PRESCRIBED BEHAVIOR OF CENTRAL SIMPLE ALGEBRAS AFTER SCALAR EXTENSION

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ABSTRACT. 1. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be central simple disjoint algebras over a field F. Let also $l_i | \exp(\mathcal{A}_i)$, $m_i | \mathrm{ind}(\mathcal{A}_i)$, $l_i | m_i$, and for each $i = 1, \dots, n$, let l_i and m_i have the same sets of prime divisors. Then there exists a field extension E/F such that $\exp(\mathcal{A}_{i E}) = l_i$ and $\mathrm{ind}(\mathcal{A}_{i E}) = m_i$, $i = 1, \dots, n$.

2. Let $\mathcal{A}$ be a central simple algebra over a field K with an involution τ of the second kind. We prove that there exists a regular field extension E/K preserving indices of central simple K-algebras such that $\mathcal{A} \otimes_K E$ is cyclic and has an involution of the second kind extending τ.

Introduction and motivations

This paper is a continuation of [18] where some properties of central simple algebras after scalar extensions were examined. In [18] we solved two problems.

1. For a given central simple K-algebra $\mathcal{A}$, some K-variety X was constructed such that for a field extension L/K the variety X has an L-rational point iff $\mathcal{A} \otimes_K L$ has some prescribed properties (e.g., being a symbol-algebra).

2. For a given central simple K-algebra $\mathcal{A}$, a regular field extension E/K was constructed preserving indices of all central simple K-algebras, such that $\mathcal{A} \otimes_K E$ becomes a cyclic algebra.

Note that if a field extension E/K preserves indices of all central simple K-algebras then E/K preserves exponents for all such K-algebras, but in some applications one needs to reduce exponents and indices of algebras in a prescribed manner.

Below we fix the following notations and conventions. Let $\mathcal{A}$ be a finite dimensional central simple algebra over a field F. By Wedderburn's theorem, there is a unique integer $m \geq 1$ and a central division F-algebra $\mathcal{D}$ which is unique up to F-isomorphism such that $\mathcal{A} \cong \mathrm{M}_m(\mathcal{D})$. The degree of $\mathcal{A}$ is defined by $\deg(\mathcal{A}) = \sqrt{\dim_F \mathcal{A}}$, the index of $\mathcal{A}$ is said to be $\mathrm{ind}(\mathcal{A}) = \deg(\mathcal{D})$.

Two central simple F-algebras $\mathcal{A} = \mathrm{M}_m(\mathcal{D})$ and $\mathcal{A}' = \mathrm{M}_{m'}(\mathcal{D}')$ are said to be Brauer equivalent if $\mathcal{D} \cong \mathcal{D}'$. In this case we write $\mathcal{A} \sim \mathcal{A}'$ and denote the equivalence class of $\mathcal{A}$ by $[\mathcal{A}]$. The tensor product of central simple algebras defines an abelian group structure on this set of equivalence classes, called the Brauer group of F and denoted by $\mathrm{Br}(F)$. The inverse of the class $[\mathcal{A}]$ is induced by the opposed algebra $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{A}$. $\mathcal{A}^m$ will denote the central simple algebra $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$ (m times).

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The neutral element is defined by the class $\mathcal{A} \sim F$, in this case we write $\mathcal{A} \sim 1$. The exponent $\exp(\mathcal{A})$ of $\mathcal{A}$ in $\mathrm{Br}(F)$ is the order of $[\mathcal{A}]$ in $\mathrm{Br}(F)$. It is known that $\exp(\mathcal{A})$ and $\mathrm{ind}(\mathcal{A})$ have the same prime divisors and $\exp(\mathcal{A})|\mathrm{ind}(\mathcal{A})$ [17, §14.4, Prop. b].
For a field extension K/F, $\mathcal{A}_K$ will denote the K-algebra $\mathcal{A} \otimes_F K$. If $[K : F]$ is coprime to $\mathrm{ind}(\mathcal{A})$, then $\mathrm{ind}(\mathcal{A}_K) = \mathrm{ind}(\mathcal{A})$ [17, §13.4, Prop.].
Let us recall three special types of central simple algebras:
Crossed products $(L/F, \mathrm{Gal}(L/F), f)$. Let L/F be a Galois field extension, $\mathrm{Gal}(L/F)$ its Galois group and f a 2-cocycle of $\mathrm{Gal}(L/F)$ with values in L^*. Then the left L-module with L-base $\{u_\tau\}_{\tau \in \mathrm{Gal}(L/F)}$ and multiplication table

$$u_s l = l^s u_s \text{ for } l \in L, \quad u_s u_t = f(s,t) u_{st} \text{ for any } s, t \in \mathrm{Gal}(L/F)$$

is a central simple F-algebra and denoted by $(L/F, \mathrm{Gal}(L/F), f)$.
Cyclic algebras $(E/F, \sigma, a)$. They are a special form of crossed products. Let E/F be a cyclic field extension of degree n, σ a generator of $\mathrm{Gal}(E/F)$ and $a \in F^*$. Then $(E/F, \sigma, a)$ is a left E-module with E-base $\{u_\sigma^i\}_{i=1,\dots,n}$ and multiplication table:

$$u_\sigma^i c = c^{\sigma^i} u_\sigma^i$$

and

$$u_\sigma^n = a$$

for any $i = 0, \dots, n-1$ and $c \in E$. The corresponding cocycle is the following

$$c_a(\sigma^i, \sigma^j) = \begin{cases} 1, & \text{if } i + j < [E : F]; \\ a, & \text{if } i + j \geq [E : F]. \end{cases}$$

Symbol algebras $(a, b)_n$. These algebras also have a simple set of generators and defining relations. Let $\rho_n \in F$ be a primitive root of unity of degree n and $a, b \in F^*$. Then $(a, b)_n$ is an n^2-dimensional vector F-space with an F-base

$$\{A^i B^j\}_{i,j=1,\dots,n}$$

and multiplication table

$$A^i B^j = \rho_n^{ij} B^j A^i, \; A^n = a, B^n = b.$$

Following some arguments from [12] we prove in this paper, for disjoint algebras (see the Definition 1.1 below), the following

Theorem 1. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be central simple disjoint algebras over F. Let also $l_i|\exp(\mathcal{A}_i)$, $m_i|\mathrm{ind}(\mathcal{A}_i)$, $l_i|m_i$ such that, for each $i = 1, \dots, n$, both numbers l_i, m_i have the same prime divisors. Then there exists a regular finitely generated field extension E/F such that $\exp(\mathcal{A}_{iE}) = l_i$ and $\mathrm{ind}(\mathcal{A}_{iE}) = m_i$, $i = 1, \dots, n$.*

The remaining part of the paper is devoted to algebras with involutions. Using ideas similar to those in [18] we prove the following

Theorem 2. *Let $\mathcal{A}$ be a central simple algebra over a field K with an involution τ of the second kind. Then there exists a regular field extension E/K preserving indices of central simple K-algebras such that $\mathcal{A}_E$ is cyclic and has an involution of the second kind extending τ.*

