APPLICATIONS OF CONICS TO CERTAIN QUADRATIC FORMS OVER THE RATIONAL FUNCTION FIELDS

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Abstract. A few results on quadratic forms over fields are obtained. In particular, we show that for any forms φ_1 and φ_2 over a field k of characteristic different from 2 and $a \in k^*$, the anisotropic part of the form $\varphi_1 \perp (t^2 - a)\varphi_2$ over the rational function field $k(t)$ is of the same type, i.e. there exist forms τ_1 and τ_2 over k such that $(\varphi_1 \perp (t^2 - a)\varphi_2)_{an} \simeq (\tau_1 \perp (t^2 - a)\tau_2)$. Also we determine the structure of certain Pfister forms over $k(t)$, and describe the behavior of quadratic forms under biquadratic extensions of k in terms of some related forms over the function field of the product of two conics over $k(x)$, or $k(x, y)$.

1. Structure of certain quadratic forms over the rational function field

Let k be a field of characteristic different from 2, $W(k)$ the Witt group of k. It is well known (see, for example, [Sch]) that the sequence of abelian groups

$$0 \to W(k) \xrightarrow{\text{res}} W(k(t)) \xrightarrow{\coprod \partial_p} \coprod_{p \in \mathbb{A}^1_k} W(k_p) \to 0,$$

is split exact. We consider here a point $p \in \mathbb{A}^1_k$ as a monic irreducible polynomial over k, $k_p = k[t]/p$ is the corresponding residue field, and $\partial_p : W(k(t)) \to W(k_p)$ is the residue homomorphism well defined by the rule

$$\partial_p(\langle f \rangle) = \begin{cases} 0 & \text{if } v_p(f) = 0 \\ \langle \overline{fp^{-1}} \rangle & \text{if } v_p(f) = 1 \end{cases}$$

The splitting map $W(k(t)) \to W(k)$ is defined by the rule $\langle f \rangle \to \langle l(f) \rangle$, where $l(f)$ is the leading coefficient of the polynomial $f \in k[t]$. Let $\varphi \in W(k(t))$. Knowing the projections of φ to all direct summands of $W(k(t))$, it is easy to determine φ itself. However, only in some specific cases it is clear how to determine the anisotropic part φ_{an} of the form φ. One situation when it is possible is the case where the form φ has an only residue, and, moreover, at the linear polynomial t. Then, in view of the exact sequence above, $\varphi = \tau_1 + t\tau_2$ for some anisotropic forms τ_1, τ_2 over k. It

Key words and phrases. .

is well known that the form $\tau_1 \perp t\tau_2$ is anisotropic, hence $\varphi_{an} \simeq \tau_1 \perp t\tau_2$. We start this paper by proving a similar result, where the linear polynomial t is replaced by the quadratic polynomial $t^2 - a$, a being a nonsquare element of k^*.

A few words about the notation. In the sequel all the fields are assumed to be of characteristic different from 2. The Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ is the product $\langle 1, -a_1\rangle \otimes \dots \otimes \langle 1, -a_n\rangle$ (take notice of the signs !). Usually, we will omit the sign $\otimes$ in products of quadratic forms. For the field extension L/F the kernel of the restriction map $W(F) \to W(L)$ is denoted by $W(L/F)$. The extension L/F is called excellent if for any form φ over F the anisotropic part of the form φ_L is defined over F. The basic statement used throughout the paper is that the extension $F(C)/F$ is excellent for any field F and a conic over F ([A], [R]).

Theorem 1.1. *Let k be a field, φ_1, φ_2 quadratic forms over k, m an odd positive integer. Then there exist forms τ_1, τ_2 over k such that over $k(t)$*

$$(\varphi_1 \perp (t^{2m} - a)\varphi_2)_{an} \simeq \tau_1 \perp (t^{2m} - a)\tau_2.$$

