GROTHENDIECK—SERRE CONJECTURE FOR GROUPS OF TYPE F_4 WITH TRIVIAL f_3 INVARIANT

V. PETROV AND A. STAVROVA

Abstract. Assume that R is a semi-local regular ring containing an infinite perfect field. Let K be the field of fractions of R. Let H be a simple algebraic group of type F_4 over R such that H_K is the automorphism group of a 27-dimensional Jordan algebra which is a first Tits construction. If $\mathrm{char}\, K \neq 2$ this means precisely that the f_3 invariant of H_K is trivial. We prove that the kernel of the map

$$\mathrm{H}^1_{\acute{e}t}(R, H) \to \mathrm{H}^1_{\acute{e}t}(K, H)$$

induced by the inclusion of R into K is trivial.

This result is a particular case of the Grothendieck—Serre conjecture on rationally trivial torsors. It continues the recent series of papers [PaSV], [Pa], [PaPS] and complements the result of Chernousov [Ch] on the Grothendieck—Serre conjecture for groups of type F_4 with trivial g_3 invariant.

1. Introduction

In the present paper we address the Grothendieck—Serre conjecture [Se, p. 31, Remarque], [Gr, Remarque 1.11] on the rationally trivial torsors of reductive algebraic groups. This conjecture states that for any reductive group scheme G over a regular ring R, any G-torsor that is trivial over the field of fractions K of R is itself trivial; in other words, the natural map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$$

has trivial kernel. It has been settled in a variety of particular cases, and we refer to [Pa] for a detailed overview. The most recent result in the area belongs to V. Chernousov [Ch] who has proved that the Grothendieck—Serre conjecture holds for an arbitrary simple group H of type F_4 over a local regular ring R containing the field of rational numbers, given that H_K has a trivial g_3 invariant. We prove that the Grothendieck—Serre conjecture holds for another natural class of groups H of type F_4, those for which H_K has trivial f_3 invariant. In fact, since our approach is characteristic-free, we establish the following slightly more general result.

Theorem 1. *Let R be a semi-local regular ring containing an infinite perfect field. Let K be the field of fractions of R. Let J be a 27-dimensional exceptional Jordan algebra over R such that J_K is a first Tits construction. Then the map*

$$\mathrm{H}^1_{\acute{e}t}(R, \mathrm{Aut}\,(J)) \to \mathrm{H}^1_{\acute{e}t}(K, \mathrm{Aut}\,(J))$$

induced by the inclusion of R into K has trivial kernel.

Corollary. *Let R be a semi-local regular ring containing an infinite perfect field k such that $\mathrm{char}\, k \neq 2$. Let K be the field of fractions of R. Let H be a simple group scheme of type F_4 over R such that H_K has trivial f_3 invarint. Then the map*

$$\mathrm{H}^1_{\acute{e}t}(R, H) \to \mathrm{H}^1_{\acute{e}t}(K, H)$$

induced by the inclusion of R into K has trivial kernel.

V. Petrov is supported by Max-Plank-Institut für Mathematatik, RFBR 08-01-00756, RFBR 09-01-90304 and RFBR 09-01-91333; V. Petrov and A. Stavrova are supported by RFBR 09-01-00878.

2. Isotopes of Jordan algebras

In the first two sections R is an arbitrary commutative ring.

A *(unital quadratic) Jordan algebra* is a projective R-module J together with an element $1 \in J$ and an operation

$$\begin{aligned} J \times J &\to J \\ (x,\, y) &\mapsto U_x y, \end{aligned}$$

which is quadratic in x and linear in y and satisfies the following axioms:

- $U_1 = \mathrm{id}_J$;
- $\{x,\, y,\, U_x z\} = U_x\{y,\, x,\, z\}$;
- $U_{U_x y} = U_x U_y U_x$,

where $\{x,\, y,\, z\} = U_{x+z}y - U_x y - U_z y$ stands for the linearization of U. It is well-known that the split simple group scheme of type F_4 can be realized as the automorphism group scheme of the split 27-dimensional exceptional Jordan algebra J_0. This implies that any other group scheme of type F_4 is the automorphism group scheme of a twisted form of J_0.

