HYPERBOLICITY OF UNITARY INVOLUTIONS

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Abstract. We prove the so-called *Unitary Hyperbolicity Theorem*, a result on hyperbolicity of unitary involutions. The analogous previously known results for the orthogonal and symplectic involutions are formal consequences of the unitary one. While the original proofs in the orthogonal and symplectic cases were based on the incompressibility of generalized Severi-Brauer varieties, the proof in the unitary case is based on the incompressibility of their Weil transfers.

1. Inroduction

We refer to [14] for terminology and basic facts concerning central simple algebras with involutions. We fix the following notation: F is a field, K/F a separable quadratic field extension, A a central simple K-algebra, σ an F-linear unitary involution on A.

Precisely as in the orthogonal and in the symplectic case, a right ideal I of the algebra A is *isotropic* (with respect to the involution σ), if $\sigma(I) \cdot I = 0$. The involution σ is *hyperbolic*, if there exists an isotropic ideal of reduced dimension $(\deg A)/2$. Note that the reduced dimension of a right ideal (or, more generally, of a right A-module) is defined as its dimension over K (not over F) divided by the degree $\deg A := \sqrt{\dim_K A}$ of A.

In this note we prove (in Section 4)

Theorem 1.1 (**Unitary Hyperbolicity Theorem**). *Assume that* $\operatorname{char} F \neq 2$. *If σ is not hyperbolic, then there exists a field extension F'/F such that $K' := K \otimes_F F'$ is a field, the central simple K'-algebra $A' := A \otimes_F F'$ is split, and the F'-linear unitary involution $\sigma' := \sigma_{F'}$ on A' is still not hyperbolic.*

Theorem 1.1 is the unitary analogue of the following result concerning orthogonal involutions:

Theorem 1.2 (**Orthogonal Hyperbolicity Theorem** [12, Theorem 1.1]). *Assume that* $\operatorname{char} F \neq 2$. *Let B be a central simple F-algebra with an orthogonal involution τ. If τ is not hyperbolic, then there exists a field extension F'/F such that the central simple F'-algebra $B' := B \otimes_F F'$ is split and the orthogonal involution $\tau_{F'}$ on B' is still not hyperbolic.*

In the case when the exponent of A is 2, Theorem 1.1 has been deduced from Theorem 1.2 in [18]. In our setting the exponent of A is arbitrary; our proof is a unitary adaptation of the proof of Theorem 1.2.

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In fact, as we show in Section 5, Theorem 1.2 can be deduced from (the exponent 2 case of) Theorem 1.1. Since the symplectic analogue [18, Theorem 1] of these hyperbolicity results is a consequence of Theorem 1.2, Theorem 1.1 turns out to be the principle hyperbolicity result. However, as the orthogonal and symplectic hyperbolicity results are consequences of the corresponding isotropy results [6, Theorem 1] and [7, Theorem 1], the most powerful result in this whole story would be the Unitary Isotropy Theorem (which imply all the results mentioned above). But so far we are not able to prove the Unitary Isotropy Theorem; the attempt to adopt to the unitary case the proof of the orthogonal case (which succeeds for the hyperbolicity business in this paper) fails for the isotropy business. So, we can only state

Conjecture 1.3 ("Unitary Isotropy Theorem"). *If σ is anisotropic over any finite odd degree field extension of F, then there exists a field extension F'/F such that $K' := K \otimes_F F'$ is a field, the central simple K'-algebra $A' := A \otimes_F F'$ is split, and the F'-linear unitary involution $\sigma' := \sigma_{F'}$ on A' is still anisotropic.*

Theorem 1.1 (as well as Conjecture 1.3) can easily be reduced to the case where the index of A is a power of 2. Indeed, first of all, the index of an arbitrary A becomes a power of 2 over an appropriate finite odd degree field extension L/F, for instance, over the field extension of F corresponding to a Sylow 2-subgroup of the Galois groups of the normal closure of E/F, where E is a separable finite odd degree field extension of K such that $\operatorname{ind}(A \otimes_K E)$ is a power of 2. According to [14, Corollary 6.16], if $[L : F]$ is odd, the involution σ_L is still non-hyperbolic (and $K \otimes_F L$ is a field). So, if Theorem 1.1 is proved for $A \otimes_F L$, we get it also for the original A.

Because of the above reduction, we always assume below that the index of A is a power of 2.

