QUATERNIONIC GRASSMANNIANS AND PONTRYAGIN CLASSES IN ALGEBRAIC GEOMETRY

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Abstract. The quaternionic Grassmannian $\mathrm{HGr}(r, n)$ is the affine open subscheme of the ordinary Grassmannian parametrizing those $2r$-dimensional subspaces of a $2n$-dimensional symplectic vector space on which the symplectic form is nondegenerate. In particular there is $\mathrm{HP}^n = \mathrm{HGr}(1, n+1)$. For a symplectically oriented cohomology theory A, including oriented theories but also hermitian K-theory, Witt groups and algebraic symplectic cobordism, we have $A(\mathrm{HP}^n) = A(\mathrm{pt})[p]/(p^{n+1})$. We define Pontryagin classes for symplectic bundles. They satisfy the splitting principle and the Cartan sum formula, and we use them to calculate the cohomology of quaternionic Grassmannians. In a symplectically oriented theory the Thom classes of rank 2 symplectic bundles determine Thom and Pontryagin classes for all symplectic bundles, and the symplectic Thom classes can be recovered from the Pontryagin classes.

The cell structure of the $\mathrm{HGr}(r, n)$ exists in the cohomology, but it is difficult to see more than part of it geometrically. The exception is HP^n where the cell of codimension $2i$ is a quasi-affine quotient of $\mathbb{A}^{4n-2i+1}$ by a nonlinear action of $\mathbb{G}_a$.

1. Introduction

The quaternionic projective spaces $\mathbb{HP}^n$ and Grassmannians $\mathbb{HG}\mathrm{r}(r, n)$ are important spaces in topology. They have cell structures like real and complex projective spaces and Grassmannians, but the dimensions of the cells are multiples of 4. In symplectically oriented cohomology theories E^*, including oriented theories but also KO^*, MSp^* and MSU^*, there is a quaternionic projective bundle theorem $E^*(\mathbb{HP}^n) = E^*(\mathrm{pt})[p]/(p^{n+1})$ which leads to a theory of Pontryagin classes of quaternionic bundles satisfying the Cartan sum formula and the Splitting Principle. These classes are used to prove a number of well-known theorems, including Conner and Floyd's description of KO^* as a quotient of MSp^* [6, Theorem 10.2]. Infinite-dimensional quaternionic Grassmannians provide models for the classifying spaces BSp_{2r} and BSp and for symplectic K-theory. For symplectically oriented cohomology theories we have $E^*(\mathrm{BSp}_{2r}) = E^*(\mathrm{pt})[[p_1, \dots, p_r]]$ and $E^*(\mathrm{BSp}) = E^*(\mathrm{pt})[[p_1, p_2, \dots]]$ where the p_i are the Pontryagin classes of the universal bundle.

In this paper we lay the foundations for a similar theory in motivic algebraic geometry. Since quaternionic Grassmannians are quotients of compact Lie groups $\mathbb{HG}\mathrm{r}(r, n) = \mathrm{U}_n(\mathbb{H})/(\mathrm{U}_r(\mathbb{H}) \times \mathrm{U}_{n-r}(\mathbb{H}))$, we take as our models the corresponding quotients of algebraic groups $\mathrm{HGr}(r, n) = \mathrm{Sp}_{2n}/(\mathrm{Sp}_{2r} \times \mathrm{Sp}_{2n-2r})$. Then $\mathrm{HGr}(r, n)(\mathbb{C})$ has the same homotopy type as $\mathbb{HG}\mathrm{r}(r, n)$ but twice the dimension and a significantly more complicated geometry.

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An alternative description of $\mathrm{HGr}(r,n)$ is that if we equip the trivial vector bundle on the base $\mathcal{O}^{2n}$ with the standard symplectic form ψ_{2n}, then $\mathrm{HGr}(r,n)$ is the open subscheme of $\mathrm{Gr}(2r,2n)$ parametrizing the subspaces $U \subset \mathcal{O}^{2n}$ of dimension $2r$ on which ψ_{2n} is nonsingular. The $\mathrm{HGr}(r,n)$ are smooth and affine of dimension $4r(n-r)$ over the base scheme with the same global units and the same Picard group as the base.

Our first results concern $\mathrm{HP}^n = \mathrm{HGr}(1,n+1)$. In Theorems 3.1, 3.2 and 3.4 we prove the following facts.

Theorem 1.1. *The scheme HP^n is smooth of dimension $4n$ over the base scheme. It has a decomposition into locally closed strata*

$$\mathrm{HP}^n = \bigsqcup_{i=0}^{n} X_{2i} = X_0 \sqcup X_2 \sqcup \cdots \sqcup X_{2n}, \tag{3.3}$$

such that each X_{2i} is of codimension $2i$, smooth and quasi-affine, but $X_0,\ldots,X_{2n-2}$ are not affine. The closure $\overline{X}_{2i} = X_{2i} \sqcup X_{2i+2} \sqcup \cdots \sqcup X_{2n}$ is a vector bundle of rank $2i$ over HP^{n-i}. Each X_{2i} is the quotient of a free action of $\mathbb{G}_a$ on $\mathbb{A}^{4n-2i+1}$.

One can study ordinary Grassmannians inductively by considering the closed embedding $\mathrm{Gr}(r,n-1) \hookrightarrow \mathrm{Gr}(r,n)$. The complement of the image is a vector bundle over $\mathrm{Gr}(r-1,n-1)$. The normal bundle of the embedding is isomorphic to the dual $\mathcal{U}^\vee_{r,n-1}$ of the tautological subbundle on $\mathrm{Gr}(r,n-1)$, and it embeds as an open subvariety of $\mathrm{Gr}(r,n)$. This gives a long exact sequence

$$\cdots \to A_{\mathrm{Gr}(r,n-1)}(\mathcal{U}^\vee_{r,n-1}) \to A(\mathrm{Gr}(r,n)) \to A(\mathrm{Gr}(r-1,n-1)) \to \cdots$$

in any cohomology theory.

There is an analogous long exact sequence for quaternionic Grassmannians, but one has to wade through a more complicated geometry to reach it. We take a symplectic bundle (E,ϕ) of rank $2n$ over a scheme S. We then let (F,ψ) be the symplectic bundle of rank $2n$ with

$$F = \mathcal{O}_S \oplus E \oplus \mathcal{O}_S, \qquad \psi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \phi & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{4.2}$$

We have the quaternionic Grassmannian bundles $\mathrm{HGr}(E) = \mathrm{HGr}_S(r;E,\phi)$ and $\mathrm{HGr}(F) = \mathrm{HGr}(r;F,\psi)$. Earlier, the complement of the open stratum $X_0 \subset \mathrm{HP}^n$ was a vector bundle $\overline{X}_2$ over HP^{n-1} rather than HP^{n-1} itself. The same thing happens now. Let

$$N^+ = \mathrm{HGr}(F) \cap \mathrm{Gr}_S(2r;\mathcal{O}_S \oplus E), \qquad N^- = \mathrm{HGr}(F) \cap \mathrm{Gr}_S(2r; E \oplus \mathcal{O}_S).$$

For any cohomology theory A we have a long exact sequence

$$\cdots \to A_{N^+}(\mathrm{HGr}(F)) \to A(\mathrm{HGr}(F)) \to A(\mathrm{HGr}(F) \smallsetminus N^+) \to \cdots. \tag{5.4}$$

Theorem. (*a*) *The loci N^+ and N^- are vector bundles over $\mathrm{HGr}(E)$ isomorphic to the tautological rank $2r$ symplectic subbundle $\mathcal{U}_{r,E}$.*

(*b*) *The vector bundle $N^+ \oplus N^-$ is the normal bundle of $\mathrm{HGr}(E) \subset \mathrm{HGr}(F)$, and it naturally embeds as an open subscheme of the Grassmannian bundle $\mathrm{Gr}_S(2r;F)$.*

(*c*) *N^+ and N^- are the loci in $\mathrm{HGr}(F)$ where certain sections s_+ and s_- of the tautological bundle $\mathcal{U}_{r,F}$ intersect the zero section transversally.*

This is part of Theorem 4.1. Since $\mathrm{HGr}(F) \subset \mathrm{Gr}_S(2r; F)$ is also an open subscheme, we deduce natural isomorphisms

$$A_{N^+}(\mathrm{HGr}(F)) \cong A_{N^+}(N^+ \oplus N^-) \cong A_{\mathrm{HGr}(r;E,\phi)}(\mathcal{U}_{r,E}).$$

The study of the open subscheme $Y = \mathrm{HGr}(F) \smallsetminus N^+$ in §5 is crucial to this paper. The main result is:

Theorem. *We have morphisms*

$$Y \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xrightarrow{q} \mathrm{HGr}_S(r-1, E, \phi)$$

over S with g_1 an $\mathbb{A}^{2r-1}$-bundle, g_2 an $\mathbb{A}^{2r-2}$-bundle and q an $\mathbb{A}^{4n-3}$-bundle. Moreover, there is an explicit symplectic automorphism of the pullback of (F, ψ) to Y_2 which induces an isometry

$$g_2^* g_1^* (\mathcal{U}_r, \phi_r) \xrightarrow{\cong} \left(\mathcal{O} \oplus q^*\mathcal{U}_{r-1} \oplus \mathcal{O}, \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & \phi_{r-1} & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right)\right),$$

where $(\mathcal{U}_r, \phi_r)$ and $(\mathcal{U}_{r-1}, \phi_{r-1})$ are the tautological symplectic subbundles on $\mathrm{HGr}(r, F, \psi)$ *and* $\mathrm{HGr}(r-1, E, \phi)$*, respectively.*

This is a simplified version of Theorem 5.1. From it we deduce that the natural closed embedding $\mathrm{HGr}_S(r-1, E, \phi) \hookrightarrow Y$ induces isomorphisms $A(Y) \cong A(\mathrm{HGr}_S(r-1, E, \phi))$ in any cohomology theory. For $r = 1$ this is an isomorphism $A(Y) \cong A(S)$. We thus get our long exact sequence of cohomology groups

$$\cdots \to A_{\mathrm{HGr}(r,E,\phi)}(\mathcal{U}_{r,E}) \to A(\mathrm{HGr}(r, F, \psi)) \to A(\mathrm{HGr}(r-1, E, \phi)) \to \cdots. \tag{5.5}$$

The geometry of Theorems 4.1 and 5.1 and the decomposition of $\mathrm{HGr}_S(r, F, \psi)$ into the vector bundle N^+ and the complementary open locus Y is recurrent throughout the paper. This seems to be a fundamental geometry of quaternionic Grasssmannian bundles.

After the initial description of the geometry, we begin to introduce symplectically oriented cohomology theories. We follow the point of view used for oriented cohomology theories in [14]. In the end (Definition 14.3) a symplectic orientation on a ring cohomology theory is a family of *Thom isomorphisms* $\mathrm{th}^{E,\phi}_X \colon A(X) \cong A_X(E)$ for every scheme X and every symplectic bundle (E, ϕ) over X. There are also isomorphisms $\mathrm{th}^{E,\phi}_Z \colon A_Z(X) \cong A_Z(E)$ for closed subsets $Z \subset X$. These isomorphisms satisfy several axioms of functoriality, of compatibility with the ring structure, and of compatibility with orthogonal direct sums $(E_1, \phi_1) \perp (E_2, \phi_2)$.

There are five ways of presenting a symplectic orientation. The Thom class $\mathrm{th}(E, \phi) \in A_X(E)$ is the image of $1_X \in A(X)$. One can present a symplectic orientation by giving either the Thom isomorphisms, the Thom classes or the Pontryagin classes of all symplectic bundles or by giving the Thom classes or the Pontryagin classes of rank 2 symplectic bundles only. In each case the classes are supposed to obey a certain list of axioms. In the end (Theorem 14.4) the five ways of presenting a symplectic orientation on a ring cohomology theory are equivalent.

We start in §7 by presenting the version where one gives the Thom classes for rank 2 symplectic bundles only. This is a *symplectic Thom structure*. With the Thom classes one can define Thom isomorphisms $A(X) \cong A_X(E)$ for rank 2 symplectic bundles and direct image maps $i_{A,\flat} \colon A(X) \to A_X(Y)$ and $i_{A,\natural} \colon A(X) \to A(Y)$ for regular embeddings $i \colon X \hookrightarrow Y$ of codimension 2 whose normal bundle is equipped with a symplectic form $(N_{X/Y}, \phi)$. The *Pontryagin class* of a rank 2 symplectic bundle (E, ϕ) on X is $p(E, \phi) = -z^A e^A \,\mathrm{th}(E, \phi)$

where $e^A \colon A_X(E) \to A(E)$ is extension of supports and $z^A \colon A(E) \cong A(X)$ is the restriction to the zero section. The main formula is that if a section of E intersects the zero section transversally in a subscheme Z, then for the inclusion $i\colon Z \hookrightarrow X$ and for all $b \in A(X)$ we have

$$i_{A,\natural} i^A b = -b \cup p(E,\phi). \tag{7.5}$$

With the geometry of Theorems 4.1 and 5.1, the long exact sequence of cohomology (5.4) and formula (7.5) we prove one of the main results of the paper.

Theorem 8.2. (Quaternionic projective bundle theorem). *Let A be a ring cohomology theory with a symplectic Thom structure. Let (E,ϕ) be a rank $2n$ symplectic bundle over a scheme S, let $(\mathcal{U}, \phi|_{\mathcal{U}})$ be the tautological rank 2 symplectic subbundle over the quaternionic projective bundle $\mathrm{HP}_S(E,\phi)$, and let $\zeta = p(\mathcal{U}, \phi|_{\mathcal{U}})$ be its Pontryagin class. Write $\pi\colon \mathrm{HP}_S(E,\phi) \to S$ for the projection. Then for any closed subset $Z \subset X$ we have an isomorphism of two-sided $A(S)$-modules $(1, \zeta, \dots, \zeta^{n-1})\colon A_Z(S)^{\oplus n} \xrightarrow{\cong} A_{\pi^{-1}(Z)}(\mathrm{HP}_S(E,\phi))$, and we have unique classes $p_i(E,\phi) \in A(S)$ for $1 \le i \le n$ such that there is a relation*

$$\zeta^n - p_1(E,\phi) \cup \zeta^{n-1} + p_2(E,\phi) \cup \zeta^{n-2} - \cdots + (-1)^n p_n(E,\phi) = 0.$$

If (E,ϕ) is trivial, then $p_i(E,\phi) = 0$ for $1 \le i \le n$.

The classes $p_i(E,\phi)$ are the *Pontryagin classes* of (E,ϕ) with respect to the symplectic Thom structure on A. One also sets $p_0(E,\phi) = 1$ and $p_i(E,\phi) = 0$ for $i > n$. Pontryagin classes are $\mathbb{A}^1$-deformation invariant and nilpotent.

The Pontryagin classes of an orthogonal direct sum of symplectic bundles $(F,\psi) \cong (E_1,\phi_1) \perp (E_2,\phi_2)$ satisfy the Cartan sum formula

$$p_i(F,\psi) = p_i(E_1,\phi_1) + \sum_{j=1}^{i-1} p_{i-j}(E_1,\phi_1) p_j(E_2,\phi_2) + p_i(E_2,\phi_2). \tag{10.6}$$

The p_1 is additive, and the top Pontryagin classes are multiplicative. The corresponding formula for Chern classes is usually proven with a geometric argument shortly after a proof of the projective bundle theorem. Unfortunately we have not found a geometric proof of the Cartan sum formula for Pontryagin classes. The geometry of quaternionic projective bundles (as we have defined them) is just not as nice as the geometry of projective bundles. The best we have done geometrically is to show that if (E,ϕ) is an orthogonal direct summand of (F,ψ) then the Pontryagin polynomial $P_{E,\phi}(t)$ divides the Pontryagin polynomial $P_{F,\psi}(t)$ (Lemma 9.3).

So we give a roundabout cohomological proof of the Cartan sum formula. The first step is to consider the scheme $\mathrm{HFlag}(1^r; n) = \mathrm{Sp}_{2n} / (\mathrm{Sp}_2^{\times r} \times \mathrm{Sp}_{2n-2r})$. It classifies decompositions

$$(\mathcal{O}^{2n}, \psi_{2n}) \cong (\mathcal{U}_n^{(1)}, \phi_n^{(1)}) \perp \cdots \perp (\mathcal{U}_n^{(r)}, \phi_n^{(r)}) \perp (\mathcal{V}_{r,n}, \psi_{r,n}) \tag{9.1}$$

of the trivial symplectic bundle of rank $2n$ into the orthogonal direct sum of r symplectic subbundles of rank 2 plus a symplectic subbundle of rank $2n-2r$. It is an iterated quaternionic projective bundle over both the base scheme and $\mathrm{HGr}(r,n)$. The weak substitute for the Cartan sum formula (Lemma 9.3) is good enough for us to show that we have

$$\varprojlim_{n\to\infty} A(\mathrm{HFlag}(1^r; n)) \cong A(k)[[y_1, \dots, y_r]] \tag{9.2}$$

with the indeterminate y_i corresponding to the element given by the system $(p(\mathcal{U}_n^{(i)}, \phi_n^{(i)}))_{n \ge r}$. Using the universality of the families of schemes $\mathrm{HFlag}(1^r; n)$ and $\mathrm{HGr}(r,n)$ and the fact that the $y_i - y_j$ are not zero divisors in $A(k)[[y_1, \dots, y_r]]$, we are able to deduce the Symplectic Splitting Principle (Theorem 10.2) from Lemma 9.3. The Cartan sum formula then follows (Theorem 10.5).

We then calculate the cohomology of quaternionic Grassmannians for any ring cohomology theory with a symplectic Thom structure. We get

$$A(\mathrm{HFlag}(1^r, n)) = A(k)[y_1, \dots, y_r]/(h_{n-r+1}, \dots, h_n), \tag{11.6}$$

$$A(\mathrm{HGr}(r, n)) = A(k)[p_1, \dots, p_r]/(h_{n-r+1}, \dots, h_n), \tag{11.7}$$

where the y_i are the Pontryagin classes of the r tautological rank 2 symplectic subbundles on $\mathrm{HFlag}(1^r, n)$, the p_i are the Pontryagin classes of the tautological rank $2r$ symplectic subbundle on $\mathrm{HGr}(r, n)$ and are the elementary symmetric polynomials in the y_i, and the h_i are the complete symmetric polynomials. It follows that for the standard embeddings the restrictions $A(\mathrm{HGr}(r, n+1)) \to A(\mathrm{HGr}(r, n))$ and $A(\mathrm{HGr}(r+1, n+1)) \to A(\mathrm{HGr}(r, n))$ are surjective, and we have isomorphisms

$$\varprojlim_{n\to\infty} A(\mathrm{HGr}(r, n)) = A(k)[[p_1, p_2, \dots, p_r]], \tag{11.12}$$

$$\varprojlim_{n\to\infty} A(\mathrm{HGr}(n, 2n)) = A(k)[[p_1, p_2, p_3, \dots]]. \tag{11.13}$$

In §§12–14 we prove the equivalence of the five ways of presenting a symplectic orientation on a ring cohomology theory. Above all, this involves using the exact sequence (5.5) and the calculations of the cohomology of Grassmannian bundles to recover the Thom classes from the Pontryagin classes.

The isomorphism $A(Y) \cong A(S)$ when $Y = X_0 \times S \subset \mathrm{HP}^n \times S$ is the open stratum is absolutely critical to the entire paper. Without it there is no proof of the quaternionic projective bundle theorem and no definition of Pontryagin classes. Therefore in §15 we give a second proof of the isomorphism using a completely different geometric argument. However, the geometry of Theorem 5.1 seems to be more generally useful than that of Theorem 15.1.

We stop here. This is already a long paper because the theory of Thom and Pontryagin classes of symplectic bundles and the calculation of the cohomology of quaternionic Grassmannian bundles are intertwined, and we do not see how to separate one from the other. We leave for other papers the discussion of symplectic orientations on specific nonoriented cohomology theories like hermitian K-theory, derived Witt groups and symplectic algebraic cobordism, as well as discussion of the 2 and 4-valued formal group laws obeyed by the symplectic Pontryagin classes.

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2. Cohomology theories

We review the notions of a cohomology theory and a ring cohomology theory as used in [14].

We fix a base scheme k. We will study ring cohomology theories on some category $\mathcal{V}$ of k-schemes. The category could be nonsingular quasi-projective varieties over $k = \operatorname{Spec} F$ with F a field, it could be nonsingular quasi-affine varieties over $k = \operatorname{Spec} F$, it could be quasi-compact semi-separated schemes with an ample set of line bundles over k, it could be regular noetherian separated schemes of finite Krull dimension over k, or it could be something else.

Which it is is unimportant as long as the definitions make sense and the constructions work. There are only some minor complications concerning deformation to the normal bundle when we go beyond smooth varieties over a field. Our theorems all hold if $\mathcal{V}$ has the following properties (partly borrowed from Levine-Morel [9, (1.1.1)]):

(1) All schemes in $\mathcal{V}$ are quasi-compact and quasi-separated and have an ample family of line bundles.
(2) The schemes k and $\varnothing$ are in $\mathcal{V}$.
(3) If X and Y are in $\mathcal{V}$, then so are $X \sqcup Y$ and $X \times_k Y$.
(4) If X is in $\mathcal{V}$, and $U \subset X$ is a quasi-compact open subscheme, then U is in $\mathcal{V}$.
(5) If X is in $\mathcal{V}$, and Y is a quasi-compact open subscheme of a Grassmannian bundle over X for which the projection map $Y \to X$ is an affine morphism, then Y is in $\mathcal{V}$.

Some of our discussions of deformation to the normal bundle and of direct images make more sense if a sixth property is also true. Recall that a *regular embedding* $Z \to X$ is a closed embedding such that locally Z is cut out by a regular sequence.

(6) For a regular embedding $Z \to X$ with X and Z in $\mathcal{V}$ the deformation to the normal bundle space $D(Z, X)$ of §6 is in $\mathcal{V}$.

We will use our language imprecisely as if all schemes were noetherian. Thus when we write "open subscheme" we really mean a quasi-compact open subscheme, and "closed subset" means the complement of a quasi-compact open subscheme.

The condition that every scheme in $\mathcal{V}$ have an ample family of line bundles is used in the proof of the symplectic splitting principle (Theorem 10.2).

Definition 2.1 ([14, Definition 2.1])**.** A *cohomology theory* on a category $\mathcal{V}$ of k-schemes is a pair (A, ∂) with A a functor assigning an abelian group $A_Z(X)$ to every scheme X and closed subset $Z \subset X$ with X and $X \smallsetminus Z$ in $\mathcal{V}$ and assigning a morphism of abelian groups $f^A \colon A_Z(X) \to A_{Z'}(X')$ to every map $f \colon X' \to X$ such that $Z' \supset f^{-1}(Z)$. One writes $A(X) = A_X(X)$. In addition one has a morphism of functors $\partial \colon A(X \smallsetminus Z) \to A_Z(X)$. Together they have the following properties.

(1) *Localization*: The functorial sequences
$$A(X \smallsetminus Z) \xrightarrow{\partial} A_Z(X) \xrightarrow{e^A} A(X) \xrightarrow{j^A} A(X \smallsetminus Z) \xrightarrow{\partial} A_Z(X),$$
with e^A and j^A the appropriate maps of A, are exact.
(2) *Etale excision*: $f^A \colon A_Z(X) \to A_{Z'}(X')$ is an isomorphism if $f \colon X' \to X$ is étale, $Z' = f^{-1}(Z)$, and $f|_{Z'} \colon Z' \to Z$ is an isomorphism,
(3) *Homotopy invariance*: the maps $\mathrm{pr}_1^A \colon A(X) \to A(X \times \mathbb{A}^1)$ are isomorphisms.

Cohomology theories have Mayer-Vietoris sequences and satisfy $A_\varnothing(X) = 0$. They have homotopy invariance for $\mathbb{A}^n$-bundles (torsors for vector bundles). Deformation to the normal bundle isomorphisms will be discussed in §6. See [14, §2.2] for these and other properties.

Zariski excision suffices for the main results of the paper. Excision is used mainly for Mayer-Vietoris and for direct images, but we use direct images only for one Grassmannian embedded in another, and then there are global Zariski tubular neighborhoods.

Definition 2.2 ([14, Definition 2.13])**.** A *ring cohomology theory* is a cohomology theory in the sense of Definition 2.1 with cup products
$$\cup \colon A_Z(X) \times A_W(X) \to A_{Z \cap W}(X)$$
which are functorial, bilinear and associative and have two other properties:

(1) There exists an element $1 \in A(k)$ such that for every scheme $\pi_X \colon X \to k$ in $\mathcal{V}$ and every closed subset $Z \subset X$, the pullback $1_X = \pi_X^A(1) \in A(X)$ satisfies $1_X \cup a = a \cup 1_X = a$ for all $a \in A_Z(X)$.
(2) For the maps $\partial \colon A(X \smallsetminus Z) \to A_Z(X)$ one has $\partial(a \cup b) = \partial a \cup b$ for all $a \in A(X \smallsetminus Z)$ and all $b \in A(X)$.

