ON THE MOTIVIC COMMUTATIVE RING SPECTRUM BO

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Abstract. We construct an algebraic commutative ring T-spectrum $\mathbf{BO}$ which is stably fibrant and $(8,4)$-periodic and such that on $\mathcal{S}m\mathcal{O}p/S$ the cohomology theory $(X,U) \mapsto \mathbf{BO}^{p,q}(X_+/U_+)$ and Schlichting's hermitian K-theory functor $(X,U) \mapsto KO^{[q]}_{2q-p}(X,U)$ are canonically isomorphic. We use the motivic weak equivalence $\mathbb{Z} \times HGr \xrightarrow{\sim} \mathbf{KSp}$ relating the infinite quaternionic Grassmannian to symplectic K-theory to equip $\mathbf{BO}$ with the structure of a commutative monoid in the motivic stable homotopy category. When the base scheme is $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$, this monoid structure and the induced ring structure on the cohomology theory $\mathbf{BO}^{*,*}$ are the unique structures compatible with the products

$$KO_0^{[2m]}(X) \times KO_0^{[2n]}(Y) \to KO_0^{[2m+2n]}(X \times Y).$$

on Grothendieck-Witt groups induced by the tensor product of symmetric chain complexes. The cohomology theory is bigraded commutative with the switch map acting on $\mathbf{BO}^{*,*}(T \wedge T)$ in the same way as multiplication by the Grothendieck-Witt class of the symmetric bilinear space $\langle -1 \rangle$.

1. Introduction

In a recent paper [16] we defined motivic versions of symplectically oriented cohomology theories A and of quaternionic Grassmannians $HGr(r,n)$. This $HGr(r,n)$ is the open subscheme of the ordinary Grassmannian $Gr(2r,2n)$ parametrizing subspaces on which the standard symplectic form on $\mathcal{O}^{\oplus 2n}$ is nondegenerate. We defined Pontryagin classes of symplectic bundles in such theories and calculated

$$A(HGr(r,n)) = A(pt)[p_1,\dots,p_r]/(h_{n-r+1},\dots,h_n) \tag{1.1}$$

where the p_i are the Pontryagin classes of the tautological bundle on $HGr(r,n)$, and the h_i are the polynomials in the p_i corresponding to the complete symmetric polynomials. This is the same formula as the one which describes the cohomology of an ordinary Grassmannian in terms of the Chern classes of an oriented cohomology theory.

In this paper we begin to apply those results to the hermitian K-theory of regular noetherian separated schemes X of finite Krull dimension with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. We write $KO^{[n]}(X,U)$ for Schlichting's hermitian K-theory space for bounded complexes of vector bundles on X which are acyclic on the open subscheme $U \subset X$ and which are symmetric with respect to the shift by n of the usual duality. We write $KO_i^{[n]}(X,U)$ for its homotopy groups (for $i \geq 0$) or for Balmer's Witt groups $W^{n-i}(X,U)$ (for $i < 0$).

One of our main results is the following.

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Theorem 1.1. *For a regular separated noetherian scheme S of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ Schlichting's hermitian K-theory is a ring cohomology theory with an SL^c Thom classes theory.*

Here *ring cohomology theory* is used in the sense of [14, Definitions 2.1 and 2.13]. An *SL^c Thom classes theory* specifies a Thom class $th(E, L, \lambda) \in KO_0^{[n]}(E, E - X)$ for every *SL^c_n-bundle*, by which we mean a triple (E, L, λ) with E a vector bundle of rank n over X, L a line bundle and $\lambda\colon L \otimes L \to \det E$ an isomorphism. These classes are functorial, multiplicative, and induces isomorphisms $\cup\, th(E, L, \lambda)\colon KO_i^{[m]}(X) \to KO_i^{[m+n]}(E, E-X)$ for all i and m. The Thom classes restrict to Euler classes $e(E, L, \lambda) \in KO_0^{[n]}(X)$. An SL^c Thom classes theory gives Thom classes for all special linear, special orthogonal and symplectic bundles. So by the theory of [16] there are Pontryagin classes $p_i(E, \phi) \in KO_0^{[2i]}(X)$ for symplectic bundles. The $p_1(E, \phi)$ of a symplectic bundle of rank $2r$ is the class corresponding to $[E, \phi] - r[\mathsf{H}] \in KSp_0(X, U) = GW^-(X)$ under the natural isomorphism $KSp \cong KO^{[2]}$. Here H is the trivial symplectic bundle of rank 2. The higher Pontryagin classes will be calculated elsewhere.

We also construct several motivic spectra representing hermitian K-theory. The first construction is a T-spectrum whose spaces $(\mathbf{KO}^{[0]}, \mathbf{KO}^{[1]}, \mathbf{KO}^{[2]}, \dots)$ are fibrant replacements of presheaves composed of Schlichting's Waldhausen-like hermitian K-theory spaces for bounded complexes of vector bundles with shifted dualities [19]. The structure maps $\mathbf{KO}^{[n]} \wedge T \to \mathbf{KO}^{[n+1]}$ are adjoint to the maps $\mathbf{KO}^{[n]}(-) \to \mathbf{KO}^{[n+1]}(- \wedge T)$ which are essentially multiplication by the Thom class $th \in \mathbf{KO}_0^{[1]}(T)$ of the trivial line bundle. Note the use of the appearance of the Thom classes in the very structure of the spectrum. For (X, U) in $\mathcal{S}m\mathcal{O}p/S$ there are functorial isomorphisms

$$KO_i^{[n]}(X, U) \cong \mathbf{KO}_i^{[n]}(X_+/U_+) \cong \mathbf{BO}^{2n-i,n}(\Sigma_T^\infty(X_+/U_+)), \tag{1.2}$$

and the boundary maps $\partial\colon KO_i^{[n]}(U) \to KO_{i-1}^{[n]}(X, U)$ and

$$\partial\colon \mathbf{BO}^{2n-i,n}(\Sigma_T^\infty(U_+)) \to \mathbf{BO}^{2n-i+1,n}(\Sigma_T^\infty(X_+/U_+))$$

correspond. Because it is based on the hermitian K-theory of chain complexes, this spectrum has advantages in certain situations over the one constructed several years ago by Hornbostel [8]. It treats all shifts/weights uniformly instead of dealing in one way with the K^h-theory of the even weights and in another way with the U-theory and V-theory of the odd weights. It naturally handles non-affine schemes X and even pairs (X, U) with $U \subset X$ open. Finally we can easily identify the Thom, Euler and Pontryagin classes in the Grothendieck-Witt groups of chain complexes $GW^{[n]}(X \text{ on } X - U)$.

We show that the Morel and Voevodsky's theorem on Grassmannians and algebraic K-theory extends to the symplectic context. There are in truth only a few things to verify for symplectic groups beyond what is in Morel and Voevodsky's paper. (Orthogonal groups are much more problematic. **However it has been done recently by M.Schlichting and Shanker Tripathi.**)

Theorem 1.2. *Let $HGr = \operatorname{colim} HGr(n, 2n)$ be the infinite quaternionic Grassmannian. Then $\mathbb{Z} \times HGr$ and KSp are isomorphic in the motivic unstable homotopy category $H_\bullet(S)$.*

Next we define the $\times$ product. The final group of theorems in the paper concerns the product structure. The groups $KO_0^{[n]}(X, U)$, which are the Grothendieck-Witt groups of

bounded chain complexes of vector bundles which are symmetric with respect to shifted but untwisted dualities, have a naive product induced by the tensor product of chain complexes

$$KO_0^{[m]}(X,U) \times KO_0^{[n]}(Y,V) \to KO_0^{[m+n]}(X \times Y, X \times V \cup U \times Y). \tag{1.3}$$

Let $\langle 1\rangle$ and $\langle -1\rangle$ in $KO_0^{[0]}(pt)$ denote the Grothendieck-Witt classes of the rank one symmetric bilinear forms. The product is $\langle -1\rangle$-*commutative* meaning that for $\alpha \in \mathbf{BO}^{p,q}(A)$ and $\beta \in \mathbf{BO}^{p',q'}(B)$ we have $\alpha \times \beta = (-1)^{pp'}\langle -1\rangle^{qq'}\sigma^*(\beta \times \alpha)$ where $\sigma\colon A \times B \to B \times A$ switches the factors. Recall that a motivic space A is called *small* if $Hom_{SH(S)}(\Sigma_T^\infty A, -)$ commutes with arbitrary coproducts.

Theorem 1.3. *The cohomology theory $(\mathbf{BO}^{*,*}, \partial)$ on the category $\mathbf{M}_\bullet^{small}(S)$ of small motivic spaces over S has a product $\times$ which is associative and $\langle -1\rangle$-commutative with the unit $1 = \langle 1\rangle \in \mathbf{BO}^{0,0}(pt_+)$, which has $\alpha \times \Sigma_{\mathbb{G}_m} 1 = \Sigma_{\mathbb{G}_m}\alpha$ and $\alpha \times \Sigma_{S^1_s} 1 = \Sigma_{S^1_s}\alpha$ for all α, and which restricts via the isomorphism* (1.2) *to the naive ring structure* (1.3) *on the groups $KO_0^{[2n]}(X)$ for $X \in \mathcal{S}m/S$. It is the unique product with these properties.*

This is Theorems 11.5 and 11.6. Restricting to pairs (X,U) with $U \subset X$ an open subscheme of a scheme smooth over X, we get the following result.

Theorem 1.4. *There is a canonical ring structure on the cohomology theory $(KO_*^{[*]}, \partial)$ on $\mathcal{S}m\mathcal{O}p/S$ which is associative and $\langle -1\rangle$-commutative with unit $\langle 1\rangle \in KO_0^{[0]}(pt)$ and which restricts to the naive product on the Grothendieck-Witt groups of chain complexes $KO_0^{[2n]}(X)$. This product and the Thom classes of SL^c-bundles make $(KO_*^{[*]}, \partial)$ ring cohomology theory with an SL^c Thom classes theory.*

Our strongest result on the product is the following theorem.

Theorem 1.5. *There exist morphisms $m\colon \mathbf{BO} \wedge \mathbf{BO} \to \mathbf{BO}$ and $e\colon \Sigma_T^\infty \mathbf{1} \to \mathbf{BO}$ in $SH(S)$ which make $(\mathbf{BO}, m, e)$ a commutative monoid in $SH(S)$ and which are compatible with the naive product in the following sense.*

(1) *For all X and Y in $\mathcal{S}m/S$ and all even integers $2p$ and $2q$ the naive products*

$$KO_0^{[2p]}(X) \times KO_0^{[2q]}(Y) \to KO_0^{[2p+2q]}(X \times Y)$$

and the product

$$\mathbf{BO}^{4p,2p}(X_+) \times \mathbf{BO}^{4q,2q}(Y_+) \to \mathbf{BO}^{4p+4q,2p+2q}(X_+ \wedge Y_+)$$

induced by m correspond under the isomorphisms (1.2).

(2) *The elements $\langle 1\rangle \in KO_0^{[0]}(pt)$ and $e \in \mathbf{BO}^{0,0}(S^{0,0})$ correspond under the isomorphisms* (1.2).

Moreover, if for the base scheme S the groups $KO_1(S)$ and $KSp_1(S)$ are finite (for example $S = \operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$), then the monoid structure (m,e) with these properties is unique.

For the proof of the theorem see Theorem 13.5.

We explain the basic ideas in the proofs of these three theorems. Gille and Nenashev's method [7] for constructing pairings in Witt groups of triangulated categories can be used in

hermitian K-theory to construct pairings between hermitian K-theory groups and Grothendieck-Witt groups

$$KO_r^{[i]}(X,U) \times KO_0^{[j]}(Y,V) \to KO_r^{[i+j]}(X \times Y, X \times V \cup U \times Y) \tag{1.4}$$

$$KO_0^{[i]}(X,U) \times KO_r^{[j]}(Y,V) \to KO_r^{[i+j]}(X \times Y, X \times V \cup U \times Y) \tag{1.5}$$

(see (5.4c)–(5.4d)). They respect the boundary maps of the cohomology theorem $KO_*^{[*]}$. This gives the hermitian K-theory groups the structure of a cohomology theory with a partial multiplicative structure with Thom classes for all SL^c bundles including symplectic bundles. Although this is less structure than we assumed while writing [16], it is enough to prove the quaternionic projective bundle theorem and the symplectic splitting principle and to calculate the cohomology of quaternionic Grassmannians. Thus formula (1.1) holds for $A = KO_*^{[*]}$ with the $p_i \in KO_0^{[2i]}(HGr(r,n))$. Because of the isomorphism (1.2) it also holds for $A = \mathbf{BO}^{*,*}$ with the $p_i \in \mathbf{BO}^{4i,2i}(HGr(r,n))$.

The isomorphism (1.2) also transplants these pairings to $\mathbf{BO}$, giving in particular pairings

$$\boxtimes\colon \mathbf{BO}^{2i-r,i}(X_+) \times \mathbf{BO}^{2j,j}(Y_+) \to \mathbf{BO}^{2i+2j-r,i+j}(X_+ \wedge Y_+) \tag{1.6}$$

Theorem 1.2 and the isomorphisms $KSp \cong KO^{[4k+2]}$ gives us a canonical elements $\tau_{4k+2} \in \mathbf{BO}^{8k+4,4k+2}(\mathbb{Z} \times HGr)$. Write $[-n,n] = \{m \in \mathbb{Z} \mid -n \le m \le n\}$ and set

$$HGr_n = [-n,n] \times HGr(n,2n).$$

We have $\mathbb{Z} \times HGr = \operatorname{colim} HGr_n$. A standard formula for homotopy colimits in triangulated categories gives us an exact sequence

$$0 \to \varprojlim{}^1 \mathbf{BO}^{8k+3,4k+2}(HGr_n) \to \mathbf{BO}^{8k+4,4k+2}(\mathbb{Z} \times HGr) \to \varprojlim \mathbf{BO}^{8k+4,4k+2}(HGr_n) \to 0$$

The inclusions $HGr_n \to HGr_{n+1}$ induce surjections on cohomology for any symplectically oriented theory. So the $\varprojlim^1$ vanish, and we have

$$\mathbf{BO}^{8k+4,4k+2}(\mathbb{Z} \times HGr) \cong \varprojlim \mathbf{BO}^{8k+4,4k+2}(HGr_n). \tag{1.7}$$

For essentially the same reasons we have isomorphisms

$$\mathbf{BO}^{16k+8,8k+4}((\mathbb{Z} \times HGr) \wedge (\mathbb{Z} \times HGr)) \cong \varprojlim \mathbf{BO}^{16k+8,8k+4}(HGr_n \wedge HGr_n). \tag{1.8}$$

Our class $\tau_{4k+2} \in \mathbf{BO}^{8k+4,4k+2}(\mathbb{Z} \times HGr)$ and the pairing (1.6) gives us a system of classes

$$\tau_{4k+2}|_{HGr_n} \boxtimes \tau_{4k+2}|_{HGr_n} \in \mathbf{BO}^{16k+8,8k+4}(HGr_n \wedge HGr_n). \tag{1.9}$$

and therefore a class

$$\tau_{4k+2} \boxtimes \tau_{4k+2} \in \mathbf{BO}^{16k+8,8k+4}(\mathbf{KO}^{[4k+2]} \wedge \mathbf{KO}^{[4k+2]}).$$

There is also an exact sequence of the form

$$0 \to \varprojlim{}^1 \mathbf{BO}^{4i-1,2i}(\mathbf{KO}^{[i]} \wedge \mathbf{KO}^{[i]}) \to \mathbf{BO}^{0,0}(\mathbf{BO} \wedge \mathbf{BO}) \to \varprojlim \mathbf{BO}^{4i,2i}(\mathbf{KO}^{[i]} \wedge \mathbf{KO}^{[i]}) \to 0.$$

The elements $\tau_{4k+2} \boxtimes \tau_{4k+2}$ define an element $\bar{m} \in \varprojlim \mathbf{BO}^{4i,2i}(\mathbf{KO}^{[i]} \wedge \mathbf{KO}^{[i]})$. This $\bar{m}$ is the unique element compatible with the naive product (1.9) and the isomorphisms (1.7) and (1.8). Lifting this element to $m \in \operatorname{Hom}_{SH(S)}(\mathbf{BO} \wedge \mathbf{BO}, \mathbf{BO})$ gives an element we can use to define the product $\times$. On small motivic spaces $\times$ depends only on $\bar{m}$ and not on the choice of m. We deduce the associativity and bigraded commutativity of $\times$ on small motivic spaces from the associativity and commutativity of the naive product on the $KO_0^{[2r]}(HGr_n)$. This gives us Theorem 1.3.

Theorem 1.5 is more subtle. The obstructions to the uniqueness of m and to the associativity, commutativity and unit property of the monoid it defines all live in certain $\varprojlim^1$ groups. We show in §13 that when $S = \operatorname{Spec} R$ with $\frac{1}{2} \in R$ and with $KO_1(R)$ and $KSp_1(R)$ finite groups, the $\varprojlim^1$ vanish. This uses the construction in §12 of three new spectra using a new motivic sphere.

The geometry used to prove the quaternionic projective bundle theorem in [16] also shows that the pointed quaternionic projective line (HP^1, x_0) is isomorphic to $T^{\wedge 2}$ in the motivic homotopy category $H_\bullet(S)$. The pointed scheme HP^{1+} which is the $\mathbf{A}^1$ mapping cone of the pointing $x_0 \colon pt \to HP^1$ is therefore also homotopy equivalent to $T^{\wedge 2}$. It is the union of HP^1 and $\mathbf{A}^1$ with $x_0 \in HP^1$ identified with $0 \in \mathbf{A}^1$, pointed at $1 \in \mathbf{A}^1$. Therefore the motivic stable homotopy categories of T-spectra and of HP^{1+}-spectra are equivalent.

We construct three HP^{1+}-spectra $\mathbf{BO}_{HP^{1+}}$, $\mathbf{BO}^{geom}$ and $\mathbf{BO}^{fin}$ representing hermitian K-theory. The spaces of $\mathbf{BO}_{HP^{1+}}$ are the even-indexed spaces $(\mathbf{KO}^{[0]}, \mathbf{KO}^{[2]}, \mathbf{KO}^{[4]}, \dots)$ of the T-spectrum. The spaces of $\mathbf{BO}^{geom}$ are alternately $\mathbb{Z} \times RGr$ and $\mathbb{Z} \times HGr$. The spaces of $\mathbf{BO}^{fin}$ are finite unions of finite-dimensional real and quaternionic Grassmannians. Here a *real Grassmannian* $RGr(r, 2n)$ is the open subscheme of the ordinary Grassmannian $Gr(r, 2n)$ where the hyperbolic quadratic form on $\mathcal{O}^{\oplus 2n}$ is nondegenerate, while $RGr = \operatorname{colim} RGr(n, 2n)$. For the details of $\mathbf{BO}^{fin}$ see Theorem 12.3. The structure maps of the spectra are all essentially multiplication with the Euler class $-p_1(\mathcal{U})$ of the tautological rank 2 symplectic subbundle on HP^1. The bonding maps $\mathbf{BO}^*_{2i} \wedge HP^{1+} \to \mathbf{BO}^*_{2i+2}$ of the two geometric spectra are morphisms of schemes or ind-schemes which are constant on the wedge $\mathbf{BO}^*_{2i} \vee HP^{1+}$. The inclusion $\mathbf{BO}^{fin} \to \mathbf{BO}^{geom}$ is a motivic stable weak equivalence, while the isomorphism $\mathbf{BO}^{geom} \cong \mathbf{BO}_{HP^1}$ in $SH(S)$ is constructed from *classifying maps*

$$\tau_{4k} \colon \mathbb{Z} \times RGr \to \mathbf{KO}^{[4k]}, \qquad \tau_{4k+2} \colon \mathbb{Z} \times HGr \xrightarrow{\sim} \mathbf{KO}^{[4k+2]},$$

in $H_\bullet(S)$. The τ_{4k+2} are the isomorphisms of Theorem 1.2, while the τ_{4k} are constructed from the τ_{4k+2}. We do not know if the τ_{4k} are isomorphisms, although stably they have a right inverse (Proposition 12.7).

The $\varprojlim^1$ calculated with $\mathbf{BO}$ and with $\mathbf{BO}^{fin}$ are the same. Calculations based on the quaternionic projective bundle theorem show that for $\mathbf{BO}^{fin}$ the $\varprojlim^1$ are over inverse systems of groups which are finite direct sums of copies of $KO_1(S)$ and $KSp_1(S)$. When those groups are finite, the $\varprojlim^1$ vanishes. This gives Theorem 1.5 for $S = \operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$. For other S one pulls the structure back from $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ using the closed motivic model structure of [15].

Theorem 1.6. *The products of Theorems 1.3, 1.4 and 1.5 are compatible with all the naive products of* (1.3) *and with the partial multiplicative structure of* (1.4) *and* (1.5).

We do not know how to prove this theorem using our construction of the hermitian K-theory product. But Marco Schlichting has described to us (oral communication) how to put a pairing on his hermitian K-theory spaces when one has a pairing of complicial exact categories with weak equivalences and duality. When applied to our situation his product is isomorphic to ours on small motivic spaces by Theorem 1.3. Since Schlichting's product is compatible with all the naive products of (1.3) and with the partial multiplicative structure of (1.4) and (1.5), therefore ours is as well.

