METABOLIC INVOLUTIONS

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Abstract. In this paper we study the conditions under which an involution becomes metabolic over a quadratic field extension. We characterise those involutions that become metabolic over a given separable quadratic extension. We further give an example of an anisotropic orthogonal involution that becomes isotropic over a separable quadratic extension.

1. Introduction

The study of the conditions under which a symmetric bilinear form becomes metabolic over a field extension is an important question in quadratic form theory. Over fields of characteristic different from 2, the concept of metabolicity of a bilinear form coincides with that of hyperbolicity of a quadratic form. Over fields of characteristic 2, however, the relationship between bilinear forms and quadratic forms is more complex, and metabolicity is a weaker condition than hyperbolicity. Hence these concepts must be studied separately.

It is well known that a symmetric or skew-symmetric bilinear form can be associated with an involution, and that many of the concepts from quadratic form theory have a natural extension to the theory of central simple algebras with involution on a split algebra. The definition of a metabolic involution was introduced in [3, Appendix A.1]. It is the natural extension of the definition of a metabolic space from the theory of symmetric bilinear forms to central simple algebras with involution. The main motivation for the introduction of this concept was to properly study hyperbolic involutions. The authors of [3] are primarily interested in the study of symplectic involutions, in order to study quadratic pairs, and for symplectic involutions, the concepts of hyperbolicity and metabolicity coincide. Apart from this work, no further study has been made on metabolic involutions.

As we shall see, a complete study of metabolicity will be concerned primarily with orthogonal involutions over fields of characteristic 2. As can be seen in [8, Chapter 1], the theory of involutions over fields of characteristic 2 is similar to the theory over fields of characteristic different from 2, but has some crucial differences. Studying involutions over fields of characteristic 2 is not only of interest in itself, but also due to their intimate connection to quadratic pairs. Not only must one study symplectic involutions to understand quadratic pairs, but more generally, much as bilinear forms naturally act on quadratic forms, algebras with involution, of both orthogonal and symplectic type, also act on quadratic pairs (see [8, Proposition 5.18]). This action is important in defining a generalisation of quadratic Pfister

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forms to quadratic pairs, and the study of metabolic involutions should be helpful in the study of this special type of quadratic pair.

2. Bilinear forms

In the following sections we will investigate quadratic extensions over which certain involutions defined over the base field become metabolic. Before defining metabolicity for involutions we recall some results from the theory of bilinear forms.

Let F be a field of arbitrary characteristic. A *bilinear space* is a pair (V, b) where V is a F–vector space and b is a F-bilinear form on V. We will sometimes refer to a bilinear space simply as a *form*. We say that a bilinear space (V, b) is *symmetric* if $b(x, y) = b(y, x)$ for all $x, y \in V$. We call a bilinear space (V, b) *alternating* if $b(x, x) = 0$ for all $x \in V$. If (V, b) is an alternating form then we have that $b(x, y) = -b(y, x)$ for all $x, y \in V$, that is, (V, b) is *skew-symmetric*. In particular every alternating form over a field of characteristic 2 is symmetric.

Let $\simeq$ denote *isometry* between bilinear spaces. The *orthogonal sum* of the symmetric or alternating bilinear spaces (V, b_1) and (W, b_2) is written $(V, b_1) \perp (W, b_2)$. A bilinear space (V, b) is said to be *isotropic* if it represents 0 non–trivially, that is $0 = b(x, x)$ for some $x \in V \setminus \{0\}$, and *anisotropic* otherwise.

Let $\lambda = \pm 1$. We put $\mathbb{H}_\lambda = (F^2, h)$ where

$$h : F^2 \times F^2 \to F \qquad (x, y) \mapsto x^t \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} y$$

and call this the λ-*hyperbolic plane* over F. For a fixed λ, we call a bilinear space (V, b) that is isometric to an orthogonal sum of λ-hyperbolic planes *hyperbolic.* Over fields of characteristic 2 where $-1 = 1$, we just speak of the hyperbolic plane.

Proposition 2.1. *Let (V, b) be an alternating bilinear space. Then* $\dim V = 2n$ *for some n and $(V, b) \simeq \perp_{i=1}^n \mathbb{H}_{-1}$, that is, (V, b) is hyperbolic.*

Proof. See [4, Proposition 1.8]. □

For $a_1, \ldots, a_n \in F^\times$ we denote by $\langle a_1, \ldots, a_n\rangle$ the symmetric bilinear space (F^n, b) where

$$b : F^n \times F^n \to F, \qquad (x, y) \mapsto \sum_{i=1}^{n} x_i a_i y_i.$$

We call such a form a *diagonal form.* A symmetric bilinear space that is isometric to a diagonal form is called *diagonalisable.*

Proposition 2.2. *Let (V, b) be a symmetric bilinear space. Then (V, b) is diagonalisable except in the case where* $\operatorname{char}(F) = 2$ *and (V, b) is hyperbolic.*

Proof. See [4, Proposition 1.17]. □

Example 2.3. Assume $\operatorname{char}(F) = 2$ and $a \in F^\times$. Then $\langle a\rangle \perp \mathbb{H} \simeq \langle a, a, a\rangle$.

Given a bilinear space (V, b) we call a subspace $W \subset V$ *totally isotropic* (with respect to b) if $b|_W = 0$. We call (V, b) *metabolic* if it has a totally isotropic subspace W with $W^\perp = W$. Note that an alternating form is always metabolic.

Proposition 2.4. *For a 2-dimensional symmetric bilinear space (V, b), the following are equivalent:*

(1) *(V, b) is isotropic.*

(2) (V, b) *is metabolic.*
(3) *Either* $(V, b) \simeq \mathbb{H}$ *or* $(V, b) \simeq \langle a, -a \rangle$ *for some* $a \in F^\times$.

Moreover, if $\mathrm{char}(F) \neq 2$*, then* $\mathbb{H} \simeq \langle a, -a \rangle$ *for all* $a \in F^\times$.

Proof. See [4, Example 1.22] and [4, Lemma 1.23] for the equivalence of (1) – (3), and [4, Corollary 1.25] for the last statement. □

We now fix (V, b) to be a symmetric bilinear space over a field F for the rest of this section. Note that (V, b) can be decomposed as $(V, b) \simeq (W_1, b_{\mathrm{an}}) \perp (W_2, b_{\mathrm{meta}})$ with (W_1, b_{an}) anisotropic and (W_2, b_{meta}) metabolic. In this decomposition (W_1, b_{an}) is uniquely determined up to isometry (see [4, Theorem 1.27]), whereas (W_2, b_{meta}) is not in general, as follows easily from (2.3).