3. Basic geometry of HP^n

We define HP^n and discuss a stratification resembling the cell decomposition of the topological HP^n. We also present quaternionic projective bundles, Grassmannians and flag varieties.

Let (V, ϕ) be a trivial symplectic bundle of rank $2n+2$ over the base scheme k. The symplectic group $\mathrm{Sp}_{2n+2} = \mathrm{Sp}(V, \phi)$ acts on the Grassmannian $\mathrm{Gr}(2, V)$ with (i) a closed orbit $\mathrm{GrSp}(2, V, \phi)$ parametrizing 2-dimensional subspaces $U \subset V$ with $\phi|_U \equiv 0$, and (ii) a complementary open orbit parametrizing 2-dimensional subspaces $U \subset V$ with $\phi|_U$ nondegenerate which we will call the *quaternionic projective space* HP^n. We will use this object as a motivic analogue of the topological $\mathbb{HP}^n$. Determining the stabilizer of a point of HP^n yields an identification $\mathrm{HP}^n = \mathrm{Sp}_{2n+2} / (\mathrm{Sp}_2 \times \mathrm{Sp}_{2n})$, which compares well with $\mathbb{HP}^n = \mathrm{U}_{n+1}(\mathbb{H}) / \mathrm{U}_1(\mathbb{H}) \times \mathrm{U}_n(\mathbb{H})$. So the manifold $\mathrm{HP}^n(\mathbb{C})$ of complex points is the complexification of the quotient of compact Lie groups $\mathbb{HP}^n$ and has the same homotopy type. It does not have the homotopy type of a complex projective manifold.

The topological $\mathbb{HP}^n$ is the union of cells of dimensions $0, 4, 8, \dots, 4n$. A related decomposition of the space HP^n may be defined. Fix a flag

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n+1} = E_{n+1}^{\perp} \subset \cdots \subset E_1^{\perp} \subset E_0^{\perp} = V \tag{3.1}$$

of subbundles of (V, ϕ) with the $E_i \cong \mathcal{O}_k^{\oplus i}$ totally isotropic and satisfying $\dim E_i = i$ and $\dim E_i^{\perp} = 2n + 2 - i$. Set

$$\overline{X}_{2i} = \mathrm{Gr}(2, E_i^{\perp}) \cap \mathrm{HP}^n, \qquad X_{2i} = \overline{X}_{2i} \smallsetminus \overline{X}_{2i+2}. \tag{3.2}$$

Choose for convenience a lagrangian supplementary to the lagrangian E_{n+1}, and let $\mathrm{GL}_{n+1} \subset \mathrm{Sp}_{2n+2}$ be the subgroup fixing the two lagrangians.

Theorem 3.1. *The scheme* HP^n *is the disjoint union of the locally closed strata*

$$\mathrm{HP}^n = \bigsqcup_{i=0}^{n} X_{2i} = X_0 \sqcup X_2 \sqcup \cdots \sqcup X_{2n}, \tag{3.3}$$

which have the following properties:

(*a*) *The scheme* X_{2i} *and its closure* $\overline{X}_{2i} = X_{2i} \sqcup X_{2i+2} \sqcup \cdots \sqcup X_{2n}$ *are smooth of relative dimension* $4n - 2i$ *over the base scheme* k. *The* $\overline{X}_{2i}$ *and* $X_{2n} = \overline{X}_{2n}$ *are affine over the base, but* $X_0, \dots, X_{2n-2}$ *are not. We have* $\mathrm{Pic}(\overline{X}_{2i}) = \mathrm{Pic}(k)$ *and* $\mathcal{O}(\overline{X}_{2i})^{\times} = \mathcal{O}(k)^{\times}$.

(*b*) *Each* $\overline{X}_{2i}$ *is the transversal intersection of* i *translates of* $\overline{X}_2$ *under the action of the subgroup* $\mathrm{GL}_{n+1} \subset \mathrm{Sp}_{2n+2}$.

(*c*) *The intersection of* $n+1$ *general translates of* $\overline{X}_2$ *under the action of* GL_{n+1} *is empty.*

In particular, since $\overline{X}_0 = \mathrm{HP}^n$ we have $\mathrm{Pic}(\mathrm{HP}^n) = \mathrm{Pic}(k)$ and $\mathcal{O}(\mathrm{HP}^n)^{\times} = \mathcal{O}(k)^{\times}$.

Proof. (a) Let $\mathcal{U}_{\mathrm{Gr}}$ be the tautological rank 2 subbundle on $\mathrm{Gr} = \mathrm{Gr}(2, V)$. Then $\mathrm{GrSp}(2, V, \phi)$ is the zero locus of a section of $\Lambda^2 \mathcal{U}_{\mathrm{Gr}}^{\vee} \cong \mathcal{O}_{\mathrm{Gr}}(1)$ induced by ϕ. So HP^n is the complement in the smooth projective scheme Gr of an irreducible ample divisor whose class generates

$\operatorname{Pic}(\mathrm{Gr})/\operatorname{Pic}(k) \cong \mathbb{Z}$. It follows that HP^n is smooth and affine of the same dimension $4n$ as the Grassmannian, and that it satisfies $\operatorname{Pic}(\mathrm{HP}^n)/\operatorname{Pic}(k) = 0$ and $\mathcal{O}(\mathrm{HP}^n)^\times = \mathcal{O}(k)^\times$.

The same argument applied to $\mathrm{Gr}(2, E_i^\perp)$ shows that the nonempty $\overline{X}_{2i}$ are smooth and affine of the same dimension $4n - 2i$ as $\mathrm{Gr}(2, E_i^\perp)$ and satisfy $\operatorname{Pic}(\overline{X}_{2i})/\operatorname{Pic}(k) = 0$ and $\mathcal{O}(\overline{X}_{2i})^\times = \mathcal{O}(k)^\times$. Moreover, $\overline{X}_{2i}$ is empty if and only if $E_i^\perp$ is totally isotropic, and this is true only for $i = n+1$. This implies $\overline{X}_{2n+2} = \varnothing$ and $\mathrm{HP}^n = \bigsqcup_{i=0}^n X_{2i}$ and that $X_{2n} = \overline{X}_{2n}$ is affine. But the other X_{2i} are obtained by removing a nonempty closed subscheme of codimension 2 from a smooth affine scheme, and such schemes are never affine.

(b) If one chooses $g_1, \dots, g_i \in \mathrm{GL}_{n+1} \subset \mathrm{Sp}_{2n+2}$ satisfying $\bigoplus_{j=1}^i g_j(E_1) = E_i$ then one has $\overline{X}_{2i} = \bigcap_{j=1}^i g_j(\overline{X}_2)$ because both parametrize subspaces $U \subset \bigcap g_j(E_1)^\perp = E_i^\perp$ with $\phi|_U \not\equiv 0$. Moreover, the intersection is transversal because the map $\bigoplus_{j=1}^i N^\vee_{g_j(\overline{X}_1)/\mathrm{HP}^n,[U]} \to T^\vee_{\mathrm{HP}^n,[U]}$ which needs to be injective can be identified with the natural map $\bigoplus_{j=1}^i \bigl(U \otimes g_j(E_1)\bigr) \to U \otimes U^\perp$.

(c) *Idem* with $i = n+1$. ☐

We have two other results which make the stratification (3.3) of HP^n resemble the cell decomposition of $\mathbb{HP}^n$. We will prove them later, but we state them now so they don't get lost amid more general results about quaternionic Grassmannian bundles.

Theorem 3.2. *There is a natural map $q\colon \overline{X}_{2i} \to \mathrm{HP}^{n-i} = \mathrm{HP}(E_i^\perp/E_i, \overline{\phi})$ which is an $\mathbb{A}^{2i}$-bundle. The stratification $\overline{X}_{2i} = X_{2i} \sqcup X_{2i+2} \sqcup \cdots \sqcup X_{2n}$ is the inverse image under q of the stratification $\mathrm{HP}^{n-i} = X'_0 \sqcup X'_2 \sqcup \cdots \sqcup X'_{2n-2i}$ associated to the flag of subspaces of $E_i^\perp/E_i$ induced by (3.1). The pullbacks to $\overline{X}_{2i}$ of the two tautological symplectic subbundles are naturally isometric $q^*(\mathcal{U}_{\mathrm{HP}^{n-i}}, \overline{\phi}|_{\mathcal{U}_{\mathrm{HP}^{n-i}}}) \cong (\mathcal{U}_{\mathrm{HP}^n}, \phi|_{\mathcal{U}_{\mathrm{HP}^n}})|_{\overline{X}_{2i}}$.*

Corollary 3.3. *There is a natural isomorphism $\overline{X}_{2n} = X_{2n} \cong \mathbb{A}^{2n}$.*

Theorem 3.4. (*a*) *The stratum X_{2i} is the quotient of the free action of $\mathbb{G}_a$ on $\mathbb{A}^{4n-2i+1} = \mathbb{A}^i \times \mathbb{A}^{2n-2i} \times \mathbb{A}^i \times \mathbb{A}^{2n-2i} \times \mathbb{A}^1$ given by*

$$t \cdot (\alpha, a, \beta, b, r) = (\alpha, a, \beta + t\alpha, b + ta, r + t(1 - \phi(a, b))) \tag{3.4}$$

where $\phi\colon \mathbb{A}^{2n-2i} \times \mathbb{A}^{2n-2i} \to \mathbb{A}^1$ is the standard symplectic form.

(*b*) *For any scheme S, pullback along the structural map $t\colon X_{2i} \to k$ induces isomorphisms $(t \times 1_S)^A\colon A(S) \xrightarrow{\cong} A(X_{2i} \times S)$ for any cohomology theory A.*

Theorem 3.2 is essentially a special case of Theorem 4.1. We will prove Theorem 3.4 in §5.

Some curious things are beginning to happen. The strata in our decomposition are cohomological cells but not affine spaces or even affine schemes. They are of the right codimension but the wrong dimension, for $\dim_{\mathbb{R}} \mathrm{HP}^n(\mathbb{C}) = 8n$ but $\dim_{\mathbb{R}} \mathbb{HP}^n = 4n$. The $\overline{X}_{2i}$ are not copies of the subspaces HP^{n-i} but vector bundles over copies of the HP^{n-i}, and the difference in dimensions is worse: $\dim_{\mathbb{R}} \mathrm{HP}^n(\mathbb{C}) = 8n - 4i$ but $\dim_{\mathbb{R}} \mathbb{HP}^{n-i} = 4n - 4i$. There is no problem cohomologically, but it is curious geometrically.

For a rank $2n+2$ symplectic bundle (E, ϕ) over a scheme S there is a *quaternionic projective bundle* $\mathrm{HP}_S(E, \phi)$. It is the open subscheme of the Grassmannian bundle $\mathrm{Gr}_S(2, E)$ whose points over $s \in S$ correspond to those $U \subset E_s$ on which ϕ is nondegenerate. Let $\pi\colon \mathrm{HP}_S(E, \phi) \to S$ be its structure map, i.e. the natural projection to S. Then the symplectic bundle $\pi^*(E, \phi)$ splits as the orthogonal direct sum

$$\pi^*(E, \phi) \cong (\mathcal{U}, \phi|_{\mathcal{U}}) \perp (\mathcal{U}^\perp, \phi|_{\mathcal{U}^\perp}) \tag{3.5}$$

of the tautological rank 2 symplectic subbundle and its orthogonal complement. This decomposition is *universal* in the sense that for any a morphism $g\colon X \to S$ any orthogonal direct sum decomposition $g^*(E,\phi) \cong (F_1,\psi_1) \perp (F_2,\psi_2)$ with $\operatorname{rk} F_1 = 2$ is the pullback along a unique morphism $f\colon X \to \mathrm{HP}_S(E,\phi)$ of the universal decomposition (3.5). The map f is said to *classify* either the decomposition or the rank 2 symplectic subbundle $(F_1,\psi_1) \subset g^*(E,\phi)$.

We define the quaternionic Grassmannians and partial flag varieties as the quotient varieties

$$\mathrm{HGr}(r,n) = \mathrm{Sp}_{2n}/(\mathrm{Sp}_{2r} \times \mathrm{Sp}_{2n-2r}),$$
$$\mathrm{HFlag}(a_1,\dots,a_r;n) = \mathrm{Sp}_{2n}/(\mathrm{Sp}_{2a_1} \times \cdots \times \mathrm{Sp}_{2a_r} \times \mathrm{Sp}_{2n-\sum 2a_i}).$$

The second family of schemes includes the first, so we discuss it. Over $\mathrm{HFlag}(a_1,\dots,a_r;n)$ there are $r+1$ universal symplectic subbundles $(\mathcal{U}_1,\phi_1),\dots,(\mathcal{U}_r,\phi_r),(\mathcal{V},\psi)$ with $\operatorname{rk}\mathcal{U}_i = 2a_i$ and $\operatorname{rk}\mathcal{V} = 2n - \sum 2a_i$ and a canonical decomposition of the trivial symplectic bundle of rank $2n$ into their orthogonal direct sum

$$(V,\phi)\otimes \mathcal{O} \cong (\mathcal{U}_1,\phi_1) \perp \cdots \perp (\mathcal{U}_r,\phi_r) \perp (\mathcal{V},\psi)$$

Moreover, any decomposition over a scheme S of the trivial symplectic bundle $(V,\phi)\otimes\mathcal{O}_S$ into an orthogonal direct sum of symplectic subbundles of the appropriate ranks is the pullback of this universal decomposition along a unique morphism $S \to \mathrm{HFlag}(a_1,\dots,a_r;n)$.

Now write $\mathcal{F}_i = \bigoplus_{j=1}^i \mathcal{U}_i$. We then have a filtration

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_r \subset V \otimes \mathcal{O} \tag{3.6}$$

Clearly $\mathrm{HFlag}(a_1,\dots,a_r;n)$ parametrizes flags of subspaces $0 \subset F_1 \subset \cdots \subset F_r \subset V$ of the appropriate dimensions such that $\phi|_{F_i}$ is nondegenerate for all i. So it is an open subscheme of a flag variety.

Theorem 3.5. *The quaternionic flag varieties are dense open subschemes of the flag varieties*

$$\mathrm{HFlag}(a_1,\dots,a_r;n) \subset \mathrm{Flag}(2m_1,2m_2,\dots,2m_r;2n).$$

with $m_i = \sum_{j=1}^i a_j$. They are smooth and affine of relative dimension $4n\sum_i a_i - 4\sum_{i\leq j} a_i a_j$ over the base k, and they satisfy $H^0(\mathrm{HFlag},\mathcal{O}^\times) = \mathcal{O}(k)^\times$ and $\mathrm{Pic}(\mathrm{HFlag}) = \mathrm{Pic}(k)$.

The proof is essentially the same as that of Theorem 3.1.

4. Normal bundles of sub-Grassmannians

For ordinary Grassmannians, there are closed embeddings $\mathrm{Gr}(r,n-1) \hookrightarrow \mathrm{Gr}(r,n)$, and the complement of the image is isomorphic to a vector bundle over $\mathrm{Gr}(r-1,n-1)$. For the cohomology there is then a long exact sequence

$$\cdots \to A_{\mathrm{Gr}(r,n-1)}(\mathrm{Gr}(r,n)) \to A(\mathrm{Gr}(r,n)) \to A(\mathrm{Gr}(r-1,n-1)) \to \cdots. \tag{4.1}$$

Moreover, the normal bundle N of $\mathrm{Gr}(r,n-1)$ in $\mathrm{Gr}(r,n)$ embeds as an open subvariety of $\mathrm{Gr}(r,n)$, and so excision gives an isomorphism $A_{\mathrm{Gr}(r,n-1)}(\mathrm{Gr}(r,n)) \cong A_{\mathrm{Gr}(r,n-1)}(N)$.

We will show that something similar happens for the cohomology of quaternionic Grassmannians but with many differences in the details.

We work in a relative situation. Our basic setup is as follows. Suppose (E,ϕ) is a symplectic bundle of rank $2n-2$ over S, and let (F,ψ) be the symplectic bundle of rank $2n$ with

$$F = \mathcal{O}_S \oplus E \oplus \mathcal{O}_S, \qquad \psi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \phi & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{4.2}$$

We will consider the natural embedding of $\mathrm{HGr}_S(r;E,\phi)$ in $\mathrm{HGr}_S(r;F,\psi)$. Let $(\mathcal{U}_E,\phi_{\mathcal{U}_E})$ and $(\mathcal{U}_F,\psi_{\mathcal{U}_F})$ be the tautological symplectic subbundles of rank $2r$. We will abbreviate

$$\mathrm{HGr}(E)=\mathrm{HGr}_S(r;E,\phi) \qquad \mathrm{HGr}(F)=\mathrm{HGr}_S(r;F,\psi). \tag{4.3}$$

We now study the embedding of $\mathrm{HGr}(E)$ in $\mathrm{HGr}(F)$.

Theorem 4.1. (*a*) *The normal bundle N of $\mathrm{HGr}(E)$ in $\mathrm{HGr}(F)$ has a canonical embedding as an open subscheme of $\mathrm{Gr}_S(2r;F)$ overlapping the open subscheme $\mathrm{HGr}_S(r;F,\psi)$.*

(*b*) *The subschemes $N^+=\mathrm{HGr}(F)\cap\mathrm{Gr}_S(2r;\mathcal{O}_S\oplus E)$ and $N^-=\mathrm{HGr}(F)\cap\mathrm{Gr}_S(2r;E\oplus\mathcal{O}_S)$ are closed in $\mathrm{HGr}(F)$ and are subbundles of N with $N=N^+\oplus N^-$. We have $N^+\cap N^-=\mathrm{HGr}(E)$.*

(*c*) *There are natural vector bundle isomorphisms $N^+\cong\mathcal{U}_E$ and $N^-\cong\mathcal{U}_E$.*

(*d*) *There is a natural section s_+ of $\mathcal{U}_F$ intersecting the zero section transversally in N^+ and similarly for N^-.*

(*e*) *Let $\pi_+\colon N^+\to\mathrm{HGr}(E)$ be the structural map. Then $\pi_+^*(\mathcal{U}_E,\phi|_{\mathcal{U}_E})$ is isometric to $(\mathcal{U}_F,\psi|_{\mathcal{U}_F})|_{N^+}$ and similarly for N^-.*

Actually in (b) and (c) the truly natural isomorphisms are $N^+\cong\mathcal{U}_E^\vee$ and $N^-\cong\mathcal{U}_E^\vee$, while s_+ is naturally a section of $\mathcal{U}_F^\vee$. But since $\mathcal{U}_E$ and $\mathcal{U}_F$ are symplectic, this does not matter.

When S is k, and (E,ϕ) is the trivial hyperbolic symplectic bundle $\mathrm{h}(\mathcal{O}_k^{\oplus n-1})$, then we are in the situation of Theorem 3.2 with $\mathrm{HGr}(F)=\mathrm{HP}^{n-1}$ and $N^+=\overline{X}_2$ and $\mathrm{HGr}(E)=X_0'$.

Proof. (a) The normal bundle N of $\mathrm{HGr}(E)$ in $\mathrm{HGr}(F)$ is isomorphic to $\mathcal{U}_E^\vee\otimes F/E\cong\mathcal{U}_E^\vee\otimes\mathcal{O}^{\oplus 2}$. Therefore N has a universal property: to give a map $T\to N$ one gives a map $t\colon T\to S$ plus a symplectic subbundle $i\colon U\subset t^*E$ of rank $2r$ plus a morphism $(\alpha_1,\alpha_2)\colon U\to\mathcal{O}_T^{\oplus 2}$. Giving such data is the same as giving $t\colon T\to S$ plus a rank $2r$ subbundle $(\alpha_1,i,\alpha_2)\colon U\hookrightarrow\mathcal{O}_T\oplus t^*E\oplus\mathcal{O}_T=t^*F$ such $i^\vee\phi i$ is everywhere of maximal rank. So N is naturally isomorphic to an open subscheme of $\mathrm{Gr}_S(2r;F)$.

(b)(e) To give a morphism $T\to N^+$ one gives a map $t\colon T\to S$ plus a subbundle of rank $2r$ of the form $(\alpha_1,i,0)\colon U\hookrightarrow\mathcal{O}_T\oplus t^*E\oplus\mathcal{O}_T=t^*F$ such that $(\alpha_1,i,0)^\vee\psi(\alpha_1,i,0)$ is nonsingular. That is equivalent to giving $t\colon T\to S$ and $i\colon U\hookrightarrow t^*E$ such that $i^\vee\phi i$ is nonsingular plus $\alpha_1\colon U\to\mathcal{O}_T\oplus 0$. So N^+ is the subbundle $\mathcal{U}_E^\vee\otimes(\mathcal{O}_S\oplus 0)$ of $N=\mathcal{U}_E^\vee\otimes\mathcal{O}_S^{\oplus 2}$. Similarly $N^-=\mathcal{U}_E^\vee\otimes(0\oplus\mathcal{O}_S)$. The other assertions of (b) are clear. The two presentations have the same bundle U and the same form $(\alpha_1,i,0)^\vee\psi(\alpha_1,i,0)=i^\vee\phi i$. Hence (e).

(c) By construction we have $N^+\cong\mathcal{U}_E^\vee$. But $\mathcal{U}_E$ is symplectic, so $\mathcal{U}_E^\vee\cong\mathcal{U}_E$.

(d) For $\mathcal{U}_F$ we have the inclusion $(\alpha_1,i,\alpha_2)\colon\mathcal{U}_F\hookrightarrow\mathcal{O}_S\oplus E\oplus\mathcal{O}_S=F$. The scheme N^+ is the zero locus of α_2 or equivalently of $\alpha_2^\vee\colon\mathcal{O}_S\to\mathcal{U}_F^\vee\cong\mathcal{U}_F$. $\square$

In the proof of Theorem 12.6 we will use the following subtle point about the geometry of Theorem 4.1. The *tautological section* of $\pi_-^*N_-=N_-\times_{\mathrm{HGr}(E)}N_-\xrightarrow{p_2}N_-$ is the diagonal Δ.

Lemma 4.2. *The isomorphism $\mathcal{U}_F|_{N^-}\cong\pi_-^*\mathcal{U}_E$ of (d) and the pullback $\pi_-^*\mathcal{U}_E\cong\pi_-^*N^-$ of the isomorphism of (b) identifies the restriction $s_+|_{N^-}$ of the section of (c) with the tautological section of $\pi_-^*N^-$.*

Both are $\alpha_2^\vee$. We leave the details to the reader.

Proposition 4.3. *In the situation of Theorem 4.1 let $f\colon N^-\hookrightarrow\mathrm{HGr}(F)$ be the inclusion. Then for any cohomology theory A the map $f^A\colon A_{N^+}(\mathrm{HGr}(F))\to A_{\mathrm{HGr}(E)}(N^-)$ is an isomorphism.*

Proof. We have a direct sum $N = N^+ \oplus N^-$ of vector bundles over $\mathrm{HGr}(E)$, so the pullback and its left inverse the restriction map $A_{N^+}(N) \to A_{\mathrm{HGr}(E)}(N^-)$ are isomorphisms by homotopy invariance. Moreover, N^+ is a closed subscheme of both N and of $\mathrm{HGr}(F)$, which are both open subschemes of $\mathrm{Gr}_S(2r; F)$. So the diagram

$$\begin{tikzcd} A_{N^+}\big(N \cap \mathrm{HGr}(F)\big) \arrow[dr] & A_{N^+}(\mathrm{HGr}(F)) \arrow[l, "\text{excision}"', "\cong"] \arrow[d, "f^A"] \\ A_{N^+}(N) \arrow[u, "\text{excision}", "\cong"'] \arrow[r, "\cong", "\text{v.b.}"'] & A_{\mathrm{HGr}(E)}(N^-) \end{tikzcd} \tag{4.4}$$

commutes, and the top and left arrows are isomorphisms by Zariski excision. Hence f^A is also an isomorphism. □

5. Geometry of the open stratum

In this section we study the geometry and cohomology of the open stratum in the localization sequence for quaternionic sub-Grassmannians. Our main results are Theorems 5.1 and 5.2. We include a proof of Theorem 3.4 as a special case of the latter. A second proof is Theorem 15.1.

Recall the basic setup of (4.2). We have a symplectic bundle (E, ϕ) of rank $2n-2$ over S, and let (F, ψ) is of rank $2n$ with $F = \mathcal{O}_S \oplus E \oplus \mathcal{O}_S$ and ψ the orthogonal direct sum of ϕ and the hyperbolic symplectic form. We will consider the natural embedding of $\mathrm{HGr}_S(r; E, \phi)$ in $\mathrm{HGr}_S(r; F, \psi)$. Let $\lambda\colon F \twoheadrightarrow \mathcal{O}_S$ be the projection onto the third factor with kernel $\mathcal{O}_S \oplus E$. For $r \le n-1$ let

$$N^+ = \mathrm{HGr}_S(r, F, \psi) \cap \mathrm{Gr}_S(2r, \mathcal{O}_S \oplus E), \qquad Y = \mathrm{HGr}_S(r, F, \psi) \smallsetminus N^+. \tag{5.1}$$

with $N^+ \subset \mathrm{HGr}_S(r, F, \psi)$ closed of codimension $2r$ and Y open. Let $(\mathcal{U}_r, \psi_r)$ be the tautological symplectic subbundle of rank $2r$ on $\mathrm{HGr}(r, F, \psi)$, and let $(\mathcal{U}_{r-1}, \phi_{r-1})$ be the tautological symplectic subbundle of rank $2r-2$ on $\mathrm{HGr}(r-1, E, \phi)$. In this section we will investigate the open stratum Y.