In particular, this theorem has applications to a unitary variant of Suslin's conjecture. To formulate this conjecture we will recall a few notions. The notion of R-equivalence in the set $X(F)$ of F-points of an algebraic variety defined over a field F was introduced by Manin in [14] and studied firstly for linear algebraic groups by Colliot-Thélène and Sansuc in [7] (See also [5], [8], [16], [22].) It is an important birational invariant of an algebraic variety defined over an arbitrary field F. For an algebraic group G defined over a field F, the subgroup $RG(F)$ of R-trivial elements in the group $G(F)$ of all F points is defined as follows. An element g belongs to $RG(F)$ if there is a rational morphism $f : \mathbb{A}^1_n \to G$ over F, defined at the points 0 and 1 such $f(0) = 1$ and $f(1) = g$. In other words, g can be connected with the identity of the group by the image of a rational curve. The subgroup $RG(F)$ is normal in $G(F)$ and the factor group $G(F)/RG(F) = G(F)/R$ is called the group of R-equivalence classes. The group G is called R-trivial, if the group of R-equivalence classes $G(L)/R$ is trivial for any field extension L/F.
Let K/F be a quadratic field extension and let $\mathcal{A}$ be a central simple algebra over K with an involution τ of the second kind trivial on F. Let $\mathrm{U}(\mathcal{A},\tau)$ be the unitary group of $\mathcal{A}$. Let also $\mathrm{SU}(\mathcal{A},\tau)$ be the special unitary group, that is, the set of elements of $\mathrm{U}(\mathcal{A},\tau)$ with reduced norm 1. It is known that if $\mathrm{ind}(\mathcal{A})$ is square-free, then $\mathrm{SU}(\mathcal{A},\tau)/R = 1$ ([5], [16], [24], [25], [26], [27]). In the case $\mathrm{ind}(\mathcal{A})$ is not square-free, a unitary variant of Suslin's conjecture states that the group $\mathrm{SU}(\mathcal{A},\tau)$ is not R-trivial. Since $\mathrm{SU}(A,\tau)_K \cong \mathrm{SL}_{1,A}$, it follows that this conjecture is true if $\mathrm{ind}(\mathcal{A})$ is divisible by 4 ([5, Remark 6.6]). The latter isomorphism says also that Suslin's conjecture about reduced Whitehead groups implies the conjecture above. Thus Theorem 2 allows to reduce the conjecture about special unitary groups to algebras of a special type.

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1. Reducing exponent

In this section we show that for disjoint algebras the exponents and indices can be reduced in a prescribed manner over some field extension.
We need the following definitions and facts.

Definition 1.1. ([11, Def.2.5]) *The central simple F-algebras $\mathcal{A}_1,\dots,\mathcal{A}_n$ are called disjoint if*

$$\mathrm{ind}(\mathcal{A}_1^{j_1}\otimes\cdots\otimes\mathcal{A}_n^{j_n}) = \mathrm{ind}(\mathcal{A}_1^{j_1})\dots\mathrm{ind}(\mathcal{A}_n^{j_n})$$

for all $j_1,\dots,j_n$.

Proposition 1.2. *Let $\mathcal{A}$ be a central simple algebra over a field F and E the function field of the generalized Severi-Brauer variety $\mathrm{SB}_n(\mathcal{A})$, $n \le \deg(\mathcal{A})$. Then*

- (i) ([2, Th.7]) *the relative Brauer group $\mathrm{Br}(E/F)$ is generated by the class of $\mathcal{A}^n$ in $\mathrm{Br}(F)$;*
- (ii) ([2, Th.3]) $\mathrm{ind}(\mathcal{A}\otimes_F E) = \gcd(n, \mathrm{ind}(\mathcal{A}))$.

Proposition 1.3. ([23, Th.2]) *Let $\mathcal{A}$ be a central simple algebra over a field F,* $\deg(\mathcal{A}) = d$ *and $s|d$. Let also E be the function field of the generalized Severi-Brauer variety* $\mathrm{SB}_s(\mathcal{A})$*. Then for any central simple F-algebra $\mathcal{D}$,*

$$\begin{aligned}\operatorname{ind}(\mathcal{D} \otimes_F E) &= \gcd\left\{\frac{s}{\gcd(i,s)} \operatorname{ind}(\mathcal{D} \otimes_F \mathcal{A}^{-i}) \;\Big|\; 1 \le i \le d\right\}\\ &= \min\left\{\frac{s}{\gcd(i,s)} \operatorname{ind}(\mathcal{D} \otimes_F \mathcal{A}^{-i}) \;\Big|\; 1 \le i \le d\right\}.\end{aligned}$$

Remark: This was proved in [23, Th.2]), however, the fact that the gcd is actually a min (which is more or less obvious in our examples below) has been pointed out in [15, see (0.3), p. 520 and (5.11), p. 565].

In order to prove Theorem 1 we need the following preliminary

Proposition 1.4. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be central simple algebras over F,* $\deg(\mathcal{A}_i) = d_i$*, $i = 1, \dots, n$, and $s_i|d_i$. Let also E_i be the function field of the generalized Severi-Brauer variety* $\mathrm{SB}_{s_i}(\mathcal{A}_i)$*, $i = 1, \dots, n$, and $E_1 \cdots E_n$ the free composite over F. Then for any central simple F-algebra $\mathcal{D}$,*

$$\begin{aligned}&\operatorname{ind}(\mathcal{D} \otimes_F E_1 \cdots E_n) =\\ &\quad = \gcd\left\{\frac{s_1}{\gcd(j_1,s_1)} \cdots \frac{s_n}{\gcd(j_n,s_n)} \operatorname{ind}(\mathcal{D} \otimes_F \mathcal{A}_1^{-j_1} \otimes_F \cdots \otimes_F \mathcal{A}_n^{-j_n}) \;\Big|\; 1 \le j_i \le d_i\right\}.\end{aligned}$$

Proof. We will use induction on n. In the case $n = 1$ the statement follows from Proposition 1.3.
Suppose that the statement of proposition is true for $n = n_0$, i.e., for any field K and central simple K-algebras $\mathcal{B}, \mathcal{C}_1, \dots, \mathcal{C}_{n_0}$, $c_i|\deg(\mathcal{C}_i)$, $1 \le i \le n_0$, the following holds:

$$\begin{aligned}&\operatorname{ind}(\mathcal{B} \otimes_K L) =\\ &= \gcd\left\{\frac{c_1}{\gcd(j_1,c_1)} \cdots \frac{c_{n_0}}{\gcd(j_{n_0},c_{n_0})} \operatorname{ind}(\mathcal{B} \otimes_K \mathcal{C}_1^{-j_1} \otimes_K \cdots \otimes_K \mathcal{C}_{n_0}^{-j_{n_0}}) \;\Big|\; 1 \le j_i \le \deg(\mathcal{C}_i)\right\}\end{aligned}$$

where L is the free composite over K of the function fields of generalized Severi-Brauer varieties $\mathrm{SB}_{c_i}(\mathcal{C}_i)$, $1 \le i \le n_0$.
Consider the case $n = n_0 + 1$. By Proposition 1.3,

$$\begin{aligned}&\operatorname{ind}(\mathcal{D} \otimes_F E_1 \cdots E_{n_0+1}) =\\ &= \gcd\left\{\frac{s_{n_0+1}}{\gcd(j_{n_0+1}, s_{n_0+1})} \operatorname{ind}(\mathcal{D}_{E_1 \cdots E_{n_0}} \otimes_{E_1 \cdots E_{n_0}} \mathcal{A}_{n_0+1}^{-j_{n_0+1}}{}_{E_1 \cdots E_{n_0}}) \;\Big|\; 1 \le j_{n_0+1} \le d_{n_0+1}\right\}\end{aligned}$$

By induction hypothesis, for a fixed j_{n_0+1},

$$\begin{aligned}&\operatorname{ind}(\mathcal{D}_{E_1 \cdots E_{n_0}} \otimes_{E_1 \cdots E_{n_0}} \mathcal{A}_{n_0+1}^{-j_{n_0+1}}{}_{E_1 \cdots E_{n_0}}) =\\ &= \gcd\left\{\frac{s_1}{\gcd(j_1,s_1)} \cdots \frac{s_{n_0}}{\gcd(j_n,s_{n_0})} \operatorname{ind}(\mathcal{D} \otimes_F \mathcal{A}_{n_0+1}^{-j_{n_0+1}} \otimes_F \mathcal{A}_1^{-j_1} \otimes_F \cdots \otimes_F \mathcal{A}_{n_0}^{-j_{n_0}}) \;\Big|\; 1 \le j_i \le d_i\right\}.\end{aligned}$$