Let K/F be a field extension. Then we write $(V, b)_K = (V \otimes_F K, b_K)$ where b_K is the extension of b to $V \otimes_F K$. The metabolicity behavior of symmetric bilinear forms over separable quadratic algebraic extensions is particularly simple in characteristic 2 and was discovered in [6, Satz 10.2.1].

Proposition 2.5. *Let* $\mathrm{char}(F) = 2$ *and let* K/F *be a separable field extension. If* (V, b) *is anisotropic then* $(V, b)_K$ *is anisotropic.*

For odd degree extensions, this is a simple corollary of Springer's theorem (see [4, Corollary 18.5]). By basic Galois Theory, this leaves only the case of a quadratic separable extension to consider. We provide a proof based on [4, Corollary 34.15] for this case, that is K/F a separable quadratic extension, for convenience.

Proof. We may write $K = F(\delta)$ with $\delta \in K \backslash F$ such that $\delta^2 + \delta + a = 0$ for some $a \in F^\times$. Suppose $v, w \in V$ are such that $b_K(v + \delta w, v + \delta w) = 0$. Expanding gives

$$0 = b(v, v) + ab(w, w) + \delta b(w, w).$$

Since $\delta \notin F$, we have $b(w, w) = 0 = b(v, v)$. Therefore if b is anisotropic, then so is b_K. □

For more details on symmetric bilinear forms we refer to [4, Chapter 1].

3. Involutions and Hermitian forms

In this section we recall the basic definitions and results we will use on central simple algebras with involution and hermitian forms. We refer to [10] for a general reference on central simple algebras.

Throughout, let F be a field and A a finite-dimensional F–algebra. If A is simple and E is the centre of A, then A can be viewed as an E–algebra and by Wedderburn's theorem, $A \simeq \mathrm{End}_D(V)$ for an F–division algebra D with centre E and a right D–vector space V. In this case $\dim_E(A)$ is a square, and the positive root of this integer is called the *degree of* A and is denoted $\deg(A)$. The degree of D is called the *index of* A and is denoted $\mathrm{ind}(A)$. We call any A with $\mathrm{ind}(A) = 1$ *split*. For any field extension K/F we will use the notation $A_K = A \otimes_F K$. We call a field extension K/F a *splitting field of* A if A_K is split. If $E = F$, then we call the F–algebra A *central simple*.

An F*–involution* on A is an F–linear map $\sigma : A \to A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$ and $\sigma^2 = \mathrm{id}_A$. An F*–algebra with involution* is a pair (A, σ) of a finite-dimensional F–algebra A and an F–involution σ of A such that, with E being the centre of A, one has $F = \{x \in E \mid \sigma(x) = x\}$, and such that either A is simple or A is a product of two simple F–algebras that are mapped to each other by σ. In this situation, there are two possibilities: either $E = F$, so that A is a central simple F–algebra, or E/F is a quadratic étale extension with σ restricting to the nontrivial F–automorphism of E. To distinguish these two situations, we speak of involutions of the first or second kind; more precisely, we say that the F–algebra with involution (A, σ) is of the *first kind* if $E = F$ and of the *second kind* otherwise. Involutions of the second kind are also known as *unitary involutions*, and we refer to [8, Section 2.B] for more details on unitary involutions. For any field extension K/F we will use the notations $\sigma_K = \sigma \otimes \mathrm{id}_K$ and $(A, \sigma)_K = (A_K, \sigma_K)$.

Let (A, σ) be an F–algebra with involution of the first kind. Then it is well known (see [8, Proposition 2.1]) that in the case where the algebra A is split, that is $A \cong \mathrm{End}_F(V)$ for some F-vector space V, each F-involution on A is adjoint to a non-singular symmetric or alternating bilinear space on V. An F–algebra with involution of the first kind is said to be *symplectic* if it becomes adjoint to an alternating bilinear space over any splitting field, and *orthogonal* otherwise.

For an F-algebra with involution (A, σ) we define the *set of alternating elements* to be

$$\mathrm{Alt}(A, \sigma) = \{a - \sigma(a) \mid a \in A\}.$$

Proposition 3.1. *Assume* $\mathrm{char}(F) = 2$. *Then* (A, σ) *is symplectic or unitary if and only if* $1 \in \mathrm{Alt}(A, \sigma)$.

Proof. See [8, Proposition 2.6] for the case of involutions of the first kind. It only remains to show that $1 \in \mathrm{Alt}(A, \sigma)$ if (A, σ) is unitary. Assume we are in this case, and let E be the centre of A. Then E is a quadratic étale extension of F and σ restricted to E is the the nontrivial F–automorphism of E/F, which we denote ι.

If E is a quadratic separable extension of F then $E = F(\alpha)$ for some $\alpha \in E$ such that $\alpha^2 + \alpha \in F$, and $\iota(\alpha) = \alpha + 1$. Hence $\iota(\alpha) + \alpha = 1 \in \mathrm{Alt}(A, \sigma)$. Otherwise, $E \simeq F \times F$ and $\iota(1, 0) + (1, 0) = (0, 1) + (1, 0) = (1, 1) \in \mathrm{Alt}(A, \sigma)$. □

An F–*quaternion algebra* is a central simple F-algebra of degree 2. Every quaternion algebra Q has a unique symplectic involution, called the *canonical involution* (see [8, Proposition 2.21]). The description of quaternion algebras in terms of a basis distinguished by cases depending on whether the characteristic of F is 2 or not is given in [11, Section 8.11]. We recall it for fields of characteristic 2.

Assume $\mathrm{char}(F) = 2$. Given any $\alpha \in F$ and $\beta \in F^\times$, there exists an F–quaternion algebra with an F–basis $(1, i, j, k)$ subject to the relations that

$$i^2 + i = \alpha,\ j^2 = \beta \ \text{ and } \ k = ij = ji + j;$$

we denote this F–quaternion algebra by $[\alpha, \beta)_F$. If $\mathrm{char}(F) = 2$, by [11, Section 8.11], every F–quaternion algebra is isomorphic to $[\alpha, \beta)_F$ for some $\alpha \in F$ and $\beta \in F^\times$. Note that $[\alpha, \beta)_F$ is split if $\alpha = u^2 + u$ for some $u \in F$. In particular, any quaternion division algebra splits over a quadratic separable extension. When $\mathrm{char}(F) = 2$ and $(1, i, j, k)$ is an F–basis of Q with relations as above, the canonical involution is given by

$$x_0 + x_1 i + x_2 j + x_3 k \longmapsto x_0 + x_1(i + 1) + x_2 j + x_3 k$$

for $x_0, x_1, x_2, x_3 \in F$.

Throughout the rest of this section, let (D, θ) be an F–division algebra with involution and let E be the centre of D. Further, fix $\lambda \in E$ such that $\lambda\theta(\lambda) = 1$.