Theorem 5.1. *In the situation of* (4.2) *and* (5.1) *we have morphisms over* S

$$Y \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xrightarrow{q} \mathrm{HGr}_S(r-1, E, \phi)$$

with g_1 an $\mathbb{A}^{2r-1}$-bundle, g_2 an $\mathbb{A}^{2r-2}$-bundle, and q an $\mathbb{A}^{4n-3}$-bundle. Moreover, $g_2^* g_1^* \mathcal{U}_r$ has two tautological sections e, f over Y_2 satisfying the properties

$$\lambda(f) = 1, \qquad \lambda(e) = 0, \qquad \psi(e, f) = 1, \tag{5.2}$$

and writing $\pi\colon Y_2 \to S$ for the projection, there is a symplectic automorphism ρ of $\pi^*(F, \psi) = (\mathcal{O}_{Y_2} \oplus \pi^* E \oplus \mathcal{O}_{Y_2}, \psi)$ with $\rho(1,0,0) = e$ and $\rho(0,0,1) = f$ and an orthogonal direct sum of symplectic subbundles of $\pi^*(F, \psi)$

$$g_2^* g_1^* \mathcal{U}_r = \langle e, f \rangle \perp \rho(q^* \mathcal{U}_{r-1}) \subset \pi^* F. \tag{5.3}$$

In (5.2) the bilinear form is $\psi(f, e) = f^\vee \psi e$ for $\psi\colon F \to F^\vee$.

Theorem 5.2. *In the situation of Theorem 5.1 with $Y = \mathrm{HGr}_S(r, F, \psi) \smallsetminus N^+$ the open locus of* (5.1)*, the following hold for any cohomology theory A.*

(a) *For $r = 1$ let $t\colon Y \to S$ be the projection map. Then $t^A\colon A(S) \to A(Y)$ is an isomorphism.*

(b) *For $1 \le r \le n$ let*

$$\mathrm{HGr}_S(r-1, E, \phi) \xrightarrow{\sigma} Y \subset \mathrm{HGr}_S(r, F, \psi)$$

be the map classifying the rank $2r$ symplectic subbundle $\mathcal{O} \oplus \mathcal{U}_{r-1} \oplus \mathcal{O}$ of the pullback of F. Then the map $\sigma^A\colon A(Y) \to A(\mathrm{HGr}_S(r-1, E, \phi))$ is an isomorphism.

The localization exact sequence

$$\cdots \to A_{N^+}(\mathrm{HGr}(F)) \to A(\mathrm{HGr}(F)) \to A(\mathrm{HGr}(F) \smallsetminus N^+) \to \cdots \tag{5.4}$$

combines with Proposition 4.3 and Theorem 5.2 to give the following result.

Corollary 5.3. *In the situation of* (4.2)–(5.1) *for any cohomology theory A the localization sequence for the closed embedding $N^+ \hookrightarrow \mathrm{HGr}(r, F, \psi)$ and the complementary open embedding $j\colon Y \hookrightarrow \mathrm{HGr}(r, F, \psi)$ is isomorphic to*

$$\cdots \to A_{\mathrm{HGr}(r,E,\phi)}(\mathcal{U}_E) \xrightarrow{e^A (f^A)^{-1}} A(\mathrm{HGr}(r, F, \psi)) \xrightarrow{\sigma^A j^A} A(\mathrm{HGr}(r-1, E, \phi)) \xrightarrow{f^A \partial (\sigma^A)^{-1}} \cdots \tag{5.5}$$

where $\mathcal{U}_E$ is the tautological rank $2r$ symplectic subbundle on $\mathrm{HGr}(r, E, \phi)$, and f is as in Proposition 4.3, σ as in Theorem 5.2, and e^A is the extension of supports operator.

We prove a series of lemmas before proving the theorems.

Lemma 5.4. *Let U be a vector bundle of rank m over S with a quotient line bundle $\lambda\colon U \twoheadrightarrow \mathcal{O}_S$ with a fixed trivialization. Then the locus*

$$Y = \{f \in U \mid \lambda(f) = 1\} \subset U$$

is an $\mathbb{A}^{m-1}$-bundle over S which has a section if and only if λ is split. Moreover, giving a morphism $T \to Y$ is equivalent to giving a pair (g, f_T) with $g\colon T \to S$ a morphism, and f_T a section of g^*U with $\lambda(f_T) = 1$.

Proof. The subbundle $\ker \lambda$ acts on U by translation in the fibers, and this action preserves Y. Over any open subscheme $S_\alpha \subset S$ where λ has a local splitting σ_α, one has $Y|_{S_\alpha} = (\ker \lambda)|_{S_\alpha} + \sigma_\alpha(1)$, so $Y \to S$ is a torsor for $\ker \lambda$. A global section $S \to Y$ is by construction a section s of U with $\lambda(s) = 1$. It exists if and only if λ is split. Finally, giving a morphism $T \to U$ is equivalent to giving a morphism $g\colon T \to S$ plus a section f_T of g^*U because of the cartesianness of

$$\begin{tikzcd} g^*U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r, "g"'] \arrow[u, bend left, "f_T"] & S \end{tikzcd}$$

The image of $T \to U$ lies in $Y \subset U$ if and only if $\lambda(f_T) = 1$. $\square$

Lemma 5.5. *Suppose that (U,ψ) is a rank $2s$ symplectic bundle over S with a quotient line bundle $\lambda\colon U \twoheadrightarrow \mathcal{O}_S$ with a fixed trivialization. Let*

$$\begin{aligned} Y_1 &= \{f \in U \mid \lambda(f) = 1\} \subset U, \\ Y_2 &= \{(e,f) \in U \oplus U \mid \lambda(f) = 1 \text{ and } \lambda(e) = 0 \text{ and } \psi(e,f) = 1\} \subset U \oplus U. \end{aligned}$$

Then the projection map $Y_1 \to S$ is an $\mathbb{A}^{2s-1}$-bundle which has a section if and only if λ is split. The projection map $Y_2 \to Y_1$ is an $\mathbb{A}^{2s-2}$-bundle which has a section, and when λ is split the composition $Y_2 \to Y_1 \to S$ is an $\mathbb{A}^{4s-3}$-bundle with a section. Moreover, giving a morphism $T \to Y$ is equivalent to giving a triple (g, e_T, f_T) with $g\colon T \to S$ a morphism and e_T, f_T sections of g^*U satisfying the three conditions (5.2).

Writing $\pi\colon Y_2 \to S$ for the projection, the universal property implies the existence of *tautological sections* e, f of π^*U satisfying the three conditions and such that the triples (g, e_T, f_T) of the lemma are the pullbacks of the tautological (π, e, f) along the classifying map $T \to Y_2$.

Proof. The statements about Y_1 follow from immediately from the previous lemma. Let $i\colon \ker\lambda \to U$ be the inclusion, and $h\colon Y_1 \to S$ the projection map. Over Y_1 we have morphisms

$$\mathcal{O}_{Y_1} \oplus h^*(\ker\lambda) \xrightarrow[\cong]{(f,i)} h^*U \xrightarrow[\cong]{\psi} h^*U^\vee \xrightarrow{f^\vee} \mathcal{O}_{Y_1}.$$

The map $f^\vee$ is surjective because $f\colon \mathcal{O}_{Y_1} \to h^*U$ is nowhere vanishing. We have $f^\vee\psi f = 0$ since ψ is alternating. So $f^\vee\psi i\colon h^*(\ker\lambda) \twoheadrightarrow \mathcal{O}_{Y_1}$ is surjective. Applying the previous lemma to $f^\vee\psi i$ gives Y_2. The projection $Y_2 \to Y_1$ is always split because by the nondegeneracy of ψ there exists a nowhere vanishing section e_0 of U such that $\lambda(?) = \psi(e_0, ?)$, and $-e_0$ gives a splitting of $f^\vee\psi i$.

When λ is split by a section $f_0\colon \mathcal{O}_S \to U$, then the two sections e_0, f_0 form a hyperbolic plane, and setting $E = \langle e_0, f_0\rangle^\perp$ and $\phi = \psi|_E$, we find that up to isometry we have $U = \mathcal{O}_S \oplus E \oplus \mathcal{O}_S$, with ψ as in (4.2), and with $\lambda\colon U \twoheadrightarrow \mathcal{O}_S$ the third projection. By construction $Y_2 \subset U \oplus U$ is the locus of points (e,f) satisfying the three conditions (5.2). The conditions $\lambda(f) = 1$ and $\lambda(e) = 0$ imply that e and f are of the forms $e = (a, u, 0)$ and $f = (b, v, 1)$ in terms of the direct sum decomposition. The condition $\psi(e,f) = 1$ is $a + \phi(u,v) = 1$, so we get $e = (1 - \phi(u,v), u, 0)$. So Y_2 is the image of the vector bundle $E \oplus E \oplus \mathcal{O}_S$ under the map sending $(u, v, b) \mapsto ((1 - \phi(u,v), u, 0), (b, v, 1))$. So it is an $\mathbb{A}^{4s-3}$-bundle with a section.

Finally, applying Lemma 5.4 twice we see that giving a morphism $T \to Y_2$ is equivalent to giving a pair (g_1, e) with $g_1\colon T \to Y_1$ a morphism and e a section of $g_1^* h^*(\ker\lambda)$ such that $f^\vee\psi i(e) = \psi(i(e), f) = 1$, and giving g_1 is equivalent to giving (g, f) with $g\colon T \to S$ a morphism and f a section of g^*U such that $\lambda(f) = 1$. Since in these equivalences one has $g = hg_1$, we see that giving $T \to Y_2$ is equivalent to giving (g, e, f) with e, f sections of g^*U satisfying (5.2). □

Suppose now that we have (E,ϕ) and (F,ψ) as in (4.2). Let $\mathrm{Sp}_S(F,\psi) \to S$ be the group scheme of automorphisms of (F,ψ). Its sections correspond to endomorphisms $\alpha\colon F \to F$ such that $\alpha^\vee\psi\alpha = \psi$. There are two maps $\rho_+\colon E \oplus \mathcal{O}_S \to \mathrm{Sp}_S(F,\psi)$ and $\rho_-\colon E \oplus \mathcal{O}_S \to \mathrm{Sp}_S(F,\psi)$ with

$$\rho_+(e,s) = \begin{pmatrix} 1_{\mathcal{O}_S} & e^\vee\phi & s \\ 0 & 1_E & e \\ 0 & 0 & 1_{\mathcal{O}_S} \end{pmatrix}, \qquad \rho_-(e_1,s_1) = \begin{pmatrix} 1_{\mathcal{O}_S} & 0 & 0 \\ e_1 & 1_E & 0 \\ s_1 & -e_1^\vee\phi & 1_{\mathcal{O}_S} \end{pmatrix} \tag{5.6}$$

These maps parametrize the unipotent radicals of the parabolic subgroups of $\mathrm{Sp}_S(F,\psi)$ associated with the two filtrations compatible with the splitting (4.2).

Lemma 5.6. *In the above situation let $\lambda = (0,0,1)\colon F = \mathcal{O}_S \oplus E \oplus \mathcal{O}_S \twoheadrightarrow \mathcal{O}_S$. Suppose we have a morphism $\pi\colon T \to S$ and two sections e, f of π^*F with $\lambda(f) = 1$ and $\lambda(e) = 0$ and $\psi(e,f) = 1$. Then e and f are of the form*

$$e = \begin{pmatrix} 1-\phi(u,v) \\ u \\ 0 \end{pmatrix}, \qquad\qquad f = \begin{pmatrix} b \\ v \\ 1 \end{pmatrix}. \tag{5.7}$$

Set $\rho = \rho_+(v,b)\rho_-(u,0)$. Then ρ is a symplectic automorphism of $\pi^*(F,\psi)$ satisfying

$$\rho\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = f, \qquad \rho\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e, \qquad \rho(E) = \langle e, f\rangle^{\perp}.$$

Proof. The conditions $\lambda(f) = 1$ and $\lambda(e) = 0$ imply that e and f can be written in the forms $e = (a, u, 0)$ and $f = (b, v, 1)$. The condition $\psi(e, f) = 1$ is $a + \phi(u, v) = 1$, so we get $e = (1 - \phi(u, v), u, 0)$. One calculates $\rho(0, 0, 1) = f$ and $\rho(1, 0, 0) = e$, and since ρ preserves the symplectic form ψ, it takes $\langle(1, 0, 0), (0, 0, 1)\rangle^{\perp} = E$ isometrically onto $\langle e, f\rangle^{\perp}$. □

Proof of Theorem 5.1. Applying Lemma 5.5 with $U = F$ and $\lambda\colon F \twoheadrightarrow \mathcal{O}_S$ the third projection, we get maps $Z_2 \to Z_1 \to S$ which are $\mathbb{A}^{2n-2}$- and $\mathbb{A}^{2n-1}$-bundles, respectively. Moreover, since λ has a splitting, the composition is an $\mathbb{A}^{4n-3}$-bundle. Over Z_2 there are tautological sections e, f of the pullback of F with a universal property.

Over Y the restriction $\lambda|_{\mathcal{U}_r}\colon \mathcal{U}_r \to \mathcal{O}_Y$ of the third projection $F \twoheadrightarrow \mathcal{O}$ is surjective. Hence Lemma 5.5, applied with $U = \mathcal{U}_r$ over Y, gives us maps $Y_2 \to Y_1 \to Y$ which are $\mathbb{A}^{2r-2}$- and $\mathbb{A}^{2r-1}$-bundles, resp., with the first one with a section, and tautological sections e, f of the pullback of $\mathcal{U}_r$ to Y_2 with a certain universal property.

We claim that the varieties Y_2 and $Z_2 \times_S \mathrm{HGr}_S(r-1, E, \phi)$ are characterized by equivalent universal properties and thus represent isomorphic functors $(\mathbf{Sm}/S)^{\mathrm{op}} \to \mathbf{Set}$. They are therefore isomorphic over S.

To give a morphism $T \to Y_2$ one gives (g, U_T, e_T, f_T) with $g\colon T \to S$ a morphism, $U_T \subset g^*F$ a symplectic subbundle of rank $2r$, and and e_T, f_T sections of U_T satisfying the three conditions (5.2).

To give a morphism $T \to Z_2 \times_S \mathrm{HP}_S(r-1, E, \phi)$ one gives (g, e_T, f_T, V_T) with $g\colon T \to S$ a morphism, e_T, f_T sections of g^*F satisfying the three conditions (5.2), and a symplectic subbundle $V_T \subset g^*E$ of rank $2r-2$.

Lemma 5.6 gives us a symplectic automorphism ρ of $g^*(F,\psi)$ with $\rho(1, 0, 0) = e_T$ and $\rho(0, 0, 1) = f_T$ and $\rho(E) = \langle e_T, f_T\rangle^{\perp}$. It follows that the data determining morphisms from T to the two varieties determine each other using the formulas

$$U_T = \langle e_T, f_T, \rho(V_T)\rangle, \qquad\qquad V_T = \rho^{-1}(U_T) \cap E.$$

So Y_2 and $Z_2 \times_S \mathrm{HP}_S(r-1, E, \phi)$ are isomorphic over S.

The last sentence of the theorem is a description of these data in the universal case $T = Y_2 \cong Z_2 \times_S \mathrm{HP}_S(r-1, E, \phi)$. □

Proof of Theorem 5.2. (b) Let $\pi_{r-1}\colon \mathrm{HGr}_S(r-1, E, \phi) \to S$ be the projection. According to the proof we have just given of Theorem 5.1 there is a morphism

$$s\colon\ \mathrm{HGr}_S(r-1, E, \phi) \to Z_2 \times_S \mathrm{HGr}_S(r-1, E, \phi)$$

corresponding to the data $(\pi_{r-1}, (1,0,0), (0,0,1), \mathcal{U}_{r-1})$. It is a section of the second projection. It composition with the isomorphism and projections

$$\mathrm{HGr}_S(r-1,E,\phi) \xrightarrow{s} Z_2 \times_S \mathrm{HGr}_S(r-1,E,\phi) \cong Y_2 \xrightarrow{g_2} Y_1 \xrightarrow{g_1} Y$$

is the map classified by the symplectic subbundle $\mathcal{O} \oplus \mathcal{U}_{r-1} \oplus \mathcal{O} \subset \pi_{r-1}^* F$, which is the σ of the statement of the theorem. By Theorem 5.1 g_1 and g_2 are affine bundles, and s is the section of an affine bundle. So g_1^A, g_2^A, s^A and their composition σ^A are isomorphisms in any cohomology theory A.

(a) For $r = 1$ we have $\mathrm{HGr}_S(0,E,\phi) = S$, and $\sigma \colon S \to Y$ is a section of $t \colon Y \to S$. By (b) σ^A is an isomorphism, and we have $1_S^A = t^A \sigma^A$, so t^A is an isomorphism. □

We conclude this section by proving Theorem 3.4.

Lemma 5.7. *The fibration $Y_1 \to Y$ of Lemma 5.5 is a torsor under the unipotent group scheme $R_u \subset \mathrm{Sp}(U,\psi)$ of automorphisms of U respecting the filtration $0 \subset (\ker\lambda)^\perp \subset \ker\lambda \subset U$ and the trivialization $U/\ker\lambda \cong \mathcal{O}_Y$ and inducing the identity on $\ker\lambda/(\ker\lambda)^\perp$.*

This is easily seen when the filtration is split with $U = F = \mathcal{O}_S \oplus E \oplus \mathcal{O}_S$ and R_u is the $\rho_+(E \oplus \mathcal{O}_S)$ of (5.6).

We now return to the decomposition $\mathrm{HP}^n = \bigsqcup_{i=0}^n X_{2i}$ of Theorem 3.1.

Proof of Theorem 3.4. (a) We first look at the open stratum X_0. Let (U,ψ) be the tautological rank 2 symplectic subbundle over $X_0 \subset \mathrm{HP}^n$. By Lemmas 5.5 and 5.7 the fibration $Y_2 = Y_1 \to X_0$ is a torsor under $R_u = \mathbb{G}_a$, and

$$Y_2 = Y_1 = \{(e,f) \mid \mathbb{A}^{2n+2} \times \mathbb{A}^{2n+2} \mid \lambda(f) = 1,\ \lambda(e) = 0,\ \psi(e,f) = 1\},$$

with the action of $\mathbb{G}_a$ given by $t \cdot (e,f) = (e, f + te)$. Writing $f = (b_1, \dots, b_{2n}, b_{2n+1}, 1)$ and $e = (a_1, \dots, a_{2n}, 1 - \sum_{j=1}^n (a_{2j-1} b_{2j} - a_{2j} b_{2j-1}), 0)$ gives the presentation of X_0 as a quotient of a free action of $\mathbb{G}_a$ on $\mathbb{A}^{4n+1}$. The argument for the other X_{2i} is similar.

(b) This follows from (a) or directly from Theorem 5.2(a). □

6. Deformation to the normal bundle

The localization sequence for Grassmannians (4.1) exists for all cohomology theories, but it is particularly well-behaved for oriented theories. (See Balmer-Calmès [4] for what can happen in a non-oriented theory.) So we should now start discussing the class of cohomology theories for which the localization sequence for quaternionic Grassmannians of Corollary 5.3 is well-behaved. But before doing so we discuss the deformation to the normal bundle construction. Our notation follows Nenashev [13].

The *deformation space* $D(Z,X)$ for a closed embedding $Z \hookrightarrow X$ of regular schemes is the complement of the strict transform of $Z \times 0$ in the blowup $\mathrm{bl}_{Z\times 0}(X \times \mathbb{A}^1)$. The inclusion and blowup maps compose to a map $D(Z,X) \to X \times \mathbb{A}^1$, so $D(Z,X)$ has projections $p \colon D(Z,X) \to X$ and $D(Z,X) \to \mathbb{A}^1$. The fiber of $D(Z,X)$ over $0 \in \mathbb{A}^1$ is $N_{Z/X}$, and its fiber over $1 \in \mathbb{A}^1$ is X. The strict transform of $Z \times \mathbb{A}^1$ is isomorphic to $Z \times \mathbb{A}^1$, and the restriction of p to it is the first projection $Z \times \mathbb{A}^1 \to Z$. We thus have maps

$$\begin{array}{ccccc} N_{Z/X} & \xrightarrow{\quad i_0 \quad} & D(Z,X) & \underset{p}{\overset{i_1}{\rightleftarrows}} & X, \\ A_Z(N_{Z/X}) & \xleftarrow{\ i_0^A\ } & A_{Z\times\mathbb{A}^1}(D(Z,X)) & \xrightarrow{\ i_1^A\ } & A_Z(X). \end{array} \tag{6.1}$$

The map $p^A\colon A(X) \to A(D(Z,X))$ makes i_0^A and i_1^A into morphisms of $A(X)$-bimodules. Because of the blowup geometry the $A(X)$-bimodule structure on $A_Z(N_{Z/X})$ factors through the restriction $A(X) \to A(Z) \cong A(N_{Z/X})$. When the maps i_0^A and i_1^A are isomorphisms, as they frequently are, their composition

$$d^A_{Z/X}\colon A_Z(N_{Z/X}) \xrightarrow{\cong} A_Z(X) \tag{6.2}$$

is the *deformation to the normal bundle isomorphism.* The functoriality of the space $D(Z,X)$ makes the deformation to the normal bundle isomorphisms functorial.

$$\begin{tikzcd} Z' = Z \times_X X' \arrow[r, "g'"] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ X' \arrow[r, "g"] & X \end{tikzcd} \qquad \begin{tikzcd} N_{Z'/X'} \cong N_{Z/X} \times_Z Z' \arrow[r, "g_N"] \arrow[d] & N_{X/Z} \arrow[d] \\ Z' \arrow[r, "g'"] & Z \end{tikzcd} \tag{6.3}$$

Proposition 6.1. *Suppose that in* (6.3) *we have a pullback square of schemes with i and i' closed embeddings and that that induces a pullback square of normal bundles. Then in any cohomology theory A for which the deformation to the normal bundle isomorphisms $d^A_{Z'/X'}$ and $d^A_{Z/X}$ exist, we have $d^A_{Z'/X'} \circ g^A_N = g^A \circ d^A_{Z/X}$.*

The main existence result is the following.

Theorem 6.2 ([14, Theorem 2.2])**.** *When $Z \hookrightarrow X$ is a closed embedding of smooth quasi-projective varieties over a field, the deformation to the normal bundle isomorphism $d^A_{Z/X}$ exists for any cohomology theory A.*

Dévissage theorems such as Quillen's for K-theory or Gille's for Witt groups [7] produce Thom isomorphisms $A(Z) \cong A_Z(X)$ directly. The compatibility of these morphisms with the i_0^A and i_1^A of (6.1) gives deformation to the normal bundle isomorphisms for all closed embedding of regular schemes for these theories.

The proof of Theorem 6.2 uses Nisnevich tubular neighborhoods which do not always exist in mixed characteristic. But we only need a simple case.

A regular closed subscheme of a regular scheme $i\colon Z \hookrightarrow X$ has a *Zariski tubular neighborhood* if there exists a open subscheme $U \subset X$ containing Z and an open embedding $U \hookrightarrow N_{Z/X}$ such that the composition $Z \hookrightarrow U \hookrightarrow N_{Z/X}$ is the zero section.

Proposition 6.3. *If $Z \hookrightarrow X$ has a Zariski tubular neighborhood, then the deformation to the normal bundle isomorphism $d^A_{Z/X}$ exists for any cohomology theory A.*

Proof. If $U \subset Y$ is an open embedding with Z closed in both U and Y, then the isomorphism $d^A_{Z/U}$ exists if and only if $d^A_{Z/Y}$ exists because $D(Z,U)$ is an open subscheme of $D(Z,Y)$ in which $Z \times \mathbb{A}^1$ is closed, so Zariski excision provides isomorphisms between the sources and targets of the different maps i_0^A and i_1^A.

Hence given that for the vector bundle $N_{Z/X}$ the deformation to the normal bundle isomorphism $d_{Z/N_{Z/X}}$ exists and is the identity by a local calculation, $d_{Z/U}$ and $d_{Z/X}$ also exist. $\square$

Proposition 6.4. *In the situation of Theorem 4.1 or* (5.1) *the deformation to the normal bundle isomorphisms $d^A(N^+, \mathrm{HP}(F,\psi))\colon A_{N^+}(\mathrm{HP}(F,\psi)) \to A_{N^+}(\mathcal{U}_F|_{N^+})$ exist for any cohomology theory A.*

Proof. First, by Theorem 4.1(a)(b) N^+ is a direct summand of the normal bundle $N = N^+ \oplus N^-$ of $\mathrm{HP}_S(E,\phi) \subset \mathrm{HP}_S(F,\psi)$, which embeds as an open subscheme $N \subset \mathrm{Gr}_S(2,F)$ overlapping the open subscheme $\mathrm{HP}_S(F,\psi)$. It follows that the projection map $N \twoheadrightarrow N^+$ is also the structural map of the normal bundle of N^+ in $\mathrm{HP}_S(F,\psi)$ and that $N \cap \mathrm{HP}_S(F,\psi)$ is a Zariski tubular neighborhood of N^+ in $\mathrm{HP}_S(F,\psi)$. Therefore by Proposition 6.3 the deformation to the normal bundles isomorphism $d^B(N^+, \mathrm{HP}_S(F,\psi))$ exists.