We will prove that Theorem 1.1 holds for F' being the function field of the Weil transfer $R_{K/F}(X)$ of the Severi-Brauer K-variety X of A. Clearly, K' is a field and A' is split for such F', so that we only need to check the non-hyperbolicity of σ'.

We start our work by writing down in Section 2 some consequences of the general motivic decompositions of [1] applied to the case of some varieties related to unitary involutions.

We start getting new results in Section 3 by establishing the unitary analogue of the results of [11] (which are about orthogonal involutions). The involution σ' is adjoint to certain (uniquely determined by σ' up to an isomorphism and a non-zero factor from F') K'/F'-hermitian form on a vector K'-space; the *Witt index* $\operatorname{ind}\sigma'$ of σ' is the Witt index of this hermitian form. The characteristic assumption $\operatorname{char} F \neq 2$ is dropped in the following result (proved in Section 3):

Theorem 1.4. *We are assuming that the (Schur) index of A is a power of 2. Let F' be the function field of the Weil transfer $R_{K/F}(X)$ of the Severi-Brauer variety X of A. Then the Witt index $\operatorname{ind}\sigma'$ of σ' is divisible by the Schur index $\operatorname{ind} A$ of A.*

2. Motivic decompositions of some isotropic varieties of unitary type

The characteristic of the base field F is arbitrary in this section.

By a *variety* we mean a separated scheme of finite type over a field. A variety is called *anisotropic*, if the degree of any its closed point is *even*.

Example 2.1. Any K-variety T, considered as an F-variety via the composition $T \to \operatorname{Spec} K \to \operatorname{Spec} F$, is anisotropic because the residue field of any point on T contains K.

Example 2.2. Let V be a finite-dimensional vector space over K with a K/F-hermitian form h. If h is anisotropic, then the F-variety of 1-dimensional totally isotropic subspaces in V is also anisotropic (that is, h remains anisotropic over any finite odd degree field extension of F). This classical fact is a consequence of the Springer theorem [3, Corollary 18.5] for quadratic forms (applied to the quadratic form $v \mapsto h(v,v) \in F$ on V considered as a vector space over F).

We write Ch for the Chow group with coefficients in $\mathbb{F}_2$ (the finite field of 2 elements). *Motives* below are the Chow motives with coefficients in $\mathbb{F}_2$, [3, §64]. We write $M(Y)$ for the motive of a smooth projective variety Y. The motive of the spectrum of the base field is denoted by $\mathbb{F}_2$. For any integer i, the motive $\mathbb{F}_2(i)$ (the ith shift of the motive $\mathbb{F}_2$) is called a Tate motive.

Note that the Krull-Schmidt principle [13, Corollary 2.2] holds for the motives of *quasi-homogeneous* varieties, [13, Definition of §2].

Let D be a central division K-algebra of a index a power of 2 (possibly of index $1 = 2^0$) with a fixed F-linear unitary involution τ. Let V' be a finite-dimensional right D-module with a hermitian (with respect to the involution τ) form. Let $\mathbb{H}$ be the hermitian hyperbolic D-plane. By definition, $\mathbb{H}$ is the right D-module $D \oplus D$ equipped with the hermitian form of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let V be the orthogonal sum of $\mathbb{H}$ and V'. Let Y be the F-variety of the totally isotropic submodules in V of reduced dimension $\deg D$ (that is, of D-dimension 1). The variety Y is a closed subvariety of the Weil transfer with respect to the field extension K/F of the K-variety of the submodules in V of the reduced dimension $\deg D$. Similarly, we define Y' as the F-variety of the totally isotropic submodules in V' of reduced dimension $\deg D$.

Lemma 2.3. *The motive of Y is the sum of two Tate motives, a shift of the motive of Y', and shifts of the motives of some anisotropic F-varieties.*

Proof. According to [9, Theorem 15.8], the variety Y is a relative cellular space (as defined in [3, §66]) over the (non-connected) variety Z of triples (I, J, N), where I and J are right ideals in D and where N is a submodule in V' such that the submodule $I \oplus J \oplus N \subset V$ is a point of Y (that is, $\tau(I) \cdot J = 0$, N is totally isotropic, and the reduced dimension of the D-module $I \oplus J \oplus N$ is equal to $\deg D$). Therefore, by [3, Corollary 66.4], the motive of Y is the sum of shifts of the motives of the components of Z.