A λ–*hermitian form* over (D,θ) is a pair (V,h) where V is a finite-dimensional right D-vector space and h is a non-degenerate bi-additive map $h : V \times V \to D$ such that

$$h(x, yd) = h(x, y)d \quad \text{and} \quad h(y, x) = \lambda\theta(h(x, y))$$

holds for all $x, y \in V$ and $d \in D$. Let $\simeq$ denote *isometry* between hermitian forms. We denote the *orthogonal sum* of two hermitian forms (V,h) and (W,h') by $(V,h)\perp(W,h')$. Note that if (D,θ) is of the first kind one must have that $\lambda = \pm 1$.

Let V be a finite dimensional right D–vector space and let $V^* = \mathrm{End}_D(V,D)$, the dual of V. We define an F–bilinear map

$$h_\lambda : (V^* \oplus V) \times (V^* \oplus V) \to D$$

by

$$h_\lambda(\varphi + x, \psi + y) = \varphi(y) + \lambda\theta(\psi(x)) \qquad \text{for } \varphi, \psi \in V^* \text{ and } x, y \in V.$$

Then $\mathbb{H}_\lambda(V) = (V^* \oplus V, h_\lambda)$ is a regular λ–hermitian form over (D,θ). We call a λ-hermitian form over (D,θ) *hyperbolic* if it is isometric to $\mathbb{H}_\lambda(V)$ for some right D–vector space V.

Let $S \subset V$. We define the *orthogonal complement* $S^\perp$ of S with respect to a fixed hermitian form h as

$$S^\perp = \{x \in V \mid h(x, s) = 0 \text{ for all } s \in S\}.$$

A hermitian space (V,h) is called *metabolic* if there exists a subspace $S \subset V$ such that $S = S^\perp$.

Let K/F be a field extension. Then we write $(V,h)_K = (V \otimes_F K, h_K)$ where $h_K = h \otimes \mathrm{id}_K$.

As with bilinear spaces, we can decompose any hermitian space into an orthogonal sum of an anisotropic hermitian space and a metabolic hermitian space (see [7, Proposition 6.1.1]). The first, called the *anisotropic part* of (V,h) is uniquely determined up to isometry by [7, Proposition 6.1.4] and denoted $(V,h)_{\mathrm{an}}$.

For $a_1, \ldots, a_n \in D$ such that $a_i = \lambda\theta(a_i)$, for $i = 1 \ldots, n$ and $\lambda = \pm 1$, we denote by $\langle a_1, \ldots, a_n\rangle_{(\mathrm{D},\theta,\lambda)}$ the λ-hermitian space (V,h) where

$$h : V \times V \to D, \qquad (x, y) \mapsto \sum_{i=1}^{n} \theta(x_i)a_i y_i.$$

We call such a form a *diagonal form*. We call a hermitian form that is isometric to a diagonal form *diagonalisable*. We have the following result.

Proposition 3.2. *Let (V,h) be a λ-hermitian space over (D,θ). Then (V,h) is diagonalisable, except when $(D,\theta) = (F, \mathrm{id_F})$ and either $\mathrm{char}(F) \neq 2$ and (V,h) is a skew-symmetric bilinear space, or $\mathrm{char}(F) = 2$ and (V,h) is a hyperbolic symmetric bilinear space.*

Proof. See [7, Proposition 6.2.4] □

We call a hermitian space (V,h) over (D,θ) *alternating* if $h(x,x) \in \mathrm{Alt}(D,\theta)$ for all $x \in V$.

Proposition 3.3. *Let (V,h) be an alternating hermitian form over (D,θ). If $(D,\theta) = (F, \mathrm{id}_F)$ then (V,h) is a hyperbolic form. Otherwise, $(V,h) \simeq \langle a_1, \ldots, a_n\rangle_{(\mathrm{D},\theta,\lambda)}$ for some $a_1, \ldots, a_n \in \mathrm{Alt}(D,\theta)$.*

Proof. Let (V,h) be an alternating form. If $(D,\theta)=(F,\mathrm{Id})$ then (V,h) is a hyperbolic bilinear form by (2.1). Otherwise (V,h) is diagonalisable by (3.2). Let $(V,h)\simeq\langle a_1,\dots,a_n\rangle_{(\mathrm{D},\theta,\lambda)}$ for some $a_1,\dots,a_n\in D^\times$.

Let $(e_1,\dots,e_n)$ be the standard basis of $V=D^n$. Then, if (V,h) is alternating, $a_i=h(e_i,e_i)\in\mathrm{Alt}(D,\theta)$ for all $i\in\{1,\dots,n\}$. On the other hand, if $a_i\in\mathrm{Alt}(D,\theta)$, then clearly (V,h) is alternating. $\square$

There is a well known correspondence between non-degenerate λ-hermitian forms on V and involutions on A.

Proposition 3.4. *Let (D,θ) be an division F-algebra with involution, V a right D–vector space and let $A=\mathrm{End}_D(V)$. For every non-degenerate λ-hermitian form (V,h), there is a unique F–algebra with involution (A,σ) such that $\sigma(a)=\theta(a)$ for all $a\in E$ and*

$$h(f(x),y)=h(x,\sigma(f)(y))\ \textit{for all}\ x,y\in V\ \textit{and}\ f\in A.$$

Proof. See, for example, [8, Theorem 4.1]. $\square$

In this situation, we call the involution σ on $\mathrm{End}_D(V)$ the *adjoint involution* to the hermitian space (V,h), and we denote it by $\mathrm{Ad}(V,h)$. This correspondence commutes with extension of the base field. That is, for a field extension K/F, we have that $\mathrm{Ad}((V,h)_K)\cong(\mathrm{Ad}(V,h))_K$.

Proposition 3.5. *Assume* $\mathrm{char}(F)=2$. *Let (V,h) be a λ–hermitian space over some F–division algebra with involution of the first kind. Then (V,h) is alternating if and only if $\lambda=1$ and $\mathrm{Ad}(V,h)$ is symplectic if (D,θ) is of the first kind.*

Proof. See [8, Theorem 4.2] if (D,θ) is of the first kind. For unitary involutions, the result follows from [8, Proposition 2.17]. $\square$

Let (A,σ) be an F–algebra with involution. We call an algebra with involution an *anisotropic part* of (A,σ), denoted $(A,\sigma)_{\mathrm{an}}$, if $(A,\sigma)_{\mathrm{an}}\cong\mathrm{Ad}((V,h)_{\mathrm{an}})$ for some hermitian space (V,h) such that $\mathrm{Ad}(V,h)\cong(A,\sigma)$.