Second, by Theorem 4.1(d) N^+ is the transversal intersection of a section s_+ of $\mathcal{U}_F$ with the zero section. So the normal bundle of N^+ in $\mathrm{HP}_S(F,\psi)$ is naturally isomorphic to $\mathcal{U}_F|_{N^+}$. □

7. Symplectic Thom structures

An *oriented cohomology theory* in the sense of Panin and Smirnov [14, Definition 3.1] is a ring cohomology theory such that for every vector bundle $E \to X$ over a nonsingular variety and closed subset $Z \subset X$ there is a distinguished *Thom isomorphism*

$$\mathrm{th}^E_Z \colon A_Z(X) \to A_Z(E)$$

of $A(X)$-bimodules satisfying several axioms. Using deformation to the normal bundle, Thom isomorphisms induce isomorphisms $A_Z(X) \cong A_Z(Y)$ and in particular $A(X) \cong A_X(Y)$ for any closed embedding of smooth varieties $X \hookrightarrow Y$ over a field for oriented cohomology theories.

Some cohomology theories have Thom isomorphisms only for vector bundles with some extra structure. For instance Balmer's derived Witt groups [3] have Thom isomorphisms $W^i(X) \to W^{i+n}_X(E)$ for triples (E, L, λ) with E a vector bundle of rank n, L a line bundle and $\lambda \colon L \otimes L \cong \det E$ an isomorphism. This is because the isomorphisms depending on E alone involve a twisted Witt group $W^i(X, \det E) \cong W^{i+n}_X(E)$ (see for instance [5, 7, 13]), and the (L,λ) are required to specify an isomorphism $W^i(X) \cong W^i(X, \det E)$. Isometric symmetric bilinear line bundles (L,λ) induce the same isomorphism, and the isometry classes of possible (L,λ) for a given E are a torsor under $H^1(X_{\text{ét}}, \boldsymbol{\mu}_2)$. The extra structure is very reminiscent of Spin and Spin^c structures in differential topology (e.g. [1, 2]) even to the point of having the same group $H^1(X, \mathbb{Z}/2)$ parametrizing the choices of Spin structure if X is a complex projective variety. Derived Witt groups might therefore be called *an* SL^c*-oriented* cohomology theory.

A cohomology theory is *symplectically oriented* if to each symplectic bundle (E,ϕ) over a variety X and each closed subset $Z \subset X$ there is an isomorphism $\mathrm{th}^{E,\phi}_Z \colon A_Z(X) \to A_Z(E)$ satisfying the properties which we will describe in detail in Definition 14.3.

Oriented cohomology theories are also symplectically oriented. Witt groups, Witt cohomology, hermitian K-theory and oriented Chow groups (also called Chow-Witt groups or hermitian K-cohomology) are symplectically oriented as are symplectic and special linear algebraic cobordism $\mathrm{MSp}^{*,*}$ and $\mathrm{MSL}^{*,*}$ defined along the lines of Voevodsky's [15, §6.3] algebraic cobordism $\mathrm{MGL}^{*,*}$.

We will actually prove our quaternionic projective bundle theorem using a structure which is weaker *a priori* but is equivalent in the end.

Definition 7.1. A *symplectic Thom structure* on a ring cohomology theory A on a category of schemes is rule which assigns to each rank 2 symplectic bundle (E,ϕ) over a scheme X in the category an $A(X)$-central element $\mathrm{th}(E,\phi) \in A_X(E)$ with the following properties:

(1) For an isomorphism $u \colon (E,\phi) \cong (E_1,\phi_1)$ one has $\mathrm{th}(E,\phi) = u^A \,\mathrm{th}(E_1,\phi_1)$.
(2) For a morphism $f \colon Y \to X$ with pullback map $f_E \colon f^*E \to E$ one has $f^A_E(\mathrm{th}(E,\phi)) = \mathrm{th}(f^*(E,\phi)) \in A_Y(f^*E)$.

(3) For the trivial rank 2 bundle $X \times \mathbb{A}^2 \to X$ with symplectic structure given by $\mathrm{h}(\mathcal{O}_X) = \big(\mathcal{O}_X^{\oplus 2}, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\big)$, the map $\cup\,\mathrm{th}(\mathrm{h}(\mathcal{O}_X)) \colon A(X) \to A_X(X \times \mathbb{A}^2)$ is an isomorphism.

Global nondegeneracy follows from the nondegeneracy condition above via local trivializations of the symplectic bundle and a Mayer-Vietoris argument.

Proposition 7.2. *Let A be a ring cohomology theory with a symplectic Thom structure. Then for any rank 2 symplectic bundle (E, ϕ) over a scheme X the map $\cup\,\mathrm{th}(E, \phi) \colon A(X) \to A_X(E)$ is an isomorphism.*

For (E, ϕ) a rank 2 symplectic bundle over a scheme X, let $e^A \colon A_X(E) \to A(E)$ be the extension of supports map, and let $z^A \colon A(E) \to A(X)$ be the restriction to the zero section (or to any section). Define the *Pontryagin class* of (E, ϕ) as

$$p(E, \phi) = -z^A e^A(\mathrm{th}(E, \phi)) \in A(X). \tag{7.1}$$

The sign follows the convention in differential topology (for instance Milnor-Stasheff [11]) where one has $p_i(\xi) = (-1)^i c_{2i}(\xi)$ for a real or quaternionic vector bundle ξ. Like the symplectic Thom classes, the Pontryagin classes are $A(X)$-central and are functorial with respect to isomorphisms of symplectic bundles over X and with respect to pullbacks along maps $X' \to X$.

Now suppose that $i \colon X \to Y$ is a closed embedding of codimension 2 of regular schemes with a normal bundle $N = N_{X/Y}$ equipped with a symplectic form ϕ. We then say that $i \colon X \to Y$ has a *symplectic normal bundle* (N, ϕ). We can compose the isomorphism of Proposition 7.2 with the deformation to the normal bundle isomorphism when the latter exists

$$i_{A,\flat} \colon A(X) \xrightarrow[\cong]{\cup\,\mathrm{th}(N,\phi)} A_X(N) \xrightarrow[\cong]{d_{X/Y}} A_X(Y). \tag{7.2}$$

The isomorphism $i_{A,\flat}$ is a *Thom isomorphism*, while its composition $i_{A,\natural} \colon A(X) \to A_X(Y) \to A(Y)$ with the extension of supports is a *direct image map*. The symbols $\flat$ and $\natural$ are placeholders indicating that the maps depend on more than i and A, namely the symplectic form ϕ and the symplectic Thom structure. The dependence on ϕ is very real in certain theories. For Witt groups, replacing ϕ by $a\phi$ with $a \in k^\times$ multiplies the direct image map by the class $\langle a \rangle \in W(k)$.

We will need several facts about the Thom isomorphisms. For a section of a rank 2 symplectic bundle (E, ϕ) the deformation to the normal bundle map is the identity. Hence:

Proposition 7.3. *Let $z \colon X \to E$ be the zero section of a rank 2 symplectic bundle (E, ϕ) over X with symplectic normal bundle $(N_{X/E}, \phi) = (E, \phi)$, then the Thom isomorphism $z_{A,\flat} \colon A(X) \to A_X(E)$ coincides with $\cup\,\mathrm{th}(E, \phi)$.*

Now suppose we have a pullback diagram

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "i'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "i"] & Y \end{tikzcd} \tag{7.3}$$

Proposition 6.1 and the functoriality of Thom classes give us the next lemma.

Proposition 7.4. *If in the pullback square* (7.3) *all the schemes are regular, and i and i' are closed embeddings of codimension 2 with symplectic normal bundles $(N_{X/Y}, \phi)$ and*

$(N_{X'/Y'}, g'^*\phi) \cong (g'^* N_{X/Y}, g'^*\phi)$ *and the deformation to the normal bundle isomorphisms exist, then we have* $g^A i_{A,\flat} = i'_{A,\flat} g'^A$.

The $A(X)$-centrality of $\mathrm{th}(N,\phi)$ means that $\cup\,\mathrm{th}(N,\phi)$ is an isomorphism of two-sided $A(X)$-modules and via i^A also of two-sided $A(Y)$-modules. The deformation to the normal bundle maps $\nu_{X/Y}\colon A_X(Y) \to A_X(N)$ are also isomorphisms of two-sided $A(Y)$-modules as discussed earlier. We therefore have the next proposition.

Proposition 7.5. *The Thom isomorphism* $i_{A,\flat}\colon A(X) \to A_X(Y)$ *is a two-sided* $A(Y)$*-module isomorphism, and* $i_{A,\flat}(1)$ *is* $A(Y)$*-central. Thus for* $a \in A_X(Y)$ *and* $b \in A(Y)$ *we have*

$$i_{A,\flat}(a \cup i^A b) = i_{A,\flat}(a) \cup b, \qquad i_{A,\flat}(i^A b \cup a) = b \cup i_{A,\flat}(a),$$
$$i_{A,\flat} i^A(b) = i_{A,\flat}(1) \cup b = b \cup i_{A,\flat}(1).$$

We now prove some key formulas.

Proposition 7.6. *Let* (E,ϕ) *be a rank* 2 *symplectic bundle over a regular scheme* Y *with a section* $s\colon Y \to E$ *meeting the zero section* z *tranversally in* X*. Suppose that* A *is a ring cohomology theory with a symplectic Thom structure, and let* $\overline{e}^A\colon A_X(Y) \to A(Y)$ *be the extension of supports map. Then we have*

$$p(E,\phi) = -\overline{e}^A s^A(\mathrm{th}(E,\phi)), \tag{7.4}$$

Moreover, if the inclusion $i\colon X \hookrightarrow Y$ *has a deformation to the normal bundle isomorphism, then for all* $b \in A(Y)$ *we have*

$$i_{A,\natural} i^A(b) = \overline{e}^A i_{A,\flat} i^A(b) = -b \cup p(E,\phi). \tag{7.5}$$

Proof. In the diagram

$$\begin{tikzcd}
A(Y) \arrow[r, "\cup\,\mathrm{th}(E,\phi)", "= z_{A,\flat}"'] \arrow[d, "i^A"] & A_Y(E) \arrow[r, "e^A"] \arrow[d, "s^A"] & A(E) \arrow[d, shift right, "s^A"'] \arrow[d, bend left=60, "z^A"] \\
A(X) \arrow[r, "i_{A,\flat}"'] & A_X(Y) \arrow[r, "\overline{e}^A"] & A(Y) \arrow[u, shift right, "\pi^A"']
\end{tikzcd}$$

the righthand rectangle commutes by functoriality. The pullbacks along the two sections of $\pi\colon E \to Y$ are left inverses of the same isomorphism π^A, so they satisfy $s^A = z^A$. We get

$$p(E,\phi) = -z^A e^A(\mathrm{th}(E,\phi)) = -s^A e^A(\mathrm{th}(E,\phi)) = -\overline{e}^A s^A(\mathrm{th}(E,\phi)).$$

The lefthand rectangle of the diagram commutes using the label $z_{A,\flat}$ by Proposition 7.4, and the equality $z_{A,\flat} = \cup\,\mathrm{th}(E,\phi)$ is Proposition 7.3. It follows that $\overline{e}^A i_{A,\flat} i^A(b) = i_{A,\natural} i^A(b)$ is the same as $\overline{e}^A s^A(b \cup \mathrm{th}(E,\phi))$. Since all the maps are two-sided $A(Y)$-modules maps, that is the same as $b \cup \overline{e}^A s^A(\mathrm{th}(E,\phi)) = -b \cup p(E,\phi)$ using formula (7.4). $\square$

Proposition 7.7. *For any ring cohomology theory with a symplectic Thom structure a rank* 2 *symplectic bundle* (E,ϕ) *with a nowhere vanishing section has* $p(E,\phi) = 0$.

Proof. The nowhere vanishing section $s\colon Y \to E$ meets the zero section in $\varnothing$. Since we have $\mathrm{th}(E,\phi) \in A_Y(E)$, we have $s^A(\mathrm{th}(E,\phi)) \in A_\varnothing(Y) = 0$. Then formula (7.4) gives $p(E,\phi) = -\overline{e}^A s^A(\mathrm{th}(E,\phi)) = 0$. $\square$

8. The quaternionic projective bundle theorem

We return to the situation where (V, ϕ) is a symplectic space of dimension $2n+2$, and $\mathrm{HP}^n = \mathrm{Gr}(2, V) \smallsetminus \mathrm{GrSp}(2, V, \phi)$ is the affine scheme parametrizing 2-dimensional subspaces $U \subset V$ such that $\phi|_U$ is nondegenerate.

Theorem 8.1 (Quaternionic projective bundle theorem for trivial bundles)**.** *Let A be a ring cohomology theory with a symplectic Thom structure. Let $(\mathcal{U}, \phi|_{\mathcal{U}})$ be the tautological rank 2 symplectic subbundle over HP^n and $\zeta = p(\mathcal{U}, \phi|_{\mathcal{U}})$ its Pontryagin class. Then for any scheme S we have $A(\mathrm{HP}^n \times S) = A(S)[\zeta]/(\zeta^{n+1})$.*

Proof. It is enough to consider the case $S = k$.

We need to show that $(1, \zeta, \dots, \zeta^n)\colon A(k)^{\oplus(n+1)} \to A(\mathrm{HP}^n)$ is an isomorphism of two-sided $A(k)$-modules and that we have $\zeta^{n+1} = 0$.

We go by induction on n. For $n = 0$ we have $\mathrm{HP}^0 = k$, and the tautological rank 2 symplectic subbundle of the trivial symplectic bundle of rank 2 over k is the trivial bundle. Its Pontryagin class verifies $\zeta = 0$ by Proposition 7.7.

For $n \geq 1$ we have morphisms smooth schemes by Theorem 3.1 or 4.1.

$$\begin{tikzcd}
N^+ = \overline{X}_2 \arrow[r, "i", "\text{closed}"'] \arrow[d, "q", "\mathbb{A}^2\text{-bundle}"'] & \mathrm{HP}^n \arrow[dr] & Y = X_0 \arrow[l, hook', "\text{open}"', "j"] \arrow[d, "f"] \\
\mathrm{HP}^{n-1} & & k
\end{tikzcd} \tag{8.1}$$

This yields a localization exact sequence and various maps of two-sided $A(k)$-modules.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial\,[=0]"] & A_{N^+}(\mathrm{HP}^n) \arrow[r, "e^A"] & A(\mathrm{HP}^n) \arrow[r, "j^A"] \arrow[dl, "i^A"] & A(Y) \arrow[r, "\partial\,[=0]"] & \cdots \\
A(\mathrm{HP}^{n-1}) \arrow[r, "q^A", "\cong"'] & A(N^+) \arrow[u, "i_{A,\flat}", "\cong"'] & & A(k) \arrow[u, "\cong", "t^A"'] \arrow[ul, "\cup 1"] &
\end{tikzcd} \tag{8.2}$$

The map $t^A\colon A(k) \to A(Y)$ is an isomorphism by Theorem 3.4 or 5.2, so j^A is an epimorphism split by $\cup 1$, the boundary map ∂ vanishes, and $(1, e^A)\colon A(k) \oplus A_{N^+}(\mathrm{HP}^n) \xrightarrow{\cong} A(\mathrm{HP}^n)$ is an isomorphism.

The map $q^A\colon A(\mathrm{HP}^{n-1}) \to A(N^+)$ is an isomorphism because q is an $\mathbb{A}^2$-bundle by Theorem 3.2 or 4.1. The locus $N^+ \subset \mathrm{HP}^n$ is the transversal intersection of the section s_+ of the rank 2 symplectic bundle $(\mathcal{U}, \psi)$ and the zero section. So it has a symplectic normal bundle $(\mathcal{U}, \psi)|_{N^+}$. In addition the deformation to the normal bundle isomorphisms $d(N^+, \mathrm{HP}^n)$ exist by Proposition 6.4. So the Thom isomorphisms $i_{A,\flat}\colon A(N^+) \to A_{N^+}(\mathrm{HP}^n)$ are defined. Writing $\tau = -e^A i_{A,\flat} q^A$, we get an isomorphism of two-sided $A(k)$-modules $(1, \tau)\colon A(k) \oplus A(\mathrm{HP}^{n-1}) \xrightarrow{\cong} A(\mathrm{HP}^n)$.

Write $(\mathcal{V}, \overline{\phi}|_{\mathcal{V}})$ for the tautological rank 2 symplectic subbundle on HP^{n-1} and $\xi = p(\mathcal{V}, \overline{\phi}|_{\mathcal{V}})$ for its Pontryagin class. By induction we have an isomorphism of two-sided $A(k)$-modules $(1, \xi, \dots, \xi^{n-1})\colon A(k)^{\oplus n} \xrightarrow{\cong} A(\mathrm{HP}^{n-1})$. We therefore have an isomorphism

$$(1, \tau(1), \tau(\xi), \dots, \tau(\xi^{n-1}))\colon A(k)^{\oplus(n+1)} \xrightarrow{\cong} A(\mathrm{HP}^n).$$

By Theorem 3.2 or 4.1 the two symplectic bundles $q^*(\mathcal{V}, \overline{\phi}|_{\mathcal{V}})$ and $i^*(\mathcal{U}, \phi|_{\mathcal{U}})$ on N^+ are isomorphic. By functoriality of the Pontryagin classes, this gives $q^A\xi = i^A\zeta$ and therefore also $q^A(\xi^\ell) = i^A(\zeta^\ell)$. We also have $-e^A i_{A,\flat} i^A(b) = \zeta \cup b$ for $b \in A(\mathrm{HP}^n)$ by Proposition 7.6. This gives us

$$\tau(\xi^\ell) = -e^A i_{A,\flat} q^A(\xi^\ell) = -e^A i_{A,\flat} i^A(\zeta^\ell) = \zeta \cup \zeta^\ell = \zeta^{\ell+1}$$

for all ℓ. This gives the desired isomorphism

$$(1, \zeta, \zeta^2, \dots, \zeta^n) \colon A(k)^{\oplus(n+1)} \xrightarrow{\cong} A(\mathrm{HP}^n).$$

Finally, by induction we have $\xi^n = 0$, which gives $\zeta^{n+1} = \tau(\xi^n) = 0$. □

If (E, ϕ) is a rank $2n$ symplectic bundle over a scheme S, we can define a quaternionic projective bundle $\mathrm{HP}_S(E, \phi) = \mathrm{Gr}_S(2, E) \smallsetminus \mathrm{GrSp}_S(2, E, \phi)$. A Mayer-Vietoris argument gives the general quaternionic projective bundle theorem.

Theorem 8.2. (Quaternionic projective bundle theorem). *Let A be a ring cohomology theory with a symplectic Thom structure. Let (E, ϕ) be a rank $2n$ symplectic bundle over a scheme S, let $(\mathcal{U}, \phi|_{\mathcal{U}})$ be the tautological rank 2 symplectic subbundle over the quaternionic projective bundle $\mathrm{HP}_S(E, \phi)$, and let $\zeta = p(\mathcal{U}, \phi|_{\mathcal{U}})$ be its Pontryagin class. Write $\pi \colon \mathrm{HP}_S(E, \phi) \to S$ for the projection. Then for any closed subset $Z \subset X$ we have an isomorphism of two-sided $A(S)$-modules $(1, \zeta, \dots, \zeta^{n-1}) \colon A_Z(S)^{\oplus n} \xrightarrow{\cong} A_{\pi^{-1}(Z)}(\mathrm{HP}_S(E, \phi))$, and we have unique classes $p_i(E, \phi) \in A(S)$ for $1 \le i \le n$ such that there is a relation*

$$\zeta^n - p_1(E, \phi) \cup \zeta^{n-1} + p_2(E, \phi) \cup \zeta^{n-2} - \cdots + (-1)^n p_n(E, \phi) = 0.$$

If (E, ϕ) is trivial, then $p_i(E, \phi) = 0$ for $1 \le i \le n$.

Definition 8.3. The classes $p_i(E, \phi)$ $(i = 1, \dots, n)$ of Theorem 8.2 are the *Pontryagin classes* of (E, ϕ) with respect to the symplectic Thom structure of the ring cohomology theory A. For $i > n$ one sets $p_i(E, \phi) = 0$, and one sets $p_0(E, \phi) = 1$ and $p_i(E, \phi) = 0$ for $i < 0$.

The Pontryagin classes are universally $A(S)$-central, because they are the components of the universally $A(S)$-central ζ^n in a two-sided $A(S)$-module decomposition $A(\mathrm{HP}_S(E, \phi)) \cong A(S)^{\oplus n}$ which is compatible with base change.

The Pontryagin classes are compatible with base change. This implies that they are $\mathbb{A}^1$-deformation invariant in the following sense.

Proposition 8.4. *Let (E_0, ϕ_0) and (E_1, ϕ_1) be symplectic bundles on a scheme S. Suppose there exists a symplectic bundle (E, ϕ) on $S \times \mathbb{A}^1$ with $(E, \phi)|_{S \times \{0\}} \cong (E_0, \phi_0)$ and $(E, \phi)|_{S \times \{1\}} \cong (E_1, \phi_1)$. Then we have $p_i(E_0, \phi_0) = p_i(E_1, \phi_1)$ for all i.*

A vector bundle L has an associated *hyperbolic symplectic bundle* $\mathrm{h}(L) = \big(L \oplus L^\vee, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\big)$.

Proposition 8.5. *Suppose that (E, ϕ) is a symplectic bundle over a scheme S with a sublagrangian subbundle $L \subset E$. Let $(E_0, \phi_0) = (L^\perp/L, \overline{\phi}) \oplus \mathrm{h}(L)$. Then we have $p_i(E, \phi) = p_i(E_0, \phi_0)$ for all i.*

This is because there exists a symplectic bundle (E_1, ϕ_1) over $S \times \mathbb{A}^1$ with $(E_1, \phi_1)|_{S \times \{t\}} \cong (E, \phi)$ for $t \neq 0$ and $(E_1, \phi_1)|_{S \times \{0\}} \cong (E_0, \phi_0)$.

Theorem 8.6 (Nilpotence)**.** *Let (E, ϕ) be a symplectic bundle on a scheme X. Then its Pontryagin classes $p_i(E, \phi) \in A(X)$ are nilpotent.*

Proof. Recall from §2 that we are assuming that our schemes are quasi-compact. So we may cover X by n open subsets U_α such that (E,ϕ) trivializes over each U_α. We show that all products $p_{i_1}(E,\phi)\cup\cdots\cup p_{i_n}(E,\phi)$ of n Pontryagin classes of (E,ϕ) vanish.

For each U_α write $Z_\alpha = X \smallsetminus U_\alpha$. The restriction of the Pontryagin class $p_{i_\alpha}(E,\phi)$ to U_α vanishes because it is a Pontryagin class of a symplectic bundle trivial on U_α. So $p_{i_\alpha}(E,\phi)$ is the image of a class in $A_{Z_\alpha}(X)$ under extension of supports. It follows that $p_{i_1}(E,\phi)\cup\cdots\cup p_{i_n}(E,\phi)$ is the image of a class in $A_{Z_1\cap\cdots\cap Z_n}(X) = A_\varnothing X = 0$ under extension of supports. □

9. Asymptotic cohomology of quaternionic flag varieties

The direct system of trivial symplectic bundles $\mathrm{h}(\mathcal{O}) \rightarrowtail \mathrm{h}(\mathcal{O}^{\oplus 2}) \rightarrowtail \mathrm{h}(\mathcal{O}^{\oplus 3}) \rightarrowtail \cdots$ over the base generates a direct system of quaternionic projective spaces $\mathrm{HP}^0 \xrightarrow{i_0} \mathrm{HP}^1 \xrightarrow{i_1} \mathrm{HP}^2 \to \cdots$ and an inverse system of cohomology rings $\cdots \to A(\mathrm{HP}^2) \to A(\mathrm{HP}^1) \to A(\mathrm{HP}^0)$. Each HP^n has a rank 2 universal subbundle $(\mathcal{U}_n,\phi_n)$, and under the inclusion maps we have isomorphisms $i_{n-1}^*(\mathcal{U}_n,\phi_n) \cong (\mathcal{U}_{n-1},\phi_{n-1})$. Theorem 8.1 and the functoriality of Pontryagin classes gives us the following theorem.