We finish the paper in §14 by giving the analogue for algebraic K-theory of the spectra $\mathbf{BO}^{fin}$ and $\mathbf{BO}^{geom}$ of hermitian K-theory of §12. The $\mathbf{BGL}^{fin}$ seems to be completely new.

We write $CGr(r,n)$ for the affine Grassmannian, $CP^1 = CGr(1,2)$ for the affine version of $\mathbf{P}^1$, and CP^{1+} for the $\mathbf{A}^1$ mapping cone of the pointing map of CP^1.

Theorem 1.7. *There are CP^{1+}-spectra $\mathbf{BGL}^{fin}$ and $\mathbf{BGL}^{geom}$ isomorphic to $\mathbf{BGL}_{CP^{1+}}$ in $SH_{CP^{1+}}(S)$ with spaces*

$$\mathbf{BGL}_n^{fin} = [-4^n, 4^n] \times CGr(4^n, 2\cdot 4^n), \qquad \mathbf{BGL}_n^{geom} = \mathbb{Z} \times CGr,$$

which are unions of affine Grassmannians. The bonding maps $\mathbf{BGL}_n^* \wedge CP^{1+} \to \mathbf{BGL}_{n+1}^*$ of the two spectra are morphisms of schemes or ind-schemes which are constant on the wedge $\mathbf{BGL}_n^* \vee CP^{1+}$.

The spectum $\mathbf{BGL}^{fin}$ can be used to give alternate proofs of the uniqueness results of [15] concerning the $\mathbf{P}^1$-spectrum representing algebraic K-theory and the commutative monoid structure on that spectrum. These proofs avoid the use of topological realization and apply to any noetherian base scheme S of finite Krull dimension with finite $K_1(S)$.

2. Cohomology theories

We fix a base scheme S which is regular noetherian separated of finite Krull dimension and with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. The hermitian K-theory of such schemes is simpler than for other schemes, and we wish to avoid the complications of negative hermitian K-theory and of characteristic 2.

Let $\mathcal{S}m/S$ be the category of smooth S-schemes of finite type. Let $\mathcal{S}m\mathcal{O}p/S$ be the category whose objects are pairs (X,U) with X in $\mathcal{S}m/S$ and $U \subset X$ an open subscheme and whose morphisms $f\colon (X,U) \to (Y,V)$ are morphisms $f\colon X \to Y$ of S-schemes having $f(U) \subset V$. We write X for $(X, \varnothing)$. The base scheme itself will often be written as $S = pt$.

A *cohomology theory* on $\mathcal{S}m\mathcal{O}p/S$ [14, Definition 2.1] is a pair (A, ∂) with A a contravariant functor from $\mathcal{S}m/S$ to the category of abelian groups having localization exact sequences and satisfying étale excision and homotopy invariance. The ∂ is a morphism of functors with components $\partial_{X,U}\colon A(U) \to A(X,U)$ which are the boundary maps of the localization exact sequences. A *ring cohomology theory* in the sense of [14, Definition 2.13] has products

$$\times\colon A(X,U) \times A(Y,V) \to A(X \times Y, (X \times V) \cup (U \times Y))$$

which are functorial, bilinear and associative and and which have a two-sided unit $1_A \in A(pt)$ and satisfy $\partial(\alpha \times \beta) = \partial\alpha \times \beta$.

A cohomology theory also defines groups $A(X,x)$ for pointed smooth schemes and their smash products such as

$$A\big((X_1,x_1) \wedge (X_2,x_2)\big) = \ker\Big(A(X_1 \times X_2) \xrightarrow{(x_1^* \times 1, 1 \times x_2^*)} A(pt \times X_2) \oplus A(X_1 \times pt)\Big). \tag{2.1}$$

A *bigraded* cohomology theory $(A^{*,*}, \partial)$ is one in which the groups are bigraded, that is $A^{*,*}(X,U) = \bigoplus_{p,q\in\mathbb{Z}} A^{p,q}(X,U)$, the pullback maps are homogeneous of bidegree $(0,0)$ and the boundary maps $\partial_{X,U}$ are homogeneous of bidegree $(1,0)$. In a *bigraded ring cohomology theory* $(A, \partial, \times, 1)$ the $\times$ products respects the bigrading and we have $1 \in A^{0,0}(pt)$.

Definition 2.1. Let $(A, \partial, \times, 1)$ bigraded ring cohomology theory, and suppose $\varepsilon \in A^{0,0}(pt) = A^{0,0}(pt)$ satisfies $\varepsilon^2 = 1$. Then $(A, \partial, \times, 1)$ is *ε-commutative* if for $\alpha \in A^{p,q}(X,U)$ and $\beta \in A^{r,s}(Y,V)$ one has $\sigma^*(\alpha \times \beta) = \beta \times \alpha \times (-1)^{pr}\varepsilon^{qs}$ where $\sigma\colon Y \times X \to X \times Y$ switches the factors.

Equivalently a bigraded ring cohomology is ε-commutative if the associated cup product satisfies $\alpha\cup\beta = (-1)^{pr}\varepsilon^{qs}\,\beta\cup\alpha$ for $\alpha \in A^{p,q}(X,U)$ and $\beta \in A^{r,s}(X,V)$. For such a cohomology theory the $A^{*,*}(X)$ are bigraded-commutative rings, and for any (X,U) the $A^{*,*}(X,U)$ and $A^{*,*}(U)$ are right and left bigraded $A^{*,*}(X)$-modules. The $\partial_{X,U}$ are morphisms of right $A^{*,*}(X)$-modules.

Sometimes it is easier to define certain products than others. For a bigraded cohomology theory $(A^{*,*},\partial)$ set $A^0(X,U) = \bigoplus_{p\in\mathbb{Z}} A^{2p,p}(X,U)$. We need the following notion.

Definition 2.2. Let $(A^{*,*},\partial)$ be a bigraded cohomology theory as above. An *ε-commutative partial multiplication* on $(A^{*,*},\partial)$ is given by

(1) pairings $\times\colon A^{p,q}(X,U)\times A^{2r,r}(Y,V) \to A^{p+2r,q+r}((X,U)\wedge(Y,V))$ which are bilinear and functorial, and
(2) elements $\mathbf{1}$ and ε in $A^{0,0}(pt)$

satisfying

(a) $\alpha\times(b\times c) = (\alpha\times b)\times c$ for $\alpha\in A^{p,q}(X,U)$, $b\in A^{2r,r}(Y,V)$, $c\in A^{2s,s}(Z,W)$;
(b) $\alpha\times\mathbf{1} = \alpha$ for $\alpha\in A^{p,q}(Y,V)$,
(c) $\varepsilon\times\varepsilon = \mathbf{1}$,
(d) $a\times b = \sigma^*(b\times a)\times\varepsilon^{rs}$ for $a\in A^{2r,r}(X,U)$, $b\in A^{2s,s}(Y,V)$ where $\sigma\colon X\times Y\to Y\times X$ switches the factors;
(e) $\partial_{Y\times X,V\times X}(\alpha\times b) = \partial_{Y,V}(\alpha)\times b$ for $\alpha\in A^{p,q}(V)$, $b\in A^{2r,r}(X)$.

If $(A^{*,*},\partial)$ has such a partial multiplication, then for $\alpha\in A^{p,q}(X,V)$ and $b\in A^{2r,r}(X,U)$ one has a *cup product*

$$\alpha\cup b = \Delta^*(\alpha\times b)\in A^{p+2r,q+r}(X,U\cup V).$$

If (A,∂) is equipped with a partial multiplicative structure $(\times,\mathbf{1},\varepsilon)$, then the functor $(X,U)\mapsto A^0(X,U)$ is an ε-commutative graded ring functor in the sense that the properties (a), (b), (c) and (d) hold for $\alpha\in A^0(Y,V)$.

Moreover, A is a bigraded right A^0-module in the same sense with ∂ a morphism of bigraded right A^0-modules which is homogeneous of bidegree $(1,0)$.

The switch $\sigma\colon X\times Y\to Y\times X$ allows us to define pairings $\times\colon A^{2r,r}(X,U)\times A^{p,q}(Y,V)\to A^{p+2r,q+r}((X,U)\times(Y,V))$ by $b\times\alpha = \sigma^*(\alpha\times b)\times\varepsilon^{qr}$. There are also cup products $b\cup\alpha = \Delta^*(b\times\alpha)$. The two pairings are compatible by (d). Thus A is a bigraded left and right A^0-module, with ∂ a morphism of right A^0-modules.

3. SL and SL^c orientations

We discuss SL oriented cohomology theories. Hermitian K-theory will turn out to be one. We also include a discussion of Thom classes for vector bundles whose structural group is the double cover SL_n^c of GL_n. It contains SL_n. We believe this is the true level at which Witt groups and hermitian K-theory are oriented.

An *SL bundle* on X is a pair (E,λ) with E a vector bundle over X and $\lambda\colon \mathcal{O}_X\cong\det E$ an isomorphism. An *isomorphism of SL bundles* $f\colon(E,\lambda)\cong(E_1,\lambda_1)$ is an isomorphism $f\colon E\cong E_1$ such that $\lambda_1 = \det f\circ\lambda$.

Definition 3.1. An *SL orientation* on a bigraded cohomology theory $A^{*,*}$ with an ε-commutative partial multiplication or ring structure is an assignment to every SL bundle (E,λ) over every X in $\mathcal{S}m/S$ of a class $th(E,\lambda)\in A^{2n,n}(E,E-X)$ for $n=\operatorname{rk}E$ satisfying the following conditions:

(1) For an isomorphism $f\colon (E,\lambda) \cong (E_1,\lambda_1)$ we have $th(E,\lambda) = f^* th(E_1,\lambda_1)$.
(2) For $u\colon Y \to X$ we have $u^* th(E,\lambda) = th(u^*E, u^*\lambda)$ in $A^{2n,n}(u^*E, u^*E - Y)$.
(3) The maps $- \cup th(E,\lambda)\colon A^{*,*}(X) \to A^{*+2n,*+n}(E, E-X)$ are isomorphisms.
(4) We have

$$th(E_1 \oplus E_2, \lambda_1 \otimes \lambda_2) = q_1^* th(E_1,\lambda_1) \cup q_2^* th(E_2,\lambda_2),$$

where q_1, q_2 are the projections from $E_1 \oplus E_2$ onto its factors.

The class $th(E,\lambda)$ is the *Thom class* of the SL bundle, and $e(E,\lambda) = z^* th(E,\lambda) \in A^{2n,n}(X)$ is its *Euler class*.

This definition is analogous to the Thom classes theory version of the definition of an orientation [14, Definition 3.32] or of a symplectic orientation [16, Definition 14.2].

The Thom and Euler classes of SL bundles are not necessarily central in contrast with the classes in the oriented and symplectically oriented theories of [14] and [16]. But for an SL bundle of rank n the Thom and Euler classes are in bidegree $(2n,n)$, and such classes need not be central when n is odd and $\varepsilon \neq 1$. Centrality occcurs for oriented theories because they have $\varepsilon = 1$ and for symplectically oriented theories because the Thom and Pontryagin classes of symplectic bundles are in bieven bidegrees $(4r, 2r)$.

Twisted versions of cohomology groups with coefficients in a line bundle can be defined for any SL oriented theory by

$$A^{p,q}(X;L) = A^{p+2,q+1}(L, L-X)$$

and more generally by $A^{p,q}(X, X-Z; L) = A^{p+2,q+1}(L, L-Z)$ for closed subsets $Z \subset X$.

Theorem 3.2. *Let E be a vector of rank n over X. Suppose that $A^{*,*}$ is an SL oriented bigraded cohomology theory. Then there are canonical isomorphisms of bigraded right $A^0(X)$ or $A^{*,*}(X)$-modules $A^{*+2n,*+n}(E, E-X) \cong A^{*,*}(X; \det E)$.*

Proof. Write $L_E = \det E$. There are canonical isomorphisms

$$\lambda_1\colon \mathcal{O}_X \cong \det(E \oplus L_E^\vee), \qquad \lambda_2\colon \mathcal{O}_X \cong \det(L_E \oplus L_E^\vee).$$

This gives us SL bundles $(E \oplus L_E^\vee, \lambda_1)$ and $(L_E \oplus L_E^\vee, \lambda_2)$ over X. The pullback of the first bundle along $q\colon L_E \to X$ gives an SL bundle whose structural map is the first projection $L_E \oplus E \oplus L_E^\vee \twoheadrightarrow L_E$. The pullback of the second bundle along $p\colon E \to X$ and permutation of the summands gives an SL bundle whose structural map is the second projection $L_E \oplus E \oplus L_E^\vee \twoheadrightarrow E$. We now have canonical isomorphisms

$$\begin{array}{ccc} A^{*+2n,*+n}(E, E-X) & \xrightarrow[\cong]{p^* th(L_E \oplus L_E^\vee, \lambda_2)\cup} & A^{*+2n+4,*+n+2}(E \oplus L_E \oplus L_E^\vee, E \oplus L_E \oplus L_E^\vee - X) \\ & & \cong \Big\downarrow \varepsilon^n \\ A^{*+2,*+1}(L_E, L_E - X), & \xrightarrow[q^* th(E \oplus L_E^\vee, \lambda_1)\cup]{\cong} & A^{*+2n+4,*+n+2}(L_E \oplus E \oplus L_E^\vee, L_E \oplus E \oplus L_E^\vee - X) \end{array}$$

and the bottom left module is $A^{*,*}(X; \det E)$ by definition. The sign ε^n is appropriate when one permutes the rank 1 bundles and the rank n bundle. □

Hermitian K-theory and Witt groups have more Thom classes than just those for SL bundles because of what are often called periodicity isomorphisms such as $W^*(X;L) \cong W^*(X; L \otimes L_1^{\otimes 2})$. However, these periodicity isomorphisms depend on choices. A good way

to structure these choices is to talk about SL^c bundles, using extra structure analogous to the $Spin^c$ structures frequently used in differential geometry.

An SL^c *vector bundle* on X is a triple (E, L, λ) with E a vector bundle, L a line bundle, and $\lambda\colon L \otimes L \cong \det E$ an isomorphism. The structural group of an SL^c bundle of rank n is SL^c_n which is the kernel of

$$GL_n \times \mathbb{G}_m \xrightarrow{(\det^{-1},\, t \mapsto t^2)} \mathbb{G}_m.$$

There is a natural exact sequence $1 \to \mu_2 \to SL^c_n \to GL_n \to 1$. The notation SL^c is in imitation of $Spin^c$. The role of this double cover of GL_n in direct images in real topological K-theory merited a mention by Atiyah [1, p. 55] nearly forty years ago.

Definition 3.3. An SL^c *orientation* on a cohomology theory $A^{*,*}$ with a ε-commutative ring structure or partial multiplication is an assignment to every SL^c bundle (E, L, λ) over every scheme X in $\mathcal{S}m/S$ of a class $th(E, L, \lambda) \in A^{2n,n}(E, E - X)$ where $n = \operatorname{rk} E$ satisfying the conditions (1)–(4) of Definition 3.1.

4. Schlichting's hermitian K-theory and the Gille-Nenashev pairing

In [19, §2.7] Schlichting defines the hermitian K-theory space of a complicial exact category with weak equivalences and duality in the style of Waldhausen's K-theory. We will denote his space by $KO(C, w, \sharp, \eta)$. More generally we write

$$KO^{[n]}(C, w, \sharp, \eta) = KO\big((C, w, \sharp, \eta)[n]\big)$$

for the hermitian K-theory space for the nth shifted duality, and $KO^{[n]}_i(C, w, \sharp, \eta)$ for its homotopy groups. A *symmetric object of degree* n in $(C, w, \sharp, \eta)$ is a pair (X, ϕ) with $\phi\colon X \to X^\sharp[n]$ a weak equivalence which is symmetric $\phi = \phi^t$ for the shifted duality. There is a natural definition of a Grothendieck-Witt group of symmetric objects of degree n, and The π_0 of the hermitian K-theory space is the Grothendieck-Witt group of degree n symmetric objects

$$KO^{[n]}_0(C, w, \sharp, \eta) = GW^n(C, w, \sharp, \eta).$$

When C is $\mathbb{Z}[\frac{1}{2}]$-linear, that is the same as the triangulated Grothendieck-Witt group of the homotopy category $Ho(C, w) = C[w^{-1}]$ for the duality $(\sharp, \eta)$, defined à la Balmer.

For a duality-preserving exact functor $(F, f)\colon (C, w, \sharp, \eta) \to (D, v, \natural, \varpi)$ there are induced maps of spaces $KO^{[n]}(C, w, \sharp, \eta) \to KO^{[n]}(D, v, \natural, \varpi)$. A weak equivalence between duality-preserving exact functors $(F, f) \simeq (G, g)$ produces a homotopy between the maps.

There are natural periodicity isomorphisms $KO^{[n]}(C, w, \sharp, \eta) \simeq KO^{[n+4k]}(C, w, \sharp, \eta)$. Moreover, we may write

$$KSp^{[n]}(C, w, \sharp, \eta) = KO^{[n]}(C, w, \sharp, -\eta)$$

because the effect of changing the sign is to interchange symmetric and skew-symmetric forms. Then there are isomorphisms $KSp^{[n]}(C, w, \sharp, \eta) \simeq KO^{[n+4k+2]}(C, w, \sharp, \eta)$ induced by the duality preserving functor $X \mapsto X[2k+1]$. However, it is more useful to use the identifications

$$\begin{array}{ccc} KSp^{[n]}_i(C, w, \sharp, \eta) & \longrightarrow & KO^{[n+4k+2]}_i(C, w, \sharp, \eta) \\ \xi & \longmapsto & -\,\xi[2k+1] \end{array} \tag{4.1}$$

because these commute with the forgetful maps to Waldhausen's K-theory $K_i(C, w)$.

Among the many important results Schlichting proves is localization. Suppose that $\overline{C}_1 \subset Ho(C, w)$ is a thick triangulated subcategory which is stable under the duality. Let $C_1 \subset C$

be the full exact subcategory with the same objects as $\overline{C}_1$. Let w_1 be the set of all morphisms in C whose mapping cone is in C_1.

Theorem 4.1 ([19, Theorem 6])**.** *If $(C, w, \sharp, \eta)$ is a complicial exact category with weak equivalences and duality, and C_1 is as above, then*

$$KO(C_1, w, \sharp, \eta) \to KO(C, w, \sharp, \eta) \to KO(C, w_1, \sharp, \eta)$$

is a fibration sequence up to homotopy.

Gille and Nenashev [7] have defined pairings for Witt groups of triangulated categories. We explain how their construction can be applied to hermitian K-theory to give pairings in the spirit of the partial multiplicative structure of Definition 2.2. A *pairing*

$$(\boxtimes, t_1, t_2, \lambda)\colon (C, w, \sharp, \eta) \times (D, v, \flat, \theta) \to (E, u, \natural, \varpi)$$

of complicial exact categories with weak equivalences and duality is an additive bifunctor $\boxtimes\colon C \times D \to E$ which commutes with the translations up to specified functorial isomorphisms $t_{1,X,Y}\colon X[1] \boxtimes Y \cong (X \boxtimes Y)[1]$ and $t_{2,X,Y}\colon X \boxtimes Y[1] \cong (X \boxtimes Y)[1]$ plus functorial weak equivalences $\lambda_{X,Y}\colon X^\sharp \boxtimes Y^\flat \to (X \boxtimes Y)^\natural$ such that for any X in C and any Y in D the functors $X \boxtimes -$ and $- \boxtimes Y$ are exact and preserve weak equivalences and such that all the conditions of [7, Definitions 1.2 and 1.11] hold.

Suppose given a symmetric object (M, ϕ) of degree r in $(C, w, \sharp, \eta)$ and a symmetric object (N, ψ) of degree s in $(D, v, \flat, \theta)$. Gille and Nenashev show how to define duality-preserving exact functors [7, Lemma 1.14]

$$\big(- \boxtimes (N, \psi), \mathfrak{R}(N, \psi)\big)\colon (C, w, \sharp, \eta) \to (E, u, \natural, \varpi)[s]$$
$$\big((M, \phi) \boxtimes -, \mathfrak{L}(M, \phi)\big)\colon (D, v, \flat, \theta) \to (E, u, \natural, \varpi)[r]$$

It follows that these induce maps of KO spaces, which we will write as

$$(- \boxtimes (N, \psi))_*\colon KO^{[n]}(C, w, \sharp, \eta) \to KO^{[n]}(E, u, \natural, \varpi)[s] \tag{4.3a}$$
$$((M, \phi) \boxtimes -)_*\colon KO^{[n]}(D, v, \flat, \theta) \to KO^{[n]}(E, u, \natural, \varpi)[r] \tag{4.3b}$$

The duality-preserving functor $(1_E, -1)$, which acts on symmetric objects by $(Z, \xi) \mapsto (Z, -\xi)$, induces a map

$$\varepsilon\colon KO^{[n]}(E, u, \natural, \varpi) \to KO^{[n]}(E, u, \natural, \varpi). \tag{4.4}$$

These *sign involutions* exist for the hermitian K-theory of any complicial exact category with weak equivalences and duality, and they satisfy $\varepsilon^2 = 1$ exactly. (In general ε is not the same as the -1 which is the inverse map for the H-space structure induced by the orthogonal direct sum.) The methods of Gille and Nenashev show [7, Lemma 1.15] that the effect of the two functors on the Grothendieck-Witt classes $[M, \phi] \in GW^r(C, w, \sharp, \eta)$ and $[N, \psi] \in GW^s(D, v, \flat, \theta)$ is

$$(- \boxtimes (N, \psi))_*[M, \phi] = \varepsilon^{rs}((M, \phi) \boxtimes -)_*[N, \psi]. \tag{4.5}$$

Proposition 4.2. *The homotopy classes of the maps* (4.3a) *and* (4.3b) *on hermitian K-theory spaces depend only on the classes $[N, \psi] \in GW^s(D, v, \flat, \theta)$ and $[M, \phi] \in GW^r(C, w, \sharp, \eta)$ respectively.*

Proof. There are three relations in the definition of the Grothendieck-Witt groups [19, Definition 1]. The maps on homotopy groups are compatible with the relations $[N,\psi]+[N_1,\psi_1]=[N\oplus N_1,\psi\oplus\psi_1]$ because the orthogonal direct sum of symmetric objects induces a monoidal structure on the $KO^{[n]}(E,u,\natural,\varpi)$ giving a naive additivity for orthogonal direct sums of duality-preserving functors.