Proposition 3.6. *Let (A,σ) be an F–algebra with involution. Then the F–algebra with involution $(A,\sigma)_{\mathrm{an}}$ is determined up to isomorphism by (A,σ).*

Proof. Let (V,h) be an hermitian space over some F–division algebra with involution (D,θ) such that $\mathrm{Ad}(V,h)\cong(A,\sigma)$. Let (V_1,h_1) and (V_2,h_2) be anisotropic hermitian spaces over (D,θ) that are adjoint to different anisotropic parts of (A,σ), say (B,τ) and (C,γ) respectively.

We have that $(B,\tau)\cong\mathrm{Ad}((W,h')_{\mathrm{an}})$ for some hermitian form (W,h') over some F–division algebra with involution (D',θ') such that $(A,\sigma)\cong\mathrm{Ad}(W,h')$. Clearly $(W,h')\perp(W,-h')_{\mathrm{an}}$ is metabolic. From this, it follows that some orthogonal sum of (A,σ) and (B,τ) (in the sense of [3, Definition 1.1]) is metabolic. This implies that $(V,h)\perp\lambda_1(V_1,h_1)$ is metabolic for some $\lambda_1\in F^\times$. Similarly $(V,h)\perp\lambda_2(V_2,h_2)$ is metabolic for some $\lambda_2\in F^\times$.

Hence both $\lambda_1(V_1,h_1)$ and $\lambda_2(V_2,h_2)$ represent inverses of the class of (V,h) in the Witt group of hermitian forms on (D,θ) (see, for example, [8, Chapter I, Section 6]). Since $\lambda_1(V_1,h_1)$ and $\lambda_2(V_2,h_2)$ are anisotropic, this implies that $\lambda_1(V_1,h_1)\simeq\lambda_2(V_2,h_2)$, and hence $(B,\tau)\cong(C,\gamma)$. $\square$

Note that, while the anisotropic part of a symplectic involution is always symplectic, the anisotropic part of an orthogonal involution may also be symplectic, as we shall see later in (5.11).

4. Hyperbolic and metabolic involutions

The concepts of isotropy, hyperbolicity and metabolicity all have analogues in the theory of algebras with involution. In this section we recall these concepts and expand on the basic results on metabolic involutions given in [3, Appendix A.1]. In particular we will establish a connection between metabolic involutions and metabolic hermitian forms. This connection was clearly known to the authors of [3], but no proof yet appears in the literature.

Recall that an F-involution (A, σ) is said to be *isotropic* if there exists $0 \neq a \in A$ such that $\sigma(a)a = 0$, and *anisotropic* otherwise. The F-involution (A, σ) is called *hyperbolic* if there exists an idempotent $e \in A$ such that $\sigma(e) = 1 - e$, and we refer to such an e as a *hyperbolic idempotent*.

We now collect some definitions and basic results from [3, Appendix A.1].

Proposition 4.1. *Let A be a central simple E–algebra of even degree and let $e, e' \in A$ be two idempotents. Any two of the following conditions imply the third one:*
(1) $e'e = 0$,
(2) $(1 - e)(1 - e') = 0$,
(3) $\dim_E eA + \dim_E e'A = \dim_E A$.
Moreover, any two of these conditions hold (and hence all three hold) if and only if $eA = (1 - e')A$.

Proof. See [3, Lemma A.1]. □

Corollary 4.2. *Let A be a central simple E-algebra and let $e, e' \in A$ be two idempotents. If $\dim_E eA = \dim_E e'A = \frac{1}{2}\dim_E A$, then $e'e = 0$ if and only if $(1 - e)(1 - e') = 0$.*

Proof. If $\dim_E eA = \dim_E e'A = \frac{1}{2}\dim_E A$ then Condition (3) in (4.1) is satisfied by e and e', so by (4.1) $e'e = 0$ if and only if $(1 - e)(1 - e') = 0$. □

Corollary 4.3. *Let A be a central simple E–algebra and let $e \in A$ be an idempotent such that $\sigma(e)e = 0$. Then $\dim_E eA = \frac{1}{2}\dim_E A$ if and only if $(1 - e)(1 - \sigma(e)) = 0$.*

Proof. By [8, Proposition 1.12] and the remarks preceding it, we have

$$\dim_E \sigma(e)A = \dim_E \sigma(Ae) = \dim_E Ae = \dim_E eA.$$

Hence, if $\dim_E eA = \frac{1}{2}\dim_E A$, then (4.1) applies with $e' = \sigma(e)$, yielding $(1 - e)(1 - \sigma(e)) = 0$. Conversely, if $(1 - e)(1 - \sigma(e)) = 0$, then $\dim_E A = \dim_E eA + \dim_E \sigma(e)A = 2\dim_E eA$. □

Let (A, σ) be an F-algebra with involution. An idempotent $e \in A$ is called *metabolic* if $\sigma(e)e = 0$ and $\dim_E eA = \frac{1}{2}\dim_E A$. Note that, by (4.3), we may substitute the condition that $\dim_E eA = \frac{1}{2}\dim_E A$ for the condition that $(1-e)(1-\sigma(e)) = 0$ in this definition. An F-algebra with involution (A, σ) is called *metabolic* (with respect to σ) if A contains an metabolic idempotent with respect to σ.

For every right ideal I, we define its *orthogonal* $I^\perp$ (with respect to σ) as the right annihilator of $\sigma(I)$, that is

$$I^\perp = \{x \in A \mid \sigma(I)x = 0\},$$

which is also a right ideal in A. By [8, Section 6.A] an F-algebra with involution (A,σ) is isotropic if and only if A contains a non-zero ideal I such that $I \subset I^{\perp}$. The following is a corresponding characterisation of metabolic involutions.

Proposition 4.4. *Let (A,σ) be an F-algebra with involution. (A,σ) is metabolic if and only if there exists a right ideal $I \subset A$ such that $I^{\perp} = I$.*

Proof. Let $I \subset A$ be a right ideal with $I^{\perp} = I$. By [8, Proposition 6.2], $\dim_E I = \frac{1}{2}\dim_E A$. By [8, Corollary 1.13] there exists an idempotent $e \in A$ such that $I = eA$. Hence $\sigma(e)e = 0$ and $\dim_E eA = \frac{1}{2}\dim_E A$.