Theorem 9.1. *Let A be a ring cohomology theory with a symplectic Thom structure. Then each map $i_{n-1}^A\colon A(\mathrm{HP}^n) \to A(\mathrm{HP}^{n-1})$ in the inverse system of cohomology rings is surjective, and we have an isomorphism $\varprojlim A(\mathrm{HP}^n) \cong A(k)[[y]]$ with the indeterminate y corresponding to the element of $\varprojlim A(\mathrm{HP}^n)$ given by the system of elements $(p(\mathcal{U}_n,\phi_n))_{n\in\mathbb{N}}$.*

More generally, let $r \geq 1$. For any $n \geq r$ we will write $\mathrm{HFlag}(1^r;n) = \mathrm{HFlag}(1,\dots,1;n)$ with the 1 repeated r times. The system of trivial symplectic bundles also generates a direct system of flag bundles

$$\mathrm{HFlag}(1^r;r) \rightarrowtail \mathrm{HFlag}(1^r;r+1) \rightarrowtail \mathrm{HFlag}(1^r;r+2) \rightarrowtail \cdots$$

and an inverse system of cohomology rings

$$\cdots \to A(\mathrm{HFlag}(1^r;r+2)) \to A(\mathrm{HFlag}(1^r;r+1)) \to A(\mathrm{HFlag}(1^r;r)).$$

Each $\mathrm{HFlag}(1^r;n)$ has r rank 2 universal symplectic subbundles $(\mathcal{U}_n^{(1)},\phi_n^{(1)}),\dots,(\mathcal{U}_n^{(r)},\phi_n^{(r)})$ plus a rank $2n-2r$ universal symplectic subbundle $(\mathcal{V}_{r,n},\psi_{r,n})$ and a decomposition as an orthogonal direct sum

$$\mathrm{h}(\mathcal{O}^{\oplus n}) \cong (\mathcal{U}_n^{(1)},\phi_n^{(1)}) \perp \cdots \perp (\mathcal{U}_n^{(r)},\phi_n^{(r)}) \perp (\mathcal{V}_{r,n},\psi_{r,n}) \tag{9.1}$$

For the inclusion maps $j_n\colon \mathrm{HFlag}(1^r;n) \to \mathrm{HFlag}(1^r;n+1)$ we have natural isomorphisms $j_n^*(\mathcal{U}_{n+1}^{(i)},\phi_{n+1}^{(i)}) \cong (\mathcal{U}_n^{(i)},\phi_n^{(i)})$ for $i=1,\dots,r$. The rest of this section will be devoted to proving the following theorem.

Theorem 9.2. *Let A be a ring cohomology theory with a symplectic Thom structure. Let $r \geq 1$ be an integer. Then the maps $j_n^A\colon A(\mathrm{HFlag}(1^r;n+1)) \to A(\mathrm{HFlag}(1^r;n))$ are surjective, and we have*

$$\varprojlim_{n\to\infty} A(\mathrm{HFlag}(1^r;n)) \cong A(k)[[y_1,\dots,y_r]] \tag{9.2}$$

with the indeterminate y_i corresponding to the element of $\varprojlim A(\mathrm{HFlag}(1^r;n))$ given by the system $(p(\mathcal{U}_n^{(i)},\phi_n^{(i)}))_{n\geq r}$.

The *Pontryagin polynomial* of a symplectic bundle (E, ϕ) of rank $2r$ is

$$P_{E,\phi}(t) = t^r - p_1(E,\phi)t^{r-1} + p_2(E,\phi)t^{r-2} - \cdots + (-1)^r p_r(E,\phi),$$

while the *total Pontryagin class* is

$$p_t(E,\phi) = 1 + p_1(E,\phi)t + \cdots + p_r(E,\phi)t^r.$$

We will need the following lemma, which is a weak version of the Cartan sum formula.

Lemma 9.3. *Suppose the symplectic bundle (E, ϕ) is an orthogonal direct summand of the symplectic bundle (F, ψ) on X. Then the Pontryagin polynomial $P_{E,\phi}(t)$ divides the Pontryagin polynomial $P_{F,\psi}(t)$ in $A(X)[t]$.*

Proof. There is an embedding $i\colon \mathrm{HP}_X(E,\phi) \subset \mathrm{HP}_X(F,\psi)$ such that the tautological rank 2 symplectic subbundle $(\mathcal{U}_1, \psi|_{U_1})$ on $\mathrm{HP}_X(F,\psi)$ restricts to the tautological rank 2 symplectic subbundle $(\mathcal{U}_2, \phi|_{U_2})$ of $\mathrm{HP}_X(E,\phi)$. Hence $i^A\colon A(\mathrm{HP}_X(F,\psi)) \to A(\mathrm{HP}_X(E,\phi))$ sends $\zeta_1 = p(\mathcal{U}_1, \psi|_{U_1}) \mapsto \zeta_2 = p(\mathcal{U}_2, \psi|_{U_2})$. So we have $P_{F,\psi}(\zeta_2) = 0$. The division of $P_{F,\psi}(t)$ by the monic polynomial $P_{E,\phi}(t)$ yields a remainder $R(t) \in A(\mathrm{HP}_X(E,\phi))[t]$ of degree at most $\frac{1}{2}\operatorname{rk} E - 1$ such that $R(\zeta_2) = 0$. So the remainder vanishes. $\square$

Proof of Theorem 9.2. To prove the theorem we will calculate the rings $A(\mathrm{HFlag}(1^r;n))$. Each of the relative quaternionic flag bundles is an iterated quaternionic projective bundle, so its cohomology ring is of the form

$$A(\mathrm{HFlag}(1^r;n)) = A(k)[y_1,\dots,y_r]/I_{r,n} \tag{9.3}$$

The maps $A(\mathrm{HFlag}(1^r;n+1) \to A(\mathrm{HFlag}(1^r;n))$ are surjective because they are maps of $A(k)$-algebras which send a set of generators onto a set of generators.

The construction of $\mathrm{HFlag}(1^r;n)$ as an iterated flag bundle gives us

$$I_{r,n} = (P_1(y_1), P_2(y_1,y_2), \dots, P_r(y_1,\dots,y_r)) \tag{9.4}$$

where each $P_i(y_1,\dots,y_i)$ is the Pontryagin polynomial in the last variable y_i of the symplectic bundle

$$(G_{n,i}, \gamma_{n,i}) = (U_n^{(i)}, \phi_n^{(i)}) \perp \cdots \perp (U_n^{(r)}, \phi_n^{(r)}) \perp (\mathcal{V}_{r,n}, \psi_{r,n}). \tag{9.5}$$

This bundle is an orthogonal direct summand of the $\mathrm{h}(\mathcal{O}^{\oplus n})$ of (9.1), so by Lemma 9.3 the polynomial $P_i(y_1,\dots,y_i)$ divides $P_{\mathrm{h}(\mathcal{O}^{\oplus n})}(y_i) = y_i^n$ in $A(k)[y_1,\dots,y_i]/(P_1,\dots,P_{i-1})$. From this we deduce

$$(y_1^n, y_2^n, \dots, y_r^n) \subset I_{r,n}. \tag{9.6}$$

Each $(G_{n,i}, \gamma_{n,i})$ is an orthogonal direct summand of $(G_{n,i-1}, \gamma_{n,i})$, so $P_{G_{n,i},\gamma_{n,i}}(t)$ divides $P_{G_{n,i-1},\gamma_{n,i-1}}(t)$. Since the difference in the degrees is 1, we have

$$P_{G_{n,i-1},\gamma_{n,i-1}}(t) = (t - z_i)P_{G_{n,i},\gamma_{n,i}}(t), \qquad z_i = p_1(G_{n,i-1},\gamma_{n,i-1}) - p_1(G_{n,i},\gamma_{n,i}) \tag{9.7}$$

We deduce $p_t(G_{n,i-1},\gamma_{n,i-1}) = (1 + z_i t)p_t(G_{n,i},\gamma_{n,i})$. Since $p_t(G_{n,1},\gamma_{n,1}) = p_t(\mathrm{h}(\mathcal{O}^{\oplus n})) = 1$ we get

$$p_t(G_{n,i},\gamma_{n,i}) = \frac{1}{\prod_{m=1}^{i-1}(1 + z_m t)}$$

in $A(k)/(P_1,\dots,P_{i-1})[t]$.

Write $\mathrm{coeff}(t^i, f(t))$ for the coefficient of t^i in the power series or polynomial $f(t)$. The Pontryagin polynomial and total Pontryagin class are related by the formula

$$P_{E,\phi}(y) = \mathrm{coeff}\left(t^{r+1}, \frac{p_{-t}(E,\phi)}{1-yt}\right)$$

where $p_{-t}(E,\phi)$ means that one substitutes $-t$ for t in the series $p_t(E,\phi)$. Hence we have

$$P_i(y_1,\dots,y_i) = \mathrm{coeff}\left(t^{n-i+1}, \frac{1}{\prod_{m=1}^{i-1}(1-z_m t)\cdot(1-y_i t)}\right).$$

Let $h_i(u_1,\dots,u_s)$ be the i^{th} *complete symmetric polynomial*, the sum of all the monomials in $u_1,\dots,u_s$ of degree i. Set $h_0 = 1$. Their generating function is

$$H(t) = \sum_{i=0}^{\infty} h_i(u_1,\dots,u_s)t^i = \frac{1}{\prod_{m=1}^{s}(1-u_m t)}.$$

Thus we have

$$P_i(y_1,\dots,y_i) = h_{n-i+1}(z_1,\dots,z_{i-1},y_i). \tag{9.8}$$

We claim that the z_i lie in the ideal $(y_1,\dots,y_r) \subset A(k)[y_1,\dots,y_r]$. That is because by the universal property of $\mathrm{HFlag}(1^r;n)$, fixing a decomposition $\mathrm{h}(\mathcal{O}^{\oplus n}) = \mathrm{h}(\mathcal{O})^{\perp r} \perp \mathrm{h}(\mathcal{O}^{\oplus n-r})$ gives a section $s\colon k \to \mathrm{HFlag}(1^r;n)$ of the structural map. Since we have $s^*(\mathcal{U}_n^{(i)}, \phi_n^{(i)}) = \mathrm{h}(\mathcal{O})$, we have $s^A y_i = p(\mathrm{h}(\mathcal{O})) = 0$. So we have $(y_1,\dots,y_r) = \ker s^A$. However, the $s^*(G_{n,i},\gamma_{n,i})$ are also trivial, so from (9.7) we have $s^A z_i = 0$, proving the claim.

It now follows from (9.8) that we have $P_i(y_1,\dots,y_i) \in (y_1,\dots,y_i)^{n-i+1}$. This gives us

$$I_{n,r} \subset (y_1,\dots,y_n)^{n-r+1} \tag{9.9}$$

The inclusions (9.6) and (9.9) together give $\varprojlim A(\mathrm{HFlag}(1^r;n)) = A(k)[[y_1,\dots,y_r]]$. $\square$

10. The splitting principle and the sum formula

The quaternionic flag varieties we studied in the previous section are iterated quaternionic projective bundles over quaternionic Grassmannians

$$\mathrm{HGr}(r,n) \leftarrow \mathrm{HFlag}(1,r-1;n) \leftarrow \cdots \leftarrow \mathrm{HFlag}(1^{r-2},2;n) \leftarrow \mathrm{HFlag}(1^r;n). \tag{10.1}$$

The quaternionic projective bundle theorem 8.2 implies that $A(\mathrm{HFlag}(1^r;n))$ is a free module over $A(\mathrm{HGr}(r,n))$ for any ring cohomology theory A with a symplectic Thom structure. More precisely, when one pulls back the universal rank $2r$ symplectic bundle $(\mathcal{U}_n,\phi_n)$ on $\mathrm{HGr}(r,n)$ to $\mathrm{HFlag}(1^r;n)$, it splits into the orthogonal direct sum of the r rank 2 universal symplectic subbundles

$$(\mathcal{U}_n^{(1)},\phi_n^{(1)}) \perp \cdots \perp (\mathcal{U}_n^{(r)},\phi_n^{(r)}) \tag{10.2}$$

on $\mathrm{HFlag}(1^r;n)$. For $1 \le i \le r-1$ the bundle $(\mathcal{U}_n^{(i)},\phi_n^{(i)})$ is the pullback to $\mathrm{HFlag}(1^r;n)$ of the tautological rank 2 symplectic subbundles of the Quaternionic Projective Bundle Theorem for the i-th projective bundle of (10.1), which is an HP^{r-i}-bundle. Writing $\overline{y}_i = p(\mathcal{U}_n^{(i)},\phi_n^{(i)})$ we see the following.

Proposition 10.1. *Let A be a ring cohomology theorem with a symplectic Thom structure. The projection $q\colon \mathrm{HFlag}(1^r;n) \to \mathrm{HGr}(r,n)$ induces a monomorphism $q^A\colon A(\mathrm{HGr}(r,n)) \to$*

$A(\mathrm{HFlag}(1^r;n))$ under which $A(\mathrm{HFlag}(1^r;n))$ is a free $A(\mathrm{HGr}(r,n))$-module of rank $r!$ with basis

$$\mathbb{B}_r = \{\overline{y}_1^{a_1}\overline{y}_2^{a_2}\cdots\overline{y}_{r-1}^{a_{r-1}} \mid 0 \le a_i \le r-i \text{ for all } i\}. \tag{10.3}$$

Let (E,ϕ) be a symplectic bundle of rank $2r$ on a scheme X, and let $q\colon \mathrm{HFlag}_X(E,\phi) \to X$ be the associated complete quaternionic flag bundle. The pullback of E splits as the orthogonal direct sum of the r universal symplectic subbundles of rank 2

$$q^*(E,\phi) \cong (\mathcal{U}_1,\psi_1) \perp (\mathcal{U}_2,\psi_2) \perp \cdots \perp (\mathcal{U}_r,\psi_r). \tag{10.4}$$

Write $u_i = p(\mathcal{U}_i,\psi_i) \in A(\mathrm{HFlag}_X(E,\phi))$. We will call the u_i the *Pontryagin roots* of (E,ϕ).

Theorem 10.2 (Symplectic splitting principle)**.** *The map $q^A\colon A(X) \to A(\mathrm{HFlag}_X(E,\phi))$ is injective and makes $A(\mathrm{HFlag}_X(E,\phi))$ into a free two-sided $A(X)$-module of rank $r!$ with basis the $\mathbb{B}_r$ of* (10.3)*. Moreover, the Pontryagin classes $p_i(E,\phi) \in A(X)$ coincide after pullback by q^A with the elementary symmetric polynomials $e_i(u_1,\dots,u_r)$ in the Pontryagin roots.*

The proof uses two lemmas.

Lemma 10.3. *Let R be a ring, and let $a_1,\dots,a_n \in R$ be central elements such that $a_i - a_j$ is not a zero divisor for all $i \neq j$. If the polynomials $t-a_1, \dots, t-a_n$ all divide $h \in R[t]$, then $\prod_{i=1}^n (t-a_i)$ also divides h.*

Proof. By induction we may assume that $h = g\prod_{i=1}^{n-1}(t-a_i)$. We then have $g(a_n)\prod(a_n-a_i) = 0$. Since the $a_n - a_i$ are not zero divisors, we have $g(a_n) = 0$, and so $t-a_n$ divides g. □

Lemma 10.4. *Let (E,ϕ) be a symplectic bundle on an affine scheme $X = \operatorname{Spec} R$. Suppose that E can be generated by n global sections. Then we can embed (E,ϕ) as a symplectic subbundle of the trivial symplectic bundle of rank $2n$.*

This lemma is well known (see for example [8]). Hypotheses like $\frac{1}{2} \in R$ are not necessary for alternating forms.

Proof of Theorem 10.2. The first sentence of the theorem is simply a relative version of Proposition 10.1. It remains only to prove the second sentence.

We first treat the special case of the tautological rank $2r$ symplectic subbundle $(\mathcal{U}_{r,n},\phi_{r,n})$ on $\mathrm{HGr}(r,n)$. Let $p_i \in \varprojlim A(\mathrm{HGr}(r,n))$ be the element corresponding to the inverse system of Pontryagin classes $\big(p_i(\mathcal{U}_{r,n},\phi_{r,n})\big)_{n\ge r}$. Recall the $y_i \in \varprojlim A(\mathrm{HFlag}(1^r;n))$ of Theorem 9.2 given by the system $(p(\mathcal{U}_n^{(i)},\phi_n^{(i)}))_{n\ge r}$. Since $(\mathcal{U}_n^{(i)},\phi_n^{(i)})$ is an orthogonal direct summand of the pullback to $\mathrm{HFlag}(1^r;n)$ of the tautological bundle $(\mathcal{U}_{r,n},\phi_{r,n})$ of $\mathrm{HGr}(r,n)$, the Pontryagin polynomial $t - p(\mathcal{U}_n^{(i)},\phi_n^{(i)})$ divides the Pontryagin polynomial

$$P_{\mathcal{U}_{r,n},\phi_{r,n}}(t) = t^r - p_1(\mathcal{U}_{r,n},\phi_{r,n})t^{r-1} + \cdots + (-1)^r p_r(\mathcal{U}_{r,n},\phi_{r,n}).$$

The quotient polynomial of degree $r-1$ at level n restricts to the quotient at level $n-1$ because the quotients and remainders for division by monic polynomials with central coefficients are unique. So the quotient polynomials also form an inverse system, and $t-y_i$ divides $P(t) = t^r - p_1t^{r-1} + \cdots + (-1)^r p_r$ in the inverse limit $A(k)[[y_1,\dots,y_r]]$. Lemma 10.3 applies, and we get $\prod_{i=1}^r (t-y_i) = P(t)$. Hence the p_i are the elementary symmetric polynomials in the y_i. It follows that the $p_i(\mathcal{U}_{r,n},\phi_{r,n})$ are the elementary symmetric polynomials in the $u_i = p(\mathcal{U}_n^{(i)},\phi_n^{(i)})$.

We next treat the case where X is affine. By Lemma 10.4 (E,ϕ) can be embedded as an orthogonal direct summand of some trivial symplectic bundle. This is classified by a map

$f\colon X \to \mathrm{HGr}(r,n)$. Since f^A and the map for the corresponding quaternionic flag bundles pull back the Pontryagin classes and roots of $(\mathcal{U}_{r,n}, \phi_{r,n})$ to those of (E,ϕ), the Pontryagin classes of (E,ϕ) are also the elementary symmetric polynomials in the Pontryagin roots.

Finally suppose X is general. Recall from §2 that we are assuming that our schemes are quasi-compact with an ample family of line bundles. Therefore there is an affine bundle $g\colon Y \to X$ with Y an affine scheme. The Pontryagin classes of $g^*(E,\phi)$ are the elementary symmetric polynomials in its Pontryagin roots. Then g^A and the induced map on the cohomology of the quaternionic flag bundles are isomorphisms and send the Pontryagin classes and roots of (E,ϕ) to those of $g^*(E,\phi)$. So the Pontryagin classes of (E,ϕ) are the elementary symmetric polynomials in its Pontryagin roots. □

Theorem 10.5 (Cartan sum formula)**.** *Suppose $(F,\psi) \cong (E_1,\phi_1) \perp (E_2,\phi_2)$ is an orthogonal direct sum of symplectic bundles over a scheme X. Then for all i we have*

$$p_t(F,\psi) = p_t(E_1,\phi_1)p_t(E_2,\phi_2), \tag{10.5}$$

$$p_i(F,\psi) = p_i(E_1,\phi_1) + \sum_{j=1}^{i-1} p_{i-j}(E_1,\phi_1)p_j(E_2,\phi_2) + p_i(E_2,\phi_2). \tag{10.6}$$

The first Pontryagin class is additive, and the top Pontryagin class is multiplicative.

Proof. Consider the fiber bundles

$$p\colon \mathrm{HFlag}_X(E_1,\phi_1) \times_X \mathrm{HFlag}_X(E_2,\phi_2) \to X, \qquad q\colon \mathrm{HFlag}_X(F,\psi) \to X$$

We have orthogonal direct sum decompositions

$$p^*(E_1,\phi_1) \cong \perp_{i=1}^{r} (\mathcal{U}_i,\phi_i), \qquad p^*(E_2,\phi_2) \cong \perp_{j=1}^{s} (\mathcal{U}'_j,\phi'_j), \qquad q^*(F,\psi) \cong \perp_{\ell=1}^{r+s} (\mathcal{V}_\ell,\psi_\ell), \tag{10.7}$$

By Theorem 10.2 over $\mathrm{HFlag}(E_1,\phi_1) \times_X \mathrm{HFlag}(E_2,\phi_2)$ we have

$$p_t(E_1,\phi_1)p_t(E_2,\phi_2) = \prod_{i=1}^{r}\bigl(1 + p(\mathcal{U}_i,\phi_i)t\bigr) \prod_{j=1}^{s}\bigl(1 + p(\mathcal{U}'_j,\phi'_j)t\bigr) \tag{10.8}$$

while over $\mathrm{HFlag}(F,\psi)$ we have $p_t(F,\psi) = \prod_{\ell=1}^{r+s}\bigl(1+p(\mathcal{V}_\ell,\psi_\ell)t\bigr)$. We also have the orthogonal direct sum $p^*(F,\psi) = \perp_{i=1}^{r} p^*(\mathcal{U}_i,\phi_i) \perp \perp_{j=1}^{s} p^*(\mathcal{U}'_j,\phi'_j)$. This decomposition is classified by a unique map

$$f\colon \mathrm{HFlag}(E_1,\phi_1) \times_X \mathrm{HFlag}(E_2,\phi_2) \to \mathrm{HFlag}_X(F,\psi)$$

such that

$$f^*(\mathcal{V}_\ell,\psi_\ell) \cong \begin{cases} p^*(\mathcal{U}_\ell,\phi_\ell) & \text{for } \ell = 1,\dots,r, \\ p^*(\mathcal{U}'_{\ell-r},\phi'_{\ell-r}) & \text{for } \ell = r+1,\dots,r+s. \end{cases}$$

It follows that when we pull $p_t(F,\psi)$ back along f, we get

$$p_t(F,\psi) = \prod_{i=1}^{r}\bigl(1 + p(\mathcal{U}_i,\phi_i)t\bigr) \prod_{j=1}^{s}\bigl(1 + p(\mathcal{U}'_j,\phi'_j)t\bigr) = p_t(E_1,\phi_1)p_t(E_2,\phi_2).$$

Equating the terms of degree i in this equality of series gives formula (10.6). □

The Cartan sum formula, Proposition 8.5, and nilpotence combine to show that the total Pontryagin class may be defined for Grothendieck-Witt classes of symplectic bundles, giving maps

$$p_t\colon GW^-(X) \to A(X)[t]^\times, \tag{10.9}$$

functorial in X, and sending sums to products. Note that since symplectic bundles have only finitely many nonzero Pontryagin classes, and they are all nilpotent, the same holds for virtual differences of bundles as well. Hence the image is in $A(X)[t]^\times$. Actually the morphism sends

$$p_t\colon \widetilde{GW}^-(X) = \frac{GW^-(X)}{\mathbb{Z}[\mathrm{h}(\mathcal{O})]} \to A(X)[t]^\times,$$

and in particular, the first Pontryagin class is an additive map $p_1\colon \widetilde{GW}^-(X) \to A(X)$.

11. Cohomology of quaternionic Grassmannians

We recall some facts about symmetric polynomials. They may mostly be found in Macdonald's book [10, Chap. 1, §§1–3].

Let $\Lambda_r \subset \mathbb{Z}[y_1, \dots, y_r]$ be the ring of symmetric polynomials in r variables. Let e_i denote the i-th elementary symmetric polynomial, and h_i the i-th *complete symmetric polynomial*, the sum of all the monomials of degree i. Set $e_0 = h_0 = 1$. We have $\Lambda_r = \mathbb{Z}[e_1, \dots, e_r]$. The generating functions are

$$E(t) = \sum_{i\geq 0} e_i t^i = \prod_{j=1}^{r}(1 + y_i t), \qquad H(t) = \sum_{i\geq 0} h_i t^i = \prod_{j=1}^{r}(1 - y_i t)^{-1}. \tag{11.1}$$

So we have

$$E(t)H(-t) = 1, \qquad h_m + \sum_{i=1}^{r}(-1)^r e_i h_{m-i} = 0. \tag{11.2}$$

So we also have $\Lambda_r = \mathbb{Z}[h_1, \dots, h_r]$. The h_i with $i > r$ are nonzero but are dependent on $h_1, \dots, h_r$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of length $l(\lambda) \leq r$. Write $\delta = (r-1, r-2, \dots, 1, 0)$ and $a_{\lambda+\delta} = \det(y_i^{\lambda_j+r-j})_{1\leq i,j\leq r}$. Then $a_{\lambda+\delta}$ is a skew-symmetric polynomial and therefore divisible by the Vandermonde determinant a_δ. The quotient $s_\lambda = a_{\lambda+\delta}/a_\delta$ is the *Schur polynomial* for λ. It is symmetric of degree $|\lambda| = \sum \lambda_i$. One has $s_{(1^i)} = e_i$ and $s_{(i)} = h_i$. The $a_{\lambda+\delta}$ with $l(\lambda) \leq r$ form a $\mathbb{Z}$-basis of the skew-symmetric polynomials in r variables, so the s_λ with $l(\lambda) \leq r$ form a $\mathbb{Z}$-basis of Λ_r. Denote by λ' the partition dual to λ. One has formulas

$$s_\lambda = \det(e_{\lambda'_i - i + j})_{1\leq i,j\leq m} = \det(h_{\lambda_i - i + j})_{1\leq i,j,\leq r}, \tag{11.3}$$

for $m \geq l(\lambda')$ and $r \geq l(\lambda)$.