Proof. First consider the case where $m = 1$. We may assume that the form $(\varphi_1 \perp (t^2 - a)\varphi_2)$ is isotropic. Let C be the affine conic associated with the quaternion algebra (a, x) over the Laurent series field $F = k((x))$, with the equation $t^2 - a = xv^2$. Notice that $k(t) \subset F(C)$. The form

$$(\varphi_1 \perp x\varphi_2)_{F(C)} \simeq (\varphi_1 \perp (t^2 - a)\varphi_2)_{F(C)}$$

is isotropic by the hypothesis. Since the extension $F(C)/F$ is excellent, and $W(F) = W(k) \oplus xW(k)$, we get

$$(\varphi_1 \perp x\varphi_2)_{F(C)_{an}} \simeq (\tau_1 \perp x\tau_2)_{F(C)}$$

for some forms τ_1 and τ_2 over k. It follows, since $W(F(C)/F) = \langle\langle a, x\rangle\rangle W(F)$ ([Sch], Ch.4, Th. 5.4), that

$$\varphi_1 + x\varphi_2 = \tau_1 + x\tau_2 + \langle\langle a, x\rangle\rangle q \in W(F) \qquad (*)$$

for some form q over F. Since $q \simeq q_1 \perp xq_2$ for some forms q_1, q_2 over k, and $-x\langle\langle a, x\rangle\rangle \simeq \langle\langle a, x\rangle\rangle$, we may assume that q is a form over k. Therefore,

$$\varphi_1 - \tau_1 - \langle\langle a\rangle\rangle q = x(\tau_2 - \varphi_2 - \langle\langle a\rangle\rangle q) \in W(F),$$

or, in other words,

$$\varphi_1 - \tau_1 - \langle\langle a\rangle\rangle q = \tau_2 - \varphi_2 - \langle\langle a\rangle\rangle q = 0 \in W(k).$$

So we have

$$\varphi_1 - \tau_1 - \langle\langle a\rangle\rangle q = (t^2 - a)(\tau_2 - \varphi_2 - \langle\langle a\rangle\rangle q) = 0 \in W(k(t)),$$

and, consequently, we can replace x by $t^2 - a$ and $k((x))$ by $k(t)$ in $(*)$, i.e.

$$\varphi_1 + (t^2 - a)\varphi_2 = \tau_1 + (t^2 - a)\tau_2 + \langle\langle a, t^2 - a\rangle\rangle q = \tau_1 + (t^2 - a)\tau_2 \in W(k(t)),$$

since $\langle\langle a, t^2-a\rangle\rangle = 0$. Notice that the form $\tau_1 \perp (t^2-a)\tau_2$ is anisotropic over $k(t)$, since $k(t) \subset F(C)$, and the form $\tau_1 \perp x\tau_2$ is anisotropic over $F(C)$.

In the general case, where m is an arbitrary odd number, consider the odd degree field extension $k(u)/k(t)$, where $u = t^{\frac{1}{m}}$. Assuming that the form $\varphi_1 \perp (u^{2m}-a)\varphi_2$ is isotropic over $k(u)$, we see by the Springer theorem ([Sch], Ch.2, Th. 5.3) that the form $\varphi_1 \perp (t^2-a)\varphi_2$ is isotropic over $k(t)$. By Theorem 1.1

$$(\varphi_1 \perp (t^2-a)\varphi_2)_{an} \simeq \tau_1 \perp (t^2-a)\tau_2$$

for some forms τ_1, τ_2 over k. Replacing t by t^m, and applying the Springer theorem again, we get

$$(\varphi_1 \perp (t^{2m}-a)\varphi_2)_{an} \simeq \tau_1 \perp (t^{2m}-a)\tau_2.$$

□

Remark. If $\varphi_2 \simeq -\varphi_1$, then the proof of Theorem 1.1 can be substantially simplified. Indeed, suppose the form $\varphi_1 \perp -(t^2-a)\varphi_1$ is isotropic, and φ_1 is anisotropic. Then the form $(\varphi_1 \perp -(t-\sqrt{a})(t+\sqrt{a})\varphi_1)_{k(\sqrt{a})((t-\sqrt{a}))}$ is isotropic, which implies that $\varphi_{1k(\sqrt{a})}$ is isotropic as well. Hence $\varphi_1 \simeq \langle\langle a\rangle\rangle\psi \perp \tau$ for some nonempty forms ψ and τ. We get

$$\varphi_1 \perp -(t^2-a)\varphi_1 = \langle\langle a\rangle\rangle\psi + \tau - (t^2-a)(\langle\langle a\rangle\rangle\psi + \tau) =$$
$$(\langle\langle a\rangle\rangle\psi - (t^2-a)\langle\langle a\rangle\rangle\psi) + (\tau - (t^2-a)\tau) = \tau - (t^2-a)\tau,$$

since $(t^2-a)\langle\langle a\rangle\rangle = \langle\langle a\rangle\rangle$, so we can finish the proof by induction on $\dim\varphi_1$.