Combining the latter formulas we obtain the statement of the proposition. $\square$

Now we are in a position to prove Theorem 1.
Proof of Theorem 1. Let E_i be the function field of the generalized Severi-Brauer variety $\mathrm{SB}_{m_i}(\mathcal{A}_i)$, and let F_i be the function field of the Severi-Brauer variety $\mathrm{SB}(\mathcal{A}_i^{l_i}) =$

$\mathrm{SB}_1(\mathcal{A}_i^{l_i})$. By Proposition 1.4,

$$\operatorname{ind}(\mathcal{A}^j_{1E_2F_2\cdots E_nF_n}) = $$
$$= \gcd\{ \frac{m_2}{\gcd(j_2,m_2)}\cdots\frac{m_n}{\gcd(j_n,m_n)} \operatorname{ind}(\mathcal{A}_1^j\otimes\mathcal{A}_2^{-j_2'-l_2j_2}\otimes\cdots\otimes\mathcal{A}_n^{-j_n'-l_nj_n}) \mid 1\le j_i'\le\deg(\mathcal{A}_i), 1\le j_i\le\deg(\mathcal{A}_i^{l_i})\ \}$$

for all j. Now,

$$\operatorname{ind}(\mathcal{A}_1^j\otimes\mathcal{A}_2^{-j_2'-l_2j_2}\otimes\cdots\otimes\mathcal{A}_n^{-j_n-l_nj_n}) = \operatorname{ind}(\mathcal{A}_1^j)\operatorname{ind}(\mathcal{A}_2^{-j_2'-l_2j_2})\ldots\operatorname{ind}(\mathcal{A}_n^{-j_n'-l_nj_n})$$

since the algebras $\mathcal{A}_1,\ldots,\mathcal{A}_n$ are disjoint. Hence $\operatorname{ind}(\mathcal{A}^j_{1E_2F_2\cdots E_nF_n}) = \operatorname{ind}(\mathcal{A}_1^j)$ for every j and therefore $\exp(\mathcal{A}_{1E_2F_2\cdots E_nF_n}) = \exp(\mathcal{A}_1)$.
Let $\operatorname{ind}(\mathcal{A}_1)$ have the prime power decomposition $\operatorname{ind}(\mathcal{A}_1) = \prod p^{\nu_p(\operatorname{ind}(\mathcal{A}_1))}$. By [12, Lemma 1.3], we obtain

$$\exp(\mathcal{A}_{1F_1E_2F_2\cdots E_nF_n}) = l_1$$

and

$$\operatorname{ind}(\mathcal{A}_{1F_1E_2F_2\cdots E_nF_n}) = \prod_{p\mid l_1} p^{\nu_p(\operatorname{ind}(\mathcal{A}_1))} = \prod_{p\mid m_1} p^{\nu_p(\operatorname{ind}(\mathcal{A}_1))}, \qquad (*)$$

the latter equation being true because the prime divisors of l_1 and m_1 are the same.
We define $E := E_1F_1E_2F_2\cdots E_nF_n$, $E' := F_1E_2F_2\cdots E_nF_n$, and apply 1.2 to the extension E/E', using the variety $\mathrm{SB}_{m_1}(A_{E'}) = \mathrm{SB}_{m_1}(A_1)\times_F E'$.
By 1.2 (i), we get $\mathrm{Br}(E/E') = \langle[\mathcal{A}_{1E'}^{m_1}]\rangle$.
Since $l_1 = \exp(\mathcal{A}_{1E'}) \mid m_1$, the latter group is trivial and hence the restriction map $\mathrm{Br}(E')\longrightarrow\mathrm{Br}(E)$ is injective. Therefore $\exp(\mathcal{A}_{1E}) = \exp(\mathcal{A}_{1E_2'}) = l_1$.
By 1.2 (ii) and by equation $(*)$ above, we obtain $\operatorname{ind}(\mathcal{A}_{1E}) = \gcd(m_1, \operatorname{ind}(A_{1E'})) = m_1$.
In view of symmetry we obtain the same results for algebras $\mathcal{A}_i$, $2\le i\le n$. □

2. Algebras after a scalar extension

The main ingredient of the proof of Theorem 2 is the following statement obtained in [18, Th. 2.11]).

Theorem 2.1. *Let $\mathcal{A}$ be a central simple algebra over a field F. Then there exists a regular field extension M/F such that*

(i) *$\mathcal{A}_M$ is cyclic,*
(ii) *for any central simple F-algebra $\mathcal{C}$, $\operatorname{ind}(\mathcal{C}_M) = \operatorname{ind}(\mathcal{C})$,*
(iii) *for any central simple F-algebra $\mathcal{C}$, $\exp(\mathcal{C}_M) = \exp(\mathcal{C})$,*
(iv) *the restriction map $\operatorname{res}: \mathrm{Br}(F)\longrightarrow\mathrm{Br}(M)$ is an injection.*

For the reader's convenience, we present a modified proof here. The original proof in [18]) based on a technical construction of a tower of field extensions with certain properties ([18, Lemmas 2.5 and 2.6]). Lemma 2.9 below allows to avoid these difficulties and leads to a slight generalization (see Theorem 2.11 below).
In order to prove Theorem 2.1 we need a few preliminary statements.

Proposition 2.2. ([3, Th. 1.3], [20, Th. 13.10]) *Let $\mathcal{D}$, $\mathcal{E}$ be central division algebras over F of indices m and n respectively. Let $\mathrm{SB}(\mathcal{E})$ be the Severi-Brauer variety of $\mathcal{E}$ and let K be its function field. Then*

$$\operatorname{ind}(\mathcal{D}\otimes_F K) = \gcd\{\operatorname{ind}(\mathcal{D}\otimes_F\mathcal{E}^i)\}$$

where i ranges from 1 to $\exp(\mathcal{E})$.

Remark 2.3. In the literature the latter formula is called the index reduction formula.

Corollary 2.4. *Let $\mathcal{D}$, $\mathcal{E}$ be central division algebras over F. Let K be the function field of the Severi-Brauer variety* $\mathrm{SB}(\mathcal{E})$. *Assume that* $\mathrm{ind}(\mathcal{D})$ *is coprime to* $\mathrm{ind}(\mathcal{E})$. *Then* $\mathrm{ind}(\mathcal{D} \otimes_F K) = \mathrm{ind}(\mathcal{D})$.

Proof. Use the index reduction formula. □

Lemma 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be central simple F-algebras. Assume* $\mathrm{ind}(\mathcal{A}) = p^m$ *and* $\mathrm{ind}(\mathcal{B}) = p^n$. *Then* $\mathrm{ind}(\mathcal{A} \otimes_F \mathcal{B}) \geq p^{|m-n|}$.

Proof. Assume for definiteness that $m \geq n$. Let E/F be a field extension of degree p^n which splits $\mathcal{B}$. Let also $\mathrm{ind}(\mathcal{A} \otimes_F \mathcal{B}) = p^s$. Assume $p^s < p^{m-n}$. Then there exists a field extension L/F of degree p^s splitting $\mathcal{A} \otimes_F \mathcal{B}$. Hence

$$1 \sim (\mathcal{A} \otimes_F \mathcal{B})_{EL} \sim \mathcal{A}_{EL} \otimes_{EL} \mathcal{B}_{EL} \sim \mathcal{A}_{EL}.$$

Thus EL is a splitting field of $\mathcal{A}$. Since $[EL : F] < p^m$, then $\mathrm{ind}(\mathcal{A}) < p^m$. Contradiction. □

Lemma 2.6. *Let $\mathcal{A}$ be a central simple F-algebra with* $\mathrm{ind}(\mathcal{A}) = p^m$. *Then* $\mathrm{ind}(\mathcal{A}^{p^t}) \leq p^{m-t}$ *for* $0 \leq t \leq m$.