Assume now that there exists an idempotent $e \in A$ such that $\sigma(e)e = 0$ and $\dim_E eA = \frac{1}{2}\dim_E A$. Let $I = eA$. Then $I \subset I^{\perp}$, and by [8, Proposition 6.2] we have $\dim_E I^{\perp} = \dim_E A - \dim_E eA = \dim_E eA$. Therefore $I = I^{\perp}$. □

Proposition 4.5. *Let (D,θ) be an F–division algebra with E the centre of D, and $\lambda \in E$ such that $\lambda\theta(\lambda) = 1$. Let (V,h) be a λ-hermitian space on (D,θ). Then (V,h) is hyperbolic if and only if $Ad(V,h)$ is hyperbolic.*

Proof. See [8, Proposition 6.7]. □

Proposition 4.6. *Let* $\mathrm{char}(F) = 2$ *and let (A,σ) be an F-algebra. Then (A,σ) becomes hyperbolic over some field extension if and only if (A,σ) is symplectic or unitary.*

Proof. First we assume that $(A\sigma)$ is of the first kind. Since the type of (A,σ) does not change under field extensions, it will be sufficient to prove the case where F is algebraically closed.

Assume that F is algebraically closed. Then (A,σ) is the adjoint involution to some bilinear space (V,b) over F. By (4.5), (A,σ) is hyperbolic if and only if (V,b) is hyperbolic. By (2.1), (V,b) is hyperbolic if and only if it is alternating. Hence the result.

It remains to show that (A,σ) becomes hyperbolic over some field extension if (A,σ) is unitary. If the centre of A is $E \simeq F \times F$, then for the idempotent $(1,0) \in E$ we have $\sigma(1,0) = (0,1) = (1,1) + (1,0)$, and hence (A,σ) is hyperbolic. Otherwise, the centre of E is a field, and by [8, Section 2.B], the centre of A_E is isomorphic to $E \times E$, and hence $(A,\sigma)_E$ is hyperbolic. □

Corollary 4.7. *Let* $\mathrm{char}(F) = 2$ *and (A,σ) be an F-algebra with involution of the first kind. Let K be a splitting field for A. Then $(A,\sigma)_K$ is hyperbolic if and only if (A,σ) is symplectic.*

We show the characterisation of metabolicity analogous to (4.5).

Theorem 4.8. *Let (D,θ) be an F–division algebra with E the centre of D, and $\lambda \in E$ such that $\lambda\theta(\lambda) = 1$. Let (V,h) be a λ-hermitian space on (D,θ) and $(A,\sigma) = Ad(V,h)$. Then (V,h) is metabolic if and only if $Ad(V,h)$ is metabolic.*

Proof. Assume that (V,h) is metabolic, that is, there exists a subspace $S \subset V$ such that $S = S^{\perp}$. Let $I = \mathrm{Hom}_D(V,S)$. Then since $S = S^{\perp}$ we have $I = I^{\perp}$ by [8, Proposition 6.2]. Hence, by (4.4), we have that $\mathrm{Ad}(V,h)$ is metabolic.

Assume now that $\mathrm{Ad}(V,h)$ is metabolic and let $e \in \mathrm{End}_D(V)$ be a metabolic idempotent with respect to the involution of the pair $\mathrm{Ad}(V,h)$. Let $S = \mathrm{Im}(e) \subset V$, so that $eA = \mathrm{Hom}_D(V,S)$ and $\dim_F S = \frac{1}{2}\dim_E V$. Then for all $s \in S$ we have

$h(s,s) = h(e(s), e(s)) = h(\sigma(e)e(s), s) = h(0, s) = 0$. Hence $S = S^\perp$, so (V,h) is metabolic. $\square$

Proposition 4.9. *Any hyperbolic involution is metabolic.*

Proof. Any idempotent $e \in A$ satisfying $\sigma(e) = 1 - e$ also clearly satisfies $\sigma(e)e = 0$ and $(1-e)(1-\sigma(e)) = 0$. $\square$

Proposition 4.10. *Let* $\mathrm{char}(F) \neq 2$. *An involution on a central simple F-algebra is metabolic if and only if it is hyperbolic.*

Proof. By (4.9) any hyperbolic involution is metabolic. Now assume that $e \in A$ is a metabolic idempotent for the involution σ on A. Take $e' = e - \frac{1}{2}e\sigma(e)$. Since $\sigma(e)e = 0$ we have that $e'^2 = e'$. Further we have

$$1 - e' - \sigma(e') = 1 - e - \sigma(e) + e\sigma(e) = (1-e)(1-\sigma(e)) = 0.$$

Hence e' is a hyperbolic idempotent for (A,σ). $\square$

Proposition 4.11. *Let (A,σ) be an F-algebra with symplectic or unitary involution. Then (A,σ) is metabolic if and only if it is hyperbolic.*

Proof. See [3, Lemma A.3]. $\square$

For a right ideal $I \subset A$ we denote the left annihilator ideal

$$I^0 = \{a \in A \,|\, ax = 0 \text{ for all } x \in I\}.$$

Lemma 4.12. *Let (A,σ) be an F-algebra with involution and $e \in A$ be a metabolic idempotent. Then $A\sigma(e) = A(1-e) = (eA)^0$ and $A = Ae \oplus A\sigma(e)$.*

Proof. We have that $A(1-e) \subset (eA)^0$, as $e = e^2$, and $A\sigma(e) \subset (eA)^0$, as $\sigma(e)e = 0$. Furthermore, since e is a metabolic idempotent, we also have $\dim_E eA = \frac{1}{2}\dim_E A$, and hence $\dim_E A\sigma(e) = \dim_E \sigma(e)A = \dim_E eA = \frac{1}{2}\dim_E A$. Similarly, since $A = A(1-e) \oplus Ae$ we have $\dim_E(1-e)A = \frac{1}{2}\dim_E A$.

Finally, $\dim_E(eA)^0 = \frac{1}{2}\dim_E A$ as $\dim_E(eA)^0 + \dim_E eA = \dim_E A$, by [8, Proposition 1.14]. Hence $A(1-e) = A\sigma(e) = (eA)^0$. The last statement is now clear, as $A = A(1-e) \oplus Ae$. $\square$

5. Involutions and separable quadratic extensions

In this section we will characterise those F-algebras with involution (A,σ) that become metabolic over K, where K is a given separable quadratic extension of F. We will then apply this characterisation to hermitian forms.

Theorem 5.1. *Let K/F be a separable quadratic extension with non-trivial F-automorphism ι. Let (A,σ) be a central simple F-algebra with symplectic or unitary involution. Then $(A,\sigma)_K$ is hyperbolic if and only if it is metabolic, if and only if there is an F-embedding $(K,\iota) \hookrightarrow (A,\sigma)$.*

Proof. See [3, Theorem 1.15] for the statement on hyperbolicity, and then apply (4.11) to obtain the full statement. $\square$

Theorem 5.2. *Assume that* $\mathrm{char}(F) \neq 2$. *Let K/F be a separable quadratic extension with non-trivial F-automorphism ι. Let (A,σ) be a central simple F-algebra with orthogonal involution. Then $(A,\sigma)_K$ is hyperbolic if and only if it is metabolic, if and only if there exists an embedding $(K,\iota) \hookrightarrow (A,\sigma)$ or $(A,\sigma) \cong \mathrm{Ad}(V,b)$, where $(V,b) \simeq \langle 1, -d\rangle \otimes (W,\tau) \perp \mathbb{H}$, and (W,τ) is some symmetric bilinear space over F.*

Proof. See [2, Theorem 3.3] and the subsequent remark for the statement on hyperbolicity, then apply (4.10) for the full statment. □

In the following results, we extend [3, Lemma A.9], which was restricted to symplectic and unitary involutions, so that it holds for involutions of arbitrary type.