Corollary 1.2. *Under the notation above the following two conditions are equivalent.*

1) *The form* $\varphi_1 \perp (t^{2m}-a)\varphi_2$ *is isotropic.*

2) *There exist forms* τ_1 *and* τ_2 *over* k *such that* $\varphi_1 - \tau_1 = \tau_2 - \varphi_2 \in W(k(\sqrt{a})/k)$, *and* $\dim\tau_1 + \dim\tau_2 < \dim\varphi_1 + \dim\varphi_2$.

Proof. 1) $\Longrightarrow$ 2). By Theorem 1.1 we have

$$\varphi_1 + (t^{2m}-a)\varphi_2 = \tau_1 + (t^{2m}-a)\tau_2,$$

and $\dim\tau_1 + \dim\tau_2 < \dim\varphi_1 + \dim\varphi_2$. Since $\varphi_1 - \tau_1 = (t^{2m}-a)(\tau_2 - \varphi_2)$, we get

$$\varphi_1 - \tau_1 = \tau_2 - \varphi_2 \in W(k(a^{\frac{1}{2m}})/k).$$

Since m is odd, by the Springer theorem we get

$$\varphi_1 - \tau_1 = \tau_2 - \varphi_2 \in W(k(\sqrt{a})/k).$$

2) $\Longrightarrow$ 1). The same argument, but in the opposite direction. □

If $\varphi_1,\ \varphi_2 \in I^n(k)$, then $(\varphi_1 \perp t\varphi_2)_{an} \simeq \tau_1 \perp \tau_2$, where $\tau_1,\ \tau_2 \in I^n(k)$, since $\tau_i \simeq \varphi_{ian}$ $(i=1,2)$. The similar statement is not true even for $n=2$ if one replaces the polynomial t by t^2-a, as the following counterexample shows.

Proposition 1.3. *Let k_0 be a field, $k = k_0(z)$, $a, b, c, u, v \in {k_0}^*$ are such that* $\mathrm{ind}((b,u)+(c,v))_{k_0(\sqrt{a})} = 4$,

$$\psi \simeq \langle 1, -b, -u, abu\rangle \perp (t^2-a)z\langle 1, -c, -v, acv\rangle \in W(k(t)).$$

Then $\psi \simeq (\varphi_1 \perp (t^2-a)\varphi_2)_{an}$ for some forms $\varphi_1, \varphi_2 \in I^2(k)$, but $\psi \not\simeq \tau_1 \perp (t^2-a)\tau_2$ for any forms $\tau_1, \tau_2 \in I^2(k)$.

Proof. Clearly, ψ is anisotropic, $\psi \in I^2(k(t))$. By [T],Prop. 2.4 we have

$$\mathrm{ind}\, C(\psi) = \mathrm{ind}((b,u)+(c,v)+(a,-bcuvz)) = 8.$$

Moreover, comparing the residues, it is easy to see that $\psi \simeq (\varphi_1 \perp (t^2-a)\varphi_2)_{an}$, where $\varphi_1 \simeq \langle 1, -b, -u, abu, -zcv, zacv\rangle$, $\varphi_2 \simeq z\langle 1, -c, -v, cv\rangle$. Obviously, $\varphi_1, \varphi_2 \in I^2(k)$. Suppose $\psi \simeq \tau_1 \perp (t^2-a)\tau_2$, where $\tau_1, \tau_2 \in I^2(k)$. Then either $\dim \tau_1 = \dim \tau_2 = 4$, or $\dim \tau_1 = 8$, $\dim \tau_2 = 0$, or $\dim \tau_1 = 0$, $\dim \tau_2 = 8$. But the first case is impossible, since $\mathrm{ind}\, C(\psi) = 8$. The second and the third cases are impossible, since $\partial_{t^2-a}(\psi) \neq 0$ and $\partial_{t^2-a}((t^2-a)\psi) \neq 0$. $\square$

Given an even number $2n \geq 4$, Theorem 1.1 is not true in general for irreducible polynomials in t of degree $2n$. To construct corresponding counterexamples we use the following statement, which immediately follows from [Si1], Prop.11.

Proposition 1.4. *Let k_0 be a field, $n > m \geq 1$, $k = k_0(x)$, $a, b \in k_0^*$, $\langle\langle a, b\rangle\rangle \neq 0$, $p(t) = at^{2n} + bt^{2m} - x$, X the affine curve over k determined by the equation $y^2 = at^{2n} + bt^{2m} - x$. Then the form $(\langle -a, -b, x, 1\rangle_{k(X)})_{an}$ is not defined over k.*

Corollary 1.5. *Under the notation of Proposition* 4

$$\langle -a, -b, x, p(t)\rangle_{an} \not\simeq \varphi_1 \perp p(t)\varphi_2$$

for any forms φ_1, φ_2 over k.

