anisotropic quadric X_φ possesses a Rost correspondence, then φ is a Pfister neighbor of dimension $2^n + 1$ for some $n \geq 1$.

Of course, by Theorem 6.16(ii), to prove the conjecture, one may assume that $\dim\varphi = 2^n + 1$ for some $n \geq 1$. Since 3-dimensional forms are always Pfister neighbors, trivially the conjecture holds in that case. The conjecture is also true in the cases $n = 2, 3$ as shown by Karpenko (see [Kar, Prop. 10.8, Th. 1.7]):

Theorem 6.18. *Let φ be an anisotropic form over F of dimension 2^n+1, $n = 2, 3$, and let $X = X_\varphi$. If X_φ possesses a Rost correspondence, then φ is a Pfister neighbor.*

It is now natural to introduce the property $RP(n)$ for $n \geq 1$:

$RP(n)$: *F has the property $RP(n)$ for $n \geq 1$ if every form φ over F of dimension $2^n + 1$ has a Rost projector.*

In view of the above, we immediately get

Proposition 6.19. *Let $n \geq 1$.*

(i) *$PN(n)$ implies $RP(n)$.*
(ii) *If $n \leq 3$, then $RP(n)$ implies $PN(n)$.*
(iii) *If Conjecture 6.17 holds, then $RP(n)$ implies $PN(n)$ for all $n \in \mathbb{N}$.*

Conjecturally and in view of Theorem 1.1, we therefore get a 'algebro-geometric' criterion for the finiteness of the Hasse number:

Corollary 6.20. *If Conjecture 6.17 holds, then $\widetilde{u}(F) < \infty$ if and only if F has property $RP(n)$ for some $n \geq 2$.*

References

[AP] Arason, J.Kr.; Pfister, A.: *Beweis des Krullschen Durchschnittsatzes für den Wittring*, Invent. Math. **12** (1971), 173–176.
[B1] Becher, K.: *Supreme Pfister Forms.* Comm. Algebra **32** (2004), 217–241.
[B2] Becher, K.: *On the u-invariant of a real function field.* Math. Ann., to appear.

[E] Elman, R.: *Quadratic forms and the u-invariant*, III. Proc. of Quadratic Forms Conference (ed. G. Orzech). Queen's Papers in Pure and Applied Mathematics No. **46** (1977), 422–444.

[EKM] Elman, R.; Karpenko, N.; Merkurjev, A.: The algebraic and geometric theory of quadratic forms. Amer. Math. Soc. Coll. Publ. **56**, Providence, RI: Amer. Math. Soc. 2008.

[EL1] Elman, R.; Lam, T.Y.: *Quadratic forms over formally real fields and pythagorean fields.* Amer. J. Math. **94** (1972), 1155–1194.

[EL2] Elman, R.; Lam, T.Y.: *Quadratic forms and the u-invariant* I. Math. Z. **131** (1973), 283–304.

[EL3] Elman, R.; Lam, T.Y.: *Quadratic forms and the u-invariant* II. Invent. Math. **21** (1973), 125–137.

[ELP] Elman, R.; Lam, T.Y.; Prestel, A.: *On some Hasse principles over formally real fields.* Math. Z. **134** (1973), 291–301.

[EP] Elman, R.; Prestel, A.: *Reduced stability of the Witt ring of a field and its Pythagorean closure.* Amer. J. Math. **106** (1983), 1237–1260.

[F] Fitzgerald, R.W.: *Witt kernels of function field extensions.* Pacific J. Math. **109** (1983), 89–106.

[GV] Gesquière, N.; Van Geel, J.: *Note on universal quadratic forms.* Bull. Soc. Math. Belg. — Tijdschr. Belg. Wisk. Gen., Ser. B, **44** (1992), 193–205.

[He] Heath-Brown, D.R.: *Zeros of systems of $\mathfrak{p}$-adic quadratic forms.* Compositio Math., to appear.

[H1] Hoffmann, D.W.: *Isotropy of quadratic forms over the function field of a quadric.* Math. Z. **220** (1995), 461–476.