The rational points $(0, D, 0)$ and $(D, 0, 0)$ of Z are components of Z which produce the two promised Tate summands. Since D is a 2-primary division algebra, any other odd degree rational point of Z lies on the component of the triples $(0, 0, N)$. This component is naturally identified with Y'. □

Remark 2.4. Since we are trying to avoid unnecessary precision, we do not determine the shifting numbers of the summands in the decomposition of Lemma 2.3. A more detailed analysis of the situation shows that the shifting numbers of the two Tate motives are 0 and $\dim Y$; the shifting number of the motive of Y' is $(\dim Y - \dim Y')/2$.

3. Characterization of Witt index of hermitian forms

In this section we still allow to the base field F to be of arbitrary characteristic (including 2). We recall that the index of the K-algebra A is a 2-power.

For any integer $i \in \{0, 1, \ldots, (\deg A)/2\}$ we write Y_i for the F-variety of the isotropic reduced dimension i ideals in A. This is a closed subvariety of the Weil transfer $R_{K/F}X_i$ of the generalized Severi-Brauer K-variety X_i of the reduced dimension i right ideals in A. The K-variety $(Y_i)_K$ is identified with the variety of the flags $I \subset J$ of right ideals I, J in A of reduced dimensions $i, \deg A - i$ (see [14, Proposition 2.15] and [9, Lemma 15.5]).

Now we assume that the algebra A is split and we study the variety Y_1. More precisely, we assume that $A = \operatorname{End}_K(V)$ for some finite-dimensional vector K-space V (of dimension $\deg A$). In this case, the involution σ is adjoint to some K/F-hermitian form h on V. By the Morita equivalence, the variety Y_1 is the variety of 1-dimensional totally isotropic with respect to h subspaces in V. So, the variety Y_1 is the unitary analogue of a projective quadric (which we have in the orthogonal case). And it turns out that the Chow group of Y_1 contains the information about the precise value of the Witt index of h exactly as in the orthogonal case the Chow group of the projective quadric contains the information about the precise value of the Witt index of the quadratic form [3, Corollary 72.6].

The K-variety $(Y_1)_K$ is a hypersurface in $\mathbb{P}(V) \times \mathbb{P}^{\#}(V)$, where $\mathbb{P}(V)$ is the projective space of V, i.e., the variety of 1-dimensional subspaces in V, and $\mathbb{P}^{\#}(V)$ is the dual projective space of V, i.e., the variety of 1-codimensional subspaces in V, which can be identified with $\mathbb{P}(V^{\#})$, the projective space of the vector space $V^{\#}$ dual to V. More precisely, $(Y_1)_K \subset \mathbb{P}(V) \times \mathbb{P}^{\#}(V)$ is the *flag* hypersurface, the hypersurface of the pairs of subspaces (U, W) of the vector space V satisfying the condition $U \subset W$.

We write C for the image of the composition

$$\operatorname{Ch}(Y_1) \to \operatorname{Ch}((Y_1)_K) \to \operatorname{Ch}(\mathbb{P}(V) \times \mathbb{P}^{\#}(V)).$$

For any variety $\mathbb{P}$ isomorphic to a projective space and any integer $i \in \{0, 1, \ldots, \dim \mathbb{P}\}$, we write l_i for the class in $\operatorname{Ch}_i(\mathbb{P})$ of an i-dimensional linear subspace in $\mathbb{P}$.

The following statement is the unitary analogue of [3, Corollary 72.6]:

Lemma 3.1. *Assume that A is split. For any $i \in \{0, 1, \ldots, (\deg A)/2 - 1\}$ the following statements are equivalent:*

(1) *the Witt index of σ is $> i$;*
(2) *$l_i \times l_i \in C$;*
(3) *the motive of Y_1 contains the Tate summand $\mathbb{F}_2(2i)$.*

Proof. (1) $\Rightarrow$ (2). If the Witt index of σ is $> i$, there exists an isotropic ideal $I \subset A$ of reduced dimension $i + 1$. The variety of the reduced dimension 1 right ideals contained in I is then a closed subvariety of Y_1 whose class in C is equal to $l_i \times l_i$.