Proof. Without loss of generality we can assume that there exists a splitting field L of $\mathcal{A}$ such that $[L : F] = \mathrm{ind}(\mathcal{A})$ and L contains a subfield K with $[L : K] = p$. (see e.g. [1, Ch.IV, Th.31]). Then $\mathrm{ind}(\mathcal{A}_K) = p$. Hence $1 = \mathrm{ind}(\mathcal{A}_K^p)$. Thus $\mathrm{ind}(\mathcal{A}^p) \leq [K : F] < [L : F] = \mathrm{ind}(\mathcal{A})$. The formula now follows by induction. □

Lemma 2.7. *Let K/F be a cyclic field extension,* $\langle\sigma\rangle = \mathrm{Gal}(K(z)/F(z))$, *and let z be transcendental over F. Also let $\mathcal{C}$ be a central division F-algebra such that $\mathcal{C}_K$ is a division algebra. Then*

$$(K(z)/F(z), \sigma, z) \otimes \mathcal{C}_{F(z)}$$

is a division $F(z)$-algebra.

Proof. See [17, §19.6, Prop.] □

In the notations of the previous lemma we have immediately the following

Corollary 2.8. (i) *For any $j \geq 1$, the algebra $(K(z)/F(z), \sigma, z^j)$ is Brauer-equivalent to $(K'(z)/F(z), \tau, z)$ for some $K' \subset K$ and some generator τ of* $\mathrm{Gal}(K'/F)$.

(ii) *Let $\mathcal{A}$ be a central simple F-algebra such that* $\mathrm{ind}(\mathcal{A}_K) = \mathrm{ind}(\mathcal{A})$. *Then for any $j \geq 1$,*

$$\mathrm{ind}\big(\,(K(z)/F(z), \sigma, z^j) \otimes \mathcal{A}_{F(z)}\big) = \mathrm{ind}((K(z)/F(z), \sigma, z^j)) \cdot \mathrm{ind}(\mathcal{A}).$$

Proof.
(i) Let $n := [K : F]$, $d := \gcd(j, n)$, $n' =: n/d$, and $j' := j/d$. As j' is relatively prime to n', there is a natural number j'' such that $j'j'' \equiv 1 \mod n'$.
Moreover, let K'/F be the subfield of K such that $[K' : F] = n'$.
We obtain

$$\begin{array}{rcll} (K(z)/F(z), \sigma, z^j) & = & (K(z)/F(z), \sigma, z^{j'd})) & \\ & \cong & (K'(z)/F(z), \sigma|_{K'(z)}, z^{j'}) & \text{by [17, §15.1, Cor. b]} \\ & \sim & (K'(z)/F(z), \sigma^{j''}|_{K'(z)}, z) & \text{by [17, §15.1, Cor. a (i)],} \end{array}$$

and of course $\tau = \sigma^{j''}|_{K'(z)}$ generates $\mathrm{Gal}(K'(z)/F(z))$.
(ii) Since $\mathrm{ind}(\mathcal{A}_K) = \mathrm{ind}(\mathcal{A})$, then $\mathrm{ind}(\mathcal{A}_{K'}) = \mathrm{ind}(\mathcal{A})$ and we may apply Lemma 2.7 to the algebra $(K'(z)/F(z), \tau, z) \otimes \mathcal{A}_{F(z)}$ obtained from (i). $\square$

Lemma 2.9. *Let F be a field and G a finite group. Then there exists a tower of field extensions*

$$F \subset K \subset E$$

such that

(i) *E/F is a finitely generated purely transcendental extension;*
(ii) *E/K is Galois with the group G;*
(iii) *for any central simple F-algebra $\mathcal{C}$, $\mathrm{ind}(\mathcal{C}_E) = \mathrm{ind}(\mathcal{C})$.*

Proof. Let E/F be a purely transcendental extension of degree $|G|$ with algebraically independent variables x_g, $g \in G$. Define an action of G on E as follows. For $h \in G$, $h(x_g) = x_{hg}$ and h is trivial on F. Let $K = E^G$ be the subfield of fixed elements. Then E/K is Galois with the group G. Moreover, since E/F is purely transcendental, then E preserves indices of central simple F-algebras. $\square$

Remark 2.10. Our proof for this Lemma in [18] was very technical and did work only for finite cyclic groups. We have to thank J.-L. Colliot-Thélène, who provided us with a much simpler and more elegant proof which works for arbitrary finite groups. Our argument above is a further simplification of his suggestion.

Proof of Theorem 2.1.
Let $\deg(\mathcal{A}) = n$. It follows from Lemma 2.9 that there exists a tower of field extensions $F \subset K \subset E$ such that E/F is a finitely generated purely transcendental extension, E/K is cyclic of degree n and E preserves indices of central simple F-algebras.
Consider the cyclic algebra

$$\mathcal{D} = (E(z)/K(z), \sigma, z),$$

where $\langle\sigma\rangle = \mathrm{Gal}(E(z)/K(z))$ and z is a transcendental variable. The algebra $\mathcal{D}$ is of exponent and index n with a maximal subfield $E(z)$.
One has

$$\mathcal{D} \sim \mathcal{D} \otimes_{K(z)} \mathcal{A}^{\mathrm{op}}_{K(z)} \otimes_{K(z)} \mathcal{A}_{K(z)}.$$

Let M be the function field of the Severi-Brauer variety $\mathrm{SB}(\mathcal{D} \otimes_{K(z)} \mathcal{A}^{\mathrm{op}}_{K(z)})$. Then $\mathcal{A}_M \sim \mathcal{D}_M$. Since $\deg(\mathcal{A}_M) = \deg(\mathcal{D}_M)$, then $\mathcal{A}_M \cong \mathcal{D}_M$.
Let $\mathcal{C}$ be a central simple F-algebra and $\mathcal{C} = \otimes_{i=1}^m \mathcal{C}_i$ the decomposition of $\mathcal{C}$ as a tensor product of algebras of relatively prime primary indices. Since $\mathrm{ind}(\mathcal{C}_M) =$

$\prod_{i=1}^{m} \mathrm{ind}(\mathcal{C}_{iM})$, then to prove the statement about preserving indices it is enough to consider the case where $\mathrm{ind}(\mathcal{C}) = p^m$ for a power of some prime p.
Using the index reduction formula we obtain

$$\mathrm{ind}(\mathcal{C}_M) = \gcd\{\mathrm{ind}(\mathcal{D}^j \otimes_{K(z)} \mathcal{A}_{K(z)}^{\mathrm{op}\,j} \otimes_{K(z)} \mathcal{C}_{K(z)})\}$$

where j ranges from 1 to n.
Consider the algebra $\mathcal{B}_j = \mathcal{D}_p^j \otimes_{K(z)} \mathcal{A}_{pK(z)}^{\mathrm{op}\,j} \otimes_{K(z)} \mathcal{C}_{K(z)}$, where $\mathcal{D}_p$ and $\mathcal{A}_p$ are p-primary parts of algebras $\mathcal{D}$ and $\mathcal{A}$. Let $p^k = \mathrm{ind}(\mathcal{D}_p)$ and $p^l = \mathrm{ind}(\mathcal{A}_p)$. Note that $k \geq l$. Since $\mathrm{ind}(\mathcal{C})$ is a power of p, then $\mathrm{ind}(\mathcal{C}_M) = \min_{j=1}^{n}\{\mathrm{ind}(\mathcal{B}_j)\}$.
Fix some j. Let $j = p^t j_1$, where p does not divide j_1. Then $\exp(\mathcal{D}_p^j) = p^{k-t}$. Hence $\mathrm{ind}(\mathcal{D}_p^j) = p^{k-t}$ in view of Lemma 2.6. Let $\mathrm{ind}(\mathcal{A}_p^{\mathrm{op}\,j}) = p^s$. Then $s \leq l - t$ by Lemma 2.6. Note that by Corollary 2.8,

$$\mathrm{ind}(\mathcal{B}_j) = \mathrm{ind}(\mathcal{D}_p^j)\mathrm{ind}(\mathcal{A}_{pK}^{\mathrm{op}\,j} \otimes_K \mathcal{C}_K).$$