Let v be an *invertible* element of J (that is, U_v is invertible). An *isotope* $J^{(v)}$ of J is a new Jordan algebra whose underlying module is J, while the identity and U-operator are given by the formulas

$$\begin{aligned} 1^{(v)} &= v^{-1}; \\ U_x^{(v)} &= U_x U_v. \end{aligned}$$

An *isotopy* between two Jordan algebras J and J' is an isomorphism $g\colon J \to J'^{(v)}$; it follows that $v = g(1)^{-1}$. We are particularly interested in *autotopies* of J; one can see that g is an autotopy if and only if

$$U_{g(x)} = gU_x g^{-1} U_{g(1)}$$

for all $x \in J$. In particular, transformations of the form U_x are autotopies. The group scheme of all autotopies is called the *structure group* of J and is denoted by $\mathrm{Str}(J)$. Obviously it contains $\mathbf{G}_{\mathrm{m}}$ acting on J by scalar transformations.

It is convenient to describe isotopies as isomorphisms of some algebraic structures. This was done by O. Loos who introduced the notion of a *Jordan pair*. We will not need the precise definition, see [Lo75] for details. It turns out that every Jordan algebra J defines a Jordan pair $(J,\, J)$, and the isotopies between J and J' bijectively correspond to the isomorphims of $(J,\, J)$ and $(J',\, J')$ ([Lo75, Proposition 1.8]). In particular, the structure group $\mathrm{Str}(J)$ is isomorphic to $\mathrm{Aut}\,((J,\, J))$. We use this presentation of $\mathrm{Str}(J)$ to show that, if J is a 27-dimensional exceptional Jordan algebra, $\mathrm{Str}(J)$ can be seen as a Levi subgroup of a parabolic subgroup of type P_7 (with the enumeration of roots as in [B]) in an adjoint group of type E_7. See also Garibaldi [Ga].

Lemma 1. *Let J be a 27-dimensional exceptional Jordan algebra over a commutative ring R. There exists an adjoint simple group G of type E_7 over R such that* $\mathrm{Str}(J)$ *is isomorphic to a Levi subgroup L of a maximal parabolic subgroup P of type P_7 in G.*

Proof. By [Lo78, Theorem 4.6 and Lemma 4.11] for any Jordan algebra J the group $\mathrm{Aut}\,((J,\, J))$ is isomorphic to a Levi subgroup of a parabolic subgroup P of a reductive group $\mathrm{PG}(J)$ (not necessarily connected; the definition of a parabolic subgroup extends appropriately). Moreover, $\mathrm{PG}(J) \cong \mathrm{Aut}\,(\mathrm{PG}(J)/P)$. If J is a 27-dimensional exceptional Jordan algebra, i.e., an Albert algebra, the group $\mathrm{PG}(J)$ is of type E_7 and P is a parabolic subgroup of type P_7. Let G be the corresponding adjoint group of type E_7. Then by [Dem, Thérème 1] we have $\mathrm{Aut}\,(\mathrm{PG}(J)/P) \cong \mathrm{Aut}\,(G) \cong G$. Hence $\mathrm{Aut}\,((J,\, J))$ is isomorphic to a Levi subgroup of a parabolic subgroup P of type P_7 in G. ☐

3. Cubic Jordan algebras and the first Tits construction

A *cubic map* on a projective R-module V consists of a function $N\colon V \to R$ and its *partial polarization* $\partial N\colon V \times V \to R$ such that $\partial N(x, y)$ is quadratic in x and linear in y, and N is cubic in the following sense:

- $N(tx) = t^3 N(x)$ for all $t \in R$, $x \in V$;
- $N(x + y) = N(x) + \partial N(x, y) + \partial N(y, x) + N(y)$ for all $x, y \in V$.

These data allow to extend N to $V_S = V \otimes_R S$ for any ring extension S of R.

A *cubic Jordan algebra* is a projective module J equipped with a cubic form N, quadratic map $\#\colon J \to J$ and an element $1 \in J$ such that for any extension S/R

- $(x^\#)^\# = N(x)x$ for all $x \in J_S$;
- $1^\# = 1$; $N(1) = 1$;
- $T(x^\#, y) = \delta N(x, y)$ for all $x, y \in J_S$;
- $1 \times x = T(x)1 - x$ for all $x \in J_S$,

where $\times$ is the linearization of $\#$, $T(x) = \partial N(1, x)$, $T(x, y) = T(x)T(y) - N(1, x, y)$, $N(x, y, z)$ is the linearization of ∂N.