(2) $\Rightarrow$ (3). If $l_i \times l_i \in C$, we write α for an element of $\operatorname{Ch}_{2i}(Y_1)$ whose class in C is equal to $l_i \times l_i$. The variety Y_1 is a hypersurface in $R_{K/F}(\mathbb{P}(V))$. For $j := \dim \mathbb{P}(V) - i$, we consider the cycle class $R_{K/F}(l_j) \in \operatorname{Ch}_{2j}(R_{K/F}(\mathbb{P}(V)))$ ($R_{K/F}$ in the expression $R_{K/F}(l_j)$ here is the Weil transfer on the algebraic cycle classes, see [10, §3]) and let $\beta \in \operatorname{Ch}_{2j-1}(Y_1)$

be the pull-back of $R_{K/F}(l_j)$ with respect to the imbedding $in : Y_1 \hookrightarrow \mathbb{P}(V)$. By the projection formula, the product $\alpha \cdot \beta$ is a 0-cycle class on Y_1 of degree 1. Indeed,

$$\deg(\alpha \cdot \beta) = \deg in_*(\alpha \cdot \beta) = \deg\big(\, in_*(\alpha) \cdot R_{K/F}(l_j)\big) = \\ \deg\big(\, in_*(\alpha)_K \cdot R_{K/F}(l_j)_K\big) = \deg\big((l_i \times l_i) \cdot (l_j \times l_j)\big) = \deg(l_0 \times l_0) = 1.$$

Therefore, α and β, considered as morphisms of motives

$$\alpha : \mathbb{F}_2(2i) \to M(Y_1) \quad \text{and} \quad \beta : M(Y_1) \to \mathbb{F}_2(2i),$$

identify $\mathbb{F}_2(2i)$ with a summand of $M(Y_1)$.

Note that the same α and β are morphisms $\beta : \mathbb{F}_2(\dim Y_1 - 2i) \to M(Y_1)$ and $\alpha : M(Y_1) \to \mathbb{F}_2(\dim Y_1 - 2i)$ showing that the Tate motive $\mathbb{F}_2(\dim Y_1 - 2i)$ is also a summand of $M(Y_1)$ (this summand is dual to the previous one, cf. [3, §65]).

(3) $\Rightarrow$ (1). Let j be the Witt index of σ. Note that $j \le (\deg A)/2$. According to Lemma 2.3 (applied j times), there exists a motivic decomposition of Y_1 containing $2j$ Tate summands and such that each of the remaining summands is a shift of the motive of an anisotropic variety (take Example 2.2 into account). According to [12, Lemma 6.3], the complete motivic decomposition of an anisotropic variety does not contain Tate summands. It follows that the complete motivic decomposition of $M(Y_1)$ contains precisely $2j$ Tate summands. But during the previous step, we have already constructed $2j$ Tate summands of $M(Y_1)$ (with pairwisely different shifting numbers):

$$\mathbb{F}_2, \mathbb{F}_2(2), \dots, \mathbb{F}_2(2j-2) \;\text{ and }\; \mathbb{F}_2(d), \mathbb{F}_2(d-2), \dots, \mathbb{F}_2(d-2j+2),$$

where $d = \dim Y_1 = 2\deg A - 3$. Note that the shifting number of every Tate summand in the first group is $\le \deg A - 2$ while the shifting number of every Tate summand in the second group is $\ge \deg A - 1$. Therefore, $M(Y_1)$ contains $\mathbb{F}_2(2i)$ with some $i < (\deg A)/2$ only if $j > i$. □

Proof of Theorem 1.4. We may assume that $\operatorname{ind}\sigma' > 0$. Let r be the biggest multiple of $\operatorname{ind} A$ with $r < \operatorname{ind}\sigma'$. We will show that $\operatorname{ind}\sigma' = r + \operatorname{ind} A$.

Let T be the Severi-Brauer variety of a central division K-algebra Brauer-equivalent to A. We write R instead of $R_{K/F}$ and we write Y instead of Y_1 for short. By Lemma 3.1, there exists $\alpha' \in \mathrm{Ch}_{2r}(Y_{F(R(T))})$ with the image $l_r \times l_r$ in $\mathrm{Ch}_{2r}(Y_{K(R(T))})$. Let

$$\alpha \in \mathrm{Ch}_{2r+2\dim T}\big(R(T) \times Y\big)$$

be a preimage of α' with respect to the epimorphism

$$\mathrm{Ch}_{2r+2\dim T}\big(R(T) \times Y\big) \twoheadrightarrow \mathrm{Ch}_{2r}(Y_{F(R(T))})$$

given by the pull-back with respect to the morphism $Y_{F(R(T))} \to R(T) \times Y$ induced by the generic point of the (integral) variety $R(T)$.