In view of Lemma 2.5,

$$\mathrm{ind}(\mathcal{B}_j) \geq p^{k-t} p^{|s-m|} = p^{k-t+|s-m|}.$$

Finally consider two cases.
(i) $s \geq m$. Then $k - t \geq l - t \geq s \geq m$ and $k - t + |s - m| \geq m$.
(ii) $s < m$. Then $k - t + |s - m| = k - t - s + m \geq l - t - s + m \geq m$.
Therefore, $\mathrm{ind}(\mathcal{B}_j) \geq p^m = \mathrm{ind}(\mathcal{C})$ for any j. Thus $\mathrm{ind}(\mathcal{C}_M) = \mathrm{ind}(\mathcal{C})$.
Note that, for a field extension M/F, preserving indices for all central simple F-algebras implies also preserving exponents of central simple F-algebras. Indeed, assume $\mathcal{C}_M^m \sim 1$ for some central simple F-algebra $\mathcal{C}$. Since

$$1 = \mathrm{ind}(\mathcal{C}_M^m) = \mathrm{ind}(\mathcal{C}^m),$$

then $\mathcal{C}^m \sim 1$. Thus $\exp(\mathcal{C}_M) = \exp(\mathcal{C})$. Moreover, preserving exponents implies, in turn, that the restriction homomorphism

$$\mathrm{res} : \mathrm{Br}(F) \longrightarrow \mathrm{Br}(M)$$

is an embedding. □

We have also the following kind of generalization of Theorem 2.1 to the case of abelian groups.

Theorem 2.11. *Let $\mathcal{A}$ be a central simple algebra of degree n over a field F and G an abelian group of order n. Then there exists a regular finitely generated field extension L/F such that*

(i) *$\mathcal{A}_L$ is a crossed product with the group G,*
(ii) $\mathrm{ind}(\mathcal{A}_L) = \mathrm{ind}(\mathcal{A})$.

Before proving this theorem, we introduce some notations and prove a preliminary lemma.
Let $F \subset K \subset E$ be a tower of field extension such that E/K is Galois with the group G and E preserves indices of F-algebras.
Let

$$G = H_1 \oplus \cdots \oplus H_m,$$

where $H_1, \dots, H_m$ are cyclic with generators respectively $\sigma_1, \dots, \sigma_m$. Let E_i be the subfield of E fixed by

$$\widehat{H}_i := \bigoplus_{\substack{j=1 \\ j \neq i}}^{m} H_j.$$

Then the extension E_i/K is cyclic with the Galois group H_i. Denote the canonical surjective homomorphism from G to H_i by τ_i.
Let $y_1, \dots, y_m$ be transcendental variables over K. Consider the cyclic algebras

$$\mathcal{D}_i = (E_i(y_1, \dots, y_m)/K(y_1, \dots, y_m), \sigma_i, y_i).$$

Note that

$$\mathcal{D}_i \sim (E(y_1, \dots, y_m)/K(y_1, \dots, y_m), G, c_i),$$

where

$$c_i(g, h) = c_{y_i}(\tau_i(g), \tau_i(h))$$

and the cocycle c_{y_i} is defined by

$$c_{y_i}(\sigma_i^k, \sigma_i^j) = \begin{cases} 1, & \text{if } k + j < [E_i : F]; \\ y_i, & \text{if } k + j \geq [E_i : F]. \end{cases}$$

(see e.g. [10, Th. 2.13.8] or [17, §14.5]).
Let

$$\mathcal{D} = \mathcal{D}_1 \otimes \dots \otimes \mathcal{D}_m.$$

Then

$$\mathcal{D} \cong (E(y_1, \dots, y_m)/K(y_1, \dots, y_m), G, c_1 \dots c_m)$$

is a crossed product with the group G and $\deg(\mathcal{D}) = n$ ([17, §14.3]).
In the notations above, we have the following

Lemma 2.12. *For any central division F-algebra $\mathcal{C}$, $\mathcal{D} \otimes \mathcal{C}_{K(y_1,\dots,y_m)}$ is a division algebra. Moreover, for any $j \geq 1$,*

$$\operatorname{ind}(\mathcal{D}^j \otimes \mathcal{C}_{K(y_1,\dots,y_m)}) = \operatorname{ind}(\mathcal{D}^j) \cdot \operatorname{ind}(\mathcal{C}).$$

Proof: We will prove the lemma using induction on m. If $m = 1$, then the statement is true in view of Lemma 2.7 and of Corollary 2.8. Assume that the statement is true for $m = m_0$. That is, for any tower of field extensions $F \subset K' \subset E'$ such that E' preserves indices of F-algebras and E'/K' is Galois with the group $H'_1 \oplus \dots \oplus H'_{m_0}$, ($H'_i$, $1 \leq i \leq m_0$, is cyclic with generator σ'_i) and any central division F-algebra $\mathcal{C}$, the algebra

$$(E'_1(y'_1, \dots, y'_{m_0})/K'(y'_1, \dots, y'_{m_0}), \sigma'_1, y'_1) \otimes \cdots$$
$$\cdots \otimes (E'_{m_0}(y'_1, \dots, y'_{m_0})/K'(y'_1, \dots, y'_{m_0}), \sigma'_{m_0}, y'_{m_0}) \otimes \mathcal{C}_{K'(y'_1,\dots,y'_{m_0})}$$

is division, where E'_i is a subfield of E' fixed by the group $\oplus_{j=1, j\neq i}^{m_0} H'_j$ and $y'_1, \dots, y'_{m_0}$ are transcendental variables over K'.

Let $m = m_0 + 1$. We have the following diagram of field extensions.

$$\begin{tikzcd}[arrows=dash]
 & E_1 \cdots E_{m_0} \cdot E_m \arrow[dd, "G_i"] \\
E_1 \cdots E_{m_0} \arrow[ur, "H_m"'] \arrow[dd, "G_i"'] & \\
 & E_i \cdot E_m \arrow[dd, "H_i"] \\
E_i \arrow[ur, "H_m"'] \arrow[dd, "H_i"'] & \\
 & E_m \\
K \arrow[ur, "H_m"'] &
\end{tikzcd}$$

for any $1 \leq i \leq m_0$, where

$$G_i = \bigoplus_{\substack{j=1 \\ j \neq i}}^{m_0} H_j.$$

Denote

$$\mathcal{B} = (E_1(y_1, \dots, y_{m_0})/K(y_1, \dots, y_{m_0}), \sigma_1, y_1) \otimes \cdots$$
$$\cdots \otimes (E_{m_0}(y_1, \dots, y_{m_0})/K(y_1, \dots, y_{m_0}), \sigma_{m_0}, y_{m_0}) \otimes \mathcal{C}_{K(y_1,\dots,y_{m_0})}.$$

Then

$$\mathcal{D} \otimes \mathcal{C}_{K(y_1,\dots,y_m)} = (E_1(y_1, \dots, y_m)/K(y_1, \dots, y_m), \sigma_1, y_1) \otimes \dots$$
$$\cdots \otimes (E_m(y_1, \dots, y_m)/K(y_1, \dots, y_m), \sigma_m, y_m) \otimes \mathcal{C}_{K(y_1,\dots,y_m)} \cong$$
$$\cong \mathcal{B}_{K(y_1,\dots,y_{m_0})(y_m)} \otimes (E_m(y_1, \dots, y_{m_0})(y_m)/K(y_1, \dots, y_{m_0})(y_m), \sigma_m, y_m).$$