Proof. Suppose $\langle -a, -b, x, p(t)\rangle_{an} \simeq \varphi_1 \perp p(t)\varphi_2$. Then $\dim \varphi_1 = \dim \varphi_2 = 1$, and

$$(\langle -a, -b, x, 1\rangle_{k(X)})_{an} \simeq (\langle -a, -b, x, p(t)\rangle_{k(X)})_{an} \simeq (\varphi_1 \perp p(t)\varphi_2)_{k(X)} \simeq (\varphi_1 \perp \varphi_2)_{k(X)},$$

a contradiction to Proposition 1.4. $\square$

The condition that m is odd is essential in Theorem 1.1, as the following example shows.

Proposition 1.6. *For any even positive integer $m = 2n \geq 2$ Theorem* 1.1 *does not remain true in general if one replaces $t^2 - a$ by $t^{2m} - a$.*

Proof. Let k_0 be a field, $k = k_0(x, y, z)$. Consider the form

$$\pi \simeq \langle\langle x(t^{2n} - y - zy^{-1}),\ (y + zy^{-1})(t^{4n} - 4z)\rangle\rangle$$

over k. Obviously, $\partial_p(\pi) = 0$ for any $p \in \mathbb{A}^1_k$ distinct from $t^{2n} - y - zy^{-1}$ and $t^{4n} - 4z$. Moreover,

$$\partial_{t^{2n}-y-zy^{-1}}(\pi) = -x\langle 1, -(y+zy^{-1})((y+zy^{-1})^2-4z)\rangle = -x\langle 1, -t^{2n}(y-zy^{-1})^2\rangle = 0,$$

$$\partial_{t^{4n}-4z}(\pi) = -(y+zy^{-1})\langle 1, -x(t^{2n}-y-zy^{-1})\rangle =$$
$$-(y+zy^{-1})\langle 1, (t^{2n}-2y)^2(4y)^{-1}x\rangle = -(y+zy^{-1})\langle 1, xy\rangle.$$

Suppose $\pi \simeq \tau_1 \perp (t^{4n}-4z)\tau_2$ for some forms τ_1, τ_2 over k. Then, obviously, $\pi \simeq \langle\langle c_1, c_2(t^{4n}-4z)\rangle\rangle$, and, consequently, $\langle\langle x, y+zy^{-1}\rangle\rangle = \langle\langle c_1, c_2\rangle\rangle$ for some $c_1, c_2 \in k^*$. Also

$$c_1 = \operatorname{disc}\partial_{t^{4n}-4z}(\pi) = -xy,$$

which implies that $-xyc_1 \in {k_{t^{4n}-4z}}^{*2}$. It is easy to see that this implies $-xyc_1 \in k^{*2} \cup zk^{*2}$, or, equivalently, $c_1 \in -xyk^{*2} \cup -xyzk^{*2}$. Since $\langle\langle x, y+zy^{-1}\rangle\rangle = \langle\langle c_1, c_2\rangle\rangle$, we conclude that either $\langle\langle x, y+zy^{-1}\rangle\rangle_{k(\sqrt{-xy})} = 0$, or $\langle\langle x, y+zy^{-1}\rangle\rangle_{k(\sqrt{-xyz})} = 0$, which is, clearly, impossible.

Summing up we see that

$$(\varphi_1 \perp (t^{4n}-4z)\varphi_2)_{an} \simeq \pi \not\simeq \tau_1 \perp (t^{4n}-a)\tau_2,$$

where

$$\varphi_1 \simeq (y+zy^{-1})\langle 1, xy\rangle \perp \langle\langle x, y+zy^{-1}\rangle\rangle,\ \varphi_2 \simeq -(y+zy^{-1})\langle 1, xy\rangle,$$

and τ_1, τ_2 are some forms over k. □

The following theorem determines the structure of certain Pfister forms over $k(t)$.