The maps are compatible with the relations $[N,\psi]=[N_2,\sigma^\flat\psi\sigma]$ for a weak equivalence $\sigma\colon N_2\to N$ because σ induces a natural weak equivalence of duality-preserving functors.

The maps are compatible with the third relation related to lagrangians because of Schlichting's Additivity Theorem [19, Theorem 5]. □

We thus get pairings

$$KO_j^{[m]}(C,w,\sharp,\eta)\times KO_0^{[s]}(D,v,\flat,\theta)\to KO_j^{[m+s]}(E,u,\natural,\varpi) \tag{4.6a}$$

$$KO_0^{[r]}(C,w,\sharp,\eta)\times KO_i^{[n]}(D,v,\flat,\theta)\to KO_i^{[n+r]}(E,u,\natural,\varpi) \tag{4.6b}$$

which we call the *right pairing* and the *corrected left pairing.* They coincide on $KO_0^{[m]}\times KO_0^{[n]}$.

The Gille-Nenashev pairings have a number of other properties such as functoriality, associativity, compatibility with localization sequences. Using the right pairing means that the boundary map on the homotopy groups in the localization sequences satisfies $\partial(\alpha\cup\xi)=\partial\alpha\cup\xi$ for $\alpha\in KO_i^{[n]}(C,w_1,\sharp,\eta)$ and $\xi\in KO_0^{[r]}(D,v,\flat,\theta)$.

In [19] Schlichting constructs a hermitian K-theory space for $(C,w,\sharp,\eta)$ but not a spectrum. So in principle what we have discussed so far does not yield any negative homotopy groups. In practice it is known that if the categories are $\mathbb{Z}[\frac{1}{2}]$-linear, then Balmer's triangulated Witt groups for the homotopy category $Ho(C,w)=C[w^{-1}]$ can function as negative homotopy groups

$$KO_i^{[n]}(C,w,\sharp,\eta)=W^{n-i}(C[w^{-1}],\sharp,\eta)\qquad\text{for } i<0.$$

This is explained in [18]. The localization sequences for the homotopy groups of the hermitian K-theory spaces and Balmer's localization sequence for triangulated Witt groups [2, Theorem 6.2] attach to each other because the π_0 are triangulated Grothendieck-Witt groups.

Two other important theorems are the following.

Theorem 4.3 (Fundamental Theorem [18])**.** *Let $(C,w,\sharp,\eta)$ be a $\mathbb{Z}[\frac{1}{2}]$-linear complicial exact category with weak equivalences and duality. Then for all n*

$$KO^{[n-1]}(C,w,\sharp,\eta)\xrightarrow{F}K(C,w)\xrightarrow{H}KO^{[n]}(C,w,\sharp,\eta)$$

is a homotopy fiber sequence, where F is the forgetful map and H the hyperbolic map.

Theorem 4.4 ([18])**.** *Let $(F,f)\colon(C,w,\sharp,\eta)\to(E,u,\natural,\varpi)$ be a duality-preserving exact functor between $\mathbb{Z}[\frac{1}{2}]$-linear complicial exact categories with weak equivalences and duality. If (F,f) induces a homotopy equivalence $K(C,w)\simeq K(E,u)$ of Waldhausen K-theory spaces and isomorphisms $W^i(C[w^{-1}],\sharp,\eta)\cong W^i(E[u^{-1}],\natural,\varpi)$ of Balmer's triangulated Witt groups, then (F,f) induces homotopy equivalences $KO^{[n]}(C,w,\sharp,\eta)\simeq KO^{[n]}(E,u,\natural,\varpi)$ for all n.*

In particular if (F,f) induces an equivalence of $\mathbb{Z}[\frac{1}{2}]$-linear triangulated categories with duality $(C[w^{-1}],\sharp,\eta)\simeq(E[u^{-1}],\natural,\varpi)$, then it induces a homotopy equivalence $KO(C,w,\sharp,\eta)\simeq KO(E,u,\natural,\varpi)$ by Thomason's theorem and by the fact that Balmer's Witt groups are a functorial over the category with objects $\mathbb{Z}[\frac{1}{2}]$-linear triangulated categories with duality and arrows isomorphism classes of duality-preserving triangulated functors [3, Lemma 4.1].

5. The cohomology theory $KO_*^{[*]}$ on the category $\mathcal{S}m\mathcal{O}p/k$

Let S be a regular noetherian separated scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. For every S-scheme X consider the category VBX of big vector bundles over X in the sense of [5, Appendix C.4]. The assignments $X \mapsto VBX$ and $(f\colon Y \to X) \mapsto f^*\colon VBX \to VBY$ then form a strict functor $(\mathcal{S}m/S)^{\mathrm{op}} \to \mathcal{C}at$ because one has equalities $(f \circ g)^* = g^* \circ f^*$ instead of simply isomorphisms.

For any $X \in \mathcal{S}m/S$, let $Ch^b(VBX)$ denote the additive category of bounded complexes of big vector bundles on X. We will consider $Ch^b(VBX)$ as a complicial exact category with weak equivalences, the conflations being the degreewise-split short exact sequences, and the weak equivalences w_X being the quasi-isomorphisms. When we further endow $Ch^b(VBX)$ with the duality consisting of the functor ${}^\vee = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ and the natural biduality maps $\eta_X\colon 1 \cong {}^{\vee\vee}$, we will write it as $Ch^b(VBX)$.

$$Ch^b(VBX) = (Ch^b(VBX), w_X, {}^\vee, \eta_X)$$

Now suppose $U \subset X$ is an open subscheme and $Z = X - U$. Let w_U be the set of chain maps whose restriction to U is a quasi-isomorphism. Let $Ch^b(VBX)^{w_U}$ be the full additive subcategory of complexes which are acyclic on U. We have two new families of complicial exact categories with weak equivalences and duality

$$Ch^b(VBX \ on\ Z) = (Ch^b(VBX)^{w_U}, w_X, {}^\vee, \eta_X),$$
$$Ch^b(VBX \ on\ U) = (Ch^b(VBX), w_U, {}^\vee, \eta_X).$$

We then have hermitian K-theory spaces

$$KO^{[n]}(X) = KO^{[n]}(Ch^b(VBX))$$
$$KO^{[n]}(X, U) = KO^{[n]}(Ch^b(VBX \ on\ Z)).$$

with $KO^{[n]}(X) = KO^{[n]}(X, \varnothing)$. Let $D^b(VBX \ on\ Z)$ be the homotopy category equipped with the triangulated duality $({}^\vee, \eta_X)$. We define the hermitian K-theory groups as

$$KO_i^{[n]}(X, U) = \begin{cases} \pi_i KO^{[n]}(X, U) & \text{for } i \geq 0, \\ W^{n-i}(D^b VBX \ on\ Z) & \text{for } i < 0. \end{cases} \tag{5.1}$$

For $f\colon (X, U) \to (Y, V)$ a morphism in $\mathcal{S}m\mathcal{O}p/S$ write $W = Y - V$. Then the functor $f^*\colon Ch^b(VBX \ on\ Z) \to Ch^b(VBY \ on\ W)$ can by made duality-preserving by equipping it with the natural isomorphism $f^*\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_Y}(f^*-, \mathcal{O}_Y)$. This gives us maps

$$f^*\colon KO^{[n]}(X, U) \to KO^{[n]}(Y, V), \tag{5.2a}$$

$$f^*\colon KO_i^{[n]}(X, U) \to KO_i^{[n]}(Y, V). \tag{5.2b}$$

By Schlichting's localization theorem (Theorem 4.1) the sequences

$$KO^{[n]}(X, U) \to KO^{[n]}(X) \to KO^{[n]}(Ch^b(VBX \ on\ U))$$

are fibration sequences up to homotopy. The restriction map $Ch^b(VBX \ on\ U) \to Ch^b(VBU)$ is a duality-preserving functor which induces an equivalence on the homotopy categories $D^b(VBX \ on\ U) \simeq D^b(VBU)$. (Schlichting actually gives a different argument in [19, §9] which is valid with fewer restrictions on the schemes.) So we get fibration sequences up to homotopy

$$KO^{[n]}(X, U) \to KO^{[n]}(X) \to KO^{[n]}(U) \tag{5.3a}$$

and therefore long exact sequences

$$\cdots \to KO_i^{[n]}(X,U) \to KO_i^{[n]}(X) \to KO_i^{[n]}(U) \xrightarrow{\partial} KO_{i-1}^{[n]}(X,U) \to \cdots \tag{5.3b}$$

Now suppose (Y,V) is in $\mathcal{S}m\mathcal{O}p/S$ and write $W = Y - V$. There is then a pairing of complicial exact categories with weak equivalences and duality

$$\boxtimes\colon Ch^b(VBX \text{ on } Z) \times Ch^b(VBY \text{ on } W) \to Ch^b(VB(X \times Y) \text{ on } Z \times W).$$

For a degree r symmetric complex (N,ψ) on Y which is acyclic on V and for a degree s symmetric complex (M,ϕ) on X which is acyclic on U we have maps of spaces

$$(- \boxtimes (N,\psi))_*\colon KO^{[n]}(X,U) \to KO^{[n+r]}(X \times Y, (X \times V) \cup (U \times Y)) \tag{5.4a}$$

$$\varepsilon^{ns}((M,\phi) \boxtimes -)_*\colon KO^{[n]}(Y,V) \to KO^{[n+s]}(X \times Y, (X \times V) \cup (U \times Y)) \tag{5.4b}$$

This leads to a right pairing and a corrected left pairing

$$KO_i^{[n]}(X,U) \times KO_0^{[r]}(Y,V) \to KO_i^{[n+r]}(X \times Y, (X \times V) \cup (U \times Y)), \tag{5.4c}$$

$$KO_0^{[n]}(X,U) \times KO_i^{[r]}(Y,V) \to KO_i^{[n+r]}(X \times Y, (X \times V) \cup (U \times Y)). \tag{5.4d}$$

Now suppose (E,L,λ) is an SL^c bundle of rank n over X. Let $p\colon E \to X$ be the structural map. We may construct Thom isomorphisms for hermitian K-theory using the same method that Nenashev used for Witt groups [13, §2]. Namely, the pullback $p^*E = E \oplus E \to E$ has a canonical section s, the diagonal. There is a Koszul complex

$$K(E) = \big(0 \to \Lambda^n p^*E^\vee \to \Lambda^{n-1}p^*E^\vee \to \cdots \to \Lambda^2 p^*E^\vee \to E^\vee \to \mathcal{O}_E \to 0\big)$$

(considered as a chain complex in homological degrees n to 0) in which each boundary map the contraction with s. It is a locally free resolution of the coherent sheaf $z_*\mathcal{O}_X$. There is a canonical isomorphism $\Theta(E)\colon K(E) \to K(E)^\vee \otimes \det p^*E^\vee[n]$ which is symmetric for the $(\det p^*E^\vee)$-twisted shifted duality. The composition

$$\Theta(E,L,\lambda)\colon K(E) \otimes p^*L \xrightarrow{\Theta(E)} K(E)^\vee \otimes \det p^*E^\vee \otimes p^*L[n] \xrightarrow{1 \otimes p^*(\lambda^\vee \otimes L)} K(E)^\vee \otimes p^*L^\vee[n]$$

is symmetric for the untwisted shifted duality. We consider the Grothendieck-Witt class

$$th(E,L,\lambda) = [K(E) \otimes p^*L, \Theta(E,L,\lambda)] \in KO_0^{[n]}(E, E - X). \tag{5.5}$$

When L is trivial this is an SL Thom class $th(E,\lambda) = [K(E), \Theta(E,\lambda)]$. Finally for g a nowhere vanishing function on X we let $\langle g \rangle = [\mathcal{O}_X, g] \in KO_0^{[0]}(X)$ be the Grothendieck-Witt class of the rank one symmetric bilinear bundle.

Theorem 5.1. *Let S be a regular noetherian separated scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Then the groups $KO_i^{[n]}(X,U)$ of* (5.1)*, the maps f^* of* (5.2b) *and ∂ of* (5.3b)*, the pairings of* (5.4c) *and* (5.4d)*, the classes $1 = \langle 1 \rangle$ and $\varepsilon = \langle -1 \rangle$ in $KO_0^{[0]}(pt)$ and the classes $th(E,L,\lambda)$ of* (5.5) *form an SL^c oriented cohomology theory with an ε-commutative partial multiplication.*

In particular the products with the Thom classes are isomorphisms

$$- \cup th(E,L,\lambda)\colon KO_i^{[m]}(X) \xrightarrow{\cong} KO_i^{[m+n]}(E, E - X) \tag{5.6}$$

Sketch of the proof. The verifications are all straightforward. For instance étale excision and homotopy invariance amount to having certain pullback maps be isomorphisms, and pullback maps are induced by duality-preserving functors. Since these duality-preserving functors give isomorphisms for Quillen's K-theory and Balmer's Witt groups, they give isomorphisms for hermitian K-theory as well under our hypotheses.

The Thom maps come from duality-preserving functors. The functor part $Ch^b(VBX) \to Ch^b(VBE)$ is given by $\mathcal{F} \mapsto \pi^*\mathcal{F} \otimes_{\mathcal{O}_E} K(E)$ with the target quasi-isomorphic to the coherent sheaf $z_*\mathcal{F}$. These produce dévissage isomorphisms in both Quillen-Waldhausen K-theory and Balmer's Witt groups [6].

For the ε-commutativity of the partial multiplicative structure, $\sigma^*(b \times a)$ is calculated by applying the right pairing for a with respect to b and then switching. That is equivalent to applying the uncorrected left pairing for a with respect to b. But that satisfies (4.5). □

We will discuss later the Pontryagin classes associated to the Thom classes.

It is sometimes inconvenient that the $KO_i^{[n]}(X, U)$ for $i < 0$ are not defined as homotopy groups. But actually they are naturally isomorphic to direct summands of homotopy groups. For we have the S-scheme $\mathbb{G}_m = \mathbf{A}^1 - 0$ (pointed by 1) and groups $KO_i^{[n]}(\mathbb{G}_m^{\wedge r} \times X, \mathbb{G}_m^{\wedge r} \times U)$ defined as in (2.1).

Lemma 5.2. *For all i and n and all $r \geq 1$ and all (X, U) in $\mathcal{S}m\mathcal{O}p/S$ there are natural isomorphisms $KO_{i+r}^{[n+r]}(\mathbb{G}_m^{\wedge r} \times X, \mathbb{G}_m^{\wedge r} \times U) \cong KO_i^{[n]}(X, U)$.*

Proof. For $r = 1$ we have a localization sequence which splits and a Thom isomorphism

$$\begin{tikzcd}
KO_{i+1}^{[n+1]}(\mathbf{A}^1 \times X, \mathbf{A}^1 \times U) \arrow[r, tail] & KO_{i+1}^{[n+1]}(\mathbb{G}_m \times X, \mathbb{G}_m \times U) \arrow[d, two heads, "\partial"] & \\
KO_{i+1}^{[n+1]}(X, U) \arrow[u, "\cong"'] & KO_i^{[n+1]}(\mathbf{A}^1 \times X, \mathbf{A}^1 \times U \cup \mathbb{G}_m \times X) \arrow[l, "(in_1 \times 1_X)^*"] & KO_i^{[n]}(X, U) \arrow[l, "th \times", "\cong"']
\end{tikzcd}$$

with $th \in KO_0^{[1]}(\mathbf{A}^1, \mathbf{A}^1 - 0)$ the Thom class of the trivial rank one SL bundle. Hence we have a natural isomorphism

$$KO_{i+1}^{[n+1]}(\mathbb{G}_m \times X, \mathbb{G}_m \times U) \cong KO_{i+1}^{[n+1]}(X, U) \oplus KO_i^{[n]}(X, U).$$

By induction $KO_{i+r}^{[n+r]}(\mathbb{G}_m^{\times r} \times X, \mathbb{G}_m^{\times r} \times U)$ is a direct sum of 2^r terms of which exactly one is $KO_i^{[n]}(X, U)$. □

Definition 5.3. The *periodicity element* $\beta_8 \in KO_0^{[4]}(pt)$ is the element corresponding to $1 \in KO_0^{[0]}(pt)$ under the periodicity isomorphisms $KO_i^{[n]} \cong KO_i^{[n+4]}$ of the hermitian K-theory of chain complexes.

Then for all X and n the periodicity isomorphisms $KO_i^{[n]}(X) \cong KO_i^{[n+4]}(X)$ coincides with $- \times \beta_8$ up to homotopy.

6. $\mathbf{KO}_*^{[*]}$ of motivic spaces

In this section we recall what the category of pointed motivic spaces is and extend the functor $KO_*^{[*]}$ to a functor $\mathbf{KO}_*^{[*]}$ on that category.

The basic definitions, constructions and model structures we use are given in [21]. A *motivic space over S* is a simplicial presheaf on the site $\mathcal{S}m/S$ of smooth S-schemes of finite type. A *pointed motivic space over S* is a pointed simplicial presheaf on the site $\mathcal{S}m/S$. We write $\mathbf{M}_\bullet(S)$ for the category of pointed motivic spaces over S.

We equip the category $\mathbf{M}_\bullet(S)$ with the *local injective model structure* [21, p. 181] and with the *motivic model structure* [21, p. 194]. In both model structures the cofibrations are the monomorphisms. The weak equivalences and fibrations of the local injective model structure are called local weak equivalences and global fibrations. Those of the motivic model structure are called motivic weak equivalences and motivic fibrations.

We write $H_\bullet(S)$ for the pointed motivic unstable homotopy category obtained by inverting the motivic weak equivalences. The homotopy category $H_\bullet(S)$ is equivalent to the motivic homotopy category of [12]. For a morphism $f\colon A \to B$ of pointed motivic spaces we will write $[f]$ for the class of f in $H_\bullet(S)$.

Notation 6.1. There is a global fibrant model functor $G\colon Id_{\mathbf{M}_\bullet(S)} \to (-)^f$ functor in $\mathbf{M}_\bullet(S)$. The natural transformation G is a local weak equivalence, but we do not require it to be injective.

Lemma 6.2. *Let $\mathbf{KO}^{[i]} = (KO^{[i]})^f$. Then the map $G\colon KO^{[i]} \to \mathbf{KO}^{[i]}$ is a schemewise weak equivalence, and the space $\mathbf{KO}^{[i]}$ is motivically fibrant.*

Proof. The space $KO^{[i]}$ satisfies Nisnevich descent because the homotopy groups $KO^{[i]}_r$ satisfy étale excision. Therefore [21, Theorem 5.21] the morphism $G\colon KO^{[i]} \to \mathbf{KO}^{[i]}$ is a schemewise weak equivalence. For every $X \in \mathcal{S}m/S$ the projection $\mathbf{A}^1_X \to X$ induces a weak equivalence of simplicial sets $\mathbf{KO}^{[i]}(X) \to \mathbf{KO}^{[i]}(\mathbf{A}^1_X)$, since this the case for the space $KO^{[i]}$ and $G : KO^{[i]} \to \mathbf{KO}^{[i]}$ is a schemewise weak equivalence. This proves [21, p. 195] that the globally fibrant space $\mathbf{KO}^{[i]}$ is motivically fibrant. □

We write S^r_s for the r-sphere $\Delta[r]/\partial\Delta[r]$ in **sSet** and in the homotopy category $H_\bullet$ of pointed simplicial sets and for the corresponding constant simplicial presheaf in $\mathbf{M}_\bullet(S)$. We write $\mathbb{G}_m$ for the pointed scheme $(\mathbf{A}^1 - 0, 1)$.