According to [11, Lemma 3.1] and since r is a multiple of $\operatorname{ind} A$ lying on the interval $[0,\ \dim X]$, there exists an element

$$\beta' \in \mathrm{Ch}_{\dim X - r}(X \times T) = \mathrm{Ch}^{r+\dim T}(X \times T)$$

with the image under the composition of the push-forward $\mathrm{Ch}(X \times T) \to \mathrm{Ch}(X)$ with respect to the projection $X \times T \to X$ followed by the change of field homomorphism

$\mathrm{Ch}(X) \to \mathrm{Ch}(X_{K(X)})$ being equal to $l_{\dim X - r} \in \mathrm{Ch}(X_{K(X)})$. Let

$$\beta \in \mathrm{Ch}^{2r+2\dim T}\big(Y \times R(T)\big)$$

be the pull-back of $R(\beta') \in \mathrm{Ch}\big(R(X) \times R(T)\big)$ to $\mathrm{Ch}\big(Y \times R(T)\big)$ with respect to the closed imbedding $Y \times R(T) \hookrightarrow R(X) \times R(T)$ induced by the closed imbedding $Y \hookrightarrow R(X)$.

The composition of correspondences

$$\beta \circ \alpha \in \mathrm{Ch}_{2\dim T}\big(R(T) \times R(T)\big)$$

is a degree 0 correspondence on $R(T)$. The multiplicity of $\beta \circ \alpha$ is 1. Indeed, the multiplicity of $\beta \circ \alpha$ coincides with the degree of the 0-cycle class $(\beta \circ \alpha)_*(l_0 \times l_0)$, where $l_0 \times l_0 \in \mathrm{Ch}_0(R(T)_{K(T)})$ is the 0-cycle class of degree 1 on $R(T)_{K(T)}$. Since $\alpha_*(l_0 \times l_0) = l_r \times l_r \in \mathrm{Ch}_{2r}(Y_{K(T)})$ and $\beta_*(l_r \times l_r) = l_0 \times l_0$ by the construction of α and β, the 0-cycle class $(\beta \circ \alpha)_*(l_0 \times l_0)$ is equal to $l_0 \times l_0$ and has the degree 1.

We have constructed morphisms of motives

$$\alpha : M\big(R(T)\big) \to M(Y)(-2r) \quad \text{and} \quad \beta : M(Y)(-2r) \to M\big(R(T)\big)$$

such that the multiplicity of the composition $\beta \circ \alpha$ is 1. It follows by [8, Lemma 2.14] that a non-zero summand of the motive of $R(T)$ is isomorphic to a summand of $M(Y)(-2r)$. On the other hand, according to [5, Theorem 1.2], the motive of the variety $R(T)$ is indecomposable. Therefore, the whole motive of $R(T)$ is isomorphic to a summand of $M(Y)(-2r)$. Since over the function field of $R(T)$, the Tate motive $\mathbb{F}_2\big(\dim R(T)\big)$ is a summand of $M\big(R(T)\big)_{F(R(T))}$, the motive of $Y_{F(R(T))}$ contains a Tate summand with the shifting number $2r + \dim R(T) = 2(r + \dim T) = 2(r + \operatorname{ind} A - 1)$. It follows by Lemma 3.1 that $\operatorname{ind} \sigma' > r + \operatorname{ind} A - 1$, that is, $\operatorname{ind} \sigma' \geq r + \operatorname{ind} A$. Now, the definition of r ensures that $\operatorname{ind} \sigma' = r + \operatorname{ind} A$. □

4. Proof of Theorem 1.1

We follow the lines of [12, §7] doing the necessary modifications. First of all we refix our setting. Our base field F is now an arbitrary field of characteristic $\neq 2$, K/F is a quadratic field extension, A is a non-split central simple K-algebra whose index is a power of 2, σ is an F-linear unitary involution on A. We assume that σ becomes hyperbolic over the function field of the Weil restriction $R(X) = R_{K/F}(X)$ of the Severi-Brauer variety X of A. Equivalently, σ becomes hyperbolic over the function field of the Weil restriction $R(T) = R_{K/F}(T)$ of the Severi-Brauer variety T of a central division K-algebra D Brauer-equivalent to A.