Lemma 5.3. *Let K/F be a quadratic extension (either separable or inseparable). Assume (A,σ) is anisotropic and $(A,\sigma)_K$ is metabolic and let $e \in A_K$ be a metabolic idempotent with respect to σ_K.*

Then for all $x \in A_K$, there is a unique element $c \in A$ such that

$$\sigma_K(e)(x - c \otimes 1) = 0,$$

and the map $\epsilon : K \to A$ defined by

$$\sigma_K(e)(1 \otimes k - \epsilon(k) \otimes 1) = 0$$

for $k \in K$ is an injective F-algebra homomorphism.

Proof. We identify the image of $A \hookrightarrow A_K$, $a \mapsto a \otimes 1$ with A in the following. Assume that $(A,\sigma)_K$ is metabolic and let $e \in A_K$ be a metabolic idempotent with respect to σ_K. Then we have $\dim_K eA_K = \frac{1}{2}\dim_K A_K$ and hence $\dim_F eA_K = \dim_F A = \frac{1}{2}\dim_F A_K$, since K/F is a quadratic extension. We also have $\sigma_K(x)x = 0$ for all $x \in eA_K$, hence,

$$A \cap eA_K = 0,$$

since (A,σ) is anisotropic. Hence

$$A_K = A \oplus eA_K.$$

Therefore, for $x \in A_K$ there is a unique $c \in A$ such that $x - (c \otimes 1) \in eA_K$, that is

$$\sigma_K(e)(x - c \otimes 1) = 0.$$

We may then define a map $\epsilon : K \to A$ as follows: for $k \in K$, $\epsilon(k) \in A$ is the unique element such that

$$\sigma_K(e)(1 \otimes k - \epsilon(k) \otimes 1) = 0.$$

Obviously $\epsilon : K \to A$ is injective and F-linear. For $k, k' \in K$ we have

$$1 \otimes k'k - \epsilon(k)\epsilon(k') \otimes 1 = (1 \otimes k' - \epsilon(k') \otimes 1)(1 \otimes k) + (1 \otimes k - \epsilon(k) \otimes 1)(\epsilon(k') \otimes 1),$$

so

$$\sigma_K(e)(1 \otimes k'k - \epsilon(k)\epsilon(k') \otimes 1) = 0$$

and therefore

$$\epsilon(kk') = \epsilon(k)\epsilon(k').$$

Hence ϵ is an F-algebra homomorphism $K \hookrightarrow A$. □

Proposition 5.4. *Let K/F be a separable quadratic extension with non-trivial F-automorphism ι. Let (A,σ) be a central simple F-algebra with anisotropic involution. If $(A,\sigma)_K$ is metabolic, then there exists an embedding $(K,\iota) \hookrightarrow (A,\sigma)$.*

Proof. Let e be a metabolic idempotent of $(A,\sigma)_K$, and let ϵ be the embedding $K \hookrightarrow A$ associated with e given in the proof of (5.3). That is, for $k \in K$, $\epsilon(k) \in A$ is the unique element such that

$$\sigma_K(e)(1 \otimes k - \epsilon(k) \otimes 1) = 0.$$

We need only show that $\epsilon \circ \iota = \sigma \circ \epsilon$.

Choose now $k \in K \setminus F$ and put

$$e' = (1 \otimes k - \epsilon(k) \otimes 1)(1 \otimes (k - \iota(k))^{-1}) \in A_K.$$

Note that e' is independent of the choice of k. Moreover it is an idempotent as it is the image under $\epsilon \otimes \mathrm{Id}_K$ of the separability idempotent of K (see [8, p285]). It satisfies

$$(\mathrm{Id}_A \otimes \iota)(e') = 1 - e'.$$

Since $\dim_K(\mathrm{Id}_A \otimes \iota)(e')K = \dim_K e'K$ and $A_K = e'A_K \oplus (1-e')A_K$, we have $\dim_K e'A_K = \frac{1}{2}\dim_K A_K$, and therefore $\dim_K e'A_K = \dim_K eA_K$. By definition of $\epsilon(k)$ we have $\sigma_K(e)e' = 0$, therefore by (4.1) we have

$$(1 - e')(1 - \sigma_K(e)) = 0.$$

Applying σ_K, we obtain

$$(1 - e)(1 - \sigma_K(e')) = 0,$$

and hence $A(1-e)(1-\sigma_K(e')) = 0$. By (4.12) we have $A(1-e) = A\sigma_K(e)$, hence $A\sigma_K(e)(1-\sigma_K(e')) = 0$, and hence

$$\sigma_K(e)(1 - \sigma_K(e')) = 0.$$

As $\sigma_K(1 \otimes k) = 1 \otimes k$ and $\sigma_K(1 \otimes (k - \iota(k))^{-1})) = 1 \otimes (k - \iota(k))^{-1}$, expanding this gives

$$\sigma_K(e)(-1 \otimes \iota(k) + \sigma_K(\epsilon(k)) \otimes 1)(1 \otimes (k - \iota(k))^{-1})) = 0,$$

or equivalently

$$\sigma_K(e)(1 \otimes \iota(k) - \sigma_K(\epsilon(k)) \otimes 1) = 0,$$

that is, $\epsilon(\iota(k)) = \sigma_K(\epsilon(k)) = \sigma(\epsilon(k))$ for all $k \in K$. □

Note that the result of (5.4) is known in the case of an involution of the first kind over a field of characteristic different from 2 (see [2, Lemma 3.2]), but the proof presented here uses different methods. One can also find a proof of this statement restricted to symplectic involutions and involutions of the second kind in [3, Theorem 1.15].

Theorem 5.5. *For (A,σ) an anisotropic F–algebra with involution and K/F a separable quadratic extension, the following conditions are equivalent:*

(1) *$(A,\sigma)_K$ is hyperbolic,*
(2) *$(A,\sigma)_K$ is metabolic,*
(3) *there exists an embedding $(K,\iota) \hookrightarrow (A,\sigma)$.*

If $\mathrm{char}(F) = 2$ *and (A,σ) is of the first kind, these conditions hold only if (A,σ) is symplectic.*

Proof. That (1) implies (2) is clear.