There is a natural structure of a quadratic Jordan algebra on J given by the formula

$$U_x y = T(x, y)x - x^\# \times y.$$

Any associative algebra A of degree 3 over R (say, commutative étale cubic algebra or an Azumaya algebra of rank 9) can be naturally considered as a cubic Jordan algebra, with N being the norm, T being the trace, and $x^\#$ being the adjoint element to x.

Moreover, given an invertible scalar $\lambda \in R^\times$, one can equip the direct sum $A \oplus A \oplus A$ with the structure of a cubic Jordan algebra in the following way (which is called the *first Tits construction*):

$$\begin{aligned}
&1 = (1, 0, 0);\\
&N(a_0, a_1, a_2) = N(a_0) + \lambda N(a_1) + \lambda^{-1} N(a_2) - T(a_0 a_1 a_2);\\
&(a_0, a_1, a_2)^\# = (a_0^\# - a_1 a_2,\ \lambda^{-1} a_2^\# - a_0 a_1,\ \lambda a_1^\# - a_2 a_0).
\end{aligned}$$

Now we state a transitivity result (borrowed from [PeR, Proof of Theorem 4.8]) which is crucial in what follows.

Lemma 2. *Let E be a cubuc étale extension of R, A is the cubic Jordan algebra obtained by the first Tits construction from E, y be an invertible element of E considered as a subalgebra of A. Then y lies in the orbit of 1 under the action of subgroup of $\mathrm{Str}(A)(R)$ generated by $\mathbf{G}_{\mathrm{m}}(R)$ and elements of the form U_x, x is an invertible element of A.*

Proof. As an element of A y equals $(y, 0, 0)$. Now a direct calculation shows that

$$U_{(0,0,1)} U_{(0,y,0)} y = N(y)1.$$

□

Over a field, Jordan algebras that can be obtained by the first Tits construction can be characterized in terms of cohomological invariants. Namely, to each J one associates a 3-fold Pfister form $\pi_3(J)$, and J is of the first Tits construction if and only if $\pi_3(J)$ is hyperbolic (see [Pe, Theorem 4.10]). Another equivalent description is that J splits over a cubic extension of the base field. If the characteristic of the base field is distinct from 2, π_3 is also known as the cohomological f_3 invariant,

$$f_3 : H^1_{ét}(-, F_4) \to H^3(-, \mu_2).$$

4. Springer form

From now on J is a 27-dimensional cubic Jordan algebra over R.

Let E be a cubic étale subalgebra of J. Denote by $E^\perp$ the orthogonal complement to E in J with respect to the bilinear form T (it exists for the restriction of T to E is non-degenerate); it is a projective R-module of rank 24. It is shown in [PeR, Proposition 2.1] that the operation

$$E \times E^\perp \to E^\perp;$$
$$(a,\, x) \mapsto -a \times x$$

equips $E^\perp$ with a structure of E-module compatible with its R-module structure. Moreover, if we write

$$x^\# = q_E(x) + r_E(x),\ q_E(x) \in E^\perp,\ r_E(x) \in E,$$

then q_E is a quadratic form on $E^\perp$, which is nondegenerate as one can check over a covering of R splitting J. This form is called the *Springer form* with respect to E.

The following lemma relates the Springer form and subalgebras of J.

Lemma 3. *Let v be an element of $E^\perp$ such that $q_E(v) = 0$ and v is invertible in J. Then v is contained in a subalgebra of J obtained by the first Tits construction from E.*

Proof. It is shown in [PeR, Proof of Proposition 2.2] that the embedding

$$(a_0,\, a_1,\, a_2) \mapsto a_0 - a_1 \times v - N(v)^{-1} a_2 \times v^\#$$

defines a subalgebra desired. □

Recall that the étale algebras of degree n are classified by $\mathrm{H}^1(R,\, S_n)$, where S_n is the symmetric group in n letters. The sign map $S_n \to S_2$ induces a map

$$\mathrm{H}^1(R,\, S_n) \to \mathrm{H}^1(R,\, S_2)$$

that associates to any étale algebra E a quadratic étale algebra $\delta(E)$ called the *discriminant* of E. The norm $N_{\delta(E)}$ is a quadratic form of rank 2.