Theorem 1.7. *Let k be a field, π an n-fold Pfister form over $k(t)$, $a \in k^* \setminus k^{*2}$. Assume that $\partial_p(\pi) = 0$ for any $p \in \mathbb{A}^1_k$ different from t^2-a, and $0 \neq \partial_{t^2-a}(\pi) \in \operatorname{res}_{k(\sqrt{a})/k} W(k)$. Then $\pi \simeq \langle\langle c_1, \ldots, c_{n-1}, c_n(t^2-a)\rangle\rangle$ for some $c_1, \ldots, c_n \in k^*$.*

Proof. Put $F = k((x))$. We have $\pi = \varphi_1 + (t^2-a)\varphi_2$ for some $\varphi_1, \varphi_2 \in W(k)$, hence

$$(\varphi_1 + x\varphi_2)_{F(C)} = (\varphi_1 + (t^2-a)\varphi_2)_{F(C)} = \pi_{F(C)}.$$

Therefore, there exists a Pfister form ρ over $F = k((x))$ such that

$$(\varphi_1 + x\varphi_2)_{F(C)} = \rho_{F(C)}$$

([ELW], Prop.2.10), or, in other words, $\varphi_1 + x\varphi_2 - \rho = \langle\langle a, x\rangle\rangle q$ for some form $q \in W(k)$. Let $\rho \simeq \langle\langle c_1, \ldots, c_{n-1}, c_n x^m\rangle\rangle$, where $c_1, \ldots, c_n \in k^*$, $m \in \{0, 1\}$. Just as in the proof of Theorem 1.1 we can replace x by t^2-a. Thus, we get

$$\varphi_1 + (t^2-a)\varphi_2 - \langle\langle c_1, \ldots, c_{n-1}, c_n(t^2-a)^m\rangle\rangle = \langle\langle a, t^2-a\rangle\rangle q = 0 \in W(k(t)),$$

or, equivalently, $\pi = \varphi_1 + (t^2-a)\varphi_2 = \langle\langle c_1, \ldots, c_{n-1}, c_n(t^2-a)\rangle\rangle$, since $\partial_{t^2-a}(\pi) \neq 0$. □

The example in Proposition 1.6 shows that the analog of Theorem 1.7 is false in general for polynomials $t^{4n}-a$. Also it cannot be generalized to arbitrary irreducible polynomial of degree $2m$, where $m \geq 3$ is odd, as the following example shows.

Proposition 1.8. *Let $m \geq 3$ be an odd number, k_0 a field, $k = k_0(x, y, z)$, $p(t) = t^{2m} + xyt^2 - x$. Then*

1) The form $\langle 1, xy, -x, -p\rangle$ over $k(t)$ is isotropic, or, in other words, $\langle 1, xy, -x, -p\rangle_{an} \simeq q\langle 1, -yp\rangle$ for some $q \in k[t]$. Moreover, since $-yp\langle 1, -yp\rangle \simeq \langle 1, -yp\rangle$, we may assume that p does not divide q.

2) If $f \in \mathbb{A}^1_k$, then

$$\partial_f(\langle\langle yp, zq\rangle\rangle) = \begin{cases} 0 & \text{if } f \neq p \\ -y\langle\langle yz\rangle\rangle & \text{if } f = p \end{cases}$$

3) $\langle\langle yp, zq\rangle\rangle \not\simeq \langle\langle c_1, c_2 p\rangle\rangle$ for any $c_1, c_2 \in k^*$.

Proof. The first statement is obvious. In particular, $yq \in {k_p^*}^2$, and $yp \in {k_r^*}^2$, where r is any prime monic divisor of q, hence the second statement follows. Now suppose that $\langle\langle yp, zq\rangle\rangle = \langle\langle c_1, c_2p\rangle\rangle$ for some $c_1, c_2 \in k^*$. Comparing the residues at p we get that $c_1yz \in {k_p^*}^2 \cap k^*$.

Lemma 1.9. ${k_p^*}^2 \cap k^* = {k^*}^2$.

Proof. Let $l = k(x, y, z)[u]/(u^m + xyu - x)$. Obviously, $k_p = l(\sqrt{u})$. Suppose there is a quadratic extension $k(\sqrt{a})$ of k containing in k_p. Then $k_p = l(\sqrt{a})$, i.e. $au \in {l^*}^2$. Hence $a^m x = N_{l/k}(au) \in {k^*}^2$, and so we may assume that $a = x$. On the other hand, we can consider k_p as the field $k(u, y, z)$, where $ux = u^{m+1}(1 - uy)^{-1}$. Since this element is not a square in $k(u, y, z)$, we come to a contradiction. □