That (2) implies (3) is the result of (5.4).

For (3) implies (1), we note first that in the case of $\mathrm{char}(F) \neq 2$, the implication is found in (5.2).

It only remains to show (3) implies (1) in the case $\mathrm{char}(F) = 2$. Assume this is the case and let $\epsilon : (K, \iota) \hookrightarrow (A, \sigma)$ be the embedding. For $k \in K \backslash F$, take

$$e' = (1 \otimes k - \epsilon(k) \otimes 1)(1 \otimes (k - \iota(k))^{-1}) \in A_K.$$

This is an idempotent, as in the proof of (5.4). In particular we see that $\sigma_K(e') = (\mathrm{Id}_A \otimes \iota)(e') = 1 - e'$, and hence (A, σ) is hyperbolic.

Moreover, this shows that $1 \in \mathrm{Alt}(A, \sigma)_K$. Hence if $\mathrm{char}(F) = 2$ and (A, σ) is of the first kind, it is symplectic by (3.1). $\square$

Corollary 5.6. *Assume that* $\mathrm{char}(F) = 2$ *and let* (A, σ) *be an* F*-algebra with anisotropic orthogonal involution. Then* $(A, \sigma)_K$ *is not metabolic for any separable quadratic field extension* K/F*.*

Theorem 5.7. *Let* $\mathrm{char}(F) = 2$ *and let* K/F *be a separable quadratic extension with non-trivial* F*-automorphism* ι*. Let* (A, σ) *be an* F*-algebra with involution.*

Then $(A, \sigma)_K$ *is metabolic if and only if* (A, σ) *is metabolic or there exists an embedding* $(K, \iota) \hookrightarrow (A, \sigma)_{\mathrm{an}}$*. If* (A, σ) *is of the first kind, then in the latter case,* $(A, \sigma)_{\mathrm{an}}$ *is symplectic.*

Proof. Let (V, h) be an hermitian form over some E–division algebra with involution (D, θ) such that D is Brauer equivalent to A and $(A, \sigma) = \mathrm{Ad}(V, h)$. Then (V, h) decomposes into its anisotropic part $(V, h)_{\mathrm{an}}$ and some metabolic part.

That $(A, \sigma)_K$ is metabolic if (A, σ) is metabolic is clear.

If there exists an embedding $(K, \iota) \hookrightarrow (A, \sigma)_{\mathrm{an}} = \mathrm{Ad}(V, h)_{\mathrm{an}}$, then we have that $((A, \sigma)_{\mathrm{an}})_K$ is hyperbolic by (5.1), and hence $((V, h)_{\mathrm{an}})_K$ is hyperbolic by (4.5). Therefore $(V, h)_K$ is metabolic, and $(A, \sigma)_K$ is metabolic by (4.8).

Assume now that $(A, \sigma)_K$ is metabolic. By (4.8) this means that $(V, h)_K$ is metabolic, and by [7, Lemma 6.1.2] we must also have that $((V, h)_{\mathrm{an}})_K$ is also metabolic. Therefore by (4.8) we must have that $(A, \sigma)_{\mathrm{an}}$ is anisotropic over F and metabolic over K. If $(A, \sigma)_{\mathrm{an}}$ is orthogonal then this contradicts (5.6). Therefore $(A, \sigma)_{\mathrm{an}}$ is symplectic and metabolic and (5.1) says that there must be an embedding $(K, \iota) \hookrightarrow (A, \sigma)_{\mathrm{an}}$ as required. $\square$

Corollary 5.8. *Assume* $\mathrm{char}(F) = 2$ *and let* $K = F(\delta)$ *where* $\delta^2 + \delta = a \in F^\times$*, with non-trivial* F*-automorphism* ι*. Fix* (D, θ) *to be a* F*–division algebra with involution. Let* (V, h) *be an anisotropic hermitian space over* (D, θ)*.*

Then $(V, h)_K$ *is metabolic if and only if either* (V, h) *is metabolic or there exists an* $r \in \mathrm{End}_D(V)$ *such that* $r^2 = r + a$ *and* $h(r(x), y) = h(x, (r + 1)(y))$ *for all* $x, y \in V$*. In this case,* (V, h) *is alternating and* $(V, h) \simeq (V, ah)$*.*

Proof. The main result follows from (5.7) and (4.8), where r is the image of α under the embedding $(K, \iota) \hookrightarrow (A, \sigma)$. That (V, h) is alternating follows as the adjoint involution to (V, h) is symplectic if and only if (V, h) is alternating by (3.5).

Assume that there exists $r \in \mathrm{End}_D(V)$ such that $r^2 = r + a$ and $h(r(x), y) = h(x, (r + 1)(y))$ for all $x, y \in V$. Since r is invertible, as $r^{-1} = (r + 1)/a$, it is an isomorphism. Furthermore we have that

$$h(r(x), r(y)) = h((r + 1)r(x), y) = h(ax, y) = ah(x, y).$$

Therefore $(V, h) \simeq (V, ah)$. $\square$

Corollary 5.9. *Let* $\operatorname{char}(F) = 2$ *and let* K/F *be a separable quadratic extension. Let* (A, σ) *be an anisotropic* F*-algebra with involution of the first kind. Suppose that* A_K *is split, then* $(A, \sigma)_K$ *is metabolic if and only if* (A, σ) *is symplectic.*

Proof. If (A, σ) becomes metabolic over K, then it is symplectic by (5.7). If (A, σ) is symplectic, then it becomes hyperbolic over K by (4.7) and therefore metabolic. $\square$

Note that in the situation of (5.9), by [5, Proposition 4.5.13], A_K is split if and only if it is a quaternion algebra.
Together with the following result, we now have a complete description of those F-algebras with involution that become metabolic over a separable extension in characteristic 2.

Theorem 5.10. *Let* K/F *be a field extension of odd degree. Let* (A, σ) *be an* F*-algebra with involution. Then* $(A, \sigma)_K$ *is metabolic if and only if* (A, σ) *is metabolic.*

Proof. See [1, Proposition 1.2] for the statement in terms of hermitian forms, and apply (4.8). $\square$

However, unlike in the case of an odd degree extension, a non-metabolic orthogonal involution can become metabolic over a quadratic separable extension, as the following example shows.

Example 5.11. Let $\operatorname{char}(F) = 2$ and let K/F be a separable quadratic extension. Let (Q, γ) be a non-split quaternion algebra over F with the canonical involution, and with basis $(1, i, j, k)$ over F as defined in Section 3, such that Q_K is split. Let $(V, h) = \langle 1, j, j\rangle_{(Q,\gamma)}$ and $(A, \sigma) = \operatorname{Ad}(V, h)$.