Definition 6.3. For any pointed motivic space A and any i define

$$\mathbf{KO}^{[i]}_r(A) = \begin{cases} Hom_{H_\bullet(S)}(A \wedge S^r_s, \mathbf{KO}^{[i]}) & \text{for } r \geq 0, \\ Hom_{H_\bullet(S)}(A \wedge \mathbb{G}_m^{\wedge(-r)}, \mathbf{KO}^{[i-r]}) & \text{for } r < 0. \end{cases}$$

Lemma 6.4. *For any r and $X \in \mathcal{S}m/S$ one has functorial isomorphisms*

$$\alpha_X : KO^{[i]}_r(X) \xrightarrow{\cong} \mathbf{KO}^{[i]}_r(X_+).$$

Proof. In fact, the following chain of isomorphisms give the desired one

$$Hom_{H_\bullet}(S^r_s, KO^{[i]}(X)) \xrightarrow{\cong} Hom_{H_\bullet}(S^r_s, \mathbf{KO}^{[i]}(X)) \xrightarrow{adj} Hom_{H_\bullet(S)}(X_+ \wedge S^r_s, \mathbf{KO}^{[i]})$$

$$\pi_0(KO^{[i-r]}(X \times \mathbb{G}_m^{\times -r})) \xrightarrow{\cong} \pi_0(\mathbf{KO}^{[i-r]}(X \times \mathbb{G}_m^{\times -r})) \xrightarrow{adj} Hom_{H_\bullet(S)}((X \times \mathbb{G}_m^{\times -r})_+, \mathbf{KO}^{[i-r]})$$

All the arrows are indeed isomorphisms by Lemma 6.2. □

Now suppose $(X, U) \in \mathcal{S}m\mathcal{O}p/S$ with $j\colon U \hookrightarrow X$ the inclusion and $Z = X - U$. Schlichting's localization sequence (5.3a)

$$KO^{[i]}(X, U) \xrightarrow{e^*} KO^{[i]}(X) \xrightarrow{j^*} KO^{[i]}(U)$$

can be described more precisely than we have done so far. The composite map j^*e^* is induced by the functor $Ch^b(VBX\ on\ Z) \to Ch^b(VBU)$ such that the morphism of functors $j^*e^* \to 0$ is a natural weak equivalence. This natural weak equivalence gives a homotopy from the map of spaces j^*e^* to the trivial map, and that gives a factorization of e^* as

$$KO^{[i]}(X,U) \xrightarrow{e_{X,U}} hofib(j^*) \xrightarrow{can_{X,U}} KO^{[i]}(X). \tag{6.1}$$

Schlichting's theorem is that $e_{X,U}$ a homotopy equivalence. This factorization is functorial in (X,U).

For a pointed motivic space A write $\mathbf{KO}^{[i]}(A) = \mathbf{hom}_\bullet(A, \mathbf{KO}^{[i]})$ for the pointed mapping space, which is the pointed simplicial set $[n] \mapsto Hom_{\mathbf{M}_\bullet(S)}(A \wedge \Delta[n]_+, \mathbf{KO}^{[i]})$. For $A = X_+$ with X a smooth scheme we have $\mathbf{KO}^{[i]}(A) = \mathbf{KO}^{[i]}(X)$. For $j\colon U \hookrightarrow X$ as above we can fill out a commutative diagram of pointed simplicial sets

$$\begin{tikzcd}
KO^{[i]}(X,U) \arrow[d, "e_{X,U}"'] & & & \\
hofib(j^*) \arrow[rr, "can_{X,U}"] \arrow[d, "G_1"'] & & KO^{[i]}(X) \arrow[r, "j^*"] \arrow[d, "G(X)"'] & KO^{[i]}(U) \arrow[d, "G(U)"] \\
\mathbf{KO}^{[i]}(Cone(j_+)) = hofib(j_+^*) \arrow[rr, "nat_{X,U}"] & & \mathbf{KO}^{[i]}(X_+) \arrow[r, "j_+^*"] & \mathbf{KO}^{[i]}(U_+) \\
\mathbf{KO}^{[i]}(X_+/U_+) \arrow[u, "q_{X,U}"] & & &
\end{tikzcd}$$

We claim that all the vertical arrows are homotopy equivalences. This is true for $G(X)$ and $G(U)$ by Lemma 6.2 and therefore for G_1 because it is the map between the homotopy fibers. The map $e_{X,U}$ is a homotopy equivalence by Schlichting's theorem. The map $q_{X,U}$ is a homotopy equivalence because it is obtained by applying $\mathbf{hom}_\bullet(-, \mathbf{KO}^{[i]})$ with $\mathbf{KO}^{[i]}$ fibrant to the schemewise weak equivalence between cofibrant objects $Cone(j_+) \to X_+/U_+$.

The diagram is functorial in (X,U). We conclude:

Theorem 6.5. *For (X,U) in $\mathcal{S}m\mathcal{O}p/S$ there is a functorial zigzag of homotopy equivalences $KO^{[i]}(X,U) \to \mathbf{KO}^{[i]}(Cone(U_+ \to X_+)) \leftarrow \mathbf{KO}^{[i]}(X_+/U_+)$ which for $(X,\varnothing)$ reduces to the $G(X)\colon KO^{[i]}(X) \to \mathbf{KO}^{[i]}(X)$ of Lemma 6.2. These induce functorial isomorphisms of groups $KO_r^{[i]}(X,U) \cong \mathbf{KO}_r^{[i]}(X_+/U_+)$ for all integers i and r.*

Notation 6.6. Denote by $\alpha : KO_*^{[i]}(-,-) \to \mathbf{KO}_*^{[i]}(-_+/-_+)$ the functor isomorphism described in Theorem 6.5.

7. The T-spectrum $\mathbf{BO}$ and the cohomology theory $\mathbf{BO}^{*,*}$

Let $T = \mathbf{A}^1/(\mathbf{A}^1 - 0)$ be the Morel-Voevodsky object. A *T-spectrum E* over S consists of a sequence $(E_0, E_1, \dots)$ of pointed motivic spaces over S plus *structure maps* $\sigma_n\colon E_n \wedge T \to E_{n+1}$. Let $SH(S)$ denote the stable homotopy category of T-spectra as described in [9]. It is canonically equivalent to the motivic stable homotopy category constructed in [20].

Here we define a T-spectrum $\mathbf{BO}$. Its spaces are $(\mathbf{KO}^{[0]}, \mathbf{KO}^{[1]}, \mathbf{KO}^{[2]}, \mathbf{KO}^{[3]}, \dots)$. We now define the structure maps.

Let $\mathbf{A}^1 \to pt$ be the trivial rank one SL bundle, and let $th \in KO_0^{[1]}(\mathbf{A}^1, \mathbf{A}^1 - 0)$ be its Thom class as defined by (5.5). Because $KO_*^{[*]}$ is SL^c oriented, the maps

$$- \times th \colon KO_r^{[n]}(X) \to KO_r^{[n+1]}(X \times \mathbf{A}^1, X \times (\mathbf{A}^1 - 0)) \tag{7.1}$$

are isomorphisms. Recall that th is defined by (5.5) as the class of the symmetric complex

$$\begin{array}{ccccccccc} K(\mathcal{O}) & & 0 & \longrightarrow & \mathcal{O}_{\mathbf{A}^1} & \xrightarrow{x} & \mathcal{O}_{\mathbf{A}^1} & \longrightarrow & 0 \\ \Theta(\mathcal{O}) \downarrow \cong & & & & -1 \downarrow & & \downarrow 1 & & \\ K(\mathcal{O})^\vee[1] & & 0 & \longrightarrow & \mathcal{O}_{\mathbf{A}^1} & \xrightarrow{-x} & \mathcal{O}_{\mathbf{A}^1} & \longrightarrow & 0 \end{array} \tag{7.2}$$

of degree 1 in $Ch^b(VB\mathbf{A}^1 \; on \; 0)$. The maps $- \times th$ are induced by the maps of spaces

$$(- \boxtimes (K(\mathcal{O}), \Theta(\mathcal{O})))_* \colon KO^{[n]}(X) \to KO^{[n+1]}(X \times \mathbf{A}^1, X \times (\mathbf{A}^1 - 0))$$

These maps are thus homotopy equivalences, and $KO^{[n]} \to KO^{[n+1]}(- \times \mathbf{A}^1, - \times (\mathbf{A}^1 - 0))$ is a schemewise weak equivalence.

From Lemma 6.2 and Theorem 6.5 we now have a zigzag

$$\begin{array}{ccccc} \mathbf{KO}^{[n]} & \xleftarrow[\sim]{G} & KO^{[n]} & \xrightarrow[\sim]{\times th} & KO^{[n+1]}(- \times \mathbf{A}^1, - \times (\mathbf{A}^1 - 0)) \\ & & & & \downarrow \sim \\ & & & & \mathbf{KO}^{[n+1]}(Cone(- \wedge (\mathbf{A}^1 - 0)_+ \to - \wedge \mathbf{A}^1_+)) \xleftarrow{\sim} \mathbf{KO}^{[n+1]}(- \wedge T) \end{array}$$

of schemewise weak equivalences in $\mathbf{M}_\bullet(S)$. Their composition is an isomorphism $\mathbf{KO}^{[n]} \cong \mathbf{KO}^{[n+1]}(- \wedge T)$ in the homotopy category.

There is a Quillen adjunction with left adjoint $- \wedge T$ and right adjoint $F(-) \mapsto F(- \wedge T)$. It follows that $\mathbf{KO}^{[n+1]}(- \wedge T)$ is **fibrant**, while $\mathbf{KO}^{[n]}$ is cofibrant, so there exists a morphism $\sigma_n^* \colon \mathbf{KO}^{[n]} \to \mathbf{KO}^{[n+1]}(- \wedge T)$ in $\mathbf{M}_\bullet(S)$ representing the same isomorphism in the homotopy category as the zigzag. Let $\sigma_n \colon \mathbf{KO}^{[n]} \wedge T \to \mathbf{KO}^{[n+1]}$ be the adjoint morphism. Since the $KO^{[n]}$ and $\mathbf{KO}^{[n]}$ are periodic modulo 4, we may choose the σ_n^* and σ_n so they are also periodic.

Definition 7.1. The T-spectrum **BO** consists of the sequence of pointed motivic spaces

$$(\mathbf{KO}^{[0]}, \mathbf{KO}^{[1]}, \mathbf{KO}^{[2]}, \mathbf{KO}^{[3]}, \dots) \tag{7.3}$$

together with the structure maps $\sigma_n : \mathbf{KO}^{[n]} \wedge T \to \mathbf{KO}^{[n+1]}$ just described.

The spaces $\mathbf{KO}^{[n]}$ are motivically fibrant and the adjoints σ_n^* of the structure maps are schemewise weak equivalences. So we have [9, Lemma 2.7]:

Theorem 7.2. *The T-spectrum* **BO** *is stably motivically fibrant.*

As explained in [20] any T-spectrum E defines a bigraded cohomology theory $(E^{*,*}, \partial)$ on the category $\mathbf{M}_\bullet(S)$ with

$$E^{p,q}(A) = Hom_{SH_\bullet(S)}(A, E \wedge S_s^{p-q} \wedge \mathbb{G}_m^{\wedge q}).$$

The differential ∂ increases the bidegree by $(1, 0)$.

For any $A \in \mathbf{M}_\bullet(S)$ the adjunction map induces isomorphisms

$$\mathbf{KO}_i^{[n]}(A) = Hom_{H_\bullet(S)}(A \wedge S_s^i, \mathbf{KO}^{[n]}) \cong Hom_{SH(S)}(A \wedge S_s^i, \mathbf{BO} \wedge T^{\wedge n}) = \mathbf{BO}^{2n-i,n}(A),$$

$$\mathbf{KO}_i^{[n]}(A) = Hom_{H_\bullet(S)}(A \wedge \mathbb{G}_m^{\wedge -i}, \mathbf{KO}^{[n-i]}) \cong Hom_{SH(S)}(A \wedge \mathbb{G}_m^{\wedge -i}, \mathbf{BO} \wedge T^{\wedge n-i}) = \mathbf{BO}^{2n-i,n}(A),$$

for $i \geq 0$ and $i < 0$ respectively. This gives us an isomorphism of functors on $\mathbf{M}_\bullet(S)$

$$\beta_A : \mathbf{KO}_i^{[n]} \xrightarrow{\cong} \mathbf{BO}^{2n-i,n}.$$

Corollary 7.3. *The composition isomorphism* $\beta|_{\mathcal{S}m\mathcal{O}p/k} \circ \alpha : KO_*^{[*]} \to \mathbf{BO}^{*,*}|_{\mathcal{S}m\mathcal{O}p/S}$ *respects the boundary homomorphisms in both cohomology theories on* $\mathcal{S}m\mathcal{O}p/S$*. So it is an isomorphism*

$$\gamma : KO_*^{[*]} \to \mathbf{BO}^{*,*}|_{\mathcal{S}m\mathcal{O}p/S}$$

of cohomology theories in the sense of [14].

Definition 7.4. Using the cohomology isomorphism γ we transplant to $\mathbf{BO}|_{\mathcal{S}m\mathcal{O}p/S}$ the partial multiplicative structure of $(KO_*^{[*]}, \partial)$ and the Thom and Pontryagin classes of its SL^c orientation described in Theorem 5.1. The unit of this partial multiplicative structure is the element $e = \gamma(\langle 1 \rangle) \in \mathbf{BO}^{0,0}(S^0)$.

Theorem 7.5 (Bott periodicity)**.** *The adjoints of the structure maps and the categorical periodicity isomorphisms give levelwise weak equivalences*

$$\mathbf{BO} \xrightarrow{\sim level} \Omega_T^4 \mathbf{BO}(4) \xrightarrow{\cong} \Omega_T^4 \mathbf{BO}.$$

8. A symplectic version of the Morel-Voevodsky theorem

In [12, Theorem 4.3.13] Morel and Voevodsky showed that $\mathbb{Z} \times Gr$ represents algebraic K-theory in the motivic unstable homotopy category. If one replaces the ordinary Grassmannians by quaternionic Grassmannians, the same holds for symplectic K-theory.

We write H for the trivial rank 2 symplectic bundle $\left(\mathcal{O}^{\oplus 2}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)$. The orthogonal direct sum $\mathsf{H}^{\oplus n}$ is the trivial symplectic bundle of rank $2n$. We will sometimes write $\mathsf{H}_X^{\oplus n}$ to designate the trivial symplectic bundle over the scheme X.

The *quaternionic Grassmannian* $HGr(r,n) = HGr(r, \mathsf{H}^{\oplus n})$ is defined as the open subscheme of $Gr(2r, 2n) = Gr(2r, \mathsf{H}^{\oplus n})$ parametrizing subspaces of dimension $2r$ of the fibers of $\mathsf{H}^{\oplus n}$ on which the symplectic form of $\mathsf{H}^{\oplus n}$ is nondegenerate. We write $\mathcal{U}_{r,n}$ for the restriction to $HGr(r,n)$ of the tautological subbundle of $Gr(2r, 2n)$. The symplectic form of $\mathsf{H}^{\oplus n}$ restricts to a symplectic form on $\mathcal{U}_{r,n}$ which we denote by $\phi_{r,n}$. The pair $(\mathcal{U}_{r,n}, \phi_{r,n})$ is the *tautological symplectic subbundle* of rank $2r$ on $HGr(r,n)$. Morphisms $X \to HGr(r,n)$ are *classified* by subbundles $E \subset \mathsf{H}_X^{\oplus n}$ of rank $2r$ such that the symplectic form of $\mathsf{H}_X^{\oplus n}$ is nondegenerate on every fiber.

More generally, given a symplectic bundle (E, ϕ) of rank $2n$ over X, the *quaternionic Grassmannian bundle* $HGr(r, E, \phi)$ is the open subscheme of the Grassmannian bundle $Gr(2r, E)$ parametrizing subspaces of dimension $2r$ of the fibers of E on which ϕ is nondegenerate.

For $r = 1$ we have *quaternionic projective spaces* and *bundles* $HP^n = HGr(1, n+1)$ and $HP(E, \phi) = HGr(1, E, \phi)$.

There are commuting morphisms

$$\begin{tikzcd} HGr(r,n) \arrow[r, "\alpha_{r,n}"] \arrow[d, "\beta_{r,n}"'] & HGr(r,n+1) \arrow[d, "\beta_{r,n+1}"] \\ HGr(r+1,n+1) \arrow[r, "\alpha_{r+1,n+1}"] & HGr(r+1,n+2) \end{tikzcd} \tag{8.1}$$

with $\alpha_{r,n}$ classified by the rank $2r$ subbundle $\mathcal{U}_{r,n} \oplus 0 \subset \mathsf{H}^{\oplus n} \oplus \mathsf{H}$ and $\beta_{r+1,n+1}$ classified by the rank $2r+2$ subbundle $\mathsf{H} \oplus \mathcal{U}_{r,n} \subset \mathsf{H} \oplus \mathsf{H}^{\oplus n}$. Composition gives us maps $HGr(n,2n) \to HGr(n+1,2n+2)$. We define $HGr = \operatorname{colim} HGr(n,2n)$. We consider $\mathbb{Z} \times HGr$ pointed by $(0, HGr(0,0))$. It has a universal property.

Theorem 8.1. *Suppose $X \in \mathcal{S}m/S$ is affine. Then for every $\xi \in GW^-(X)$ there is a morphism of ind-schemes $f\colon X \to \mathbb{Z} \times HGr$ such that $\xi = f^*\tau$. Moreover f is unique up to naive $\mathbf{A}^1$-homotopy.*

This is the equivalence $\pi_0\mathcal{L}_6 \cong \pi_0\mathrm{GW}^-$ of [4, Proposition 6.2.1.5] plus the isomorphism $\pi_0\mathrm{GW}^- = \mathrm{GW}^-$ which happens for our schemes which are regular with $\frac{1}{2}$.

For a smooth S-scheme X let $KSp(X) = KO(Ch^b(VBX), w_X, {}^\vee, -\eta)$ be its symplectic K-theory space. There are natural isomorphisms $KSp \cong KO^{[4n+2]}$.

Theorem 8.2. *The objects $\mathbb{Z} \times HGr$ and KSp are isomorphic in the motivic unstable homotopy category $H_\bullet(S)$.*

The proof of this theorem is identical to Morel and Voevodsky's proof for ordinary K-theory except for one point. Let $HU(r,n) \to HGr(r,n)$ be the principal Sp_{2r}-bundle associated to the tautological rank $2r$ symplectic subbundle on $HGr(r,n)$. There is an isomorphism $HU(r,n) \cong Sp_{2n}/Sp_{2n-2r}$. To establish Theorem 8.2 by following the proof of Morel and Voevodsky exactly we would need the $HU(r,n)$ to form an *admissible gadget* in the sense of [12, Definition 4.2.1]. It does not seem as if they do. Nor can we use the admissible gadget used by Morel and Voevodsky, for that would substitute for $HU(r,n)$ the principal $GL(2r)$-bundle associated to the tautological subbundle on $Gr(2r,2n)$, called $U_{2r,2n}$ in [12]. (This $U_{2r,2n}$ is the space of $2r \times 2n$ matrices of maximal rank.) But the quotient $U_{2r,2n}/Sp_{2r}$ is not the quaternionic Grassmannian $HGr(r,n)$, and its cohomology risks being much less tractable.

So we give a new definition based on the property actually used in the proof of [12, Proposition 4.2.3]. For a commutative ring B let

$$\Delta^n_B = \operatorname{Spec} B[t_1,\dots,t_n], \qquad \partial\Delta^n_B = \operatorname{Spec} B[t_1,\dots,t_n]/(t_1t_2\cdots t_n(1-\textstyle\sum t_i)) \subset \Delta^n_B$$

Definition 8.3. An *acceptable gadget* over an S-scheme X is a sequence of smooth quasi-projective X-schemes $(U_i)_{i\in\mathbb{N}}$ and closed embeddings $U_i \to U_{i+1}$ of X-schemes such that for any henselian regular local ring B and any commutative square

$$\begin{tikzcd} \partial\Delta^n_B \arrow[r] \arrow[d, "\text{inclusion}"'] & U_i \arrow[d, "\text{projection}"] \\ \Delta^n_B \arrow[r] & X \end{tikzcd}$$

there exists a $j \geq i$ and a map $\Delta^n_B \to U_j$ making the following diagram commute.

$$\begin{tikzcd}
\partial\Delta^n_B \arrow[r] \arrow[d, "\text{inclusion}"'] & U_i \arrow[r, "\text{gadget map}"] & U_j \arrow[d, "\text{projection}"] \\
\Delta^n_B \arrow[urr] \arrow[rr] & & X
\end{tikzcd}$$

Sections of the principal bundle $HU(r,n) \to HGr(r,n)$ are given by symplectic frames (i.e. symplectic bases) of the tautological symplectic subbundle. Therefore giving a morphism $V \to HU(r,n)$ is equivalent to giving an embedding $\mathsf{H}_V^{\oplus r} \subset \mathsf{H}_V^{\oplus n}$ such that the symplectic form on $\mathsf{H}_V^{\oplus r}$ is the restriction of the symplectic form on $\mathsf{H}_V^{\oplus n}$. This is also equivalent by duality to giving $2n$ sections $s_1, \ldots, s_{2n}$ of the bundle of linear forms $\mathsf{H}_V^{\oplus r\vee}$ such that $\sum_{i=1}^n s_{2i-1} \wedge s_{2i}$ is equal to the symplectic form of $\mathsf{H}_V^{\oplus r}$.