According to Theorem 1.4, $\operatorname{coind} A := (\deg A)/(\operatorname{ind} A)$ is $2n$ for some integer $n \geq 1$. We assume that Theorem 1.1 is already proved for all algebras (over all fields) of index $< \operatorname{ind} A$ as well as for all algebras of index $= \operatorname{ind} A$ and coindex $< 2n$.[1]

Let us fix an arbitrary F-linear unitary involution τ on D and an isomorphism of F-algebras $A \simeq \mathrm{End}_D(D^{2n})$. Let h be a hermitian (with respect to τ) form on the right D-module D^{2n} such that σ is adjoint to h. Then $h_{F(R(T))}$ is hyperbolic. Since the anisotropic kernel of h also becomes hyperbolic over $F(R(T))$, our induction hypotheses

[1] We are inducting on the index and on the coindex of A. The index induction base is the trivial case of the index 1. The coindex induction base is the case of the coindex 1 proved by Theorem 1.4 (see also [4, Example 4.5] for a simpler proof).

ensure that h is anisotropic. Moreover, h_L is hyperbolic for any field extension L/F such that $K \otimes_F L$ is field and h_L is isotropic (we avoid the case where $K \otimes_F L$ is not a field because we did not give a definition of hyperbolic in this case). It follows by [14, Corollary 6.16] that h_L is anisotropic for any finite odd degree field extension L/F.

Let Y be the variety of totally isotropic submodules in D^{2n} of reduced dimension $n \deg D$ (that is, of D-dimension n). The variety Y is a closed subvariety of the Weil transfer with respect to the field extension K/F of the K-variety of the submodules in D^{2n} of the reduced dimension $n \deg D$. Note that by [15], the $F(Y)$-algebra $D_{F(Y)}$ is still a division algebra. Also note that $Y(K) \neq \emptyset$ because $\operatorname{coind} A$ is even.

We are going to apply to the motive of Y the following slightly modified version of [12, Proposition 4.6]. Here and below, for any irreducible smooth projective variety S we write $U(S)$ for the *upper motive* of S defined as the indecomposable upper motivic summand of S, see [8].

Proposition 4.1. *Let S be a geometrically split variety satisfying the nilpotence principle and let M be a motive. Assume that there exists a field extension E/F such that*

(1) *S_E is irreducible and the motive $U(S_E)$ is* lower [8, Definition 2.10]*;*
(2) *the field extension $E(S)/F(S)$ is purely transcendental;*
(3) *the upper motive $U(S_E)$ of the variety S_E is a summand of M_E.*

Then the upper motive $U(S)$ of the variety S is a summand of M.

Proof. The only difference with [12, Proposition 4.6] (besides of the particular current choice $\mathbb{F}_2$ for the coefficient ring in place of an arbitrary finite connected ring) is that now we do not require S to be geometrically irreducible and we only require instead that S is irreducible over E. This change does not affect the original proof. □

We will apply Proposition 4.1 in the case where the variety S is *quasi-homogeneous* in the sense of [13, §2]. Such S is geometrically split and satisfies the nilpotence principle by [13, Theorem 2.1]. In Example 4.2 below, we describe a sufficient amount of varieties for which Condition (1) of Proposition 4.1 is satisfied.

Example 4.2. Let i be a 2-power of the interval $[1,\ \deg D]$. Let S be the generalized Severi-Brauer K-variety of the reduced dimension i right ideals in D. According to [5], the upper motive $U\big(R_{K/F}(S)\big)$ is lower (this fact is equivalent to the 2-incompressibility of the variety $R_{K/F}(S)$ established in [5, Theorem 1.1], c.f. [4, Lemma 2.7 or Theorem 5.1]). The K-motive $U(S)$ is also lower by [8, Theorem 4.1]. Since the F-motive $\operatorname{cor}_{K/F} U(S)$ is indecomposable by [13, Proposition 3.1] (see [13, §3] for the definition of the motivic functor $\operatorname{cor}_{K/F}$), we have $U(\operatorname{cor}_{K/F} S) \simeq \operatorname{cor}_{K/F} U(S)$ and it follows that the motive $U(\operatorname{cor}_{K/F} S)$ is also lower. (The F-variety $\operatorname{cor}_{K/F} S$ is simply the K-variety S considered as an F-variety via the composition $S \to \operatorname{Spec} K \to \operatorname{Spec} F$.)