Then (A, σ) is a non-metabolic orthogonal involution and $(A, \sigma)_K$ is metabolic.

Proof. Firstly, note that (V, h) cannot be metabolic as it is of odd dimension over V, therefore (A, σ) is not metabolic by (5.7). Now note that (V, h) is non-alternating, as $j \notin \operatorname{Alt}(Q, \gamma)$. Hence (A, σ) is orthogonal by (3.5).

Finally, note that $\langle 1\rangle_{(Q,\gamma)}$ is alternating as (Q, γ) is symplectic. Since Q_K is split, $(\langle 1\rangle_{(Q,\gamma)})_K$ is metabolic by (5.9). Hence $(V, h)_K$ is the sum of two metabolic forms, and hence is metabolic. Therefore $(A, \sigma)_K$ is metabolic. $\square$

6. Isotropy over separable quadratic extensions

Throughout this section we will assume that $\operatorname{char}(F) = 2$ and that all involutions are of the first kind. It is natural to ask, when does an F-algebra with anisotropic involution become isotropic over a quadratic separable field extension? In particular, does the result of (2.5) directly generalise to the case of an algebra with involution? This is clear in the split case and involution of the first kind.

Proposition 6.1. *Let* (A, σ) *be an anisotropic* F*-algebra with involution of the first kind. If* A *is split then* $(A, \sigma)_K$ *is anisotropic for all separable quadratic extensions* K/F*.*

Proof. If (A, σ) is symplectic, then it is adjoint to an alternating bilinear form, and by (2.1) it is therefore hyperbolic and in particular isotropic. If (A, σ) is orthogonal then the result follows from (2.5). $\square$

Does a similar result hold in the non-split case? We shall show, in (6.6), that in general it does not. Recall the definition of a quaternion algebra from Section 3.

Proposition 6.2. *Let $\alpha \in F$,$\beta \in F^\times$. Take an F-quaternion algebra $Q = [\alpha, \beta)$ with the canonical involution γ. The following are equivalent*

(1) *(Q, γ) is isotropic,*
(2) *(Q, γ) is hyperbolic,*
(3) *Q is split.*

Proof. That (2) $\Rightarrow$ (1) is clear.

(3) $\Rightarrow$ (2): If (Q, γ) is split and symplectic then it is hyperbolic by (4.7).

(1) $\Rightarrow$ (3): If Q is not split, then it is a division algebra as $\deg(Q) = 2$, and any involution on Q is anisotropic. Hence (Q, γ) is anisotropic. $\square$

Theorem 6.3. *Let Q be an F-quaternion algebra with anisotropic involution of the first kind σ, and let $K = F(\delta)$ where $\delta^2 + \delta = a \in F^\times$. Then $(Q, \sigma)_K$ is metabolic if and only if σ is the canonical involution and $Q \cong [a, \beta)_F$ for some $\beta \in F^\times$.*

Proof. By Theorem 5.7, if $(Q, \sigma)_K$ is metabolic then there exists an embedding $K \hookrightarrow Q$. This gives that $Q \cong [a, \beta)$, for some $\beta \in F^\times$, by [9, Observation (9)]. That σ is symplectic also follows from (5.7), therefore it must be the canonical involution.

On the other hand, the projective conic $ax^2 + \beta y^2 = z^2 + zx$ has a rational point over K. Therefore, if $Q \cong [a, \beta)$ and σ is the canonical involution, then $(Q, \sigma)_K$ is hyperbolic by (6.2), and hence metabolic by [5, Chapter 1, Exercise 4]. $\square$

Lemma 6.4. *Let $A = M_2(Q)$ and σ be the involution given by*

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \gamma(a) & \gamma(c)j^{-1} \\ j\gamma(b) & j\gamma(d)j^{-1} \end{pmatrix}$$

for $a, b, c, d \in Q$. Then (A, σ) is an orthogonal F-algebra with involution.

Proof. That σ is an involution on A can be seen by direct computation. Note that $(A, \sigma) \cong \mathrm{Ad}(\langle 1, j\rangle_{(Q,\gamma)})$ and hence (A, σ) is orthogonal by (3.5), as $j \notin \mathrm{Alt}(Q, \gamma)$. $\square$

Lemma 6.5. *Let $Q = [\alpha, \beta)_F$, with $\alpha, \beta \in F$. Let $A = M_2(Q)$, σ as in (6.4). Then (A, σ) is isotropic if and only if (Q, γ) is isotropic.*

Proof. Assume (Q, γ) is isotropic and let $a \in Q$ be an isotropic element of (Q, γ). Then $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ is an isotropic element of $(A, \sigma)_K$.

Assume that (Q, γ) is anisotropic, then Q is not split by (6.2), that is, Q is division.

The involution (A, σ) is adjoint to the hermitian form $(V, h) = \langle 1, j\rangle_{(Q,\gamma)}$. Note that $j \notin \mathrm{Alt}(Q, \gamma)$, and therefore $\langle j\rangle_{(Q,\gamma)}$ cannot represent any alternating elements. On the other hand, $\langle 1\rangle_{(Q,\gamma)}$ only represents alternating elements. Therefore (V, h) is anisotropic and hence so is (A, σ) by (4.8). $\square$

Example 6.6. Let F and $\alpha \in F$, $\beta \in F^\times$ be such that $Q = [\alpha, \beta)$ is division over F. Let $K = F(\delta)$ where $\delta^2 + \delta + \alpha = 0$. Let $A = M_2(Q)$ and σ as in (6.4). Then (A, σ) is an anisotropic orthogonal F-algebra with involution such that $(A, \sigma)_K$ is isotropic.

Proof. (A, σ) is orthogonal by (6.4).

By (6.2), Q is split if and only if (Q, γ) is anisotropic, and this is equivalent to (A, σ) anisotropic by (6.5). Therefore (A, σ) is anisotropic, and $(A, \sigma)_K$ is isotropic by (6.3). □

Remark 6.7. Note that we can easily adapt the above method to construct examples of anisotropic orthogonal involutions that become isotropic over a quadratic field extension.

Take an anisotropic hermitian form over some F–division algebra with involution that does not represent any alternating elements, and call it (V, h). We could take, for example, any 1-dimensional form representing an non-alernating element. We then take the sum $(V, h) \perp (W, b)$, where (W, b) is an alternating form that becomes metabolic over some quadratic field extension.

Then $(V, h) \perp (W, b)$ is anisotropic, as can be argued as in (6.5), and clearly non-alternating. So $\mathrm{Ad}((V, h) \perp (W, b))$ is an anisotropic orthogonal F–algebra with involution that becomes isotropic over some quadratic field extension.

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