Returning to the proof of Proposition 1.8 we get

$$\langle\langle c_1, c_2p\rangle\rangle = \langle\langle yp, zq\rangle\rangle = \langle 1, -yp\rangle - zq\langle 1, -yp\rangle = \langle 1, -yp\rangle - z\langle 1, -x, xy, -p\rangle = -z\langle\langle x, y\rangle\rangle - yz\langle\langle yp, yz\rangle\rangle,$$

which implies, in view of $c_1yz \in {k^*}^2$, that

$$0 = \langle\langle c_1, c_2p\rangle\rangle_{k(\sqrt{yz})} = (-z\langle\langle x, y\rangle\rangle - yz\langle\langle yp, yz\rangle\rangle)_{k(\sqrt{yz})} = -z\langle\langle x, y\rangle\rangle_{k(\sqrt{yz})},$$

which is, clearly, impossible. □

The main idea in Theorem 1.1 permits to obtain a short proof of Theorem 4.1 from ([RST]), which we formulate here in a bit different way.

Theorem 1.10. *Let k be a field, $a, b \in k^*$, $a \notin {k^*}^2$, $ab \notin {k^*}^2$. Let D be a finite-dimensional central division algebra over k. Then the following conditions are equivalent.*

1) $D \otimes_{k(t)} (a, t^2 - b)$ is a division algebra over $k(t)$.

2) $D_{k(\sqrt{a})}$ and $D_{k(\sqrt{ab})}$ are division algebras.

Proof. The implication 1) $\Longrightarrow$ 2) follows from the fact that

$$\operatorname{ind} D \otimes_{k(t)} (a, t^2 - b)_{k(\sqrt{a})} = \operatorname{ind} D_{k(\sqrt{a})}$$

and

$$\operatorname{ind} D \otimes_{k(t)} (a, t^2 - b)_{k(\sqrt{ab})} = \operatorname{ind} D_{k(\sqrt{ab})}.$$

As for the implication 2) $\Longrightarrow$ 1) assume that $D_{k(\sqrt{a})}$ and $D_{k(\sqrt{ab})}$ are division algebras. Then by [T], Prop. 2.4 the algebra $D \otimes (a, x)$ is a division algebra over $k(x)$. Let C be the conic over $k(x)$ with the equation $t^2 - b = xv^2$. If the algebra $D \otimes (a, x)_{k(x)(C)}$ is a division algebra, we are done. If not, then by [M] $D \otimes (a, x) \simeq (b, x) \otimes D'$ for some division algebra D' over $k(x)$. Hence $D \otimes (ab, x) \simeq D' \otimes M_2(k(x))$ is not a division algebra. Therefore, again by [T], Prop. 2.4 $D_{k(\sqrt{ab})}$ is not a division algebra, a contradiction to the hypothesis. $\square$

2. Behavior of quadratic forms under biquadratic extensions

Our next purpose is to study the behavior of quadratic forms under a biquadratic extension. In particular, it turns out that a nonexcellent biquadratic extension over a field k gives rise to a nonexcellent extension of the field $k(x, y)$ (resp. $k((x))((y))$) determined by the function field of the product of two conics over $k(x, y)$ (resp. $k((x))((y))$). More precisely, let $a, b \in k^*$, x, y indeterminates, C_a, C_b be the affine conics associated with the quaternion algebras (a, x) and (b, y) and determined by the equations $t^2 - a = xv^2$, $u^2 - b = yw^2$.

Lemma 2.1. *Let φ a form over k. The following conditions are equivalent:*

1) *The form $\varphi_{k(\sqrt{a}, \sqrt{b})}$ is isotropic.*
2) *The form $\varphi\langle\langle t^2 - a, u^2 - b\rangle\rangle_{k(t,u)}$ is isotropic.*
3) *The form $\varphi\langle\langle x, y\rangle\rangle_{k(x,y)(C_a \times C_b)}$ is isotropic.*
4) *The form $\varphi\langle\langle x, y\rangle\rangle_{k((x))((y))(C_a \times C_b)}$ is isotropic.*

Proof. 1) $\Longrightarrow$ 2). Since $\varphi_{k(\sqrt{a}, \sqrt{b})}$ is isotropic, either $\varphi_{k(\sqrt{b})}$ is isotropic, or $\varphi_{k(\sqrt{b})} \simeq \alpha\langle\langle a\rangle\rangle \perp \psi$ for some $\alpha \in k(\sqrt{b})^*$ and a form ψ over $k(\sqrt{b})$. Since $\langle\langle a, t^2 - a\rangle\rangle = 0$, in both cases the form $\varphi\langle\langle t^2 - a\rangle\rangle_{k(t)(\sqrt{b})}$ is isotropic. Hence, either $\varphi\langle\langle t^2 - a\rangle\rangle_{k(t)}$ is isotropic, or $\varphi\langle\langle t^2 - a\rangle\rangle \simeq \beta\langle\langle b\rangle\rangle \perp \tau$ for some $\beta \in k(t)^*$ and a form τ over $k(t)$. Again, in both cases the form $\varphi\langle\langle t^2 - a, u^2 - b\rangle\rangle$ is isotropic over $k(t, u)$.