We need the $HU(r,n)$ to form an acceptable gadget in the relative case as well (cf. [12, Lemma 4.2.8]). Given a symplectic bundle (E,ϕ) of rank $2r$ over X, the relative $HU(E,\phi;n)$ is constructed by taking the Sp_{2r}-torsor $P \to X$ associated to (E,ϕ) and forming the quotient $(P \times HU(r,n))/Sp_{2r}$ by the diagonal action of Sp_{2r} on the two torsors. This gives us a fiber bundle $HU(E,\phi;n) \to X \times HGr(r,n)$ with fibers Sp_{2r} and structural group $Sp_{2r} \times Sp_{2r}$ acting on the fibers by left and right translation. Giving a morphism $V \to HU(E,\phi;n)$ is equivalent to giving a triple (f,g,ι) with $f\colon V \to X$ and $g\colon V \to HGr(r,n)$ morphisms of schemes and $\iota\colon f^*(E,\phi) \cong g^*(\mathfrak{U}_{r,n},\phi_{r,n})$ an isometry of symplectic bundles. This is equivalent to giving $(f,u_1,\ldots,u_{2n})$ with $f\colon V \to X$ a map and the u_i sections of $f^*E^\vee$ such that $\sum_{i=1}^n u_{2i-1} \wedge u_{2i} = f^*\phi$.

Lemma 8.4. *Let R be a commutative ring, let $g \in R$ and let $\overline{R} = R/(g)$. Let (E,ϕ) a symplectic R-module, and let $(\overline{E},\overline{\phi})$ be the associated symplectic $\overline{R}$-module. Let $u_1,\ldots,u_{2n} \in E^\vee$ be linear forms such that $\sum_{i=1}^n \overline{u}_{2i-1} \wedge \overline{u}_{2i} = \overline{\phi}$ in $\Lambda^2\overline{E}^\vee$. Then there exist linear forms $v_1,\ldots,v_{2n}$ and $w_1,\ldots,w_{2m}$ in $E^\vee$ such that*

$$\sum_{i=1}^n (u_{2i-1} + gv_{2i-1}) \wedge (u_{2i} + gv_{2i}) + \sum_{j=1}^m gw_{2j-1} \wedge gw_{2j} = \phi.$$

Proof. The $\overline{u}_1,\ldots,\overline{u}_{2n}$ generate $\overline{E}^\vee$. So there exist $v_1,\ldots,v_{2n} \in E^\vee$ such that

$$\phi - \sum_{i=1}^n u_{2i-1} \wedge u_{2i} \equiv g \sum_{j=1}^{2n} u_j \wedge (-1)^j v_j \pmod{g^2}. \qquad \square$$

Proposition 8.5. *For any symplectic bundle (E,ϕ) of rank $2r$ over a scheme X, the schemes $(HU(E,\phi;n))_{n\geq r}$ together with the closed embeddings $HU(E,\phi;n) \to HU(E,\phi;n+1)$ induced by the inclusions $\mathsf{H}^{\oplus n} \subset \mathsf{H}^{\oplus n} \oplus \mathsf{H}$ form an acceptable gadget.*

Proof. Let $R = B[t_1,\ldots,t_n]$ and $g = t_1t_2\cdots t_n(1-\sum t_i) \in R$. Giving the first diagram of Definition 8.3 is then equivalent to giving the (E,ϕ) and $(\overline{u}_1,\ldots,\overline{u}_{2n})$ of Lemma 8.4. The map $\Delta^n_B \to U_j$ is then given by $(u_1+gv_1,\ldots,u_{2n}+gv_{2n},gw_1,\ldots,gw_{2m})$. The top triangle of the second diagram commutes because modulo g this last vector is $(\overline{u}_1,\ldots,\overline{u}_{2n},0,\ldots,0)$. The lower triangle commutes because of the equation involving ϕ. $\square$

Substituting the acceptable gadgets $HU(r,n) \to HU(r,n+1) \to \cdots$ and the quaternionic Grassmannians $HGr(i,n)$ for the admissible gadgets $U_{n,i} \to U_{n,i+1} \to \cdots$ and Grassmannians

$Gr(i,n)$ of Morel and Voevodsky, the proof of [12, Theorem 4.3.13] can be used to prove Theorem 8.2. Proposition 8.5 substitutes for the last paragraph of the proof of Proposition 4.2.3 and for Lemma 4.2.8. We get $\operatorname{hocolim}_n HU(r,n) \cong pt$ and

$$BSp_{2r} \cong B_{et}Sp_{2r} \cong HGr(r,\infty)$$

in $H(S)$, and we get the theorem. At the end one needs the equivalence of hermitian K-theories based on group completion and of Schlichting's Waldhausen-like construction. But Schlichting has shown that each is equivalent to the hermitian Q-construction: see [17, Theorem 4.2] and [19, Proposition 6].

Similar but less definitive results can be proven for the orthogonal group. Let $\mathsf{H}_+^{\oplus n}$ denote the trivial orthogonal bundle $(\mathcal{O}_S^{\oplus 2n}, q_{2n})$ with the split quadratic form $q_{2n} = \sum_{i=1}^n x_{2i-1}x_{2i}$. Let $RGr(r,2n) = RGr(r,\mathsf{H}_+^{\oplus n})$ be the open subscheme of $Gr(r,2n)$ parametrizing subspaces of rank r on which q_{2n} is nondegenerate. Then over $RGr(r,2n)$ we have a tautological rank r orthogonal subbundle $(U|_{RGr}, q_{2n}|_{RGr})$ whose structural group scheme is the orthogonal group scheme $O(r, q_{2n}|_{RGr}) \to RGr(r,2n)$. The associated principal bundle is $RU(r,2n) \to RGr(r,2n)$. To give a morphism $V \to RU(r,2n)$ one gives a quadratic bundle (E,q) of rank r over V and $2n$ sections $s_1,\dots,s_{2n}$ of $E^\vee$ such that $q = \sum_{i=1}^n s_{2i-1}s_{2i}$. The data $(E,q,s_1,\dots,s_{2n})$ and $(E_1,q_1,t_1,\dots,t_{2n})$ define the same morphism if and only if there is an isomorphism $\phi\colon E \cong E_1$ such that $q = \phi^* q_1$ and $s_i = \phi^* t_i$ for all i. The relative case is like the relative symplectic case described earlier.

Proposition 8.6. (*a*) *For any quadratic bundle (E,q) the $RU(E,q,2n)$ and the inclusions $RU(E,q,2n) \to RU(E,q,2n+2)$ corresponding to $(E,q,s_1,\dots,s_{2n}) \mapsto (E,q,s_1,\dots,s_{2n},0,0)$ form an acceptable gadget, as do their relative versions.*

(*b*) *For any quadratic bundle (E,q) of rank r over X the fiber bundles $RU(E,q,2n) \to X$ have sections for $n \geq r$ over any open subscheme over which the vector bundle E trivializes.*

Part (b) of the proposition is of concern [12, Proposition 4.1.20 and Definition 4.2.4.3]. It is not an issue for symplectic bundles because symplectic bundles are locally trivial in the Zariski topology. It holds because to give a section of $RU(E,q,2n) \to X$ is to give sections $s_1,\dots,s_{2n}$ of $E^\vee$ such that $q = \sum s_{2i-1}s_{2i}$. If E trivializes over U with coordinates $x_1,\dots,x_r$ and $q = \sum_{i\leq j} a_{ij}x_ix_j$, then one can give a local section over U by $s_{2i-1} = x_i$ and $s_{2i} = \sum_{j=i}^n a_{ij}x_j$ for $1 \leq i \leq r$ and $s_i = 0$ for $i > 2r$.

The results of Proposition 8.6 and of the arguments of Morel and Voevodsky are isomorphisms $B_{et}O_r \cong RGr(r,\infty)$ and $B_{et}O \cong RGr$ in $H_\bullet(S)$. However, neither of the natural maps $\mathbb{Z}\times BO \to \mathbb{Z}\times B_{et}O$ or $\mathbb{Z}\times BO \to \mathbf{KO}^{[0]}$ are isomorphisms in $H_\bullet(S)$ because orthogonal bundles are not always locally trivial in the Nisnevich topology.

We also do not know how to calculate $KO_*^{[*]}(RGr)$.

9. The cohomology of quaternionic Grassmannians

We review the calculation of the cohomology of quaternionic Grassmannians of [16]. We reformulate the definitions and some of the theorems for a bigraded ε-commutative partial multiplication. In [16] we assumed we had a full ring structure. We do not redo the proofs because no change is needed: all the needed products are with the Thom and Pontryagin classes of symplectic bundles or with pullbacks of such classes, and those lie in the $A^{4i,2i}$.

Definition 9.1 ([16, Definition 7.1])**.** A *symplectic Thom structure* on a bigraded cohomology theory $(A^{*,*},\partial)$ with an ε-commutative partial multiplication or ring structure is

a rule assigning to every rank 2 symplectic bundle (E,ϕ) over an X in $\mathcal{S}m/S$ an element $th(E,\phi)\in A^{4,2}(E,E-X)$ with the following properties:

(1) For an isomorphism $u:(E,\phi)\cong(E_1,\phi_1)$ one has $th(E,\phi)=u^*\,th(E_1,\phi_1)$.
(2) For a morphism $f:Y\to X$ with pullback map $f_E:f^*E\to E$ one has $f_E^*\,th(E,\phi)=th(f^*E,f^*\phi)$.
(3) For the trivial rank 2 bundle H over pt the map

$$-\times th(\mathsf{H})\colon A^{*,*}(X)\to A^{*,*}(X\times\mathbf{A}^2,X\times(\mathbf{A}^2-\{0\}))$$

is an isomorphism for all X.

The *Pontryagin class* of (E,ϕ) is $p(E,\phi)=-z^*\,th(E,\phi)\in A^{4,2}(X)$ where $z\colon X\to E$ is the zero section.

From Mayer-Vietoris one sees that for any rank 2 symplectic bundle

$$\cup\, th(E,\phi)\colon A^{*,*}(X)\xrightarrow{\cong}A^{*,*}(E,E-X)$$

is an isomorphism. The sign in the Pontryagin class is simply conventional. It is chosen so that if $A^{*,*}$ is an oriented cohomology theory with an additive formal group law, then the Chern and Pontryagin classes satisfy the traditional formula $p_i(E,\phi)=(-1)^ic_{2i}(E)$. The following is [16, Theorem 8.2].

Theorem 9.2 (Quaternionic projective bundle theorem)**.** *Let $(A^{*,*},\partial)$ be a bigraded cohomology theory with an ε-commutative partial multiplication or ring structure and a symplectic Thom structure. Let $(\mathcal{U},\phi|_{\mathcal{U}})$ be the tautological rank 2 symplectic subbundle over the quaternionic projective bundle $HP(E,\phi)$, and let $t=p(\mathcal{U},\phi|_{\mathcal{U}})$ be its Pontryagin class. Then we have an isomorphism of graded $A^0(X)$-modules*

$$(1,t,\dots,t^{n-1})\colon A^{*,*}(X)\oplus A^{*,*}(X)\oplus\cdots\oplus A^{*,*}(X)\to A^{*,*}(HP_X(E,\phi))$$

and an isomorphism of graded modules over $A^0(X)=\bigoplus_p A^{2p,p}(X)$:

$$(1,t,\dots,t^{n-1})\colon A^0(X)\oplus A^0(X)\oplus\cdots\oplus A^0(X)\to A^0(HP_X(E,\phi)).$$

Definition 9.3. Under the hypotheses of Theorem 9.2 there are unique elements $p_i(E,\phi)\in A^{4i,2i}(X)$ for $i=1,2,\dots,n$ such that

$$t^n-p_1(E,\phi)\cup t^{n-1}+p_2(E,\phi)\cup t^{n-2}-\cdots+(-1)^np_n(E,\phi)=0.$$

The classes $p_i(E,\phi)$ are called the *Pontryagin classes* of (E,ϕ) with respect to the symplectic Thom structure of the cohomology theory (A,∂). For $i>n$ and $i<0$ one sets $p_i(E,\phi)=0$, and one sets $p_0(E,\phi)=1$.

The quaternionic projective bundle theorem has consequences the symplectic splitting principle and the Cartan sum formula for Pontryagin classes [16, Theorems 10.2, 10.5]. With them one can compute the cohomology of quaternionic Grassmannians. Let e_i denote the ith elementary symmetric polynomial, and let h_i be the ith complete symmetric polynomial, the sum of all monomials of degree i. There are formulas relating them, including the recurrence relation $h_k+\sum_{i\geq1}(-1)^ie_ih_{k-i}=0$. Let $\Pi_{r,n-r}$ be the set of all partitions whose Young diagrams fit inside an $r\times(n-r)$ box. (More formally these are partitions λ with $l(\lambda)=\lambda_1'\leq r$ and $\lambda_1\leq n-r$.) Associated to any such partition is a *Schur polynomial*, which can be written in terms of the e_i.

Theorem 9.4. *For any bigraded ring cohomology theory $A^{*,*}$ with an ε-commutative partial multiplication or ring structure and a symplectic Thom structure and any X the map*

$$A^{*,*}(X)[e_1,\dots,e_r]/(h_{n-r+1},\dots,h_n) \xrightarrow{\cong} A^{*,*}(HGr(r,n)\times X) \tag{9.1}$$

sending $e_i \mapsto p_i(\mathcal{U}_{r,n},\phi_{r,n})$ for all i is an isomorphism, as is the map

$$A^{*,*}(X)^{\oplus\binom{n}{r}} \xrightarrow{(s_\lambda(\mathcal{U}_{r,n},\phi_{r,n}))_{\lambda\in\Pi_{r,n-r}}} A^{*,*}(HGr(r,n)\times X). \tag{9.2}$$

Theorem 9.5 ([16, Theorem 11.4])**.** *For any bigraded ring cohomology theory $A^{*,*}$ with an ε-commutative partial multiplication or ring structure and a symplectic Thom structure and any X the $\alpha_{r,n}$ and $\beta_{r,n}$ of* (8.1) *induces split surjections*

$$(\alpha_{r,n}\times 1_X)^*\colon A(HGr(r,n+1)\times X) \twoheadrightarrow A(HGr(r,n)\times X)$$
$$(\beta_{r,n}\times 1_X)^*\colon A(HGr(r+1,n+1)\times X) \twoheadrightarrow A(HGr(r,n)\times X)$$

which the isomorphisms (9.2) *identify with the surjections $A(X)^{\oplus\binom{n+1}{r}} \twoheadrightarrow A(X)^{\oplus\binom{n}{r}}$ and $A(X)^{\oplus\binom{n+1}{r+1}} \twoheadrightarrow A(X)^{\oplus\binom{n}{r}}$ which are the identity on the summands corresponding to $\lambda\in\Pi_{n,r}$ and which vanish on the summands corresponding to $\lambda\notin\Pi_{n,r}$. We have isomorphisms*

$$A^{*,*}(X)[[p_1,\dots,p_r]]^{homog} \xrightarrow{\cong} \varprojlim_{n\to\infty} A^{*,*}(HGr(r,n)\times X) \tag{9.3}$$

$$A^{*,*}(X)[[p_1,p_2,p_3,\dots]]^{homog} \xrightarrow{\cong} \varprojlim_{n\to\infty} A^{*,*}(HGr(n,2n)\times X) \tag{9.4}$$

with each variable p_i sent to the inverse system of ith Pontryagin classes $(p_i(\mathcal{U}_{r,n}))_{n\geq r}$ or $(p_i(\mathcal{U}_{n,2n}))_{n\in\mathbb{N}}$.

The notation on the left in (9.3)–(9.4) is the bigraded ring of homogeneous power series. We have a simple lemma.

Lemma 9.6. *If A is a T-spectrum, then for any pointed motivic spaces X and Y the canonical map $X\times Y\to X\wedge Y$ induces a split injection $A^{r,s}(X\wedge Y)\hookrightarrow A^{r,s}(X\times Y)$. The image of the injection coincides with the kernel of the map*

$$[(id_X\times y)^*,(x\times id_Y^*)]\colon A^{r,s}(X\times Y)\to A^{r,s}(X\times y)\oplus A^{r,s}(x\times Y).$$

We write $[-n,n]=\{i\in\mathbb{Z}\mid -n\leq i\leq n\}$. We have a sequential colimit of pointed spaces

$$(\mathbb{Z}\times HGr,(0,x_0)) = \operatorname{colim}(([-n,n]\times HGr(n,2n),(0,x_0))$$

to which Theorem 10.1 applies. Theorem 9.5 and Lemma 9.6 now give the following result.

Theorem 9.7. *Let A be a T-spectrum whose associated cohomology theory $(A^{*,*},\partial)$ has an ε-commutative partial multiplication or ring structure and a symplectic Thom structure. Then the natural map*

$$A^{*,*}((\mathbb{Z}\times HGr),(0,x_0))\to \varprojlim A^{*,*}([-n,n]\times HGr(n,2n),(0,x_0))$$

is an isomorphism. More generally for any r and s the natural map

$$A^{*,*}((\mathbb{Z}\times HGr,(0,x_0))^{\wedge r}\wedge(HP^1,x_0)^{\wedge s})\to \varprojlim A^{*,*}(([-n,n]\times HGr(n,2n),(0,x_0))^{\wedge r}\wedge(HP^1,x_0)^{\wedge s})$$

is an isomorphism.

We complete our review of parts of [16] by looking at the geometry of $HP^1 = HGr(1,\mathsf{H}^{\oplus 2})$.

Theorem 9.8. *In $H_\bullet(S)$ we have a canonical isomorphism $\eta\colon (HP^1_+, pt) \cong T^{\wedge 2}$.*

Proof. By definition HP^1 is by the open subscheme of $Gr(2,4)$ parametrizing 2-dimensional symplectic subspaces of the 4-dimensional trivial symplectic bundle. It contains two distinguished points $x_0 = [\mathsf{H} \oplus 0]$ and $x_\infty = [0 \oplus \mathsf{H}]$. The x_∞ is the origin of an open cell $\mathbf{A}^4 \subset Gr(2,4)$ of the usual Grassmannian consisting of subspaces with basis of the form $(y_1, y_2, 1, 0)$ and $(y_3, y_4, 0, 1)$. Within the $\mathbf{A}^4$ there are two transversal loci $N^+ \cong \mathbf{A}^2$ defined by $y_2 = y_4 = 0$ and $N^- \cong \mathbf{A}^2$ defined by $y_1 = y_3 = 0$. They are closed in HP^1.

The complement $HP^1 - N^+$ is the quotient of $\mathbf{A}^5$ by a free action of $\mathbb{G}_a$ [16, Theorem 3.4]. Consequently the structural map $HP^1 - N^+ \to pt$ and its section $x_0\colon pt \to HP^1 - N^+$ are motivic weak equivalences. This gives us motivic weak equivalences

$$\begin{tikzcd}
T^{\wedge 2} \cong N^-/(N^- - 0) \arrow[r, "\text{2 out of 3}"] \arrow[d, "\mathbf{A}"'] \arrow[dr, "\text{2 out of 3}"] & HP^1/(HP^1 - N^+) & (HP^1, x_0) \arrow[l, "/\mathbf{A}"'] \\
\mathbf{A}^4/(\mathbf{A}^4 - N^+) & (\mathbf{A}^4 \cap HP^1)/((\mathbf{A}^4 \cap HP^1) - N^+) \arrow[l, "\text{excision}"] \arrow[u, "\text{excision}"'] &
\end{tikzcd} \tag{9.5}$$

The zigzag on the top line is the canonical isomorphism in $H_\bullet(S)$. □

A symplectic bundle (E, ϕ) is naturally a special linear bundle (E, λ_ϕ) with λ_ϕ the inverse of the Pfaffian of ϕ. Hence special linear Thom classes of hermitian K-theory (Theorem 5.1) give $\mathbf{BO}^{*,*}$ a symplectic Thom structure. So there are Pontryagin classes for symplectic bundles in hermitian K-theory, and all the formulas of this section are valid for them.

We may compute the Pontryagin classes induced by the Thom classes of this particular symplectic Thom structure. We need the isomorphism of (4.1), which becomes the isomorphism

$$\begin{array}{ccc} GW^-(X) & \xrightarrow{\cong} & KO_0^{[2]}(X) \\ [X, \phi] & \longmapsto & -\big[(X, \phi)[1]\big]. \end{array} \tag{9.6}$$

The sign makes the isomorphism commute with the forgetful maps to $K_0(X)$. For a rank 2 symplectic bundle (E, ϕ) has Pontryagin class $p_1(E, \phi) = -z^* th(E, \mathcal{O}_X, \lambda_\phi)$, which is the image under the isomorphism of $[E, \phi] - [\mathsf{H}] \in GW^-(X)$. The symplectic splitting principle [16, Theorem 10.2] now gives the following.

Proposition 9.9. *Let (F, ψ) be a symplectic bundle of rank $2r$ on X. Its first Pontryagin class $p_1(F, \psi) \in KO_0^{[2]}(X)$ is the image under the isomorphism* (9.6) *of $[F, \psi] - r[\mathsf{H}] \in GW^-(X)$.*

Formulas for higher Pontryagin classes in terms of exterior powers of (F, ψ) will be given in [22].

10. The strategy for putting a ring structure on $A^{*,*}$

We explain our strategy for turning our partial multiplicative structure on $\mathbf{BO}^{*,*}$ into a full ring structure.

We will need the following standard facts about spectra. Recall that a motivic space X is *small* if $Hom_{SH(S)}(\Sigma_T^\infty X, -)$ commutes with arbitrary coproducts. We write $\mathbf{M}_\bullet^{small}(S)$ for the full subcategory of small motivic spaces.