To prove that the latter algebra is division it is enough by Lemma 2.7 to show that the algebra $\mathcal{B}_{E_m(y_1,\dots,y_{m_0})}$ is division. But this follows from the induction hypothesis (take $K' = E_m$, $E' = E_1 \cdots E_{m_0} \cdot E_m$, $E'_i = E_i \cdot E_m$) since

$$\mathcal{B}_{E_m(y_1,\dots,y_{m_0})} \cong (E_1 \cdot E_m(y_1, \dots, y_{m_0})/E_m(y_1, \dots, y_{m_0}), \sigma_1, y_1) \otimes \cdots$$
$$\cdots \otimes (E_{m_0} \cdot E_m(y_1, \dots, y_{m_0})/E_m(y_1, \dots, y_{m_0}), \sigma_{m_0}, y_{m_0}) \otimes \mathcal{C}_{E_m(y_1,\dots,y_{m_0})}.$$

Thus for any central division F-algebra $\mathcal{C}$, $\mathcal{D} \otimes \mathcal{C}_{K(y_1,\dots,y_m)}$ is a division algebra. Hence

$$\mathrm{ind}(\mathcal{D} \otimes \mathcal{C}_{K(y_1,\dots,y_m)}) = \deg(\mathcal{D} \otimes \mathcal{C}_{K(y_1,\dots,y_m)}) = \deg(\mathcal{D}) \cdot \deg(\mathcal{C}) = \mathrm{ind}(\mathcal{D}) \cdot \mathrm{ind}(\mathcal{C}).$$

Since $\mathcal{D}^j \sim \mathcal{D}_1^j \otimes \cdots \otimes \mathcal{D}_m^j$, then using properties of cyclic algebras (see proof of Corollary 2.8) we obtain the formula

$$\mathrm{ind}(\mathcal{D}^j \otimes \mathcal{C}_{K(y_1,\dots,y_m)}) = \mathrm{ind}(\mathcal{D}^j) \cdot \mathrm{ind}(\mathcal{C}).$$

$\square$

Now we are in a position to prove Theorem 2.11.

Proof of Theorem 2.11:
Given a field F and a finite group G of order n, it follows from Lemma 2.9 that there exists a tower of field extensions $F \subset K \subset E$ such that E/F is a finitely generated purely transcendental extension and E/K is Galois with the group G. Let

$$G = H_1 \oplus \cdots \oplus H_m,$$

where $H_1, \ldots, H_m$ are cyclic. One can construct a corresponding division algebra $\mathcal{D}$ (see the text before Lemma 2.12). Then $\deg(\mathcal{D}) = \mathrm{ind}(\mathcal{D}) = n \geq \mathrm{ind}(\mathcal{A})$. One has

$$\mathcal{D} \sim \mathcal{D} \otimes \mathcal{A}^{\mathrm{op}}_{K(y_1,\ldots,y_m)} \otimes \mathcal{A}_{K(y_1,\ldots,y_m)}.$$

Let L be the function field of the Severi-Brauer variety $\mathrm{SB}(\mathcal{D} \otimes \mathcal{A}^{\mathrm{op}}_{K(y_1,\ldots,y_m)})$. Then $\mathcal{A}_L \cong \mathcal{D}_L$, i.e., $\mathcal{A}_L$ is a crossed product with the group G.
To finish the proof we need to show that $\mathrm{ind}(\mathcal{A}_L) = \mathrm{ind}(\mathcal{A})$. Let $\mathcal{A}_p$ and $\mathcal{D}_p$ be the p-primary parts of $\mathcal{A}$ and $\mathcal{D}$. It is enough to prove that $\mathrm{ind}(\mathcal{A}_{p_L}) = \mathrm{ind}(\mathcal{A}_p)$. By the index reduction formula,

$$\mathrm{ind}(\mathcal{A}_{p\,L}) = \gcd\{\mathrm{ind}(\mathcal{A}_{p_{K(y_1,\ldots,y_m)}} \otimes {\mathcal{D}_p}^j \otimes {\mathcal{A}_p^{\mathrm{op}}}^{\,j}_{K(y_1,\ldots,y_m)})\},$$

where j ranges from 1 to n.
By Lemma 2.12, for any $1 \leq j \leq n$,

$$\mathrm{ind}(\mathcal{A}_{p_{K(y_1,\ldots,y_m)}} \otimes {\mathcal{D}_p}^j \otimes {\mathcal{A}_p^{\mathrm{op}}}^{\,j}_{K(y_1,\ldots,y_m)}) = \mathrm{ind}({\mathcal{D}_p}^j) \cdot \mathrm{ind}({\mathcal{A}_p^{\mathrm{op}}}^{\,j-1}).$$

If p does not divide j, then $\mathrm{ind}({\mathcal{D}_p}^j) = \mathrm{ind}(\mathcal{D}_p) \geq \mathrm{ind}(\mathcal{A}_p)$. If p divides j, then $\mathrm{ind}({\mathcal{A}_p^{\mathrm{op}}}^{\,j-1}) = \mathrm{ind}(\mathcal{A}_p)$.
Hence obtain that $\mathrm{ind}(\mathcal{A}_{p_L}) = \mathrm{ind}(\mathcal{A}_p)$.

□

3. Algebras with involutions after a scalar extension

Proof of Theorem 2.
Let $F \subset K$ be the subfield of fixed elements of τ and $\tau|K = \sigma$.
By Theorem 2.1, there exists a regular field extension M/K preserving indices of central simple K-algebras such that $\mathcal{A}_M$ is cyclic.
For the following constructions, in particular for the construction of the transfer of a regular field extension, we refer to [19, p. 220]). The automorphism $\sigma : K \longrightarrow K$ can be extended to an isomorphism of M and another regular extension of K denoted by M_σ. That is, the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, "\sigma"] & M \arrow[d, "\sigma"] \\ K \arrow[r, hook] & M_\sigma \end{tikzcd}.$$

Let $E = MM_\sigma$ be the free composite over K of M and M_σ (see definition in [9, p. 203]). Then E is a regular extension of K. The automorphism σ can be extended to an automorphism $\bar{\sigma}$ of E. Let $T = T_{K/F}(M)$ be the transfer of M with respect to the ground field descent $F \subset K$, i.e., the subfield of E of elements fixed under the action of $\bar{\sigma}$. Thus T is a subfield of E of degree 2. Then the composite TK coincides with E.

The algebra $\mathcal{A}_E$ is cyclic. Moreover, the latter algebra has an involution of the second kind defined by the formula

$$\overline{\tau}(a \otimes e) = \tau(a) \otimes \overline{\sigma}(e),$$

where $a \in \mathcal{A}$, $e \in E$ and $\overline{\sigma}$ is an automorphism of E extending σ.
As for preserving indices note that the field E can be constructed using the same procedure as for the field M (see proof of Theorem 2.1). We just replace the ground field K by M_σ. Hence E preserves indices of central simple M_σ-algebras, but M_σ preserves indices of all central simple K-algebras. □

Using above results we prove immediately the following.