The ring of *symmetric functions* Λ in an infinite number of variables is the inverse limit of the Λ_r in the category of graded rings. It is a polynomial ring in infinitely many indeterminates

$$\Lambda = \mathbb{Z}[e_1, e_2, \dots] = \mathbb{Z}[h_1, h_2, \dots]$$

It has an involution ω with $\omega(e_i) = h_i$ and $\omega(h_i) = e_i$ for all i and $\omega(s_\lambda) = s_{\lambda'}$ for all partitions λ. The quotient map $\Lambda \twoheadrightarrow \Lambda_r$ is the quotient by the ideal $(e_{r+1}, e_{r+2}, \dots)$. The ring Λ has as $\mathbb{Z}$-basis the Schur functions s_λ with λ ranging over all partitions. If, however, $l(\lambda) = \lambda'_1 > r$, then the first row of the first determinant of (11.3) is $(e_{\lambda'_1}, e_{\lambda'_1}+1, \dots, e_{\lambda'_1}+m-1)$. All these entries are sent to 0 in Λ_r. Set

$$\Pi_{r,n-r} = \{\text{partitions } \lambda \text{ with length } l(\lambda) = \lambda'_1 \leq r \text{ and with } \lambda_1 \leq n-r\}$$

The set $\Pi_{r,n-r}$ has $\binom{n}{r}$ members. For the ideal $I_{n-r} = (h_{n-r+1}, h_{n-r+2}, \dots, h_n) \subset \Lambda_r$ the quotient map $\Lambda_r \twoheadrightarrow \Lambda_r/I_{n-r}$ sends $s_\lambda \mapsto 0$ for all $\lambda \notin \Pi_{r,n-r}$. The quotient Λ_r/I_{n-r} is free over $\mathbb{Z}$ with basis $\{s_\lambda \mid \lambda \in \Pi_{r,n-r}\}$.

Now suppose (E,ϕ) is a symplectic bundle of rank $2r$ over S. Let $q\colon \mathrm{HFlag}_S(E,\phi) \to S$ be the associated complete quaternionic flag bundle, let

$$q^*(E,\phi) \cong (\mathcal{U}_1,\psi_1) \perp (\mathcal{U}_2,\psi_2) \perp \cdots \perp (\mathcal{U}_r,\psi_r).$$

be the splitting of the pullback of E as the orthogonal direct sum of the r universal symplectic subbundles of rank 2 on $\mathrm{HFlag}_S(E,\phi)$, and let $y_i = p(\mathcal{U}_i,\psi_i) \in A(\mathrm{HFlag}_S(E,\phi))$ be the Pontryagin roots of (E,ϕ). For any partition λ of length $l(\lambda) \leq r$ write

$$s_\lambda(E,\phi) = s_\lambda(y_1,\dots,y_r) = \det(p_{\lambda'_i-i+j}(E,\phi))_{1\leq i,j\leq m} \in A(S) \tag{11.4}$$

with $m \geq l(\lambda')$.

We now complete the calculation of the cohomology of $\mathrm{HFlag}(1^r;n)$ begun in §9.

Theorem 11.1. *On $\mathrm{HFlag}(1^r;n)$ let $y_1,\dots,y_r$ be the first Pontryagin classes of the r tautological rank 2 subbundles. Then $A(\mathrm{HFlag}(1^r;n)) = A(k)[y_1,\dots,y_r]/I_{r,n}$ with*

$$I_{r,n} = (h_n(y_1), h_{n-1}(y_1,y_2), \dots, h_{n-r+1}(y_1,\dots,y_r)), \tag{11.5}$$

and if we now write h_i for the complete symmetric polynomial in all r variables $y_1,\dots,y_r$, then we also have

$$A(\mathrm{HFlag}(1^r;n)) = A(k)[y_1,\dots,y_r]/(h_n, h_{n-1}, \dots, h_{n-r+1}). \tag{11.6}$$

Proof. Using the Cartan sum formula, we see that the classes z_i of (9.7) are actually $z_i = y_i$. Hence (9.8) can be rewritten as $P_i(y_1,\dots,y_i) = h_{n-i+1}(y_1,\dots,y_i)$, and we get (11.5).

Since the complete symmetric polynomials are the sums of all monomials of a given degree, we have

$$h_m(y_1,\dots,y_{i-1},y_i) = h_m(y_1,\dots,y_{i-1}) + y_i h_{m-1}(y_1,\dots,y_{i-1},y_i).$$

This formula can be used to show that one can add one variable to each of the generators of $I_{r,n}$ which have less than r variables without changing the ideal. One repeats this until all the generators have r variables. This gives (11.6). □

Theorem 11.2. *Let $(\mathcal{U},\phi)$ be the tautological symplectic bundle of rank $2r$ on $\mathrm{HGr}(r,n)$, and let $(\mathcal{U}^\perp,\psi)$ be its orthogonal complement. Then for any ring cohomology theory A with a symplectic Thom structure and any S the map*

$$A(S)[e_1,\dots,e_r]/(h_{n-r+1},\dots,h_n) \xrightarrow{\cong} A(\mathrm{HGr}(r,n)\times S) \tag{11.7}$$

sending $e_i \mapsto p_i(\mathcal{U},\phi)$ for all i is an isomorphism of rings, the map

$$A(S)^{\oplus\binom{n}{r}} \xrightarrow{(s_\lambda(\mathcal{U},\phi))_{\lambda\in\Pi_{r,n-r}}} A(\mathrm{HGr}(r,n)\times S) \tag{11.8}$$

is an isomorphism of two-sided $A(S)$-modules, and for all partitions λ we have

$$s_\lambda(\mathcal{U},\phi) = (-1)^{|\lambda|} s_{\lambda'}(\mathcal{U}^\perp,\psi). \tag{11.9}$$

Proof. The proof is much like the case of ordinary Grassmannians. It is enough to consider $S = k$. Write $A_{r,n} = A(k)[e_1, \dots, e_r]/(h_{n-r+1}, \dots, h_n)$. Since $A(\mathrm{HGr}(r,n)) \to A(\mathrm{HFlag}(1^r;n))$ is injective, the complete symmetric polynomials h_i with $i > n-r$ vanish in $A(\mathrm{HGr}(r,n))$ as well. We thus get the map $\gamma\colon A_{r,n} \to A(\mathrm{HGr}(r,n))$ sending $e_i \mapsto p_i(\mathcal{U},\phi)$. By Proposition 10.1

$$\mathbb{B}_r = \{y_1^{a_1} y_2^{a_2} \cdots y_{r-1}^{a_{r-1}} \mid 0 \le a_i \le r-i \text{ for all } i\}.$$

is a basis of $A(\mathrm{HFlag}(1^r;n))$ as a two-sided $A(\mathrm{HGr}(r,n))$-module, and it is also a basis of $\mathbb{Z}[y_1, \dots, y_r]$ as a free module over the ring of symmetric polynomials $\mathbb{Z}[e_1, \dots, e_r]$ and therefore also a basis of $A(k)[y_1, \dots, y_r]/(h_{n-r+1}, \dots, h_n)$ as a two-sided free $A_{r,n}$-module. By Theorem 11.1 the map of free modules $A(k)[y_1, \dots, y_r]/(h_{n-r+1}, \dots, h_n) \to A(\mathrm{HFlag}(1^r;n))$ is an isomorphism. So $\gamma\colon A_{r,n} \to A(\mathrm{HGr}(r,n))$ is also an isomorphism.

Over the quaternionic Grassmannian $\mathrm{HGr}(r,n)$ the orthogonal direct sum $(\mathcal{U},\phi) \perp (\mathcal{U}^\perp,\psi)$ is the trivial symplectic bundle $\mathrm{h}(\mathcal{O})^{\oplus n}$ and has vanishing Pontryagin classes. So the Cartan sum formula (10.5) gives $p_t(\mathcal{U},\phi)p_t(\mathcal{U}^\perp,\psi) = 1$. For the generating series of (11.1) we have $\gamma(E(t)) = p_t(\mathcal{U},\phi)$. So from the identity $E(t)H(-t) = 1$ of (11.2), we see we have $\gamma(H(-t)) = p_t(\mathcal{U}^\perp,\psi)$ and thus $(-1)^i\gamma(h_i) = p_i(\mathcal{U}^\perp,\psi)$. The formula $s_\lambda(\mathcal{U},\phi) = (-1)^{|\lambda|}s_{\lambda'}(\mathcal{U}^\perp,\psi)$ now follows from (11.3). The sign change $h_i \mapsto (-1)^i h_i$ comes from the involution of Λ_r sending $f \mapsto (-1)^{\deg(f)}f$ for all homogeneous symmetric polynomials. Hence the sign $(-1)^{|\lambda|}$ in front of $s_{\lambda'}(\mathcal{U}^\perp,\psi)$. □

The usual Mayer-Vietoris argument now gives the following generalization of Theorem 11.2.

Theorem 11.3. *Let (E,ϕ) be a symplectic bundle of rank $2n$ over S. Let $(\mathcal{U},\phi|_{\mathcal{U}})$ be the tautological subbundle of rank $2r$ on $\mathrm{HGr}_S(r,E,\phi)$, and let $(\mathcal{U}^\perp,\phi|_{U^\perp})$ be its orthogonal complement. Then for any ring cohomology theory with a symplectic Thom structure*

$$A(S)^{\oplus\binom{n}{r}} \xrightarrow[\cong]{(s_\lambda(\mathcal{U},\phi|_{\mathcal{U}}))_{\lambda\in\Pi_{r,n-r}}} A(\mathrm{HGr}_S(r,E,\phi)),$$
$$A(S)^{\oplus\binom{n}{r}} \xrightarrow[\cong]{(s_\lambda(\mathcal{U}^\perp,\phi|_{\mathcal{U}^\perp}))_{\lambda\in\Pi_{n-r,r}}} A(\mathrm{HGr}_S(r,E,\phi))$$

are isomorphisms of two-sided $A(S)$-modules.

Let $(\mathcal{U}_{r,n},\phi_{r,n}) \hookrightarrow \mathrm{h}(\mathcal{O})^{\oplus n}$ be the universal tautological symplectic bundle of rank $2r$ on $\mathrm{HGr}(r,n)$. Let $\eta_{r,n}\colon \mathcal{U}_{r,n} \subset \mathrm{h}(\mathcal{O})^{\oplus n}$ be the inclusion. Consider the inclusions of symplectic subbundles

$$\begin{aligned} \mathcal{U}_{r,n} &\xhookrightarrow{\begin{pmatrix}\eta_{r,n}\\ 0\end{pmatrix}} \mathrm{h}(\mathcal{O})^{\oplus n} \oplus \mathrm{h}(\mathcal{O}) = \mathrm{h}(\mathcal{O})^{\oplus n+1}, \\ \mathrm{h}(\mathcal{O}) \oplus \mathcal{U}_{r,n} &\xhookrightarrow{\begin{pmatrix}1 & 0\\ 0 & \eta_{r,n}\end{pmatrix}} \mathrm{h}(\mathcal{O}) \oplus \mathrm{h}(\mathcal{O})^{\oplus n} = \mathrm{h}(\mathcal{O})^{\oplus n+1}. \end{aligned} \tag{11.10}$$

They are classified by maps $\alpha_{r,n}\colon \mathrm{HGr}(r,n) \to \mathrm{HGr}(r,n+1)$ and $\beta_{r,n}\colon \mathrm{HGr}(r,n) \to \mathrm{HGr}(r+1,n+1)$ respectively. Let $\gamma_{r,n} = \beta_{r,n+1}\alpha_{r,n} = \alpha_{r+1,n+1}\beta_{r,n}\colon \mathrm{HGr}(r,n) \to \mathrm{HGr}(r+1,n+2)$. We have direct systems of quaternionic Grassmannians

$$\begin{gathered} \mathrm{HGr}(r,r) \xrightarrow{\alpha_{r,r}} \mathrm{HGr}(r,r+1) \xrightarrow{\alpha_{r,r+1}} \mathrm{HGr}(r,r+2) \to \cdot \to \mathrm{HGr}(r,n) \xrightarrow{\alpha_{r,n}} \cdots \\ \mathrm{HGr}(0,0) \xrightarrow{\gamma_{0,0}} \mathrm{HGr}(1,2) \xrightarrow{\gamma_{1,2}} \mathrm{HGr}(2,4) \to \cdots \to \mathrm{HGr}(n,2n) \xrightarrow{\gamma_{n,2n}} \cdots \end{gathered} \tag{11.11}$$

Theorem 11.4. *For any ring cohomology theory A with a symplectic Thom structure and for any S the maps*

$$(\alpha_{r,n} \times 1_S)^A \colon A(\mathrm{HGr}(r, n+1) \times S) \to A(\mathrm{HGr}(r, n) \times S)$$
$$(\beta_{r,n} \times 1_S)^A \colon A(\mathrm{HGr}(r+1, n+1) \times S) \to A(\mathrm{HGr}(r, n) \times S)$$

are split surjective, and we have isomorphisms

$$A(S)[[p_1, \dots, p_r]] \xrightarrow{\cong} \varprojlim_{n\to\infty} A(\mathrm{HGr}(r, n) \times S) \tag{11.12}$$

$$A(S)[[p_1, p_2, p_3, \dots]] \xrightarrow{\cong} \varprojlim_{n\to\infty} A(\mathrm{HGr}(n, 2n) \times S) \tag{11.13}$$

with each variable p_i sent to the inverse system of i^{th} Pontryagin classes $(p_i(\mathcal{U}_{r,n}))_{n\geq r}$ or $(p_i(\mathcal{U}_{n,2n}))_{n\in\mathbb{N}}$.

The theorem follows from the explicit generators and relations for the $A(\mathrm{HGr}(r, n) \times S)$ given in Theorem 11.2 in the same way as for ordinary Grassmannians.

12. Recovering Thom classes from Pontryagin classes

In this section we show that a symplectic Thom structure is determined by its system of Pontryagin classes. This section is joint work with Alexander Nenashev.

Definition 12.1. A *Pontryagin structure* on a ring cohomology theory B on a category of schemes is a rule assigning to every rank 2 symplectic bundle (E, ϕ) over a scheme S in the category a central element $p(E, \phi) \in B(S)$ with the following properties:

(1) For $(E_1, \phi_1) \cong (E_2, \phi_2)$ we have $p(E_1, \phi_1) = p(E_2, \phi_2)$.
(2) For a morphism $f \colon Y \to S$ we have $f^B(p(E, \phi)) = p(f^*(E, \phi))$.
(3) For the tautological rank 2 symplectic subbundle $(\mathcal{U}, \phi|_{\mathcal{U}})$ on HP^1 the maps

$$(1, p(\mathcal{U}, \phi|_{\mathcal{U}})) \colon B(S) \oplus B(S) \to B(\mathrm{HP}^1 \times S)$$

are isomorphisms.
(4) For a rank 2 symplectic space (V, ϕ) viewed as a trivial symplectic bundle over k we have $p(V, \phi) = 0$ in $B(k)$.

The Pontryagin classes associated to a symplectic Thom structure by formula (7.1) form a Pontryagin structure because of the functoriality of the Thom classes, the Quaternionic Projective Bundle Theorem 8.1, and Proposition 7.7.

Theorem 12.2. *Every Pontryagin structure on a ring cohomology theory is the system of Pontryagin classes of a unique symplectic Thom structure whose classes are given by formula* (12.4) *below.*

The rest of this section is devoted to proving this theorem. Our strategy goes as follows. Let (E, ϕ) be a rank 2 symplectic bundle over S. We will study the HP^1 bundle associated to the rank 4 symplectic bundle (F, ψ) with

$$F = \mathcal{O}_S \oplus E \oplus \mathcal{O}_S, \qquad \psi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \phi & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{12.1}$$

The bundle (F, ψ) is of the form studied in Theorem 4.1, so the HP^1 bundle has the properties of that proposition. Write

$$P = \mathrm{HP}^1_S(F, \psi), \qquad N^+ = \mathrm{HP}^1_S(F, \psi) \cap \mathrm{Gr}_S(2, \mathcal{O}_S \oplus E),$$
$$N^- = \mathrm{HP}^1_S(F, \psi) \cap \mathrm{Gr}_S(2, E \oplus \mathcal{O}_S), \qquad S = N^+ \cap N^- = \mathrm{HP}^0_S(E, \phi).$$

Write $(U, \psi|_U)$ for the universal rank 2 symplectic subbundle over P. In the closed embedding $\mathrm{HP}^0_S(E, \phi) \subset \mathrm{HP}^1_S(F, \psi)$ of Theorem 4.1, the small quaternionic projective bundle is now an HP^0 bundle over S and is therefore S itself. The restriction of $(U, \psi|_U)$ to this $S = \mathrm{HP}^0(E, \phi)$ is the tautological rank 2 symplectic subbundle of the rank 2 symplectic bundle (E, ϕ). So it is (E, ϕ) itself. It now follows from Theorem 4.1 that the two summands of the normal bundle N_1 and N_2 satisfy

$$N^+ \cong E^\vee \cong E \qquad\qquad N^- \cong E^\vee \cong E \tag{12.2}$$

We have the following diagram.

$$\begin{tikzcd}
E = N^- \arrow[r, "f"] \arrow[d, shift left, "\pi"] & P \arrow[r, "s"] \arrow[dl, "h"'] & U \\
S \arrow[u, shift left, "z"] \arrow[r, hook, "g"'] & N^+ \arrow[u, "i"] \arrow[r, hook, "i"'] & P \arrow[u, "z_U"']
\end{tikzcd} \tag{12.3}$$

The loci S, N^+, N^-, P and the vector bundles $U \to P$ and $E \to S$ are as above, the maps i, f, g, and the map $S \to N^-$ are the inclusions, and h, π, and the map $N^- \to S$ are the projections to S. We have $hf = \pi$. The section $s\colon P \to U$ corresponds to the vector bundle map $s_+\colon \mathcal{O}_P \to U$ of Theorem 4.1(d) whose zero locus is N^+. In particular, s is transversal to the zero section z_U of U. The locus N^- can be identified with the vector bundle E by the isomorphism of (12.2), and z and π correspond to the zero section and structural map of the vector bundle. The intersections $z_U(P) \cap s(P) = N^+$ and $N^+ \cap N^- = S$ are transversal.

We now prove a series of lemmas. The first one applies to a slightly more general situation.

Lemma 12.3. *Let B be a ring cohomology theory with a Pontryagin structure. Let (G, ω) be a rank 4 symplectic bundle over a scheme S, and let $(\mathcal{U}, \omega|_{\mathcal{U}})$ be its tautological rank 2 symplectic subbundle. Then the map $(1, p(\mathcal{U}, \omega|_{\mathcal{U}}))\colon B(S) \oplus B(S) \to B(\mathrm{HP}^1_S(G, \omega))$ is an isomorphism.*

Proof. The lemma is true for a trivial (G, ω), which has $\mathrm{HP}^1_S(G, \omega) = \mathrm{HP}^1 \times S$, by the axioms of a Pontryagin structure. The general case follows by a Mayer-Vietoris argument. □

Lemma 12.4. *Let B be a ring cohomology theory with a Pontryagin structure. In the situation of (12.1)–(12.3) let $\overline{e}^B\colon B_X(P) \to B(P)$ be the supports extension map. Then there exists a unique element $\theta_{E,\phi} \in B_X(P)$ satisfying $\overline{e}^B(\theta_{E,\phi}) = p(U, \psi|_U)$. Moreover, $\theta_{E,\phi}$ is $B(S)$-central, and the map $\theta\colon B(S) \to B_X(P)$ sending $\alpha \mapsto h^B(\alpha) \cup \theta_{E,\phi}$ is an isomorphism.*

Proof. We are in the situation of Theorem 5.1. This gives us two pieces of information. First, write $Y = P \smallsetminus N^+$, let $j\colon Y \hookrightarrow P$ be the inclusion, and let $q = hj\colon Y \to S$ be the projection. Theorem 5.2(a) then says that $q^B\colon B(S) \to B(Y)$ is an isomorphism.

Second, by Theorem 5.1 itself there are maps

$$Y \xleftarrow{g_1} Y_1 \underset{\cong}{\xleftarrow{g_2}} Y_2 \xrightarrow{q} \mathrm{HGr}_S(0, E, \phi) = S$$

with g_1 an $\mathbb{A}^1$-bundle, g_2 an $\mathbb{A}^0$-bundle, and q an $\mathbb{A}^5$-bundle. Moreover, $g_2^* g_1^*(U|_Y)$ has two tautological sections e, f over Y_2 which together give an isometry with the trivial rank 2 symplectic bundle $\big(\mathcal{O}_S^{\oplus 2}, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\big)$. Therefore $g_2^B g_1^B j^B(p(U, \psi|_U)) = p(g_2^* g_1^* j^*(U, \psi|_U))$ is the pullback by $Y_2 \to k$ of the Pontryagin class of the trivial symplectic bundle over k, which vanishes by the axioms of a Pontryagin structure. So we have $g_2^B g_1^B j^B(p(U, \psi|_U)) = 0$. Moreover, since g_1 and g_2 are affine bundles, $g_2^B g_1^B$ is an isomorphism. So we have $j^B(p(U, \psi|_U)) = 0$.

Now consider the diagram of two-sided $B(S)$-modules.

$$\begin{tikzcd}
0 \arrow[r] & B(S) \arrow[r, "{(0,1)}"] \arrow[d, dotted, "\theta"', "\cong"] & B(S) \oplus B(S) \arrow[r, "{(1,0)}"] \arrow[d, "{(1, p(U,\psi|_U)) \circ h^B}"', "\cong"] & B(S) \arrow[r] \arrow[d, "q^B"', "\cong"] & 0 \\
\cdots \arrow[r, "\partial = 0"'] & B_{N^+}(P) \arrow[r, "\overline{e}^B"'] & B(P) \arrow[r, "j^B"'] & B(Y) \arrow[r, "\partial = 0"'] & \cdots
\end{tikzcd}$$

The bottom row is the localization exact sequence. The middle vertical arrow is an isomorphism by Lemma 12.3, and q^B is an isomorphism by the discussion above. Since we have $q^B = j^B h^B$, that also implies that j^B is surjective, and therefore ∂ vanishes. The righthand square commutes because we have $j^B(p(U, \psi|_U)) = 0$. It now follows that the diagram can be completed by a unique isomorphism θ making the lefthand square commute. Setting $\theta_{E,\phi} = \theta(1_S)$, we get an element with all the desired properties because θ is an isomorphism of two-sided $B(S)$-modules. $\square$

Theorem 12.5. *Let B be a ring cohomology theory with a Pontryagin structure. In the situation of* (12.1)–(12.3) *let $\theta_{E,\phi} \in B_X(P)$ be the unique element satisfying $\overline{e}^B(\theta_{E,\phi}) = p(U, \psi|_U)$ of Lemma 12.4. Then the assignment*

$$\operatorname{th}(E, \phi) = -f^B(\theta_{E,\phi}) \in B_S(E) \tag{12.4}$$

defines a symplectic Thom structure on B whose Pontryagin classes are the given Pontryagin structure.

Proof. We verify the axioms of a symplectic Thom structure.

First, the $\operatorname{th}(E, \phi)$ are supposed to be $B(S)$-central. By Lemma 12.3 the element $\theta_{E,\phi} \in B_{N^+}(P)$ is $B(S)$-central, and since $f\colon E = N^- \to P$ is compatible with the maps π and h to S, the map $f^B\colon B_{N^+}(P) \to B_S(E)$ is a map of two-sided $B(S)$-modules. So $f^B(\theta_{E,\phi}) = -\operatorname{th}(E, \phi)$ is $B(S)$-central.

The functoriality conditions on the $\operatorname{th}(E, \phi)$ follow from the functoriality of the constructions of (12.1)–(12.3).

The maps $\cup \operatorname{th}(E, \phi)\colon B(S) \to B_S(E)$ are isomorphisms because they are compositions

$$B(S) \xrightarrow[\cong]{-\theta} B_{N^+}(P) \xrightarrow[\cong]{f^B} B_S(E)$$

of maps which are isomorphisms because of Lemma 12.4 and Proposition 4.3.

Hence the $\operatorname{th}(E, \phi)$ define a symplectic Thom structure.

Finally, the Pontryagin classes $-z^B e^B(\operatorname{th}(E,\phi))$ defined by the symplectic Thom structure are the original ones because using the commutative diagram

$$\begin{tikzcd} B_X(P) \arrow[r, "\overline{e}^B"] \arrow[d, "f^B"] & B(P) \arrow[d, "f^B"] & \\ B_S(E) \arrow[r, "e^B"] & B(E) \arrow[r, "z^B"] & B(S) \end{tikzcd}$$

we see that we have

$$-z^B e^B(\operatorname{th}(E,\phi)) = z^B e^B f^B(\theta_{E,\phi}) = z^B f^B \overline{e}^B(\theta_{E,\phi}) = z^B f^B(p(U,\psi)) = p(E,\phi)$$

using functoriality of the Pontryagin classes. □

Theorem 12.6. *Let B be a ring cohomology theory with a Pontryagin structure. Suppose that the system of Pontryagin classes are those of some symplectic Thom structure with classes* $\operatorname{th}(E,\phi)$. *In the situation of* (12.1)–(12.3) *let $\theta_{E,\phi} \in B_{N^+}(P)$ be the unique element satisfying* $\overline{e}^B(\theta_{E,\phi}) = p(U,\psi|_U)$ *of Lemma 12.4. Then we have* $\operatorname{th}(E,\phi) = -f^B(\theta_{E,\phi}) \in B_S(E)$.