Over a field, the Springer form can be computed explicitly in terms of $\pi_3(J)$ and $\delta(E)$. We will need the following particular case:

Lemma 4. *Let J be a Jordan algebra over a field K with $\pi_3(J) = 0$. Then*

$$q_E = N_{\delta(E)_E} \perp \mathbf{h}_E \perp \mathbf{h}_E \perp \mathbf{h}_E,$$

$\mathbf{h}$ *stands for the hyperbolic form of rank* 2.

Proof. Follows from [PeR, Theorem 3.2]. □

We will also use the following standard result.

Lemma 5. *Let J be a Jordan algebra over an algebraically closed field F. Then any two cubic étale subalgebras E and E' of J are conjugate by an element of* $\mathrm{Aut}\,(J)(F)$.

Proof. Present E as $Fe_1 \oplus Fe_2 \oplus Fe_3$, where e_i are idempotents whose sum is 1; do the same with E'. By [Lo75, Theorem 17.1] there exists an element $g \in \mathrm{Str}(J)(F)$ such that $ge_i = e'_i$. But then g stabilizes 1, hence belongs to $\mathrm{Aut}\,(J)(F)$. □

5. Proof of Theorem 1

Proof of Theorem 1. Set $H = \mathrm{Aut}\,(J)$. It is a simple group of type F_4 over R. We may assume that H_K is not split, otherwise the result follows from [Pa, Theorem 1.0.1]. Let J be the Jordan algebra corredponding to H; we have to show that if J' is a twisted form of J such that $J'_K \simeq J_K$ then $J' \simeq J$. Set $L = \mathrm{Str}(J)$; then L is a Levi subgroup of a parabolic subgroup of type P_7 of an adjoint simple group scheme G of type E_7 by Lemma 1. By [SGA, Exp. XXVI Cor. 5.10 (i)] the map

$$\mathrm{H}^1_{\acute{e}t}(R,\, L) \to \mathrm{H}^1_{\acute{e}t}(K,\, G)$$

is injective. Since G is isotropic, by [Pa, Theorem 1.0.1] the map

$$\mathrm{H}^1_{\acute{e}t}(R,\, G) \to \mathrm{H}^1_{\acute{e}t}(K,\, G)$$

has trivial kernel, and so does the map

$$\mathrm{H}^1_{\acute{e}t}(R,\, L) \to \mathrm{H}^1_{\acute{e}t}(K,\, L).$$

But $(J'_K,\, J'_K) \simeq (J_K,\, J_K)$, therefore $(J',\, J') \simeq (J,\, J)$, that is J' is isomorphic to $J^{(y)}$ for some invertible $y \in J$. It remains to show that y lies in the orbit of 1 under the action of $\mathrm{Str}(J)(R)$.

Present the quotient of R by its Jacobson radical as a direct product of the residue fields $\prod k_i$. An argument in [PeR, Proof of Theorem 4.8] shows that for each i one can find an invertible element $v_i \in J_{k_i}$ such that the discriminant of the generic polynomial of $U_{v_i} y_{k_i}$ is nonzero. Lifting v_i to an element $v \in J$ and changing y to $U_v y$ we may assume that the generic polynomial $f(T) \in R[T]$ of y has the property that $R[T]/(f(T))$ is an étale extension of R. In other words, we may assume that y generates a cubic étale subalgebra E in J.

Note that E_K is a cubic field extension of K; otherwise J_K is reduced, hence split, for $\pi_3(J_K) = 0$ (see [Pe, Theorem 4.10]). Consider the form

$$q = N_{\delta(E)_E} \perp \mathbf{h}_E \perp \mathbf{h}_E \perp \mathbf{h}_E;$$

then by Lemma 4 $q_K = q_{E_K}$. By the analog of the Grothendieck—Serre conjecture for étale quadratic algebras (which follows, for example, from [EGA2, Corollaire 6.1.14]), q and q_E have the same discriminant. So q_E is a twisted form of q given by a cocycle $\xi \in \mathrm{H}^1(E,\, \mathrm{SO}(q))$. Now ξ_K is trivial, and [Pa, Theorem 1.0.1] imply that ξ is trivial itself, that is $q_E = q$. In particular, q_E is isotropic. Let us show that there is an *invertible* element v in J such that $q_E(v) = 0$.