Theorem 10.1 ([15, Lemma A.34])**.** *Let $D^{(0)} \to D^{(1)} \to D^{(2)} \to \cdots$ be a sequence of morphisms in $SH(S)$ with* $\operatorname{hocolim} D^{(i)} = D$*, let X be a small motivic space, and let A be a T-spectrum. Then there is a canonical isomorphism*

$$Hom_{SH(S)}(\Sigma_T^\infty X, D) = \operatorname{colim} Hom_{H_\bullet(S)}(X, D^{(n)}).$$

and a canonical short exact sequence:

$$0 \to \varprojlim{}^1 A^{p-1,q}(D^{(i)}) \to A^{p,q}(D) \to \varprojlim A^{p,q}(D^{(i)}) \to 0.$$

Particular cases of this are the following.

Theorem 10.2 ([16, Theorem 5.2])**.** *Let X be a small motivic space and A a T-spectrum. Then we have*

$$Hom_{SH(S)}(\Sigma_T^\infty X, A) = \operatorname{colim} Hom_{H_\bullet(S)}(X \wedge T^{\wedge n}, A_n).$$

Theorem 10.3 ([15, Corollaries 3.4, 3.5])**.** *Let A and E be T-spectra. Then for any r there is a canonical short exact sequences*

$$0 \to \varprojlim{}^1 A^{*-2rn-1,*-rn}(E_n^{\wedge r}) \to A^{*,*}(E^{\wedge r}) \to \varprojlim A^{*-2rn,*-rn}(E_n^{\wedge r}) \to 0.$$

Definition 10.4. An *almost commutative monoid* in $SH(S)$ is a triple (A, μ, e) with A a T-spectrum and $\mu\colon A \wedge A \to A$ and $e\colon \Sigma_T^\infty \mathbf{1} \to A$ morphisms in $SH(S)$ such that

(1) the morphism $\mu \circ (\mu \wedge 1) - \mu \circ (1 \wedge \mu) \in Hom_{SH(S)}(A \wedge A \wedge A, A)$ lies in the subgroup $\varprojlim{}^1 A^{*-6n-1,*-3n}(A_n \wedge A_n \wedge A_n)$,
(2) for $\sigma\colon A \wedge A \to A \wedge A$ the morphism switching the two factors, the morphism $\mu - \mu \circ \sigma \in \operatorname{Hom}_{SH(S)}(A \wedge A, A)$ lies in the subgroup $\varprojlim{}^1 A^{*-4n-1,*-2n}(A_n \wedge A_n)$,
(3) the map $1 - \mu \circ (1 \wedge e) \in \operatorname{Hom}_{SH(S)}(A, A)$ lies in the subgroup $\varprojlim{}^1 A^{*-2n-1,*-n}(A_n)$.

An almost commutative monoid (A, μ, e) defines pairings

$$\times\colon A^{p,q}(X) \times A^{r,s}(Y) \to A^{p+r,q+s}(X \wedge Y) \tag{10.1}$$

for X and Y in $\mathbf{M}_\bullet(S)$ as follows. For $\alpha : \Sigma_T^\infty X \to A \wedge S^{p,q}$ and $\beta : \Sigma_T^\infty Y \to A \wedge S^{r,s}$ define $\alpha \times \beta \in A^{p+r,q+s}(X \wedge Y)$ as the composition

$$\Sigma_T^\infty(X \wedge Y) \cong \Sigma_T^\infty X \wedge \Sigma_T^\infty Y \xrightarrow{\alpha\wedge\beta} A \wedge S^{p,q} \wedge A \wedge S^{r,s} \cong A \wedge A \wedge S^{p+r,q+s} \xrightarrow{m\wedge 1} A \wedge S^{p+r,q+s}.$$

The unit $e \in Hom_{SH(S)}(\Sigma_T^\infty pt_+, A)$ defines an element $1 \in A^{0,0}(pt_+)$. There is then a unique element $\varepsilon \in A^{0,0}(pt_+)$ such that $\Sigma_T \varepsilon \in Hom_{SH(S)}(T, A \wedge T)$ is the composition of the endomorphism of T induced by the endomorphism $x \mapsto -x$ of $\mathbf{A}^1$ with $e \wedge 1_T$.

Theorem 10.5. *For an almost commutative monoid (A, μ, e) in $SH(S)$ the cohomology theory $(A^{*,*}, \partial)$ with the pairing $\times$ of* (10.1) *and the element $1 \in A^{0,0}(pt_+)$ form an ε-commutative ring cohomology theory on $\mathbf{M}_\bullet(S)^{small}$ and on $\mathcal{S}m\mathcal{O}p/S$.*

Proof. There are canonical elements $a_n\colon \Sigma_T^\infty A_n(-n) \to A$. The definition of an almost commutative monoid is equivalent to having (A, μ, e) such that the induced pairing satisfies $(a_n \times a_n) \times a_n = a_n \times (a_n \times a_n)$ and $\sigma^*(a_n \times a_n) = \varepsilon^n a_n \times a_n$ and $a_n \times e = a_n$ for all n. It then also satisfies

$$(\Sigma^{p,q} a_n \times \Sigma^{p',q'} a_n) \times \Sigma^{p'',q''} a_n = \Sigma^{p,q} a_n \times (\Sigma^{p',q'} a_n \times \Sigma^{p'',q''} a_n)$$

for all (p,q), (p',q') and (p'',q''). By Theorem 10.2 for a small motivic space X any morphism $\Sigma^\infty_T X \to A \wedge S^{p,q}$ factors as

$$\Sigma^\infty_T X \to \Sigma^\infty_T A_n(-n) \wedge S^{p,q} \xrightarrow{\Sigma^{p,q} a_n} A \wedge S^{p,q}$$

for some n. So on small motivic spaces the multiplication is associative. The ε-commutativity and the unit property are treated similarly. The signs in the commutativity come from permuting the spheres. □

11. The universal elements

The isomorphism $\tau\colon (\mathbb{Z} \times HGr, (0,x_0)) \xrightarrow{\cong} \mathbf{KSp}$ of Theorem 8.2 is classified by an element which has restrictions

$$\tau|_{\{i\}\times HGr(n,2n)} = [\mathfrak{U}_{n,2n}, \phi_{n,2n}] + (i-n)[\mathsf{H}] \in KSp_0(HGr(n,2n)). \tag{11.1}$$

Its image under the isomorphism (9.6) is a class $\tau_2 \in \mathbf{KO}^{[2]}_0(\mathbb{Z} \times HGr, (0,x_0))$ with

$$\tau_2|_{\{i\}\times HGr(n,2n)} = p_1(\mathfrak{U}_{n,2n}, \phi_{n,2n}) + i\mathsf{h} \in \mathbf{KO}^{[2]}_0(HGr(n,2n))$$

according to Proposition 9.9. Here $\mathsf{h} \in \mathbf{KO}^{[2]}_0(pt) = \mathbf{BO}^{4,2}(pt)$ is the class corresponding to $[\mathsf{H}] \in GW^-(pt)$ under the isomorphism. By Theorem 9.5 we have an isomorphism

$$\mathbf{BO}^{*,*}(\mathbb{Z} \times HGr) \cong \prod_{i\in\mathbb{Z}} \mathbf{BO}^{*,*}(pt)[[p_1,p_2,p_3,\dots]]^{homog}$$

with the product taken in the category of graded rings. Setting

$$p_1 = (p_1)_{i\in\mathbb{Z}} \in \mathbf{BO}^{*,*}(\mathbb{Z} \times HGr), \qquad \tfrac{1}{2}\,\mathrm{rk} = (i1_{\mathbf{BO}})_{i\in\mathbb{Z}} \in \mathbf{BO}^{*,*}(\mathbb{Z} \times HGr)$$

we see we have $\tau_2 = p_1 + (\frac{1}{2}\,\mathrm{rk})\mathsf{h}$.

For any k we have a composition of isomorphisms in the homotopy category $H_\bullet(S)$

$$(\mathbb{Z} \times HGr, (0,x_0)) \xrightarrow{\tau} \mathbf{KSp} \xrightarrow{trans_{2k+1}} \mathbf{KO}^{[4k+2]} \xrightarrow{-1} \mathbf{KO}^{[4k+2]}$$

where the $trans_{2k+1}$ comes from translation and the -1 is the inverse operation of the H-space structure, which we add as in (4.1) so that the forgetful maps to K-theory should commute up to homotopy. (The inverse in the H-space structure comes from the Ω_T-spectrum structure and to the authors' knowledge not from a duality-preserving functor.)

Definition 11.1 (Universal element)**.** We denote by

$$\tau_{4k+2} \in \mathbf{KO}^{[4k+2]}_0(\mathbb{Z} \times HGr, (0,x_0)) \cong \mathbf{BO}^{8k+4,4k+2}(\mathbb{Z} \times HGr, (0,x_0))$$

the element corresponding to the composition. It is given by $\tau_{4k+2} = (p_1 + (\frac{1}{2}\,\mathrm{rk})\mathsf{h}) \cup \beta_8^k$.

We write $[-n,n] = \{i \in \mathbb{Z} \mid -n \le i \le n\}$. The class $\tau \in GW^-(\mathbb{Z} \times HGr)$ giving the isomorphism $\mathbb{Z} \times HGr \cong \mathbf{KSp}$ has a universal property.

Lemma 11.2. *There are unique μ and μ_{8k+4} in $H_\bullet(S)$ making the diagram commute*

$$\begin{tikzcd}
(\mathbb{Z} \times HGr, (0,x_0)) \wedge (\mathbb{Z} \times HGr, (0,x_0)) \arrow[r, "\mu"] \arrow[d, "\tau_{4k+2}\wedge\tau_{4k+2}"', "\cong"] & \mathbf{KO}^{[0]} \arrow[d, "\cong"', "\text{translation}"] \\
\mathbf{KO}^{[4k+2]} \wedge \mathbf{KO}^{[4k+2]} \arrow[r, "\mu_{8k+4}"'] & \mathbf{KO}^{[8k+4]}
\end{tikzcd}$$

and such that for each i, j and n the restriction of μ

$$(\{i\} \times HGr(n,2n)) \times (\{j\} \times HGr(n,2n) \to \mathbf{KO}^{[0]}$$

is the class in $H_\bullet(S)$ of the morphisms of ind-schemes represing the orthogonal Grothendieck-Witt class which is

$$([\mathcal{U}_{n,2n}, \phi_{n,2n}] + (i-n)[\mathsf{H}]) \boxtimes ([\mathcal{U}_{n,2n}, \phi_{n,2n}] + (j-n)[\mathsf{H}]).$$

The μ with the asserted restrictions exists and is unique because of Theorem 9.7. It factors through the wedge space because of Lemma 9.6. Then μ_{8k+4} is the map making the diagram commute.

Lemma 11.3. *The following diagram commutes in $H_\bullet(S)$.*

$$\begin{tikzcd}
\mathbf{KO}^{[4k-2]} \wedge \mathbf{KO}^{[4k-2]} \wedge T^{\wedge 8} \arrow[r, "\mu_{8k-4}\wedge 1"] \arrow[d, "\tau_{4k+2}\wedge\tau_{4k+2}"', "\cong"] & \mathbf{KO}^{[8k-4]} \wedge T^{\wedge 8} \arrow[dd, "\text{structure maps}"] \\
\mathbf{KO}^{[4k-2]} \wedge T^{\wedge 4} \wedge \mathbf{KO}^{[4k-2]} \wedge T^{\wedge 4} \arrow[d, "\text{structure maps}"] & \\
\mathbf{KO}^{[4k+2]} \wedge \mathbf{KO}^{[4k+2]} \arrow[r, "\mu_{8k+4}"'] & \mathbf{KO}^{[8k+4]}
\end{tikzcd}$$

Proof. Because of Theorem 9.7 it is enough to observe that the compositions of the two paths of arrows with the composition

$$\begin{tikzcd}
(\{i\} \times HGr(n,2n)) \times (\{j\} \times HGr(n,2n)) \times (HP^1, x_0)^{\times 4} \arrow[d, hook] & \\
(\mathbb{Z} \times HGr) \times (\mathbb{Z} \times HGr) \times (HP^1, x_0)^{\times 4} \arrow[r, "\tau_{4k-2}\wedge\tau_{4k-2}\wedge\eta^{\wedge 4}", "\cong"'] & \mathbf{KO}^{[4k-2]} \wedge \mathbf{KO}^{[4k-2]} \wedge T^{\wedge 8}
\end{tikzcd}$$

are the same for all i, j and n. □

By Theorem 10.3 we have a surjection

$$Hom_{SH(S)}(\mathbf{BO} \wedge \mathbf{BO}, \mathbf{BO}) \to \varprojlim \mathbf{BO}^{16k+8,8k+4}(\mathbf{KO}^{[4k+2]} \wedge \mathbf{KO}^{[4k+2]}) \to 0$$

the compositions

$$\Sigma_T^\infty(\mathbf{KO}^{[4k+2]} \wedge \mathbf{KO}^{[4k+2]})(-8k-4) \xrightarrow{\mu_{4k+2}} \Sigma_T^\infty \mathbf{KO}^{[8k+4]}(-8k-4) \xrightarrow{\text{canonical}} \mathbf{BO}$$

form a system of elements of the groups in the inverse limit. They are compatible with the connecting maps of the inverse limit because the diagrams of Lemma 11.3 commute. Let

$$\bar{m} = (\mu_{8k+4})_{k\in\mathbb{N}} \in \varprojlim \mathbf{BO}^{16k+8,8k+4}(\mathbf{KO}^{[4k+2]} \wedge \mathbf{KO}^{[4k+2]}) \tag{11.2}$$

and let

$$m \in Hom_{SH(S)}(\mathbf{BO} \wedge \mathbf{BO}) \tag{11.3}$$

be an element lifting it. As in (10.1) m defines a pairing

$$\times \colon \mathbf{BO}^{p,q}(A) \times \mathbf{BO}^{r,s}(B) \to \mathbf{BO}^{p+r,q+s}(A \wedge B). \tag{11.4}$$

Theorem 11.4. *Suppose S satisfies the hypotheses of Theorem 13.1. For X and Y in $\mathcal{S}m/S$ and all p and q the pairing*

$$\mathbf{BO}^{4p,2p}(X_+) \times \mathbf{BO}^{4q,2q}(Y_+) \to \mathbf{BO}^{4p+4q,2p+2q}(X_+ \wedge Y_+)$$

induced by m is identified via the isomorphism γ of Corollary 7.3 with the naive pairing

$$KO_0^{[2p]}(X) \times KO_0^{[2q]}(Y) \to KO_0^{[2p+2q]}(X \times Y).$$

Proof. Because of Jouanolou's trick it is enough to consider affine X and Y.

Let $\alpha \in KO_0^{[2p]}(X)$ and $\beta \in KO_0^{[2q]}(Y)$.

When we have $2p = 2q = 4k+2$ and $X = Y = [-n,n] \times HGr(n,2n)$ and $\alpha = \beta$ is the restriction of the universal class τ_{4k+2}, then we have the identification by the construction of m.

When we have $2p = 2q = 4k+2$ but X, Y, α and β are general then by Theorem 8.1 there exists an n and morphisms $f_\alpha \colon X \to [-n,n] \times HGr(n,2n)$ and $f_\beta \colon Y \to [-n,n] \times HGr(n,2n)$ in $\mathcal{S}m/S$ such that $f_\alpha^*(\tau_{4k+2}|_{[-n,n]\times HGr(n,2n)}) = \alpha$ and $f_\beta^*(\tau_{4k+2}|_{[-n,n]\times HGr(n,2n)}) = \beta$. The identification of the two pairings for α and β now follows from that for the restrictions of the universal classes because both pairings are functorial for morphisms in $\mathcal{S}m/S$.

For general p and q pick $4k+2 \geq \max(2p,2q)$. The identification of the two pairings on α and β follows from the identification of the two pairings on

$$\alpha \times p_1(U,\phi)^{\times 2k+1-p} \in KO_0^{[4k+2]}(X \times (HP^1)^{\times 2k+1-p}),$$
$$\beta \times p_1(U,\phi)^{\times 2k+1-q} \in KO_0^{[4k+2]}(Y \times (HP^1)^{\times 2k+1-q}).$$

□

Theorem 11.5. *Let m be as in (11.3), and let $e \in \mathbf{BO}^{0,0}(pt_+)$ be the unit of the partial multiplicative structure on $(\mathbf{BO}^{*,*}, \partial)$ of Definition 7.4. Then $(\mathbf{BO}, m, e)$ is an almost commutative monoid in $SH(S)$, and the $\times$ and 1 induced by m and e make $(\mathbf{BO}^{*,*}, \partial, \times, 1)$ a $\langle -1 \rangle$-commutative bigraded ring cohomology theory on $\mathbf{M}_\bullet^{small}(S)$ and on $\mathcal{S}m\mathcal{O}p/S$.*

Proof. We prove associativity. The morphisms

$$\mathbf{BO} \wedge \mathbf{BO} \wedge \mathbf{BO} \underset{m\circ(id_{\mathbf{BO}}\wedge m)}{\overset{m\circ(m\wedge id_{\mathbf{BO}})}{\rightrightarrows}} \mathbf{BO}$$

define two elements of $\mathbf{BO}^{0,0}(\mathbf{BO} \wedge \mathbf{BO} \wedge \mathbf{BO})$ with images in

$$\varprojlim \mathbf{BO}^{24k+12,12k+6}(\mathbf{KO}^{[4k+2]} \wedge \mathbf{KO}^{[4k+2]} \wedge \mathbf{KO}^{[4k+2]}).$$

This last group is isomorphic to

$$\varprojlim \mathbf{BO}^{24k+12,12k+6}((\mathbb{Z} \times HGr, (0,x_0))^{\wedge 3})$$

So it is enough to show that the corresponding elements $(\tau_{4k+2} \boxtimes \tau_{4k+2}) \boxtimes \tau_{4k+2}$ and $\tau_{4k+2} \boxtimes (\tau_{4k+2} \boxtimes \tau_{4k+2})$ in each group of the inverse system are equal. However, since we have

$$\mathbf{BO}^{24k+12,12k+6}((\mathbb{Z} \times HGr, (0,x_0))^{\wedge 3}) \cong \varprojlim \mathbf{BO}^{24k+12,12k+6}(([-n,n] \times HGr(n,2n), (0,x_0))^{\wedge 3})$$

by Theorem 9.7 it is enough to show that the restrictions to the smooth affine schemes $[-n,n] \times HGr(n,2n)$ are equal. But then by Theorem 11.4 the pairings coincide with the naive products, and those are associative.

The proofs of commutativity and of the unit property are similar.

Thus $(\mathbf{BO}, m, e)$ is an almost commutative monoid in $SH(S)$. The rest of the statement of the theorem follows from that fact by Theorem 10.5 except for the value of ε. For that note that $\Sigma_T 1 \in \mathbf{BO}^{2,1}(T)$ is the class in $GW^{[1]}(\mathbf{A}^1, \mathbf{A}^1 - 0)$ of the symmetric complex $(K(\mathcal{O}), \Theta(\mathcal{O}))$ of (7.2). The pullback of the complex along the endomorphism $x \mapsto -x$ of $\mathbf{A}^1$ is isometric to $(K(\mathcal{O}), -\Theta(\mathcal{O}))$. So we have $\varepsilon = \langle -1 \rangle$. □

Theorem 11.6. *Suppose $\times$ and $\times'$ are two products on $(\mathbf{BO}^{*,*}, \partial)$ on $\mathbf{M}_\bullet^{small}(S)$ which associative and $\langle -1 \rangle$-commutative with unit $1 = \langle 1 \rangle$, and such that $\alpha \times \Sigma_{\mathbb{G}_m} 1 = \Sigma_{\mathbb{G}_m} \alpha$ and $\alpha \times \Sigma_{S^1_s} 1 = \Sigma_{S^1_s} \alpha$ for all α. If $\times$ and $\times'$ are both compatible with the products $KO_0^{[2m]}(X) \times KO_0^{[2n]}(Y) \to KO_0^{[m+n]}(X \times Y)$ induced on Grothendieck-Witt groups of smooth schemes by the tensor product, then we have $\times = \times'$.*

Proof. Suppose first that we have $\alpha \in \mathbf{BO}^{2i,i}(A)$ and $\beta \in \mathbf{BO}^{2j,j}(B)$ with A and B small pointed motivic spaces. By Theorems 10.1 and 10.2 there exist m and n such that $\Sigma_T^{4m+2-i}\alpha \in \mathbf{BO}^{8m+4,4m+2}(A \wedge T^{\wedge 2m})$ has a factorisation

$$A \wedge T^{\wedge 4m+2} \to ([-n,n] \times HGr(n,2n), (0,x_0)) \hookrightarrow \mathbb{Z} \times HGr \cong \mathbf{KO}^{[4m+2]} \to \mathbf{BO}(4m+2)$$

and such that $\Sigma_T^{4m+2-j}\beta$ has a similar factorization. Since $\Sigma_T^{4m+2-i}\alpha$ and $\Sigma_T^{4m+2-j}\beta$ are thus pullbacks of classes on which $\times$ and $\times'$ agree, the products agree on $\Sigma_T^{4m+2-i}\alpha$ and $\Sigma_T^{4m+2-j}\beta$. The compatibility of the products with the suspension and their ε-commutativity imply that we also have $\Sigma_T^{8m+4-i-j}(\alpha \times \beta) = \Sigma_T^{8m+4-i-j}(\alpha \times' \beta)$. Since the suspension operation is a bijection on cohomology groups, we have $\alpha \times \beta = \alpha \times' \beta$.