2) $\Longrightarrow$ 3). Obvious, in view of the equations $t^2 - a = xv^2$, $u^2 - b = yw^2$.

3) $\Longrightarrow$ 4). Obvious.

4) $\Longrightarrow$ 1). Since the extension $k(\sqrt{a}, \sqrt{b})((x))((y))(C_a \times C_b)/k(\sqrt{a}, \sqrt{b})((x))((y))$ is purely transcendental, and the form $\varphi\langle\langle x, y\rangle\rangle_{k(\sqrt{a}, \sqrt{b})((x))((y)))(C_a \times C_b)}$ is isotropic, the form $\varphi\langle\langle x, y\rangle\rangle_{k(\sqrt{a}, \sqrt{b})((x))((y))}$ is isotropic as well. Hence the form $\varphi_{k(\sqrt{a}, \sqrt{b})}$ is also isotropic. $\square$

Under the notation of Lemma 2.1 we have the following

Theorem 2.2. *The following conditions are equivalent:*

1) *The form $\varphi_{k(\sqrt{a}, \sqrt{b})_{an}}$ is defined over k.*
2) *The form $\varphi\langle\langle x, y\rangle\rangle_{k(x,y)(C_a \times C_b)_{an}}$ is defined over $k(x, y)$.*
3) *The form $\varphi\langle\langle x, y\rangle\rangle_{k((x))((y))(C_a \times C_b)_{an}}$ is defined over $k((x))((y))$.*

Moreover, if these conditions are fulfilled, and $(\varphi_{k(\sqrt{a}, \sqrt{b})})_{an} \simeq \tau$, where τ is a form over k, then

$$\varphi\langle\langle x, y\rangle\rangle_{k(x,y)(C_a \times C_b)_{an}} \simeq \tau\langle\langle x, y\rangle\rangle_{k(x,y)(C_a \times C_b)}$$

and

$$\varphi\langle\langle x, y\rangle\rangle_{k((x))((y))(C_a \times C_b)_{an}} \simeq \tau\langle\langle x, y\rangle\rangle_{k((x))((y))(C_a \times C_b)}.$$

Proof. We induct on $\dim\varphi$. If $\dim\varphi = 0$, i.e. the form φ is empty, then all the conditions hold. 1) $\Longrightarrow$ 2). Suppose $(\varphi_{k(\sqrt{a},\sqrt{b})})_{an} \simeq \tau_{k(\sqrt{a},\sqrt{b})}$, where τ is a form over k. Then $\varphi - \tau \in W(k(\sqrt{a},\sqrt{b})/k)$, i.e. $\varphi - \tau = \langle\langle a\rangle\rangle\psi_1 + \langle\langle b\rangle\rangle\psi_2$, where ψ_1, ψ_2 are forms over k ([ELW]). Then, obviously,

$$\varphi\langle\langle x,y\rangle\rangle_{k(x,y)(C_a\times C_b)} = \tau\langle\langle x,y\rangle\rangle_{k(x,y)(C_a\times C_b)}.$$

By Lemma 2.1 the form $\tau\langle\langle x,y\rangle\rangle_{k(x,y)(C_a\times C_b)}$ is anisotropic, which implies

$$\varphi\langle\langle x,y\rangle\rangle_{k(x,y)(C_a\times C_b)_{an}} \simeq \tau\langle\langle x,y\rangle\rangle_{k(x,y)(C_a\times C_b)}.$$

1) $\Longrightarrow$ 3). The same argument with replacement of $k(x,y)$ by $k((x))((y))$.