Theorem 3.1. *Suslin's conjecture about special unitary groups is true iff it is true for all cyclic division algebras.*

We can also prove the following

Theorem 3.2. *Let $\mathcal{A}$ be a central simple algebra over a field K with an involution τ of the second kind. Assume that p^2 divides* $\mathrm{ind}(\mathcal{A})$ *for some prime number p. Then there exists a regular field extension M/K such that $\mathcal{A}_M$ is an algebra of index p^2, which is Brauer equivalent to a bicyclic algebra of degree p^2 and has an involution of the second kind extending τ.*

Proof: Let $F \subset K$ be the subfield of fixed elements of τ and $\tau|K = \sigma$. Denote by G the group $\mathbb{Z}/p \oplus \mathbb{Z}/p$. Let ϕ_1 be a generator of the first summand, and ϕ_2 of the second one. Then by Lemma 2.9, there exists a tower of field extensions $F \subset F_0 \subset E$ such that E/F_0 is Galois with the group G and E is a purely transcendental extension of F.
Let E_2 be the subfield of E fixed by the first summand of G and E_1 the subfield fixed by the second summand. Then the extensions E_i/F_0, i=1,2, are cyclic of degree p. Let $F_0(y_1, y_2)$ be a purely transcendental extension of F_0 of degree 2. Consider the cyclic algebras

$$\mathcal{D}_i = (E_i(y_1, y_2)/F_0(y_1, y_2), \phi_i, y_i),\ i = 1, 2.$$

Set $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_2$. By Lemma 2.12, $\mathcal{D}$ is a division algebra. Let L be the function field of the Severi-Brauer variety $\mathrm{SB}(\mathcal{D}_{KF_0(y_1,y_2)} \otimes \mathcal{A}^{\mathrm{op}}_{KF_0(y_1,y_2)})$. Then $\mathcal{A}_L \sim \mathcal{D}_L$. Thus $\mathrm{ind}(\mathcal{A}_L)|p^2$.
By means of the index reduction formula we obtain that $\mathrm{ind}(\mathcal{A}_L) = p^2$. Indeed,

$$\mathrm{ind}(\mathcal{A}_L) = \gcd\{\mathrm{ind}(\mathcal{A}_{KF_0(y_1,y_2)} \otimes \mathcal{D}^j_{KF_0(y_1,y_2)} \otimes \mathcal{A}^{\mathrm{op}\,j}_{KF_0(y_1,y_2)}\},$$

where j ranges from 1 to $\mathrm{ind}(\mathcal{A})$. By Lemma 2.12,

$$\mathrm{ind}(\mathcal{A}_{KF_0(y_1,y_2)} \otimes \mathcal{D}^j_{KF_0(y_1,y_2)} \otimes \mathcal{A}^{\mathrm{op}\,j}_{KF_0(y_1,y_2)}) = \mathrm{ind}(\mathcal{D}^j_{KF_0(y_1,y_2)}) \cdot \mathrm{ind}(\mathcal{A}^{\mathrm{op}\,j-1}_{KF_0(y_1,y_2)}).$$

If p does not divide j, then $\mathrm{ind}(\mathcal{D}^j_{KF_0(y_1,y_2)}) = p^2$. If $p|j$, then $p^2|\mathrm{ind}(\mathcal{A}^{\mathrm{op}\,j-1}_{KF_0(y_1,y_2)})$. So, in both cases $\mathrm{ind}(\mathcal{A}_{KF_0(y_1,y_2)} \otimes \mathcal{D}^j_{KF_0(y_1,y_2)} \otimes \mathcal{A}^{\mathrm{op}\,j}_{KF_0(y_1,y_2)})$ is divisible by p^2. Thus, $\mathrm{ind}(\mathcal{A}_L) = p^2$. Note also that L preserves the index of the $F_0(y_1, y_2)$-algebra $\mathcal{D}$.
The automorphism $\sigma : K \longrightarrow K$ can be extended to an isomorphism of L and another regular extension of K denoted by L_σ. Let $T = T_{K/F}(L)$ be the transfer of L with

respect to the ground field descent $F \subset K$. Then the composite TK over F coincides with the free composite LL_σ over K. Thus $\mathcal{A}_{LL_\sigma}$ has an involution of the second kind extending τ. To finish the proof we need to show that $\mathrm{ind}(\mathcal{A}_{LL_\sigma}) = p^2$.
The isomorphism $\sigma : L \longrightarrow L_\sigma$ can be extended in such a way that the following diagram commutes

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, "\sigma"] & KF_0(y_1,y_2) \arrow[r, hook] \arrow[d, "\sigma"] & L \arrow[d, "\sigma"] \\ K \arrow[r, hook] & KF_0(y_1,y_2) \arrow[r, hook] & L_\sigma \end{tikzcd}$$

where the middle arrow σ acts as σ on K and trivially on $F_0(y_1, y_2)$.
We have also the following commutative diagram.

$$\begin{tikzcd} & KF_0(y_1,y_2) \arrow[r, hook] \arrow[d, "\sigma"] & L \arrow[d, "\sigma"] \\ F_0(y_1,y_2) \arrow[ur, hook] \arrow[dr, hook] & & \\ & KF_0(y_1,y_2) \arrow[r, hook] & L_\sigma. \end{tikzcd}$$

As we noted before, L preserves the index of the $F_0(y_1, y_2)$-algebra $\mathcal{D}$ and σ is trivial on $F_0(y_1, y_2)$. Hence L_σ also preserves the index of $\mathcal{D}$.
Now consider the free composite LL_σ. It can be constructed using the same procedure as for the field L. We just replace the ground field K by L_σ. Thus LL_σ can be constructed as follows. Instead of the tower of fields extensions $K \subset KF_0 \subset KE$ and the algebra $\mathcal{D}_{KF_0(y_1,y_2)}$ in the same way we construct the tower of field extensions $L_\sigma \subset L_\sigma(KF_0) \subset L_\sigma(KE)$ and the algebra $\bar{\mathcal{D}}$ over $\bar{K}(z_1, z_2)$ where $\bar{K} = L_\sigma(KF_0)$ and $\bar{K}(z_1, z_2)$ is a purely transcendental extension of $\bar{K}$ of degree 2. Then LL_σ is the function field of the Severi-Brauer variety

$$\mathrm{SB}(\bar{\mathcal{D}} \otimes \mathcal{A}^{\mathrm{op}}_{\bar{K}(z_1,z_2)}).$$

Note that $L_\sigma(KE)$ is a purely transcendental extension of L_σ. Hence $\bar{K}$ as a subfield of $L_\sigma(KE)$ preserves the index of $\mathcal{D}$. Then $\bar{K}(z_1, z_2)$ also preserves this index.
Moreover, we conclude that $\mathrm{ind}(\mathcal{A}_{L_\sigma}) = p^2$. Indeed, assume that $\mathrm{ind}(\mathcal{A}_{L_\sigma}) < p^2$. Then the p-primary component of $\mathcal{A}^p_{L_\sigma}$ is trivial. By the index reduction formula,

$$\mathrm{ind}(\mathcal{D}_{LL_\sigma}) = \gcd\{\mathrm{ind}((\bar{\mathcal{D}}^j \otimes \mathcal{A}^{\mathrm{op}}\,{}^j_{\bar{K}(z_1,z_2)}) \otimes \mathcal{D}_{\bar{K}(z_1,z_2)})\}$$

where j ranges from 1 to $\mathrm{ind}(\bar{\mathcal{D}} \otimes \mathcal{A}^{\mathrm{op}}_{\bar{K}(z_1,z_2)})$.
Note that since $L_\sigma(KE)$ is purely transcendental extension of L_σ, then it preserves indices of central simple L_σ-algebras. By Lemma 2.12,

$$\mathrm{ind}((\bar{\mathcal{D}}^j \otimes \mathcal{A}^{\mathrm{op}}\,{}^j_{\bar{K}(z_1,z_2)}) \otimes \mathcal{D}_{\bar{K}(z_1,z_2)}) = \mathrm{ind}(\bar{\mathcal{D}}^j) \cdot \mathrm{ind}(\mathcal{A}^{\mathrm{op}}\,{}^j_{\bar{K}(z_1,z_2)} \otimes \mathcal{D}_{\bar{K}(z_1,z_2)}).$$

If p does not divide j, then $\operatorname{ind}(\bar{\mathcal{D}}^j_{\bar{K}(z_1,z_2)}) = p^2$. If p divides j, then the p-primary part $\mathcal{A}^{\mathrm{op}\ j}_{p\ \bar{K}(z_1,z_2)}$ of $\mathcal{A}^{\mathrm{op}\ j}_{\bar{K}(z_1,z_2)}$ is trivial. Hence

$$\operatorname{ind}(\mathcal{A}^{\mathrm{op}\ j}_{p\ \bar{K}(z_1,z_2)} \otimes \mathcal{D}_{\bar{K}(z_1,z_2)}) = \operatorname{ind}(\mathcal{D}_{\bar{K}(z_1,z_2)}) = \operatorname{ind}(\mathcal{D}_{L_\sigma}) = p^2.$$