For $\alpha \in \mathbf{BO}^{p,q}(A)$ and $\beta \in \mathbf{BO}^{p',q'}(B)$ with for instance $p < 2q$ and $p' > 2q'$ the products agree on $\Sigma_{S^1_s}^{2q-p}\alpha \in \mathbf{BO}^{2q,q}(A)$ and $\Sigma_{\mathbb{G}_m}^{p'-2q'}\beta \in \mathbf{BO}^{2p'-2q',p'-q'}(B)$ by the previous case. By the same sort of argument they also agree on α and β. The other cases are similar. □

Theorem 11.7. *The assertions of Theorem 1.3 and 1.4 hold.*

This follows from Theorems 11.5 and 11.6.

12. Spectra of finite and infinite real and quaternionic Grassmannians

We now wish to switch our sphere from T to the pointed HP^1. This is possible because the pointed HP^1 is isomorphic to $T^{\wedge 2}$ in $H_\bullet(S)$. The result is a spectrum $\mathbf{BO}_{HP^1}$ very similar to **BO** except that the structural maps are induced by a product with the Pontryagin class of a symplectic bundle on a pointed scheme rather than a Thom class on a Thom space. This is important because the universal property of $\mathbb{Z} \times HGr$ deals with Grothendieck-Witt classes of bundles on schemes not of chain complexes on an X acyclic over U.

The zigzag (9.5) also gives us an equivalence between the stable homotopy categories $SH_{T^{\wedge 2}}(S)$ and $SH_{(HP^1,x_\infty)}(S)$ [9, Proposition 2.13]. There is also a Quillen equivalences between $Spt_T(S)$ and $Spt_{T^{\wedge 2}}(S)$ given by the forgetful functor and its adjoints.

Theorem 12.1. *The stable homotopy categories of T-spectra and of (HP^1, x_0)-spectra are equivalent.*

The class $-p_1(\mathcal{U}_{HP^1}, \phi_{HP^1}) \in \mathbf{KO}_0^{[2]}(HP^1, x_\infty)$ corresponds to $th^{\times 2} \in \mathbf{KO}_0^{[2]}(\mathbf{A}^2/\mathbf{A}^2 - 0)$ under the identifications. So we may define our new spectrum.

Definition 12.2. The HP^1-spectrum $\mathbf{BO}_{HP^1}$ corresponding to the T-spectrum $\mathbf{BO}$ has spaces $(\mathbf{KO}^{[0]}, \mathbf{KO}^{[2]}, \mathbf{KO}^{[4]}, \dots)$ and bonding maps adjoint to the maps

$$- \times -p_1(\mathcal{U}_{HP^1}, \phi_{HP^1})\colon \mathbf{KO}^{[2n]}(-) \to \mathbf{KO}^{[2n+2]}(- \wedge (HP^1, x_\infty))$$

The purpose of this section is to prove the following result. For parallelism with the labelling of real and complex Grassmannians, we write $HGr'(2r, 2n) = HGr(r, n)$ so that $RGr(2r, 2n)$ and $HGr'(2r, 2n)$ are both open subschemes of the ordinary Grassmannian $Gr(2r, 2n)$ where a certain bilinear form is nondegenerate. We also write

$$[-n, n]' = \{i \in \mathbb{Z} \mid -n \le i \le n \text{ and } i \equiv n \bmod 2\}.$$

For even n the scheme $[-n, n]' \times RGr(n, 2n)$ is pointed in the component corresponding to $0 \in [-n, n]'$ by the point corresponding to $\mathsf{H}_+^{\oplus n/2} \oplus 0 \subset \mathsf{H}_+^{\oplus n}$. For odd n we either do not use a base point or we use a disjoint base point. To compactify our notations we write

$$HGr'_{2n} = [-n, n] \times HGr'(2n, 4n), \tag{12.1}$$

$$RGr_{2n} = ([-2n, 2n]' \times RGr(2n, 4n)) \cup ([-2n+1, 2n-1]' \times RGr(2n-1, 4n-2)). \tag{12.2}$$

The first step in the proof of this theorem is the following general construction. For (X, x) a pointed scheme over S, let $(X, x)^+$ be the pushout of

$$\mathbf{A}^1 \xleftarrow[\sim]{0} pt \xrightarrow{x} X \tag{12.3}$$

pointed by $1 \in \mathbf{A}^1(pt)$. (This is essentially the $\mathbf{A}^1$ mapping cone of the inclusion $x\colon pt \to X$.) The natural projection $(X, x)^+ \to (X, x)$ which is the identity on X and sends $\mathbf{A}^1 \to x$ is a motivic weak equivalence.

We abbreviate $HP^{1+} = (HP^1, x_0)^+$. We will actually consider HP^{1+}-spectra. The natural functor $SH_{(HP^1, x_0)}(S) \to SH_{(HP^{1+}}(S)$ is an equivalence, and let $\mathbf{BO}_{HP^{1+}}$ be the $(HP^1, x_0)^+$-spectrum corresponding to $\mathbf{BO}$.

Theorem 12.3. *There are HP^{1+}-spectra $\mathbf{BO}^{fin}$ and $\mathbf{BO}^{geom}$ which are isomorphic to $\mathbf{BO}_{HP^{1+}}$ in $SH_{HP^{1+}}(S)$ with spaces*

$$\mathbf{BO}^{fin}_{2i} = \begin{cases} RGr_{2\cdot 8^i} & \text{for even } i, \\ HGr'_{2\cdot 8^i} & \text{for odd } i, \end{cases} \tag{12.4}$$

$$\mathbf{BO}^{geom}_{2i} = \begin{cases} \mathbb{Z} \times RGr & \text{for even } i, \\ \mathbb{Z} \times HGr & \text{for odd } i, \end{cases} \tag{12.5}$$

which are unions of real and quaternionic Grassmannians. The bonding maps $\mathbf{BO}^*_{2i} \wedge HP^{1+} \to \mathbf{BO}^*_{2i+2}$ with $* =$ fin or geom are morphisms of schemes and ind-schemes, respectively, which are constant on $\mathbf{BO}^*_{2i} \vee HP^{1+}$.

Proposition 12.4. *Let (X, x), (Y, y) and (Z, z) be pointed schemes. There is a natural bijection between*

1. *the set of morphisms of pointed schemes $g\colon (X, x) \times (Y, y)^+ \to (Z, z)$ which restrict to the constant map $\big(X \times 1\big) \cup \big(x \times (Y, y)^+\big) \to z$, and*
2. *the set of pairs (f, h) where*
   (a) *$f\colon (X \times Y, x \times y) \to (Z, z)$ is a morphism of schemes which restricts to the constant map $x \times Y \to z$ and*

(b) *$h\colon (X,x)\times\mathbf{A}^1 \to (Z,z)$ is a pointed $\mathbf{A}^1$-homotopy between $f|_{X\times y}$ and the constant map $X \to z$.*

The idea is that $(X,x)\times(Y,y)^+$ is the union of two subschemes, and the restrictions of g to these subschemes are $g|_{X\times Y} = f$ and $g|_{X\times\mathbf{A}^1} = h$.

Let U_{2n} and U be the tautological symplectic subbundles on $HGr'(2n,4n)$ and $HP^1 = HGr'(2,4)$ resp., and let V_{16n} be the tautological orthogonal subbundle on $RGr(16n,24n)$. Let H be the trivial rank 2 symplectic bundle, and let H_+ be the trivial rank 2 orthogonal bundle for the split quadratic form $q(x_1,x_2) = x_1x_2$.

Lemma 12.5. *There exist morphisms of pointed schemes*

$$f_{2n}\colon \big([-n,n]\times HGr'(2n,4n)\big)\times HP^1 \to RGr(16n,32n)$$

such that the Grothendieck-Witt classes satisfy

$$f_{2n}^*([V_{16n}] - 8n[\mathsf{H}_+]) = ([U_{2n}] - (n-i)[\mathsf{H}])\boxtimes([U]-[\mathsf{H}]) \tag{12.6}$$

(where $i\in[-n,n]\subset\mathbb{Z}$ is the index of the component) and such that $f_{2n}|_{pt\times HP^1}$ is constant, and $f_{2n}|_{([-n,n]\times HGr'(2n,4n))\times pt}$ is pointed $\mathbf{A}^1$-homotopic to a constant map. These maps and homotopies are compatible with the inclusions $HGr'(2n,4n)\hookrightarrow HGr'(2(n+1),4(n+1))$ and $RGr(16n,32n)\hookrightarrow RGr(16(n+1),32(n+1))$.

Proof. There are orthogonal direct sums $U_{2n}\oplus U_{2n}^\perp \cong \mathsf{H}^{\oplus 2n}$ and $U\oplus U^\perp\cong\mathsf{H}^{\oplus 2}$. Consequently we have

$$([U_{2n}]-(n-i)[\mathsf{H}])\boxtimes([U]-[\mathsf{H}]) = [U_{2n}\boxtimes U] + [\mathsf{H}^{\oplus n-i}\boxtimes U^\perp] + [U_{2n}^\perp\boxtimes\mathsf{H}] - (6n-2i)[\mathsf{H}_+].$$

We have inclusions of orthogonal bundles

$$U_{2n}\boxtimes U\subset \mathsf{H}^{\oplus 2n}\boxtimes U, \qquad U_{2n}^\perp\boxtimes\mathsf{H}\subset\mathsf{H}^{\oplus 2n}\boxtimes\mathsf{H},$$
$$\mathsf{H}^{\oplus n-i}\boxtimes U^\perp\subset\mathsf{H}^{\oplus 2n}\boxtimes U^\perp, \qquad \mathsf{H}_+^{\oplus 2n+2i}\subset\mathsf{H}_+^{\oplus 4n}$$

Consequently the orthogonal subbundle

$$(U_{2n}\boxtimes U)\oplus(\mathsf{H}^{\oplus n-i}\boxtimes U^\perp)\oplus(U_{2n}^\perp\boxtimes\mathsf{H})\oplus\mathsf{H}_+^{\oplus 2n+2i} \tag{12.7}$$

of

$$\big(\mathsf{H}^{\oplus 2n}\boxtimes(U\oplus U^\perp\oplus\mathsf{H})\big)\oplus\mathsf{H}_+^{\oplus 4n} = \mathsf{H}_+^{\oplus 16n}$$

is classified by a map $f_{2n}\colon([-n,n]\times HGr'(2n,4n))\times HP^1\to RGr(16n,32n)$.

When we restrict to $pt\times HP^1$, we have $i=0$, and the direct sum $U_{2n}\oplus U_{2n}^\perp$ becomes $\mathsf{H}^{\oplus n}\oplus\mathsf{H}^{\oplus n}$. The orthogonal direct sum (12.7) becomes

$$\big((\mathsf{H}^{\oplus n}\oplus 0)\boxtimes(U\oplus U^\perp\oplus 0)\big)\oplus\big((0\oplus\mathsf{H}^{\oplus n})\boxtimes(0\oplus 0\oplus\mathsf{H})\big)\oplus(\mathsf{H}_+^{\oplus 2n}\oplus 0).$$

Since $U\oplus U^\perp\oplus 0 = \mathsf{H}\oplus\mathsf{H}\oplus 0$, this is the same as the subbundle

$$\big((\mathsf{H}^{\oplus n}\oplus 0)\boxtimes(\mathsf{H}\oplus\mathsf{H}\oplus 0)\big)\oplus\big((0\oplus\mathsf{H}^{\oplus n})\boxtimes(0\oplus 0\oplus\mathsf{H})\big)\oplus(\mathsf{H}_+^{\oplus 2n}\oplus 0). \tag{12.8}$$

corresponding to the pointing of $RGr(16n,32n)$. So $f_{2n}(pt\times HP^1) = pt$.

A similar argument shows that f_{2n} is compatible with the inclusions of the Grassmannians in higher-dimensional Grassmannians.

When we restrict f_{2n} to $([-n,n]\times HGr'(2n,4n))\times pt$, the direct sum over $pt\subset HP^1$ becomes $U\oplus U^\perp = \mathsf{H}\oplus\mathsf{H}$. The orthogonal direct sum (12.7) becomes

$$\big(U_{2n}\boxtimes(\mathsf{H}\oplus 0\oplus 0)\big)\oplus\big(\mathsf{H}^{\oplus n-i}\boxtimes(0\oplus\mathsf{H}\oplus 0)\big)\oplus\big(U_{2n}^\perp\boxtimes(0\oplus 0\oplus\mathsf{H})\big)\oplus\mathsf{H}_+^{\oplus 2n+2i} \tag{12.9}$$

This direct sum decomposition of the subbundle of rank $16n$ is compatible with the orthogonal direct sum decomposition of the bundle of rank $32n$ into two summands

$$\mathsf{H}^{\oplus 2n} \boxtimes (\mathsf{H} \oplus 0 \oplus \mathsf{H}) = (U_{2n} \oplus U_{2n}^{\perp}) \boxtimes (\mathsf{H} \oplus 0 \oplus \mathsf{H}) = (\mathsf{H}^{\oplus n} \oplus \mathsf{H}^{\oplus n}) \boxtimes (\mathsf{H} \oplus 0 \oplus \mathsf{H}) \tag{12.10}$$

and

$$\big(\mathsf{H}^{\oplus 2n} \boxtimes (0 \oplus \mathsf{H} \oplus 0)\big) \oplus \mathsf{H}_+^{4n} = \mathsf{H}_+^{8n}. \tag{12.11}$$

Because

$$M_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a product of matrices which are elementary symplectic and elementary orthogonal, there is a morphism $M\colon \mathbf{A}^1 \to Sp_4 \cap O_4$ with $M(0) = I$ and $M(1) = M_1$, namely

$$M(t) = \begin{pmatrix} 1-t^2 & 0 & -t & 0 \\ 0 & 1-t^2 & 0 & -2t+t^3 \\ 2t-t^3 & 0 & 1-t^2 & 0 \\ 0 & t & 0 & 1-t^2 \end{pmatrix}$$

When we restrict to $pt \times pt$ the direct sum $U \oplus U^{\perp}$ becomes $\mathsf{H} \oplus \mathsf{H}$. We may see that

$$\big((1_{\mathsf{H}^{\oplus n} \oplus 0} \boxtimes 1_{\mathsf{H}^{\oplus 2}}) \oplus (1_{0 \oplus \mathsf{H}^{\oplus n}} \boxtimes M)\big)^{-1} \big((1_{U_{2n}} \boxtimes 1_{\mathsf{H}^{\oplus 2}}) \oplus (1_{U_{2n}^{\perp}} \boxtimes M)\big)$$

is a pointed $\mathbf{A}^1$-homotopy between the subbundle

$$\big(U_{2n} \boxtimes (\mathsf{H} \oplus 0 \oplus 0)\big) \oplus \big(U_{2n}^{\perp} \boxtimes (0 \oplus 0 \oplus \mathsf{H})\big)$$

and the subbundle

$$\big((\mathsf{H}^{\oplus n} \oplus 0) \boxtimes (\mathsf{H} \oplus 0 \oplus 0)\big) \oplus \big((0 \oplus \mathsf{H}^{\oplus n}) \boxtimes (0 \oplus 0 \oplus \mathsf{H})\big)$$

One may construct a similar pointed $\mathbf{A}^1$-homotopy between the subbundles

$$\big((\mathsf{H}^{\oplus n-i} \oplus 0) \boxtimes (0 \oplus \mathsf{H} \oplus 0)\big) \oplus (\mathsf{H}_+^{\oplus 2n+2i} \oplus 0)$$

and

$$\big((\mathsf{H}^{\oplus n} \oplus 0) \boxtimes (0 \oplus \mathsf{H} \oplus 0)\big) \oplus (\mathsf{H}_+^{\oplus 2n} \oplus 0)$$

Combining the two gives a pointed $\mathbf{A}^1$-homotopy between (12.9) and (12.8) and thus between $f_{2n}|_{([-n,n] \times HGr'(2n,4n)) \times pt}$ and the constant map.

The compatibility of the homotopies with the inclusions of Grassmannians is relatively straightforward. □

The next lemma is proven in the same way as the last one.

Lemma 12.6. *There exist morphisms of pointed schemes*

$$g_n\colon ([-n,n]' \times RGr(n,2n)) \times HP^1 \to HGr'(8n, 16n)$$

such that the Grothendieck-Witt classes satisfy

$$g_n^*([U_{8n}] - 4n[\mathsf{H}]) = ([V_n] - \tfrac{1}{2}(n-i)[\mathsf{H}_+]) \boxtimes ([U] - [\mathsf{H}]) \tag{12.12}$$

(where $i \in [-n,n]' \subset \mathbb{Z}$ is the index of the component) and such that $g_n|_{pt \times HP^1}$ is constant, and $g_n|_{([-n,n]' \times RGr(n,2n)) \times pt}$ is pointed $\mathbf{A}^1$-homotopic to a constant map. These maps

and homotopies are compatible with the inclusions $RGr(n,2n) \hookrightarrow RGr(n+2,2n+4)$ *and* $HGr'(8n,16n) \hookrightarrow HGr'(8(n+2),16(n+2))$.

Proof of Theorem 12.3. By Proposition 12.4 the maps and homotopies of Lemmas 12.5 and 12.6 give us maps

$$F_n\colon HGr'_{2n} \wedge HP^{1+} \to RGr_{16n},$$
$$G_n\colon RGr_{2n} \wedge HP^{1+} \to HGr'_{16n}.$$

We now define $\mathbf{BO}^{fin}$ to be the HP^1-spectrum with spaces as in the statement of Theorem 12.3 and with bonding maps the compositions

$$RGr_{2\cdot 8^{2i}} \wedge HP^{1+} \to HGr'_{2\cdot 8^{2i+1}} \to HGr'_{2\cdot 8^{2i+2}}$$
$$HGr'_{2\cdot 8^{2i}} \wedge HP^{1+} \to RGr_{2\cdot 8^{2i+1}} \to RGr_{2\cdot 8^{2i+2}}$$

of the appropriate F_n or G_n with the maps induced by the inclusions of Grassmannians.

We define $\mathbf{BO}^{geom}$ to be the HP^1-spectrum with spaces

$$\mathbf{BO}^{geom}_{2i} = \begin{cases} \operatorname{colim}_n RGr_{2n} = \mathbb{Z} \times RGr & \text{for even } i, \\ \operatorname{colim}_n HGr'_{2n} = \mathbb{Z} \times HGr & \text{for odd } i, \end{cases}$$

and with bonding maps induced by the F_n and G_n.

We claim that the inclusion map $\mathbf{BO}^{fin} \to \mathbf{BO}^{geom}$ is a stable weak equivalence. To show this we need to show that the maps $\operatorname{colim}_i \Omega^i_{HP^1}(\mathbf{BO}^{fin}_{2i+2j})^f \to \operatorname{colim}_i \Omega^i_{HP^1}(\mathbf{BO}^{geom}_{2i+2j})^f$ are weak equivalences for all j. The $(-)^f$ denotes fibrant replacement. This is because we have two $\mathbb{N}^2$-indexed families of spaces

$$E_{n,i} = \begin{cases} \Omega^i_{HP^1}(RGr_{2\cdot 8^i})^f & \text{for even } i, \\ \Omega^i_{HP^1}(HGr'_{2\cdot 8^i})^f & \text{for odd } i, \end{cases}$$
$$E'_{n,i} = \begin{cases} \Omega^i_{HP^1}(HGr'_{2\cdot 8^i})^f & \text{for even } i, \\ \Omega^i_{HP^1}(RGr_{2\cdot 8^i})^f & \text{for odd } i, \end{cases}$$

The inclusions of Grassmannians the F_n and G_n give us maps $E_{n,i} \to E_{n+1,i}$ and $E_{n,i} \to E_{n+1,i+1}$ which commute, and similarly for the $E'_{n,i}$. Thus the $E_{n,i}$ and $E'_{n,i}$ are filtered systems of spaces indexed by a category with set of objects $\mathbb{N}^2$ such that there is a unique arrow $(n,i) \to (n',i')$ if and only if $n \le n'$ and $i - n \le i' - n'$. By cofinality we have isomorphisms $\operatorname{colim}_i E_{i,2i+2j} = \operatorname{colim}_{n,i} E_{n,i} = \operatorname{colim}_i(\operatorname{colim}_n E_{n,i})$ for all j and similarly for $E'_{n,i}$. These are the required weak equivalences for even and odd j respectively.

We now construct an isomorphism $\mathbf{BO}^{geom} \cong \mathbf{BO}_{HP^{1+}}$ in $SH_{HP^{1+}}(S)$. By Theorem 10.3 we have an exact sequence

$$0 \to \varprojlim{}^1 \mathbf{BO}^{4n-1,2n}_{HP^{1+}}(\mathbf{BO}^{geom}_{2n}) \to \mathbf{BO}^{0,0}_{HP^{1+}}(\mathbf{BO}^{geom}) \to \varprojlim \mathbf{BO}^{4n,2n}_{HP^{1+}}(\mathbf{BO}^{geom}_{2n}) \to 0.$$

For every odd $n = 2k+1$ the universal element of Definition 11.1 gives an isomorphism $\tau_{4k+2}\colon \mathbb{Z} \times HGr \cong \mathbf{KO}^{[4k+2]}$ in $H_\bullet(S)$ and by adjunction a $\tau'_{4k+2} \in \mathbf{BO}^{8k+4,4k+2}(\mathbf{BO}^{geom}_{4k+2})$.