Corollary 4.3. *The motive $U(Y)$ has the following property: $U(Y)_{F(Y)}$ is a sum of Tate motives.*

Proof. According to [13], each summand of the complete motivic decomposition of $Y_{F(Y)}$ is a shift of the upper motive of $R(S)_{F(Y)}$ or of $\operatorname{cor}(S)_{F(Y)}$, where S/K is a generalized Severi-Brauer variety of Example 4.2, $R = R_{K/F}$, and $\operatorname{cor} = \operatorname{cor}_{K/F}$.

Let us neglect the shifts. Proposition 4.1 with $E = F(Y)$ allows to split off from the motive of Y (over F) the summand $U(R(S))$ for each motivic summand $U(R(S)_{F(Y)})$ of $Y_{F(Y)}$ with $S \neq \operatorname{Spec} K$, and it allows to split off the summand $U(\operatorname{cor}(S))$ for each motivic summand $U(\operatorname{cor}(S)_{F(Y)})$ of $Y_{F(Y)}$ with any S. Indeed, Condition (1) of Proposition 4.1 is satisfied by Example 4.2 (and because D_E is still a division algebra); Condition (2) is satisfied because the variety Y_K is rational and also for any $S \neq \operatorname{Spec} K$ the variety $Y_{R(S)}$ is rational.

The remaining part of $M(Y)$ is an upper motivic summand of Y which, if considered over $F(Y)$, becomes a sum of Tate motives. Therefore, the upper motive of Y has the same property. $\square$

It will turn out below, that in fact the upper motive of Y is *binary*: $U(Y)_{F(Y)}$ is the sum of *two* Tate motives.

Let us consider a minimal right D-submodule $V \subset D^{2n}$ such that V becomes isotropic over a finite odd degree field extension of $F(Y)$. We set $v = \dim_D V$. Clearly, $v \geq 2$ (because $D_{F(Y)}$ is a division algebra). Let Y' be the variety of totally isotropic submodules in V of reduced dimension $\deg D$ (that is, of D-dimension 1). Writing $\tilde{F}$ for an odd degree field extension of $F(Y)$ with isotropic $V_{\tilde{F}}$, we have $Y'(\tilde{F}) \neq \emptyset$ (because $D_{\tilde{F}}$ is a division algebra). Therefore there exists a correspondence of odd multiplicity (that is, of multiplicity $1 \in \mathbb{F}_2$) $\alpha \in \operatorname{Ch}_{\dim Y}(Y \times Y')$.

The variety Y' is projective homogeneous (in particular, irreducible) of dimension

$$\dim Y' = (\deg D)^2(2v - 3)$$

which is not a power of 2 minus 1 (because even and positive). Moreover, the variety Y' is anisotropic (because the hermitian form h is anisotropic and remains anisotropic over any finite odd degree field extension of the base field). Therefore, Lemma 4.4 below contradicts [12, Corollary 5.14] thus proving Theorem 1.1.[2]

Lemma 4.4. *There is a Rost projector (*[12, Definition 5.1]*) on Y'.*

Proof. As explained above, there exists a correspondence of odd multiplicity (that is, of multiplicity $1 \in \mathbb{F}_2$) $\alpha \in \operatorname{Ch}_{\dim Y}(Y \times Y')$. On the other hand, since $h_{F(Y')}$ is isotropic, $h_{F(Y')}$ is hyperbolic and therefore there exist a rational map $Y' \dashrightarrow Y$ and a multiplicity 1 correspondence $\beta \in \operatorname{Ch}_{\dim Y'}(Y' \times Y)$ (e.g., the class of the closure of the graph of the rational map). It follows that the upper motives of the varieties Y and Y' are isomorphic. In particular, $U(Y')_{\tilde{F}}$, where $\tilde{F}/F(Y)$ is a finite odd degree field extension with isotropic $V_{\tilde{F}}$, is a sum of Tate motives.

On the other hand, $(h|_V)_{\tilde{F}}$ is an orthogonal sum of a hyperbolic $D_{\tilde{F}}$-plane and a hermitian form h' such that h'_L is anisotropic for any finite odd degree field extension $L/\tilde{F}$ of the field $\tilde{F}$. Indeed, otherwise – if h'_L is isotropic for some such L, the module V_L contains a totally isotropic submodule W of D-dimension 2; any D-hyperplane $V' \subset V$, considered over L, meets W non-trivially; it follows that V'_L is isotropic and this contradicts the

[2] We recall that [12, Corollary 5.14] is due to M. Rost [17]; its proof makes use of Steenrod operations on Ch which are available only over fields of characteristic $\neq 2$. This explains the characteristic assumption of Theorem 1.1.

minimality of V. (This is a very standard argument in the theory of quadratic forms over field which we applied now to a hermitian form over a division algebra.)