[H2] Hoffmann, D.W.: *A note on simple decomposition of quadratic forms over linked fields.* J. Algebra **175** (1995), 728–731.

[H3] Hoffmann, D.W.: *On Elman and Lam's filtration of the u-invariant.* J. Reine Angew. Math. **495** (1998), 175–186.

[H4] Hoffmann, D.W.: *Pythagoras numbers of fields.* J. Amer. Math. Soc. **12** (1999), 839–848.

[H5] Hoffmann, D.W.: *Isotropy of quadratic forms and field invariants.* Quadratic forms and their applications (Dublin, 1999), Contemp. Math. **272**, Providence, RI: Amer. Math. Soc. 2000, 73–102.

[H6] Hoffmann, D.W.: *Dimensions of anisotropic indefinite quadratic forms, I.* Proceedings of the Conference on Quadratic Forms and Related Topics (Baton Rouge, LA, 2001). Doc. Math. 2001, Extra Vol., 183–200.

[Hor1] Hornix, E.A.M.: *Totally indefinite quadratic forms over formally real fields.* Indag. Math. **47** (1985), 305–312.

[Hor2] Hornix, E.A.M.: *Formally real fields with prescribed invariants in the theory of quadratic forms.* Indag. Math., New Ser. **2** (1991), 65–78.

[I] Izhboldin, O.: *Fields of u-invariant* 9. Ann. of Math. **154** (2001), 529–587.

[IV] Izhboldin, O.; Vishik, A.: *Quadratic forms with absolutely maximal splitting.* Quadratic forms and their applications (Dublin, 1999), Contemp. Math. **272**, Providence, RI: Amer. Math. Soc. 2000, 103–125.

[Kar] Karpenko, N.A.: *Characterization of minimal Pfister neighbors via Rost projectors.* J. Pure Appl. Algebra **160** (2001), 195–227.

[L1] Lam, T.Y.: Orderings, valuations and quadratic forms. CBMS Regional Conference Series in Mathematics **52**, Providence, Rhode Island: American Mathematical Society 1983.

[L2] Lam, T.Y.: *Some consequences of Merkurjev's work on function fields.* Unpublished manuscript (1989).

[L3] Lam, T.Y.: Introduction to Quadratic Forms over Fields. Graduate Studies in Mathematics **67**, Providence, Rhode Island: American Mathematical Society 2005.

[M1] Merkurjev, A.S.: *On the norm residue symbol of degree 2.* Dokladi Akad. Nauk. SSSR **261** (1981) 542–547. (English translation: Soviet Math. Doklady **24** (1981) 546–551.)

[M2] Merkurjev, A.S.: *Simple algebras and quadratic forms.* Izv. Akad. Nauk. SSSR **55** (1991) 218–224. (English translation: Math. USSR Izvestiya **38** (1992) 215–221.)

[PaS] Parimala, R., Suresh, V.: *The u-invariant of the function fields of p-adic curves.* preprint, arXiv:0708.3128v1 (2007).

[P1] Prestel, A: *Quadratische Semi-Ordnungen und quadratische Formen.* Math. Z. **133** (1973), 319–342.

[P2] Prestel, A: *Remarks on the Pythagoras and Hasse number of real fields.* J. reine angew. Math. **303/304** (1978), 284–294.

[PW] Prestel, A; Ware, R.: *Almost isotropic quadratic forms.* J. London Math. Soc. **19** (1979), 241–244.

[S] Scharlau, W.: Quadratic and Hermitian Forms. Grundlehren **270**, Berlin, Heidelberg, New York, Tokyo: Springer 1985.

[T] Tignol, J.-P.: *Réduction de l'indice d'une algèbre simple centrale sur le corps des fonctions d'une quadrique.* Bull. Soc. Math. Belgique Sér. A **42** (1990) 735–745.

[V] Vishik, A.: *Fields of u-invariant* $2^r + 1$. Algebra, Arithmetic and Geometry — Manin Festschrift (ed. Yu.Tschinkel), Birkhäuser, 2009, 655–679, in press.

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