The inverse system sends $\tau'_{4k+2} \mapsto \tau'_{4k-2}$ because in the diagram

$$\begin{array}{ccc} (\mathbb{Z} \times HGr) \wedge HP^{1+} \wedge HP^{1+} & \longrightarrow & \mathbb{Z} \times HGr \\ {\scriptstyle \tau_{4k-2}\wedge 1\wedge 1}\downarrow \cong & & {\scriptstyle \tau_{4k+2}}\downarrow \cong \\ \mathbf{KO}^{[4k-2]} \wedge HP^{1+} \wedge HP^{1+} & \longrightarrow & \mathbf{KO}^{[4k+2]} \end{array} \tag{12.13}$$

both horizontal maps are a $\times$ product with $p_1(U)^{\times 2} = ([U] - [\mathsf{H}])^{\times 2}$. So the τ'_{4k+2} define an inverse system $\tau' \in \varprojlim \mathbf{BO}^{8k+4,4k+2}_{HP^1}(\mathbf{BO}^{geom}_{4k+2})$. Since the τ'_{4k+2} are all isomorphisms, any element of $\mathbf{BO}^{0,0}_{HP^1}(\mathbf{BO}^{geom})$ lifting τ' is an isomorphism by general facts about homotopy colimits of sequential direct systems in triangulated categories. □

The inverse system τ' also lies in $\varprojlim \mathbf{BO}^{4i,2i}_{HP^1}(\mathbf{BO}^{geom}_{2i})$. So it also gives us maps $\tau_{4k} \colon \mathbb{Z} \times RGr \to \mathbf{KO}^{[4k]}$ in $H_\bullet(S)$. Essentially these are the compositions of τ_{4k+2} and the maps in the motivic unstable homotopy category induced by the adjoint bonding maps of the two spectra

$$\mathbb{Z} \times RGr \to \Omega_{HP^{1+}}(\mathbb{Z} \times HGr) \xrightarrow{\sim} \Omega_{HP^{1+}}\mathbf{KO}^{[4k+2]} \xleftarrow{\sim} \mathbf{KO}^{[4k]}.$$

We do not know whether these are isomorphisms in $H_\bullet(S)$. The best we know how to do is:

Proposition 12.7. *The morphism $\Omega_{HP^1}(\tau_{4k})$ has a right inverse in $H_\bullet(S)$.*

This is because in the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \times HGr & \longrightarrow & \Omega_{HP^{1+}}(\mathbb{Z} \times RGr) \\ \downarrow \sim & & \downarrow \\ \mathbf{KO}^{[4k-2]} & \xrightarrow{\sim} & \Omega_{HP^{1+}}\mathbf{KO}^{[4k]} \end{array}$$

the arrows on the left side and bottom of the square are weak equivalences by Theorem 8.2 and 7.2 respectively.

13. The commutative monoid structure in $SH(S)$

The main technical result of this section is the following theorem. We use it to show that the almost commutative monoid structure on the T-spectrum $\mathbf{BO}$ we constructed in Theorem 11.5 is actually a commutative monoid for $S = \operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$.

Theorem 13.1. *Let S be a regular noetherian separated $\mathbb{Z}[\frac{1}{2}]$-scheme of finite Krull dimension. Suppose that $KO_1(S)$ and $KSp_1(S)$ are finite. Then for all m the natural map*

$$\mathbf{BO}^{0,0}(\mathbf{BO}^{\wedge m}) \to \varprojlim \mathbf{BO}^{2mi,mi}((\mathbf{KO}^{[i]})^{\wedge m})$$

is an isomorphism.

Proof. We prove the theorem for the HP^1-spectrum $\mathbf{BO}_{HP^1}$. The theorem then follows for the T-spectrum $\mathbf{BO}$.

By Theorem 10.3 and Theorem 12.3 there is a commutative diagram with exact rows

$$\begin{tikzcd}
\varprojlim_i{}^1 \mathbf{BO}^{4mi-1,2mi}_{HP^1}((\mathbf{KO}^{[2i]})^{\wedge m}) \arrow[r, tail] \arrow[d] & \mathbf{BO}^{0,0}_{HP^1}(\mathbf{BO}^{\wedge m}_{HP^1}) \arrow[r, two heads] \arrow[d, "\cong"] & \varprojlim_i \mathbf{BO}^{4mi,2mi}_{HP^1}((\mathbf{KO}^{[2i]})^{\wedge m}) \arrow[d, "\cong \text{ by cofinality}"] \\
\varprojlim_i{}^1 \mathbf{BO}^{4mi-1,2mi}_{HP^1}((\mathbf{BO}^{fin}_{2i})^{\wedge m}) \arrow[r, tail] & \mathbf{BO}^{0,0}_{HP^1}((\mathbf{BO}^{fin})^{\wedge m}) \arrow[r, two heads] & \varprojlim_i \mathbf{BO}^{4mi,2mi}_{HP^1}((\mathbf{BO}^{fin}_{2i})^{\wedge m})
\end{tikzcd}$$

The middle vertical arrow is an isomorphism because the morphism $(\mathbf{BO}^{fin}_{2i})^{\wedge m} \to \mathbf{BO}^{\wedge m}_{HP^1}$ is a stable weak equivalence. The righthand vertical arrow is an isomorphism by a cofinality argument as in the proof of Theorem 12.3. It follows that the lefthand vertical arrow is also an isomorphism.

The $\varprojlim^1$ in the lower row is isomorphic to

$$\varprojlim_{i \text{ odd}}{}^1 KO_1({HGr'_{2N_i}}^{\wedge m}) \qquad \text{or} \qquad \varprojlim_{i \text{ odd}}{}^1 KSp_1({HGr'_{2N_i}}^{\wedge m})$$

for even m and odd m, respectively, where $N_i = 8^{2i}$. By Theorem 9.4 each group in the system is a direct sum of a finite number of copies of $KO_1(S)$ and of $KSp_1(S)$. By the hypothesis it follows that each group in this system is finite, and so the $\varprojlim^1$ in the lower row of the diagram vanishes. Therefore the $\varprojlim^1$ in the upper row also vanishes, proving the isomorphism. □

Theorem 13.2. *Let R be a Euclidean domain. Then we have $KSp_1(R) = 0$.*

This is classical. It is proven essentially by showing that the action of the group $ESp_{2n}(R)$ on unimodular vectors is transitive.

Theorem 13.3. *Let R be a Euclidean domain with $\frac{1}{2} \in R$. Then we have $KO_1(R) \cong \mathbb{Z}/2\mathbb{Z} \times R^\times/R^{\times 2}$.*

This must be very well known to the experts. We include the proof for completeness' sake.

Proof of Theorem 13.3. We use the long exact sequences of Karoubi's fundamental theorem [11, 18]

$$\cdots \to KO_i^{[n]}(R) \xrightarrow{F} K_i(R) \xrightarrow{H} KO_i^{[n+1]}(R) \xrightarrow{\eta} KO_{i-1}^{[n]}(R) \to \cdots .$$

with F the forgetful map and H the hyperbolic map. This amounts to four exact sequences including

$$\cdots \to {}_{-1}V \to K_1 \to KO_1 \to {}_1U \xrightarrow{0} K_0 \hookrightarrow GW^+ \to W^0 \to 0 \tag{13.1}$$

$$\cdots \to KSp_1 \to K_1 \twoheadrightarrow {}_{-1}V \xrightarrow{0} GW^- \hookrightarrow K_0 \to {}_1U \to W^{-1} \to 0 \tag{13.2}$$

with the GW^+ and GW^- the Grothendieck-Witt groups of symmetric and skew-symmetric bilinear forms respectively, the ${}_{-1}V$ and ${}_1U$ the groups of [10, Appendices 2 et 3], and the W^i the Witt groups with cohomological indexing *à la* Balmer.

For a principal ideal domain containing $\frac{1}{2}$ we have $W^i = 0$ for $i \equiv 2$ or $3 \pmod 4$, while the the forgetful map $GW^- \to K_0$ of (13.2) is the inclusion $2\mathbb{Z} \subset \mathbb{Z}$, and the hyperbolic map

$K_0 \to GW^+$ of (13.1) is injective. It then follows from the two sequences that we have a short exact sequence

$$0 \to \operatorname{coker}(K_1 \twoheadrightarrow {}_{-1}V \to K_1) \to KO_1 \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

By [11, §4.1] or [4, §4.5] the composition $K_1 \twoheadrightarrow {}_{-1}V \to K_1$ is $[x] \mapsto [x] - [{}^t\bar{x}^{-1}]$. For a euclidean domain with trivial involution, this is $[x] \mapsto [x^2]$. So we have an exact sequence $1 \to R^\times/R^{\times 2} \to KO_1 \to \mathbb{Z}/2\mathbb{Z} \to 0$. The image of $[b] \in R^\times/R^{\times 2}$ in KO_1 is the class of the auto-isometry $\left[\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right]$ of the hyperbolic quadratic form $q(x_1, x_2) = x_1x_2$. The class of the auto-isometry $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is not in that image because its determinant is -1. It provides a splitting of the surjection in the exact sequence. □

For a nice discussion of the group ${}_1V$ and a bit of the other two exact sequences of Karoubi's fundamental theorem see [4, §4.5].

Theorem 13.4. *Suppose $KO_1(S)$ and $KSp_1(S)$ are finite, for instance $S = \operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$. Let $m \in Hom_{SH(S)}(\mathbf{BO} \wedge \mathbf{BO}, \mathbf{BO})$ be the morphism of* (11.3)*. Let $e \in Hom_{SH(S)}(pt_+, \mathbf{BO}) = \mathbf{BO}^{0,0}(pt_+)$ be the element corresponding to $\langle 1 \rangle \in GW^+(pt) = KO_0^{[0]}(pt)$.*

(a) Then $(\mathbf{BO}, m, e)$ is a commutative monoid in $SH(S)$.

(b) The map m is the unique element of $Hom_{SH(S)}(\mathbf{BO} \wedge \mathbf{BO}, \mathbf{BO})$ defining a pairing which, when restricted to pairing

$$\mathbf{BO}^{4p,2p}(X_+) \times \mathbf{BO}^{4q,2q}(Y_+) \to \mathbf{BO}^{4p+4q,2p+2q}(X_+ \wedge Y_+)$$

with $X, Y \in \mathcal{S}m/S$ coincides with the tensor product pairing

$$KO_0^{[2p]}(X) \times KO_0^{[2q]}(Y) \to KO_0^{[2p+2q]}(X \times Y)$$

of Grothendieck-Witt groups.

Proof. (a) By Theorem 11.5 (B, m, e) is an almost commutative monoid in $SH(S)$. By Definition 10.4 the obstructions to (B, m, e) being a commutative monoid are three classes in the kernels of the maps of Theorem 13.1 for $m = 1, 2, 3$. Those classes vanish.

(b) The product on the $KO_0^{[2p]}(X_+)$ determined uniquely the $\bar{m}$ of (11.2). By Theorem 13.1 m is the unique element of $Hom_{SH(S)}(\mathbf{BO} \wedge \mathbf{BO}, \mathbf{BO})$ mapping onto $\bar{m}$. □

We now wish to use the closed motivic model structure of [15, Appendix A]. Among its properties are:

(1) The closed motivic model structure and the local injective motivic model structure used in §§6–7 have the same weak equivalences, but the closed motivic model structure has fewer cofibrations and more fibrations than the local injective motivic model structure.

(2) A pointed smooth S-scheme (X, x_0) is cofibrant in the closed motivic model structure. More generally, a closed embedding $Z \rightarrowtail X$ in $\mathcal{S}m/S$ induces a cofibration $Z_+ \rightarrowtail X_+$ [15, Lemma A.10].

Hence the pointed scheme HP^{1+} is cofibrant for the closed motivic model structure, so we may define levelwise and stable closed motivic model structures for HP^{1+}-spectra.

(3) For any morphism $u\colon S \to S'$ of noetherian schemes of finite Krull dimension, the pullback $u^*\colon \mathbf{M}_\bullet(S') \to \mathbf{M}_\bullet(S)$ is a strict symmetric monoidal left Quillen functor for the closed motivic model structure [15, Theorem A.17]. Consequently $\mathbf{L}u^*\colon SH(S') \to$

$SH(S)$ can be computed by taking levelwise closed cofibrant replacements and then applying u^*.

To extend m to other base schemes S, we need to discuss base change for morphisms $u\colon S \to S'$. For any $X \in \mathcal{S}m/S'$ there is a duality-preserving pullback functor inducing morphisms of hermitian K-theory spaces $(1 \times u)^*\colon KO^{[n]}(X) \to KO^{[n]}(X \times_{S'} S)$. This gives us maps $KO^{[n]}_{S'} \to u_* KO^{[n]}_S$ and adjoint maps $u^* KO^{[n]}_{S'} \to KO^{[n]}_S$. These maps are compatible with Thom isomorphisms, inducing maps of spectra. The maps $u^*\mathbf{BO}^{geom}_{S'} \to \mathbf{BO}^{geom}_S$ are isomorphisms in $SH(S)$ because u^* acts as base change on the quaternionic and real Grassmannians and their direct colimits. The maps $\mathbf{L}u^*\mathbf{BO}^{geom}_{S'} \to u^*\mathbf{BO}^{geom}_{S'}$ are isomorphisms $SH(S)$ because for the closed motivic model structure $\mathbf{BO}^{geom}_{S'}$ is levelwise cofibrant and u^* is a levelwise left Quillen functor.

Setting $S' = \operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ with $u\colon S \to \operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ the canonical map, we can now define the monoidal structure on $\mathbf{BO}_S$ in $SH(S)$ as in [15, Definition 3.7] as the composition

$$m_S\colon \mathbf{BO}_S \wedge \mathbf{BO}_S \cong u^*\mathbf{BO}_{\mathbb{Z}[\frac{1}{2}]} \wedge u^*\mathbf{BO}_{\mathbb{Z}[\frac{1}{2}]} \cong u^*(\mathbf{BO}_{\mathbb{Z}[\frac{1}{2}]} \wedge \mathbf{BO}_{\mathbb{Z}[\frac{1}{2}]}) \xrightarrow{u^* m_{\mathbb{Z}[\frac{1}{2}]}} u^*\mathbf{BO}_{\mathbb{Z}[\frac{1}{2}]} \cong \mathbf{BO}_S$$

Theorem 13.5. *The assertions of Theorem 1.5 hold.*

For $S = \operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ this is part of Theorem 13.4. For other S it is deduced by base change from $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$.

Theorem 13.6. *The assertions of Theorem 1.1 hold.*

Theorem 5.1 shows that hermitian K-theory is an SL^c-oriented cohomology theory with a partial multiplicative structure. The ring structure is given by Theorem 13.5. The compatibility of the two multiplications is Theorem 11.4.

Schlichting's multiplicative structure, which we mentioned when discussing Theorem 1.6, could replace our partial multiplicative structure for Theorems 1.1, 1.3, 1.4, etc. However, as we understand it, Schlichting's product is defined in unstable homotopy theory. To get our main Theorem 1.5 with the monoid structure for T-spectra, we need our argument with the $\varprojlim^1$.

14. CP^{1+}-spectra $\mathbf{BGL}^{fin}$ and $\mathbf{BGL}^{geom}$ for algebraic K-theory

The HP^{1+}-spectra constructed in §12 have an analogue for ordinary algebraic K-theory: the CP^1-spectra $\mathbf{BGL}^{fin}$ and $\mathbf{BGL}^{geom}$. We sketch their construction. The first can be used to show that the uniqueness results concerning the algebraic K-theory spectrum $\mathbf{BGL}$ and its $\times$ product of [15, Remark 2.19 and Theorem 3.6] hold for any base scheme S which is noetherian of finite Krull dimension with finite $K_1(S)$ and not just for $S = \operatorname{Spec}\mathbb{Z}$.

We use the affine Grassmannians which can be defined as

$$CGr(m,n) = GL_n/(GL_m \times GL_{n-m})$$

or as the open subscheme

$$CGr(m,n) \subset Gr(m,n) \times Gr(n-m,n)$$

where the two tautological subbundles of $\mathcal{O}^{\oplus n}$ are supplementary or as the closed subscheme of the space on $n \times n$ matrices parametrizing projectors of rank m. Each $CGr(m,n)$ is affine over the base scheme and an $\mathbf{A}^{m(n-m)}$-bundle over the ordinary Grassmannnian $Gr(m,n)$. Morphisms $V \to CGr(m,n)$ are in bijection with direct sum decompositions $\mathcal{O}_V^{\oplus n} = U'_m \oplus$

U''_{n-m} with U'_m and U''_{n-m} subbundles of ranks m and $n-m$ respectively. We let $CGr = \operatorname{colim}_n CGr(n,2n)$.

In particular $CP^1 = CGr(1,2) \cong \mathbf{P}^1 \times \mathbf{P}^1 - \Delta$ is an $\mathbf{A}^1$-bundle over $\mathbf{P}^1$. We may point $CGr(1,2)$ by $CGr(0,0)$. Let CP^{1+} then be the pointed scheme constructed in (12.3). The motivic stable homotopy categories of $\mathbf{P}^1$-spectra, of CP^1-spectra and of CP^{1+}-spectra are equivalent. In particular there is a CP^{1+}-spectrum $\mathbf{BGL}_{CP^{1+}}$ corresponding to the $\mathbf{P}^1$-spectrum $\mathbf{BGL}$ of [15]. For any smooth S-scheme X we write $n = n[\mathcal{O}_X] \in K_0(X)$.

Lemma 14.1. *There exist morphisms of pointed schemes*

$$h_n \colon \big([-n,n] \times CGr(n,2n)\big) \times CP^1 \to CGr(4n,8n)$$

such that the classes in K_0 satisfy

$$h_n^*([U'_{4n}] - 4n) = ([U'_n] - (n-i)) \boxtimes ([U'_1] - 1) \tag{14.1}$$

(where $i \in [-n,n] \subset \mathbb{Z}$ is the index of the component) and such that $h_n|_{pt \times CP^1}$ is constant, and $h_n|_{([-n,n]\times CGr(n,2n))\times pt}$ is pointed $\mathbf{A}^1$-homotopic to a constant map. Moreover, these maps and homotopies are compatible with the inclusions $CGr(n,2n) \hookrightarrow CGr(n+1,2(n+1))$ and $CGr(4n,8n) \hookrightarrow CGr(4(n+1),8(n+1))$.

This lemma is proven in the same way as Lemma 12.5 using the equality

$$([U'_n] - (n-i)) \boxtimes ([U'_1] - 1) = [U'_n \boxtimes U'_1] + [\mathcal{O}^{\oplus n-i} \boxtimes U''_1] + [U''_n \boxtimes \mathcal{O}] - (3n-i)[\mathcal{O} \boxtimes \mathcal{O}]$$

in $K_0([-n,n] \times CGr(n,2n))$ and the direct sum decompositions of vector bundles

$$(U'_n \boxtimes U'_1) \oplus (U''_n \boxtimes U'_1) = \mathcal{O}^{\oplus 2n} \boxtimes U'_1,$$
$$(\mathcal{O}^{\oplus n-i} \boxtimes U''_1) \oplus (\mathcal{O}^{\oplus n+i} \boxtimes U''_1) = \mathcal{O}^{\oplus 2n} \boxtimes U''_1,$$
$$(U''_n \boxtimes \mathcal{O}) \oplus (U'_n \boxtimes \mathcal{O}) = \mathcal{O}^{\oplus 2n} \boxtimes \mathcal{O},$$
$$(\mathcal{O}^{\oplus n+i} \boxtimes \mathcal{O}) \oplus (\mathcal{O}^{\oplus n-i} \boxtimes \mathcal{O}) = \mathcal{O}^{\oplus 2n} \boxtimes \mathcal{O},$$

yielding a decomposition of the trivial bundle of rank $8n$ on $([-n,n] \times CGr(n,2n)) \times CP^1$ as the direct sum of two subbundles of rank $4n$.

Theorem 14.2. *There are CP^{1+}-spectra $\mathbf{BGL}^{fin}$ and $\mathbf{BGL}^{geom}$ isomorphic to $\mathbf{BGL}_{CP^1}$ in $SH_{CP^1}(S)$ with spaces*

$$\mathbf{BGL}_n^{fin} = [-4^n, 4^n] \times CGr(4^n, 2 \cdot 4^n) \qquad \mathbf{BGL}_n^{geom} = \mathbb{Z} \times CGr$$

which are unions of affine Grassmannians. The bonding maps $\mathbf{BGL}_n^* \wedge CP^{1+} \to \mathbf{BGL}_{n+1}^*$ of the two spectra are morphisms of schemes or ind-schemes which are constant on the wedge $\mathbf{BGL}_n^* \vee CP^{1+}$.

This theorem is proven in the same way as Theorem 12.3.

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