The projective quadric over E defined by q_E is isotropic, hence has an open subscheme $U \simeq \mathbb{A}^n_E$. Denote by U' the open subscheme of $R_{E/R}(U)$ consisting of invertible elements. It suffices to show that $U'(k_i)$ is non-empty for each i, or, since the condition on R implies that k_i is infinite, that $U'(\bar{k}_i)$ is non-empty.

But $J_{\bar{k}_i}$ splits, and, in particular, it is obtained by a first Tits construction from a split Jordan algebra of 3×3 matrices over $\bar{k}_i$. The diagonal matrices in this matrix algebra constitute a cubic étale subalgebra of $J_{\bar{k}_i}$. By Lemma 5 we may assume that this étale subalgebra coincides with $E_{\bar{k}_i}$. By [PeR, Proposition 2.2] there exists an invertible element $v_i \in E^{\perp}_{\bar{k}_i}$ such that $q_{E_{\bar{k}_i}}(v_i) = 0$. Thus the scheme of invertile elements intersects the quadric over $\bar{k}_i$, hence, $U'(\bar{k}_i)$ is non-empty.

Finally, Lemma 3 and Lemma 2 show that y belongs to the orbit of 1 under the group generated by $\mathbf{G}_{\mathrm{m}}(R)$ and elements of the form U_x. So $J' \simeq J^{(y)} \simeq J$, and the proof is completed. □

References

[B] N. Bourbaki, Groupes et algèbres de Lie. Chapitres 4, 5 et 6, Masson, Paris, 1981.

[Ch] V. Chernousov, Variations on a theme of groups splitting by a quadratic extension and Grothendieck—Serre conjecture for group schemes F_4 with trivial g_3 invariant. Preprint (2009), http://www.math.uni-bielefeld.de/LAG/man/354.html

[Dem] M. Demazure, Automorphismes et déformations des variétes de Borel, *Inv. Math.* **39** (1977), 179–186.

[EGA2] A. Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : II. Étude globale élémentaire de quelques classes de morphismes, *Inst. Hautes Études Sci. Publ. Math.* **8** (1961), 5–222.

[SGA] M. Demazure, A. Grothendieck, Schémas en groupes, Lecture Notes in Math., **151–153**, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[Ga] R.S. Garibaldi, Structurable algebras and groups of type E_6 and E_7, *J. Algebra* **236** (2001), 651–691.

[Gr] A. Grothendieck, Le groupe de Brauer II, Sém. Bourbaki **297** (1965/66).

[Lo75] O. Loos, Jordan pairs, Lecture Notes in Math. **460**, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[Lo78] O. Loos, Homogeneous algebraic varieties defined by Jordan pairs, *Mh. Math.* **86** (1978), 107–129.

[Pa] I. Panin, On Grothendieck—Serre's conjecture concerning principal G-bundles over reductive group schemes:II, Preprint (2009), http://www.math.uiuc.edu/K-theory/

[PaPS] I. Panin, V. Petrov, A. Stavrova, Grothendieck—Serre conjecture for adjoint groups of types E_6 and E_7, Preprint (2009), available from http://www.arxiv.org/abs/0905.1427

[PaSV] I. Panin, A. Stavrova, N. Vavilov, On Grothendieck—Serre's conjecture concerning principal G-bundles over reductive group schemes:I, Preprint (2009), http://www.math.uiuc.edu/K-theory/

[Pe] H. Petersson, Structure theorems for Jordan algebras of degree three over fields of arbitrary characteristic, *Comm. in Algebra* **32** (2004), 1019–1049.

[PeR] H. Petersson, M. Racine, Springer form and the first Tits construction of exceptional Jordan division algebras, *Manuscripta Math.* **45** (1984), 249–272.

[PS] V. Petrov, A. Stavrova, Tits indices over semilocal rings, Preprint (2008), available from http://www.arxiv.org/abs/0807.2140

[Se] J.-P. Serre, Espaces fibrés algébriques, in *Anneaux de Chow et applications*, Séminaire Chevalley, 2-e année, Secrétariat mathématique, Paris, 1958.