It follows by Lemma 2.3 that the complete motivic decomposition of $Y'_{\tilde{F}}$ has one copy of $\mathbb{F}_2$, one copy of $\mathbb{F}_2(\dim Y')$, and no other Tate summands. The anisotropy of the variety Y' implies now that

$$U(Y')_{\tilde{F}} \simeq \mathbb{F}_2 \oplus \mathbb{F}_2(\dim Y'),$$

that is, that the projector giving $U(Y')$ is a Rost projector on Y'. □

5. Relation with the orthogonal case

The aim of this Section is to explain the relationship between Theorem 1.1 and Theorem 1.2. It turns out that Theorem 1.2 is "contained" in Theorem 1.1 in the following sense: Theorem 1.2 is equivalent to Theorem 1.1 with $\exp A = 2$.

It has been already shown in [18] that Theorem 1.2 implies Theorem 1.1 for algebras A of exponent 2. The inverse implication is based on the following construction (similar to the symplectic case construction of [18]).

Let F be a field of characteristic $\neq 2$, B a central simple F-algebra with an orthogonal involution τ, x a variable, $\tilde{F}/F(x)$ the quadratic extension $\tilde{F} = F(x)(\sqrt{x})$ (the "generic quadratic extension over F"). Let $\tilde{B} := B \otimes_F \tilde{F}$ and let $\tilde{\tau}$ be the $F(x)$-linear unitary involution on $\tilde{B}$ obtained as the tensor product of τ by the non-trivial automorphism of $\tilde{F}/F(x)$.

Lemma 5.1 (cf. [18, Proposition 1])**.** *If τ is anisotropic, then $\tilde{\tau}$ is also anisotropic. If for some field extension $L/F(x)$ such that $\tilde{L} := L \otimes_{F(x)} \tilde{F}$ is a field, the involution τ_L is hyperbolic, then the involution $\tilde{\tau}_L$ is also hyperbolic.*

Proof. If $\tilde{\tau}$ is isotropic, there exists a non-zero element $a \in \tilde{B}$ with $\tilde{\tau}(a) \cdot a = 0$ which is a polynomial in $t := \sqrt{x}$, $a = a_n t^n + \cdots + a_0$ for some $a_n, \ldots, a_0 \in B$ such that $a_n \neq 0$. The coefficient of t^{2n} in the polynomial $\tilde{\tau}(a) \cdot a$ is $\pm\tau(a_n) \cdot a_n$. Therefore $\tau(a_n) \cdot a_n = 0$ and τ is isotropic.

If for some field extension $L/F(x)$ such that $\tilde{L} := L \otimes_{F(x)} \tilde{F}$ is a field, the involution τ_L is hyperbolic, the algebra B_L contains an isotropic ideal I of the reduced dimension $(\deg B)/2$. The tensor product $I \otimes_L \tilde{L}$ is then an isotropic ideal of the algebra $\tilde{B}_L = B_L \otimes_L \tilde{L}$ of the same reduced dimension $(\deg B)/2 = (\deg \tilde{B})/2$. Hence $\tilde{\tau}_L$ is also hyperbolic. □

Now Theorem 1.1 with $A = \tilde{B}$ implies Theorem 1.2 as follows. We may assume that τ is anisotropic. Then, by Lemma 5.1, the unitary involution $\tilde{\tau}$ is also anisotropic. In particular, $\tilde{\tau}$ is not hyperbolic and it follows by Theorem 1.1 that there exists a field extension $L/F(x)$ such that $L \otimes_{F(x)} \tilde{F}$ is a field, the algebra $\tilde{B} \otimes_{F(x)} L$ is split, and $\tilde{\tau}_L$ is not hyperbolic. By Lemma 5.1, the involution τ_L is also not hyperbolic. Since the algebra B_L splits over the quadratic extension $\tilde{L}/L$, $\operatorname{ind} B_L \leq 2$. We finish by the orthogonal hyperbolicity theorem for index 2 algebras proved in [2] and independently in [16].

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