Thus the symplectic Thom structure inducing the Pontryagin structure is unique.

Proof. In Proposition 7.6 we saw that $p(U,\psi|_U)$ is the image of $-\operatorname{th}(U,\psi|_U)$ under the composition

$$B_P(U) \xrightarrow{s^B} B_X(P) \xrightarrow{\overline{e}^B} B(P).$$

Since $\theta_{E,\phi}$ is the unique element with $\overline{e}^B(\theta_{E,\phi}) = p(U,\psi|_U)$, we have $\theta_{E,\phi} = -s^B(\operatorname{th}(U,\psi|_U))$. To complete the proof of the lemma, we need to prove $f^B s^B(\operatorname{th}(U,\psi|_U)) = \operatorname{th}(E,\phi)$.

Now consider the diagram.

$$\begin{tikzcd} E \arrow[r, "i_1"] \arrow[d, "\pi"] & E \oplus E \arrow[r, "f_1"] \arrow[d, shift left, "p_2"] & U \arrow[d, shift left] \\ S \arrow[r, "z"] & E \arrow[r, "f"] \arrow[u, shift left, "\Delta"] & P \arrow[u, shift left, "s"] \end{tikzcd}$$

According to Theorem 4.1(e), the pullbacks to N^- of the two bundles $U \to P$ and $E \to S$ are isomorphic. Pulling further back along the isomorphism $E \cong W$ gives an isomorphism $j_2^* f^* U \cong \pi^* E = E \oplus E$ of bundles over E. According to Lemma 4.2, the section s of U whose zero locus is X pulls back to the tautological diagonal section Δ of $\pi^* E = E \oplus E$. This gives the righthand square. The lefthand square is clear. The functoriality of the Thom classes gives $\operatorname{th}(E,\phi) = i_1^B f_1^B(\operatorname{th}(U,\psi)) \in B_S(E)$. The first projection $p_1 \colon E \oplus E \to E$ is an $\mathbb{A}^2$-bundle with $p_1^{-1}(S) = 0 \oplus E$, and it has sections i_1 and Δ. It follows that the maps i_1^B and $\Delta^B \colon B_{0\oplus E}(E \oplus E) = B_E(\pi^* E) \to B_S(E)$ are equal. Substituting, and using also $sf = f_1 \Delta$, we get the desired formula

$$\operatorname{th}(E,\phi) = i_1^B f_1^B(\operatorname{th}(U,\psi)) = \Delta^B f_1^B(\operatorname{th}(U,\psi)) = f^B s^B(\operatorname{th}(U,\psi)) = -f^B(\theta_{E,\phi}). \quad □$$

Proof of Theorem 12.2. Theorem 12.5 showed that every Pontryagin structure consists of the Pontryagin classes associated to the symplectic Thom structure given by formula (12.4), and Theorem 12.6 showed that this was the only symplectic Thom structure inducing the given Pontryagin structure. □

13. Thom classes of higher rank bundles

In this section we define Thom classes for higher rank symplectic bundles and prove some of their properties. The construction proceeds as in §12 only in higher rank and using top Pontryagin classes. But now that Theorem 12.2 has been proven establishing the equivalence of symplectic Thom structures and Pontryagin structures, we may return to using a ring cohomology theory A with a symplectic Thom structure. So we now have at our disposal our results on quaternionic Grassmannians.

Let (E, ϕ) be a symplectic bundle of rank $2r$ over S, and let (F, ψ) be the rank $2r + 2$ symplectic bundle of (4.2) or (12.1). Set

$$\begin{aligned} P &= \mathrm{HGr}_S(r, F, \psi), & N^+ &= \mathrm{HGr}_S(r, F, \psi) \cap \mathrm{Gr}_S(2r, \mathcal{O}_S \oplus E), \\ S = N^+ \cap N^- &= \mathrm{HGr}_S(r, E, \phi), & N^- &= \mathrm{HGr}_S(r, F, \psi) \cap \mathrm{Gr}_S(2r, E \oplus \mathcal{O}_S). \end{aligned} \tag{13.1}$$

Write $(U, \psi|_U)$ for the universal rank $2r$ symplectic subbundle over P. The restriction of $(U, \psi|_U)$ to $S = \mathrm{HGr}_S(r, E, \phi)$ is again (E, ϕ). We have the same isomorphisms $N^+ \cong E$ and $N^- \cong E$ as in (12.2) and the same commutative diagram (12.3). The main difference is that the relative dimensions over S of the various loci and bundles are now

$$\dim_S S = 0, \qquad \dim_S E = \dim_S N^+ = \dim_S N^- = 2r, \qquad \dim_S P = 4r, \qquad \dim_S U = 6r.$$

Proposition 13.1. *Let A be a ring cohomology theory with a symplectic Thom structure. In the situation above let $\overline{e}^A \colon A_{N^+}(P) \to A(P)$ be the supports extension map. Then there exists a unique element $\theta_{E,\phi} \in A_{N^+}(P)$ satisfying $\overline{e}^A(\theta_{E,\phi}) = p_r(U, \psi|_U)$. Moreover, $\theta_{E,\phi}$ is $A(S)$-central, and the map $\theta \colon A(S) \to A_{N^+}(P)$ sending $\alpha \mapsto h^A(\alpha) \cup \theta_{E,\phi}$ is an isomorphism.*

Proof. We claim we have an isomorphism of two-sided $A(S)$-modules

$$(1, p_1(U, \psi|_U), \ldots, p_r(U, \psi|_U)) \colon A(S)^{\oplus r+1} \xrightarrow{\cong} A(P).$$

This is because Theorem 11.3 establishes that $A(P)$ is a free two-sided $A(S)$-module with basis $s_\lambda(\mathcal{U}, \psi|_U)$ for $\lambda \in \Pi_{r,1} = \{(1^i) \mid 0 \leq i \leq r\}$. But those particular Schur polynomials are $1, e_1, \ldots, e_r$, so the characteristic classes are $1, p_1, \ldots, p_r$.

Similarly, if we write $(\mathcal{U}_{r-1}, \phi|_{\mathcal{U}_{r-1}})$ for the tautological rank $2r - 2$ symplectic subbundle on $\mathrm{HGr}_S(r - 1, E, \phi)$, we have an isomorphism of two-sided $A(S)$-modules

$$(1, p_1(\mathcal{U}_{r-1}, \phi|_{\mathcal{U}_{r-1}}), \ldots, p_{r-1}(\mathcal{U}_{r-1}, \phi|_{\mathcal{U}_{r-1}})) \colon A(S)^{\oplus r} \xrightarrow{\cong} A(\mathrm{HGr}_S(r - 1, E, \phi)).$$

Let $Y = P \smallsetminus N^+$ and let $j \colon Y \hookrightarrow P$ be the inclusion. By Theorem 5.2(b) there is a map $\sigma \colon A(\mathrm{HGr}_S(r-1, E, \phi)) \to Y \subset P$ classifying the rank $2r$ symplectic subbundle $\mathcal{O} \oplus \mathcal{U}_{r-1} \oplus \mathcal{O}$ of the pullback of F, and the pullback map

$$\sigma^A \colon A(Y) \xrightarrow{\cong} A(\mathrm{HGr}_S(r - 1, E, \phi))$$

is an isomorphism. Composing with the restriction gives a map $\sigma^A j^A \colon A(P) \to A(Y) \cong A(\mathrm{HGr}_S(r - 1, E, \phi))$ sending $1 \mapsto 1$ and (in simplified notation)

$$p_i(U) \mapsto p_i(\mathcal{O} \oplus \mathcal{U}_{r-1} \oplus \mathcal{O}) = \begin{cases} p_i(\mathcal{U}_{r-1}) & \text{for } i = 1, \ldots, r - 1, \\ 0 & \text{for } i = r, \end{cases}$$

the equalities coming from the Cartan sum formula. Therefore in the localization sequence

$$\cdots \xrightarrow{\partial} A_{N^+}(P) \xrightarrow{\overline{e}^A} A(P) \xrightarrow{j^A} A(Y) \xrightarrow{\partial} \cdots$$

the map j^A is split surjective, ∂ vanishes, and $\overline{e}^A$ is split injective with image the free direct summand $A(S) \cdot p_r(U, \psi|_U)$. So there exists indeed a unique element $\theta_{E,\phi} \in A_X(P)$ satisfying $\overline{e}^A(\theta_{E,\phi}) = p_r(U, \psi|_U)$. Moreover, $\theta_{E,\phi}$ is $A(P)$-central like the Pontryagin class and therefore also $A(S)$-central, and the map $\theta \colon A(S) \to A_X(P)$ sending $\alpha \mapsto h^A(\alpha) \cup \theta_{E,\phi}$ is indeed an isomorphism. $\square$

Theorem 13.2. *Let A be a ring cohomology theory with a symplectic Thom structure. For a symplectic bundle (E, ϕ) of rank $2r$ on S, let P, N^+, N^-, f, etc., be as in (13.1) and (12.3), and let $\theta_{E,\phi} \in A_{N^+}(P)$ be the unique element satisfying $\overline{e}^A(\theta_{E,\phi}) = p(U, \psi|_U)$ of Proposition 13.1. Then the assignment*

$$\mathrm{th}(E, \phi) = (-1)^r f^A(\theta_{E,\phi}) \in A_S(E) \tag{13.2}$$

gives a system of classes with the following properties:

1. *Each $\mathrm{th}(E, \phi) \in A_S(E)$ is $A(S)$-central.*
2. *For an isomorphism $\gamma \colon (E, \phi) \cong (E_1, \phi_1)$ we have $\mathrm{th}(E, \phi) = \gamma^A \,\mathrm{th}(E_1, \phi_1)$.*
3. *For $u \colon T \to S$, writing $u_E \colon u^*E \to E$ for the pullback, we have $u_E^A(\mathrm{th}(E, \phi)) = \mathrm{th}(u^*(E, \phi)) \in A_T(u^*E)$.*
4. *The maps $\cup\, \mathrm{th}(E, \phi) \colon A(S) \to A_S(E)$ are isomorphisms.*
5. *We have $\mathrm{th}\big((E_1, \phi_1) \perp (E_2, \phi_2)\big) = q_1^A \,\mathrm{th}(E_1, \phi_1) \cup q_2^A \,\mathrm{th}(E_2, \phi_2)$, where q_1, q_2 are the projections from $E_1 \oplus E_2$ onto its factors.*

Moreover, for $e^A \colon A_S(E) \to A(E)$ the extension of supports map, and $z^A \colon A(E) \to A(S)$ the restriction to the zero section, we have

$$p_r(E, \phi) = (-1)^r z^A e^A(\mathrm{th}(E, \phi)), \tag{13.3}$$

while for (E, ϕ) of rank 2 the class $\mathrm{th}(E, \phi)$ just defined is the same as the class in the symplectic Thom structure.

The classes $\mathrm{th}(E, \phi)$ are called the *symplectic Thom classes*.

First part of the proof of Theorem 13.2. The proof of (1)–(4) and (13.3) is identical to the proof of Theorem 12.5. The coincidence of the two symplectic Thom classes for rank 2 bundles was proven in Theorem 12.6. We will prove (5) at the end of this section. $\square$

We may now generalize the direct image maps of (7.2). When $i \colon X \to Y$ is a closed embedding of smooth varieties of even codimension $2r$ whose normal bundle $N = N_{X/Y}$ is equipped with a specified symplectic form ϕ, we say that $i \colon X \to Y$ has a *symplectic normal bundle* (N, ϕ). We can compose the isomorphism of Theorem 13.2(4) with the deformation to the normal bundle isomorphisms when they exist and extension supports maps gives *direct image maps.*

$$i_{A,\flat} \colon A(X) \xrightarrow[\cong]{\cup\,\mathrm{th}(N,\phi)} A_X(N) \xrightarrow[\cong]{d_{X/Y}} A_X(Y), \qquad\qquad i_{A,\natural} \colon A(X) \xrightarrow{i_{A,\flat}} A_X(Y) \to A(Y)$$

The analogues of Propositions 7.3, 7.4, 7.5 and 7.6 hold for these more general direct image maps except that for a symplectic bundle (E, ϕ) of rank $2r$ equations (7.4) and (7.5) should read

$$p_r(E, \phi) = (-1)^r \overline{e}^A s^A \,\mathrm{th}(E, \phi), \tag{13.4}$$

$$i_{A,\natural} i^A(b) = \overline{e}^A i_{A,\flat} i^A(b) = (-1)^r b \cup p_r(E, \phi). \tag{13.5}$$

Now let (E,ϕ) be symplectic bundle of rank $2n-2$ on S, and let $(F,\psi) = (E,\phi) \perp \mathrm{h}(\mathcal{O}_S)$ be as in (4.2). Let $N^+ = \mathrm{HGr}_S(r,F,\psi) \cap \mathrm{Gr}(2r, \mathcal{O}_S \oplus E)$ and $Y = \mathrm{HGr}_S(r,F,\psi) \smallsetminus N^+$. Let $u\colon N^+ \to \mathrm{HGr}_S(r,E,\phi)$ be the bundle map of Theorem 4.1, and let $i\colon N^+ \hookrightarrow \mathrm{HGr}_S(r,F,\psi)$ be the inclusion. Let $\sigma\colon \mathrm{HGr}_S(r-1,E,\phi) \to Y \hookrightarrow \mathrm{HGr}_S(r,F,\psi)$ be the map of Theorem 5.2. Let $\tau = e^A q^A i_{A,\flat}$.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial=0"] & A_{N^+}(\mathrm{HGr}_S(r,F,\psi)) \arrow[r, "e^A"] & A(\mathrm{HGr}_S(r,F,\psi)) \arrow[r] \arrow[dr, dashed, "\sigma^A"'] & A(Y) \arrow[r, "\partial=0"] \arrow[d, "\cong"] & \cdots \\
& A(\mathrm{HGr}_S(r,E,\phi)) \arrow[u, "i_{A,\flat}q^A", "\cong"'] \arrow[ur, dashed, "\tau"] & & A(\mathrm{HGr}_S(r-1,E,\phi)) &
\end{tikzcd} \tag{13.6}$$

Write $\mathcal{U}_r$ for the tautological bundles on $\mathrm{HGr}_S(r,E,\phi)$, write $\mathcal{U}_{r-1}$ for the one on $\mathrm{HGr}_S(r-1,E,\phi)$, and $\mathcal{V}_r$ for the one on $\mathrm{HGr}_S(r,F,\psi)$. Write $s_\lambda(\mathcal{U}_r)$ for the Schur polynomials in the Pontryagin classes as in (11.4).

Theorem 13.3. *The maps τ and σ act on the $A(S)$-bases of the cohomology of the quaternionic Grassmannian bundles by*

$$\tau\colon s_\lambda(\mathcal{U}_r) \mapsto (-1)^r p_r(\mathcal{V}_r) s_\lambda(\mathcal{V}_r) = (-1)^r s_{\lambda+(1^r)}(\mathcal{V}_r) \qquad \text{for } \lambda \in \Pi_{r,n-1-r}$$

$$\sigma\colon s_\lambda(\mathcal{V}_r) \mapsto \begin{cases} s_\lambda(\mathcal{U}_{r-1}) & \text{for } \lambda \in \Pi_{r-1,n-r}, \\ 0 & \text{for } \lambda \in \Pi_{r,n-r} \smallsetminus \Pi_{r-1,n-r}. \end{cases}$$

Proof. For a partition λ of length $l(\lambda) \le r$, one has equality $l(\lambda) = r$ if and only if the dual partition is of the form $\lambda' = (r, \lambda'_2, \dots, \lambda'_m)$. When that occurs formula (11.3) gives

$$s_\lambda = \begin{vmatrix} e_r & 0 & \cdots & 0 \\ e_{\lambda'_2-1} & e_{\lambda'_2} & \cdots & e_{\lambda'_2+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_m-m+1} & e_{\lambda'_m-m+2} & \cdots & e_{\lambda'_m} \end{vmatrix} = e_r s_{\lambda-(1^r)}.$$

in Λ_r. Therefore, keeping in mind that the pullbacks of the tautological subbundles along the projection $q\colon N^+ \to \mathrm{HGr}_S(r,E,\phi)$ and the inclusion $i\colon N^+ \to \mathrm{HGr}_S(r,F,\psi)$ are isometric, we have

$$\tau(s_\lambda(\mathcal{U}_r)) = e^A i_{A,\flat} q^A(s_\lambda(\mathcal{U}_r)) = e^A i_{A,\flat} i^A(s_\lambda(\mathcal{V}_r)) = (-1)^r p_r(\mathcal{V}_r) s_\lambda(\mathcal{V}_r) = (-1)^r s_{\lambda+(1^r)}(\mathcal{V}_r),$$

using (13.5).

The map σ classifies the rank $2r$ symplectic bundle $\mathcal{O} \oplus \mathcal{U}_{r-1} \oplus \mathcal{O}$ on $\mathrm{HGr}_S(r-1,E,\phi)$, so we have

$$\sigma^A(s_\lambda(\mathcal{V}_r)) = s_\lambda(\mathcal{O} \oplus \mathcal{U}_{r-1} \oplus \mathcal{O}) = s_\lambda(\mathcal{U}_{r-1})$$

because the Pontryagin classes of $\mathcal{U}_{r-1}$ and $\mathcal{O} \oplus \mathcal{U}_{r-1} \oplus \mathcal{O}$ are equal by the Cartan sum formula. For the $\lambda \in \Pi_{r-1,n-r}$ this is one of the elements of the basis of $A(\mathrm{HGr}_S(r-1,E,\phi)$ as a two-sided $A(S)$-module. For $\lambda \in \Pi_{r,n-r} \smallsetminus \Pi_{r-1,n-r}$, i.e. those with $l(\lambda) = r$, we have $s_\lambda(\mathcal{U}_{r-1}) = p_r(\mathcal{U}_{r-1}) s_{\lambda-(1^r)}(\mathcal{U}_{r-1}) = 0$, since the tautological symplectic subbundle on $\mathrm{HGr}_S(r-1,E,\phi)$ is of rank $2r-2$ and has $p_r(\mathcal{U}_{r-1}) = 0$. $\square$

End of the proof of Theorem 13.2. We now prove property (5) of the theorem. Suppose $\operatorname{rk} E_1 = 2r_1$ and $\operatorname{rk} E_2 = 2r_2$. Let

$$\begin{aligned}
P_{r_1+r_2} &= \mathrm{HGr}_S\big(r_1+r_2, (E_1,\phi_1) \perp (E_2,\phi_2) \perp \mathrm{h}(\mathcal{O}_S)\big),\\
P_{r_1} &= \mathrm{HGr}_S\big(r_1, (E_1,\phi_1) \perp \mathrm{h}(\mathcal{O}_S)\big),\\
P_{r_2} &= \mathrm{HGr}_S\big(r_2, (E_2,\phi_2) \perp \mathrm{h}(\mathcal{O}_S)\big),\\
F_{r_1,r_2} &= \mathrm{HFlag}_S\big(r_1,r_2; (E_1,\phi_1) \perp (E_2,\phi_2) \perp \mathrm{h}(\mathcal{O}_S)\big),\\
G_{r_1} &= \mathrm{HGr}_S\big(r_1, (E_1,\phi_1) \perp (E_2,\phi_2) \perp \mathrm{h}(\mathcal{O}_S)\big),\\
G_{r_2} &= \mathrm{HGr}_S\big(r_2, (E_1,\phi_1) \perp (E_2,\phi_2) \perp \mathrm{h}(\mathcal{O}_S)\big)
\end{aligned}$$

We have maps

$$\begin{tikzcd}
P_{r_1} \arrow[r, "t_1"] & F_{r_1,r_2} \arrow[r, "\rho_1"] \arrow[d, "\rho"] \arrow[dr, "\rho_2"] & G_{r_1} \\
P_{r_2} \arrow[ur, "t_2"] & P_{r_1+r_2} & G_{r_2}
\end{tikzcd}$$

defined as follows. Over F_{r_1,r_2} there are orthogonal tautological symplectic subbundles U_1, U_2 of ranks $2r_1$ and $2r_2$ respectively. The maps ρ_i are classified by the U_i and ρ by $U_1 \oplus U_2$. All three projections are quaternionic Grassmannian bundles.

Over P_{r_1} there is a tautological rank $2r_1$ subbundle $\mathcal{V}_1 \subset E_1 \oplus \mathrm{h}(\mathcal{O})$. The map t_1 is classified by the pair of orthogonal symplectic subbundles $\mathcal{V}_1 \perp E_2 \subset E_1 \oplus E_2 \oplus \mathrm{h}(\mathcal{O})$. The map t_2 is defined analogously, reversing the roles of E_1 and E_2.

In P_{r_1}, P_{r_2} and $P_{r_1+r_2}$ there are the loci and maps of (12.3)

$$\begin{tikzcd}
E_1 = N_1^- \arrow[r, "f_1"] \arrow[d, shift left, "\pi_1"] & P_{r_1} \arrow[r, "s_1"] \arrow[dl, "h_1"'] & U_1 \\
S_1 \arrow[u, shift left, "z_1"] \arrow[r, hook, "g_1"'] & N_1^+ \arrow[u, "i_1"] \arrow[r, hook, "i_1"'] & P_{r_1} \arrow[u, "z_{U_1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
E_2 = N_2^- \arrow[r, "f_2"] \arrow[d, shift left, "\pi_2"] & P_{r_2} \arrow[r, "s_2"] \arrow[dl, "h_2"'] & U_2 \\
S_2 \arrow[u, shift left, "z_2"] \arrow[r, hook, "g_2"'] & N_2^+ \arrow[u, "i_2"] \arrow[r, hook, "i_2"'] & P_{r_2} \arrow[u, "z_{U_2}"']
\end{tikzcd}$$

$$\begin{tikzcd}
E_1 \oplus E_2 = N^- \arrow[r, "f"] \arrow[d, shift left, "\pi"] & P_{r_1+r_2} \arrow[r, "s"] \arrow[dl, "h"'] & U \\
S \arrow[u, shift left, "z"] \arrow[r, hook, "g"'] & N^+ \arrow[u, "i"] \arrow[r, hook, "i"'] & P_{r_1+r_2} \arrow[u, "z_U"']
\end{tikzcd}$$

Writing $\mu_1 \colon \mathcal{O}_{E_1} \to \pi_1^* E_1$ and $\mu_2 \colon \mathcal{O}_{E_2} \to \pi_2^* E_2$ for the tautological sections. The compositions

$$\gamma_1 \colon E_1 = N_1^- \xrightarrow{f_1} P_{r_1} \xrightarrow{t_1} F_{r_1,r_2} \qquad\qquad \gamma_2 \colon E_2 = N_2^- \xrightarrow{f_2} P_{r_2} \xrightarrow{t_2} F_{r_1,r_2}$$

are the maps classified by the orthogonal pairs of subbundles which are the images of

$$\pi_1^* E_1 \xrightarrow{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \mu_1^\vee \phi_1 \end{pmatrix}} \mathcal{O}_{E_1} \oplus \pi_1^* E_1 \oplus \pi_1^* E_2 \oplus \mathcal{O}_{E_1} \xleftarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}} \pi_1^* E_2$$

$$\pi_2^* E_1 \xrightarrow{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}} \mathcal{O}_{E_2} \oplus \pi_2^* E_1 \oplus \pi_2^* E_2 \oplus \mathcal{O}_{E_2} \xleftarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \\ \mu_2^\vee \phi_2 \end{pmatrix}} \pi_2^* E_2$$

The restriction of $\rho\colon F_{r_1,r_2} \to P_{r_1+r_2}$ to the inverse image of $E_1 \oplus E_2 = N^-$ has a section $\gamma\colon N^- \to \rho^{-1}(N^-) \subset F_{r_1,r_2}$ classified by the orthogonal pair of subbundles

$$\pi^* E_1 \xrightarrow{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \mu_1^\vee \phi_1 \end{pmatrix}} \mathcal{O}_{E_1\oplus E_2} \oplus \pi^* E_1 \oplus \pi^* E_2 \oplus \mathcal{O}_{E_1\oplus E_2} \xleftarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \\ \mu_2^\vee \phi_2 \end{pmatrix}} \pi^* E_2$$

The restriction of $\gamma\colon E_1 \oplus E_2 \to F_{r_1,r_2}$ to each factor E_i thus coincides with $\gamma_i\colon E_i \to F_{r_1,r_2}$.

In G_{r_1} there is the locus $\overline{N}_1^+ = G_{r_1} \cap \mathrm{Gr}_S(2r_1, \mathcal{O}_S \oplus E_1 \oplus E_2)$ and there is an analogous locus $\overline{N}_2^+ \subset G_{r_2}$. We have $\rho_1^{-1}(\overline{N}_1^+) \cap \rho_2^{-1}(\overline{N}_2^+) = \rho^{-1}(N^+)$.