Thus $\operatorname{ind}(\mathcal{D}_{LL_\sigma}) = \operatorname{ind}(\mathcal{A}_{LL_\sigma}) > \operatorname{ind}(\mathcal{A}_{L_\sigma})$ and we have a contradiction.
Hence $\operatorname{ind}(A_{L_\sigma}) = p^2$, then $\operatorname{ind}(A_{\bar{K}(z_1,z_2)}) = p^2$. Using the index reduction formula we obtain that $\operatorname{ind}(\mathcal{A}_{LL_\sigma}) = p^2$. The proof of the latter equality is the same as for the index of $\mathcal{A}_L$. □

Corollary 3.3. *Let $\mathcal{A}$ be a central simple algebra over a field K with an involution τ of the second kind. Assume that p^2 divides* $\operatorname{ind}(\mathcal{A})$ *for some prime number p and the primitive p-th root of unity belongs to K. Then there exists a regular field extension M/K such that $\mathcal{A}_M$ is an algebra of index p^2, which is Brauer equivalent to a tensor product of two symbol algebras and has an involution of the second kind extending τ.*

Combining Theorems 3.1 and 3.2 one can prove:

Theorem 3.4. *Suslin's conjecture about special unitary groups is true iff it is true for all cyclic division algebras which are bicyclic algebras of degree p^2 for any prime p.*

For abelian crossed products, we have the following

Theorem 3.5. *Let $\mathcal{A}$ be a central simple algebra over a field K of degree n with an involution τ of the second kind, and let G be an abelian group of order n. Then there exists a regular field extension E/K preserving the index of $\mathcal{A}$ such that $\mathcal{A}_E$ is a crossed product with the group G and $\mathcal{A}_E$ has an involution of the second kind extending τ.*

Proof: Let $F \subset K$ be the subfield of fixed elements of τ and $\tau|K = \sigma$. It follows from Lemma 2.9 that there exists a tower of field extensions $F \subset M \subset N$ such that N/F is a finitely generated purely transcendental extension and N/M is Galois with the group G. Let $G = H_1 \oplus \cdots \oplus H_r$ be the decomposition of G as a sum of cyclic subgroups. As in the proof of Theorem 2.11 we can construct an algebra $\mathcal{D}$ over a purely transcendental extension $M(y_1, \ldots, y_r)$ of degree r of M which is a crossed product with the group G. Let L be the function field of the Severi-Brauer variety $\mathrm{SB}(\mathcal{A}^{\mathrm{op}}_{KM(y_1,\ldots,y_r)} \otimes \mathcal{D}_{KM(y_1,\ldots,y_r)})$. Then L preserves the index of $\mathcal{A}$ (the proof is analogous to that of Theorem 2.1) and $\mathcal{A}_L$ is a crossed product with the group G. Further, as in the proof of Theorem 3.2 we construct a free composite LL_σ and prove that $\mathcal{A}_{LL_\sigma}$ has prescribed properties. □

References

[1] A. A. Albert, *Structure of algebras*, rev. printing, Amer. Math. Soc., Providence, 1961.

[2] A. Blanchet, *Function fields of generalized Brauer-Severi varieties*, Comm. Algebra **19** (1991), no. 1, 97–118.

[3] M. Van den Berg, A. Schofield, *The index of a Brauer class on a Brauer-Severi variety*, Trans. AMS. **341** (1994), 505–517.

[4] N. Bourbaki, *Éléments de mathématiques / Livre 2, Algèbre, ch 8, Modules et anneaux semi-simples*, Actualités Sci. Indust. No. 1261, Hermann, Paris, 1958.

[5] V.I. Chernousov, A.S. Merkurjev, *R-equivalence and special unitary groups*, J. Algebra **209** (1998), no. 1, 175-198.

[6] P. M. Cohn. *Free rings and their relations*, London Mathematical Society Monographs, No. 2. Academic Press, London-New York, 1971.

[7] J.-L. Colliot-Thélène, J.-J. Sansuc, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 175–229.

[8] P. Gille *Le problème de Kneser-Tits*, Séminaire Bourbaki. Vol. 2007/2008. Astérisque No. 326 (2009), Exp. No. 983, vii, 39-81 (2010).

[9] N. Jacobson, *Lectures in abstract algebra. Vol III: Theory of fields and Galois theory* Inc., Princeton, N.J.-Toronto, Ont.-London-New York 1964.

[10] N. Jacobson, *Finite-dimensional division algebras over fields*, New York-Heidelberg-Berlin: Springer-Verlag, 1996.

[11] O.T. Izhboldin, N.A. Karpenko *Generic splitting fields of generic simple algebras: Galois cohomology and nonexcellence* , Algebras and Representation Theory **2** (1999), 19–59.

[12] I. Kersten, U. Rehmann, *Excellent algebraic groups. I*, J. Algebra **200** (1998), no. 1, 334–346.

[13] B. È. Kunyavskiĭ, L. H. Rowen, S. V. Tikhonov, and V. I. Yanchevskiĭ. *Bicyclic algebras of prime exponent over function fields,* Trans. Amer. Math. Soc. **358** (2006), no. 6, 2579-2610.

[14] Yu. I. Manin, *Cubic forms*, North-Holland, Amsterdam, 1974.

[15] A. S. Merkurjev, I. A. Panin and A. R. Wadsworth, *Index Reduction Formulas for Twisted Flag Varieties, I*, K-Theory **10** (1996), 517-596.

[16] A. P. Monastyrnyĭ, V. I. Yanchevskiĭ, *Whitehead groups of spinor groups*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 1, 60–96, 221; translation in Math. USSR-Izv. **36** (1991), no. 1, 61–100.

[17] R. S. Pierce, *Associative algebras*, Springer, New York, 1982.

[18] U. Rehmann, S.V. Tikhonov, V.I. Yanchevskiĭ, *Symbols and cyclicity of algebras after a scalar extension*, Journal of Mathematical Sciences **164**, n° 1, 131-142.

[19] P. Roquette, *Isomorphisms of generic splitting fields of simple algebras*, J. Reine Angew. Math. **214/215**, (1964), 207-226.

[20] D. J. Saltman, *Lectures on division algebras*, Amer. Math. Soc., Providence, RI, 1999.

[21] A. A. Suslin, *SK_1 of division algebras and Galois cohomology*, Adv. Soviet Math. **4**, Amer. Math. Soc., Providence, RI (1991), 75-100.

[22] V. E. Voskresenskiĭ, *The reduced Whitehead group of a simple algebra*, Uspehi Mat. Nauk **32** (1977), no. 6(198), 247-248.

[23] A. R. Wadsworth, *The index reduction formula for generic partial splitting varieties*, Comm. Algebra **21** (1993), 1063–1070.

[24] V. I. Yanchevskiĭ, *Simple algebras with involutions, and unitary groups*, Mat. Sb. **22** (1974), 372–385.

[25] V. I. Yanchevskiĭ, *Reduced unitary K-theory and division algebras over Henselian discretely valued fields*, Izv. Akad. Nauk SSSR Ser. Mat. 42 (1978), no. 4, 879–918; translation in Math. USSR-Izv. **13** (1979), 175–213.

[26] V. I. Yanchevskiĭ, *The inverse problem of reduced K-theory*, Mat. Zametki 26 (1979), no. 3, 475–482; translation in Math. Notes. **26** (1979), 728–731.

[27] V. I. Yanchevskiĭ, *Reduced unitary K-theory: applications to algebraic groups.*, Mat. Sb. **38** (1981), 533–561.

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