Call U_1 all the tautological bundles of rank $2r_1$, call U_2 all the tautological bundles of rank $2r_2$, and call U all the tautological bundles of rank $2r_1 + 2r_2$.

By Proposition 13.1 and Theorem 13.3 the extension of supports maps $A_{\overline{N}_i^+}(G_{r_i}) \to A(G_{r_i})$ and $A_{N_i^+}(P_{r_i}) \to A(P_{r_i})$ and $A_{N^+}(P_{r_1+r_2}) \to A(P_{r_1+r_2})$ are all split injective. Since the $\rho_i\colon F_{r_1,r_2} \to G_{r_i}$ and $\rho\colon F_{r_1,r_2}$ are quaternionic Grassmannian bundles, we may pull back along the ρ_i and ρ and see that the extension of supports maps $A_{\rho_i^{-1}(\overline{N}_i^+)}(F_{r_1,r_2}) \to A(F_{r_1,r_2})$ and $A_{\rho^{-1}(N^+)}(F_{r_1,r_2})$ are also split injective.

In the notation of (13.6) let

$$\overline{\theta}_{E_1,\phi_1} = (-1)^r i_{A,\flat} q^A (1_{\overline{N}_1^+ \cap \overline{N}_1^-}) \in A_{\overline{N}_1^+}(G_{r_1}).$$

Then by Theorem 13.3 and the functoriality of the Pontryagin classes

$$\overline{\theta}_{E_1,\phi_1} \in A_{\overline{N}_1^+}(G_{r_1}), \qquad \rho_1^A(\overline{\theta}_{E_1,\phi_1}) \in A_{\rho_1^{-1}(\overline{N}_1^+)}(F_{r_1,r_2}), \qquad t_1^A \rho_1^A(\overline{\theta}_{E_1,\phi_1}) \in A_{N_1^+}(P_{r_1}),$$

are the unique classes in their respective cohomology groups whose images under the extension of supports maps are the Pontryagin classes $p_{r_1}(U_1,\phi_1)$. Therefore $t_1^A\rho_1^A(\overline{\theta}_{E_1,\phi_1})$ is the θ_{E_1,ϕ_1} of Theorem 12.6, and $\mathrm{th}(E_1,\phi_1) = \gamma_1^A \rho_1^A(\overline{\theta}_{E_1,\phi_1}) \in A_S(E_1)$, while $q_1^A \,\mathrm{th}(E_1,\phi_1) = \gamma^A\rho_1^A(\overline{\theta}_{E_1,\phi_1}) \in A_{E_2}(E_1 \oplus E_2)$.

There are analogous classes

$$\overline{\theta}_{E_2,\phi_2} \in A_{\overline{N}_2^+}(G_{r_2}), \qquad \rho_2^A(\overline{\theta}_{E_2,\phi_2}) \in A_{\rho_2^{-1}(\overline{N}_2^+)}(F_{r_1,r_2}),$$

such that $\rho_2^A(\overline{\theta}_{E_2,\phi_2})$ is the unique class whose extension of supports is $p_{r_2}(U_2,\phi_2) \in A(F_{r_1,r_2})$ and $\gamma^A\rho_2^A(\overline{\theta}_{E_2,\phi_2}) = q_2^A \,\mathrm{th}(E_2,\phi_2)$.

Similarly there are classes

$$\theta_{E_1\oplus E_2,\phi_1\oplus\phi_2} \in A_{N^+}(P_{r_1+r_2}), \qquad \rho^A(\theta_{E_1\oplus E_2,\phi_1\oplus\phi_2}) \in A_{\rho^{-1}(N^+)}(F_{r_1,r_2}),$$

which are the unique classes whose images under the extension of supports maps are the Pontryagin classes $p_{r_1+r_2}(U)$. We have

$$\mathrm{th}(E_1 \oplus E_2, \phi_1 \oplus \phi_2) = f^A \theta_{E_1\oplus E_2,\phi_1\oplus\phi_2} = \gamma^A\rho^A\theta_{E_1\oplus E_2,\phi_1\oplus\phi_2}.$$

Now

$$\rho_1^A(\overline{\theta}_{E_1,\phi_1})\rho_2^A(\overline{\theta}_{E_2,\phi_2}) \in A_{\rho_1^{-1}(\overline{N}_1^+)\cap\rho_2^{-1}(\overline{N}_2^+)}(F_{r_1,r_2}) = A_{\rho^{-1}(N^+)}(F_{r_1,r_2})$$

is a class whose image under the extension of supports map is $p_{r_1}(U_1,\phi_1)p_{r_2}(U_2,\phi_2) = p_{r_1+r_2}(U,\phi)$. (Top Pontryagin classes multiply.) Since $\rho^A(\theta_{E_1\oplus E_2,\phi_1\oplus\phi_2})$ was the unique class with that property we have

$$\rho_1^A(\overline{\theta}_{E_1,\phi_1})\rho_2^A(\overline{\theta}_{E_2,\phi_2}) = \rho^A(\theta_{E_1\oplus E_2,\phi_1\oplus\phi_2}).$$

Applying γ^A now gives

$$q_1^A(\mathrm{th}(E_1,\phi_1))\, q_2^A(\mathrm{th}(E_2,\phi_2)) = \mathrm{th}(E_1 \oplus E_2, \phi_1 \oplus \phi_2). \qquad \square$$

14. Symplectic orientations

A ring cohomology theory can be symplectically oriented by any of five structures satisfying different axioms. We have already seen two of them: a symplectic Thom structure (Definition 7.1) and a Pontryagin structure (Definition 12.1). Here are the other three (cf. [14, Definitions 3.26, 3.32, 3.1]).

Definition 14.1. A *Pontryagin classes theory* on a ring cohomology theory A on a category of schemes is a system of assignments to every symplectic bundle (E,ϕ) over every scheme S in the category of elements $p_i(E,\phi) \in A(S)$ for all $i \geq 1$ satisfying

1. For $(E_1,\phi_1) \cong (E_2,\phi_2)$ we have $p_i(E_1,\phi_1) = p_i(E_2,\phi_2)$ for all i.
2. For a morphism $f\colon Y \to S$ we have $f^A(p_i(E,\phi)) = p_i(f^*(E,\phi))$ for all i.
3. For the tautological rank 2 symplectic subbundle $(\mathcal{U}, \phi|_{\mathcal{U}})$ on HP^1 the maps
$$(1, p_1(\mathcal{U}, \phi|_{\mathcal{U}}))\colon A(S) \oplus A(S) \to A(\mathrm{HP}^1 \times S)$$
are isomorphisms for all S.
4. For a rank 2 symplectic space (V,ϕ) viewed as a trivial symplectic bundle over k we have $p_1(V,\phi) = 0$ in $A(k)$.
5. For an orthogonal direct sum of symplectic bundles $(E,\phi) \cong (E_1,\phi_1) \perp (E_2,\phi_2)$ we have $p_i(E,\phi) = p_i(E_1,\phi_1) + \sum_{j=1}^{i-1} p_{i-j}(E_1,\phi_1)p_j(E_2,\phi_2) + p_i(E_2,\phi_2)$ for all i.
6. For (E,ϕ) of rank $2r$ we have $p_i(E,\phi) = 0$ for $i > r$.

One may also set $p_0(E,\phi) = 1$ and even $p_i(E,\phi) = 0$ for $i < 0$.

Definition 14.2. A *symplectic Thom classes theory* on a ring cohomology theory A on a category of schemes is a system of assignments to every symplectic bundle (E,ϕ) over every scheme X in the category of an element $\mathrm{th}(E,\phi) \in A_X(E)$ satisfying conditions (1)–(5) of Theorem 13.2.

Definition 14.3. A *symplectic orientation* on a ring cohomology theory A on a category of schemes is a system of assignments to every symplectic bundle (E,ϕ) over every scheme X in the category and every closed subset $Z \subset X$ with $X \smallsetminus Z$ in the category of an isomorphism $\mathrm{th}_Z^{E,\phi}\colon A_Z(X) \to A_Z(E)$ with the following properties.

1. Let $\pi\colon E \to X$ be the structure map, and let $\pi^A\colon A_Z(X) \to A_{\pi^{-1}(Z)}(E)$ be the pullback. Then for all $a \in A(X)$ and $b \in A_Z(X)$ one has
$$\mathrm{th}_Z^{E,\phi}(a \cup b) = \mathrm{th}_X^{E,\phi}(a) \cup \pi^A b, \qquad \mathrm{th}_Z^{E,\phi}(b \cup a) = \pi^A b \cup \mathrm{th}_X^{E,\phi}(a).$$
2. For every isometry of symplectic bundles $\phi\colon (E,\phi) \to (F,\psi)$ the following diagram commutes
$$\begin{tikzcd} A_Z(X) \arrow[r, "\mathrm{th}_Z^{F,\psi}", "\cong"'] \arrow[d, equal, "1"'] & A_Z(F) \arrow[d, "\phi^A", "\cong"'] \\ A_Z(X) \arrow[r, "\mathrm{th}_Z^{E,\phi}", "\cong"'] & A_Z(E). \end{tikzcd}$$

(3) For every morphism $f\colon X' \to X$ with $Z' \subset X'$ closed and $f^{-1}(Z) \subset Z'$, then for $(E', \phi') = f^*(E, \phi)$ and $g\colon E' \to E$ the pullback of f along $\pi\colon E \to X$, the following diagram commutes

$$\begin{tikzcd} A_Z(X) \arrow[r, "\mathrm{th}_Z^{E,\phi}", "\cong"'] \arrow[d, "f^A"'] & A_Z(E) \arrow[d, "g^A"] \\ A_{Z'}(X') \arrow[r, "\mathrm{th}_{Z'}^{E',\phi'}", "\cong"'] & A_{Z'}(E'). \end{tikzcd}$$

(4) For every pair of sympectic bundles (E_1, ϕ_1) and (E_2, ϕ_2) over a scheme X, with structural maps $p_i\colon E_i \to X$, the following diagram commutes

$$\begin{tikzcd} A_Z(X) \arrow[r, "\mathrm{th}_Z^{E_1,\phi_1}", "\cong"'] \arrow[d, "\mathrm{th}_Z^{E_2,\phi_2}"', "\cong"] & A_Z(E_1) \arrow[d, "\cong"', "\mathrm{th}_Z^{p_1^*E_2,p_1^*\phi_2}"] \\ A_Z(E_2) \arrow[r, "\mathrm{th}_Z^{p_2^*E_1,p_2^*\phi_1}", "\cong"'] & A_Z(E_1 \oplus E_2). \end{tikzcd}$$

Property (1) of Definition 14.3 implies that the $\mathrm{th}_Z^{E,\phi}$ are $A(X)$-bimodule maps, but at least formally it is somewhat stronger ([14, Lemma 3.33] did not seem obvious to the second author with the formally weaker property).

Theorem 14.4. *Let A be a ring cohomology theory.*
(a) *There are inverse bijections*

$$\{\textit{symplectic Thom structures on } A\} \longleftrightarrow \{\textit{Pontryagin structures on } A\}$$

given by the formulas (7.1) *and* (12.4)*.*
(b) *There are inverse bijections*

$$\{\textit{Pontryagin class structures on } A\} \longleftrightarrow \{\textit{Pontryagin structures on } A\}$$

given by forgetfulness and the map which assigns to a Pontryagin structure the Pontryagin classes assigned by Definition 8.3 to the associated symplectic Thom structure.
(c) *There are inverse bijections*

$$\{\textit{symplectic Thom class structures on } A\} \longleftrightarrow \{\textit{symplectic Thom structures on } A\}$$

given by forgetfulness and Theorem 13.2.
(d) *There are inverse bijections*

$$\{\textit{symplectic Thom class structures on } A\} \longleftrightarrow \{\textit{symplectic orientations on } A\}$$

given by the formulas $\mathrm{th}_Z^{E,\phi}(b) = \pi^A b \cup \mathrm{th}(E, \phi)$ *and* $\mathrm{th}(E, \phi) = \mathrm{th}_X^{E,\phi}(1_X)$.

Proof. (a) This is Theorem 12.2.

(b) The right-to-left map is well-defined because the Pontryagin classes associated to a symplectic Thom structure satisfy the functoriality axioms (1)(2) by simple arguments, axiom (3) concerning HP^1 by Theorem 8.1, the triviality axiom (4) by Proposition 7.7, the Cartan sum formula (5) by Theorem 10.5, and the dimension axiom (6) holds by definition. The right-to-left-to-right roundtrip assigns to a given Pontryagin structure the Pontryagin structure associated to the associated Thom structure. This is the identity map because of Theorem

12.5. The left-to-right map is injective because of the splitting principle (Theorem 10.2) and the Cartan sum formula (cf. [14, Theorem 3.27]).

(c) The right-to-left map is well-defined by Theorem 13.2, and the right-to-left-to-right roundtrip is the identity by the last phrase in that theorem. The left-to-right map is injective because of the splitting principle and the multiplicativity formula (5) for the symplectic Thom classes (cf. [14, Lemma 3.34]).

(d) Left to the reader. □

15. More on the cohomology of the open stratum

This is a version of our original proof of Theorem 3.4(b). We use the following geometry, with (V,ϕ) a symplectic space of dimension $2n+2$ over a field and $\mathrm{GrSp}(2,V,\phi)\subset \mathrm{Gr}(2,V)$ the closed subvariety parametrizing totally isotropic subspaces.

Theorem 15.1. *The open stratum $X_0\subset \mathrm{HP}^n$ is a dense open subvariety of an open subvariety $Y_0\subset \mathrm{Gr}(2,V)$ with complement $Z_0=Y_0\smallsetminus X_0=Y_0\cap \mathrm{GrSp}(2,V,\phi)$, and there is an open subvariety $Y_0'\subset Y_0$ containing Z_0 for which there exists a diagram in which the arrows have the stated properties and the horizontal arrows commute with each other and with all upward arrows and with the two solid downward arrows.*

$$
\begin{tikzcd}
Z_0 \arrow[r, "\text{closed}"'] \arrow[rr, bend left, "\text{closed}"] \arrow[d, shift left, "\text{restriction of the } \mathbb{A}^{2n}\text{-bundle}"] & Y_0' \arrow[r, hook, "\text{open}"'] \arrow[d, shift left, "\mathbb{A}^{2n}\text{-bundle}"] & Y_0 \arrow[d, shift left, dotted, "\text{different } \mathbb{A}^{2n}\text{-bundle}"] & X_0 = Y_0\smallsetminus Z_0 \arrow[l, hook', "\text{open}"'] \\
\mathbb{P}^{2n-1} \arrow[u, shift left] \arrow[r, "\text{closed}"] \arrow[rr, bend right, "\text{hyperplane}"'] & \mathbb{P}^{2n}\smallsetminus 0 \arrow[u, shift left] \arrow[r, hook, "\text{open}"] & \mathbb{P}^{2n} \arrow[u, shift left] & \mathbb{A}^{2n} \arrow[l, hook', "\text{open}"] \arrow[u]
\end{tikzcd}
\tag{15.1}
$$

Proof. In (V,ϕ) we have a 1-dimensional subspace $E=\langle e\rangle$, a vector f with $\phi(e,f)=1$, and the ϕ-nondegenerate subspace $F=\langle e,f\rangle^\perp$. We have $E^\perp=E\oplus F$. We set

$$
\begin{aligned}
X_0 &= \{U\subset V \mid U\not\subset E^\perp \text{ and } \phi|_U \text{ is nondegenerate}\},\\
Y_0 &= \{U\subset V \mid U\not\subset E^\perp\},\\
Z_0 &= Y_0\smallsetminus X_0 = Y_0\cap \mathrm{GrSp}(2,V,\phi),\\
Y_0' &= \{U\subset V \mid U\not\subset E^\perp \text{ and } U\cap E=\{0\}\}.
\end{aligned}
$$

Clearly Y_0 is an open subvariety of the Grassmannian, and X_0 and Y_0' are open in Y_0. Moreover, any U in $Y_0\smallsetminus Y_0'$ is of the form $U=\langle u,v\rangle$ with $0\neq u\in E$ and $v\notin E^\perp$. We therefore have (up to nonzero multiples) $u=e$ and $v=f+v_0$ with $v_0\in E^\perp$. We then have $\phi(u,v)=1$, so $\phi|_U$ is nondegenerate. So we have $Y_0\smallsetminus Y_0'\subset X_0$ and $Z_0\subset Y_0'$.

For the vector bundle structures, we use Białynicki-Birula's theory of cell decompositions induced by $\mathbb{G}_m$-actions. Consider the one-parameter subgroup $\lambda_1\colon \mathbb{G}_m\hookrightarrow \mathrm{GL}(V)$ given by

$$
\lambda_1(t)\cdot v = \begin{cases} v & \text{for } v\in E^\perp = \langle e\rangle\oplus F,\\ t^{-1}v & \text{for } v\in\langle f\rangle. \end{cases}
$$

The fixed-point locus of the induced action on $\mathrm{Gr}(2,V)$ has two components

$$
\Gamma_0=\mathrm{Gr}(2,E^\perp), \qquad\qquad \Gamma_1=\{\langle v,f\rangle \mid v\in E^\perp\}\cong \mathbb{P}(E^\perp)=\mathbb{P}^{2n}
$$

For each component there is a stratum $\Upsilon_i = \{x \in \mathrm{Gr}(2,V) \mid \lim_{t\to 0} \lambda_1(t)x \in \Gamma_i\}$. The assignment $x \mapsto \lim_{t\to 0} \lambda_1(t)x$ gives maps $\pi_1 \colon \Upsilon_i \to \Gamma_i$ which send any $U \in \mathrm{Gr}(2,V)$ to its associated graded space with respect to the filtration on U induced by $0 \subset E^\perp \subset V$. The maps $\pi_1 \colon \Upsilon_i \to \Gamma_i$ are vector bundles identifiable with the normal bundles N_{Γ_i/Υ_i}. In this case we have

$$\Upsilon_0 = \mathrm{Gr}(2,E^\perp) = \Gamma_0, \qquad \Upsilon_1 = Y_0.$$

The fiber of $\pi_1 \colon Y_0 \to \mathbb{P}^{2n}$ over $\langle v,f\rangle \in \mathbb{P}^{2n}$ consists of points corresponding to spaces $\langle v, u+f\rangle$ with $u \in E^\perp$. These points are parametrized by the classes $\overline{u} \in E^\perp/\langle v\rangle$. So Y_0 is isomorphic to the tautological rank $2n$ quotient bundle $\mathcal{Q}$ on $\mathbb{P}^{2n}$.

Now λ_1 does not preserve the symplectic form ϕ because it does not act on e and f with opposite weights. So to study the part of Y_0 in $\mathrm{GrSp}(2,V,\phi)$ we replace it by $\lambda_2 \colon \mathbb{G}_m \hookrightarrow \mathrm{Sp}(2,V,\phi)$ acting linearly on V with

$$\lambda_2(t)\cdot v = \begin{cases} tv & \text{for } v \in \langle e\rangle, \\ v & \text{for } v \in F, \\ t^{-1}v & \text{for } v \in \langle f\rangle. \end{cases}$$

The induced action of λ_2 on $\mathrm{Gr}(2,V)$ comes with four fixed-point loci

$$\Gamma_0' = \{\langle e,v\rangle \mid v \in F\}, \quad \Gamma_1' = \mathrm{Gr}(2,F), \quad \Gamma_2' = \{\langle e,f\rangle\}, \quad \Gamma_3' = \{\langle u,f\rangle \mid u \in F\} = \mathbb{P}^{2n-1}$$

and vector bundles $\pi_2 \colon \Upsilon_i' \to \Gamma_i'$ much as before. This time, however, we have $Y_0 = \Upsilon_2' \cup \Upsilon_3'$ and $Y_0' = \Upsilon_3'$. We have $Y_0 \smallsetminus Y_0' = \Upsilon_2' = \{\langle e, v+f\rangle \mid v \in F\} \cong \mathbb{A}^{2n}$, and we may see that it intersects $\Gamma_1 = \mathbb{P}^{2n}$ in the unique point $\langle e,f\rangle = \Gamma_2'$. We will call this point $0 \in \mathbb{P}^{2n}$.

Now the Białynicki-Birula vector bundle $Y_0' = \Upsilon_3' \to \Gamma_3' = \mathbb{P}^{2n-1}$ is of rank $2n+1$. The base, the fixed point set $\mathbb{P}^{2n-1} = \{\langle u,f\rangle \mid u \in \mathbb{P}(F)\}$, is entirely within $\mathrm{GrSp}(2,V,\phi)$. Since the one-parameter subgroup respects $\mathrm{GrSp}(2,V,\phi)$, the locus Z_0 is also a bundle over $\mathbb{P}^{2n-1}$, indeed a subbundle of Y_0' of rank $2n$.

As a bundle Y_0' is isomorphic to $N_{\mathbb{P}^{2n-1}/Y_0'}$. At any point $U_{\overline{u}} = \langle u,f\rangle \in \mathbb{P}^{2n-1}$, the fiber of the normal bundle is canonically isomorphic to the quotient of the subspace of $\phi \in \mathrm{Hom}_k(U_{\overline{u}}, V/U_{\overline{u}}) = T_{U_{\overline{u}}}\mathrm{Gr}(2,V)$ corresponding to first order deformations $\langle u+\phi(u)\varepsilon, f+\phi(f)\varepsilon\rangle$ which lie within Y_0' modulo those lying within $\mathbb{P}^{2n-1}$. These first-order deformations are thus the $\langle u+\alpha e\varepsilon, f+(v+\beta e)\varepsilon\rangle$ with $\overline{v} \in F/\langle u\rangle$. Thus Y_0' is isomorphic to a vector bundle $N \to \mathbb{P}^{2n-1}$ which we can split as $N = N_0 \oplus L$ with

$$N_0 \xrightarrow[\cong]{(\overline{v},\beta)} \mathcal{Q} \oplus \mathcal{O}, \qquad L \xrightarrow[\cong]{\alpha} \mathcal{O}(1),$$

The subbundle of N corresponding to Z_0 corresponds to the first-order deformations satisfying

$$\phi(u+\alpha e\varepsilon, f+(v+\beta e)\varepsilon) = (\alpha + \phi(u,v))\varepsilon = 0.$$

So it is defined by the equation $\alpha = -\phi(u,v)$. This bundle is thus the graph of a vector bundle map $\psi \colon N_0 \to L$ which fiberwise is $(\overline{v},\beta) \mapsto -\phi(u,v)$ (with u a function of the base). Let $N_1 \subset N$ be the graph of ψ. We still have a direct sum $N = N_1 \oplus L$. We now have a quotient map $N \twoheadrightarrow N/N_1 \cong L$ which makes $Y_0' \cong N$ into a rank $2n$ vector bundle over L, with $Z_0 \cong N_1$ the subbundle lying over the zero section $\mathbb{P}^{2n-1}$ of L. Finally the subbundle L consists of the points $\langle \alpha e + u, f\rangle$ with $\overline{u} \in \mathbb{P}^{2n-1}$, which is $\mathbb{P}^{2n} \smallsetminus 0$. $\square$

Second proof of Theorem 3.4(b). By Theorem 3.2 it is enough to treat the open stratum X_0. By Theorem 15.1 $X_0 \subset Y_0$ is an open subvariety with complement Z_0, and there is a closed embedding of pairs $(\mathbb{P}^{2n}, \mathbb{A}^{2n}) \to (Y_0, X_0)$. This yields a morphism of localization long exact sequences

$$\begin{tikzcd} A_{Z_0}(Y_0) \arrow[r] \arrow[d] & A(Y_0) \arrow[r] \arrow[d, "\cong"'] & A(X_0) \arrow[r, "\partial"] \arrow[d, "i^A"'] & A_{Z_0}(Y_0) \arrow[r] \arrow[d] & A(Y_0) \arrow[d, "\cong"'] \\ A_{\mathbb{P}^{2n-1}}(\mathbb{P}^{2n}) \arrow[r] & A(\mathbb{P}^{2n}) \arrow[r] & A(\mathbb{A}^{2n}) \arrow[r, "\partial"] & A_{\mathbb{P}^{2n-1}}(\mathbb{P}^{2n}) \arrow[r] & A(\mathbb{P}^{2n}) \end{tikzcd} \tag{15.2}$$

The map $A(Y_0) \to A(\mathbb{P}^{2n})$ is an isomorphism because $Y_0 \to \mathbb{P}^{2n}$ is a vector bundle. By excision $A_{Z_0}(Y_0) \to A_{\mathbb{P}^{2n-1}}(\mathbb{P}^{2n})$ is isomorphic to $A_{Z_0}(Y_0') \to A_{\mathbb{P}^{2n-1}}(\mathbb{P}^{2n} \smallsetminus 0)$, which again is an isomorphism because of the vector bundles in Theorem 15.1 and strong homotopy invariance. The map labeled i^A is therefore also an isomorphism by the five lemma. Since $i^A t^A \colon A(k) \to A(\mathbb{A}^{2n})$ is an isomorphism by homotopy invariance, it follows that t^A is an isomorphism. $\square$

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