3) $\Longrightarrow$ 1). We may assume that the form $\varphi_{k(\sqrt{a},\sqrt{b})}$ is isotropic. By Lemma 2.1 the form $\varphi\langle\langle t^2-a, u^2-b\rangle\rangle$ is isotropic, hence the form $\varphi\langle\langle x,y\rangle\rangle_{k((x))((y))(C_a\times C_b)}$ is isotropic as well. Therefore, by the hypothesis

$$\varphi\langle\langle x,y\rangle\rangle_{k((x))((y))(C_a\times C_b)} = \Phi_{k((x))((y))(C_a\times C_b)},$$

where Φ is a form over $k((x))((y))$, $\dim\Phi < 4\dim\varphi$. Since

$$W(k((x))((y))) = W(k)\oplus xW(k)\oplus yW(k)\oplus xyW(k),$$

we have

$$\Phi \simeq \psi_1 \perp x\psi_2 \perp y\psi_3 \perp xy\psi_4,$$

where ψ_i are forms over k. On the other hand,

$$\varphi\langle\langle x,y\rangle\rangle - \Phi \in W(k((x))((y))(C_a\times C_b)/k((x))((y))).$$

Therefore, in view of [ELW] we get

$$\varphi\langle\langle x,y\rangle\rangle - \Phi = \langle\langle a,x\rangle\rangle\Phi_a + \langle\langle b,y\rangle\rangle\Phi_b, \qquad (*)$$

where Φ_a and Φ_b are some forms over $k((x))((y))$. Since

$$\Phi_a \simeq \tau_1 \perp x\tau_2 \perp y\tau_3 \perp xy\tau_4,$$

$$\Phi_b \simeq \rho_1 \perp x\rho_2 \perp y\rho_3 \perp xy\rho_4,$$

for some forms τ_i and ρ_i over k, it is easy to see that just as in Theorem 1.1 we can replace in $(*)$ x, y and $k((x))((y))$ by t^2-a, u^2-b and $k(t,u)$ respectively. It follows that

$$\varphi\langle\langle t^2-a, u^2-b\rangle\rangle = \psi_1 \perp (t^2-a)\psi_2 \perp (u^2-b)\psi_3 \perp (t^2-a)(u^2-b)\psi_4. \quad (**)$$

Notice that $\sum_{i=1}^{4}\dim\psi_i < 4\dim\varphi$. Multiplying if needed all the parts of equality $(**)$ by $-(t^2-a)$, or $-(u^2-b)$, or $(t^2-a)(u^2-b)$ we may assume that $\dim\psi_4 < \dim\varphi$. We have

$$\varphi_{k(\sqrt{a},\sqrt{b})} = \partial_{t^2-a,u^2-b}(\varphi\langle\langle t^2-a, u^2-b\rangle\rangle) = \psi_4,$$

where $\partial_{t^2-a,u^2-b} : W(k(t,u)) \to W(k(\sqrt{a},\sqrt{b}))$ is the composition of the residues maps at t^2-a and u^2-b. It follows by [ELW] that

$$\varphi - \psi_4 = \langle\langle a\rangle\rangle f_1 + \langle\langle b\rangle\rangle f_2$$

for some forms f_1, f_2 over k. Hence

$$\psi_4\langle\langle x,y\rangle\rangle_{k((x))((y))(C_a\times C_b)} = \varphi\langle\langle x,y\rangle\rangle_{k((x))((y))(C_a\times C_b)}.$$

By the induction hypothesis the form $(\psi_{4k(\sqrt{a},\sqrt{b})})_{an}$ is defined over k. Since $(\psi_{4k(\sqrt{a},\sqrt{b})})_{an} \simeq (\varphi_{k(\sqrt{a},\sqrt{b})})_{an}$, we are done.

2) $\Longrightarrow$ 1). We may assume that the form $\varphi_{k(\sqrt{a},\sqrt{b})}$ is isotropic. Then by Lemma 2.1 the form $\varphi\langle\langle x,y\rangle\rangle_{k(x,y)(C_a\times C_b)}$ is isotropic, hence by the hypothesis of 2)

$$\varphi\langle\langle x,y\rangle\rangle_{k((x))((y))(C_a\times C_b)} = \Phi_{k((x))((y))(C_a\times C_b)}$$

for some form Φ defined over $k((x))((y))$, $\dim\Phi < 4\dim\varphi$. The argument in the proof of implication 3) $\Longrightarrow$ 1) shows that the form $\left(\varphi_{k(\sqrt{a},\sqrt{b})}\right)_{an}$ is defined over k. $\square$

Now let φ be a 4-dimensional anisotropic form over k, $a,b \in k^*$, C_1, C_2 the affine conics over $k(x)$ associated with the quaternion algebras (a,x), (b,x) and determined by the equations $t^2-a = x\alpha^{-2}$, $u^2-b = x\beta^{-2}$.

Theorem 2.3. *The following two conditions are equivalent:*

1) *The form $\varphi_{k(\sqrt{a},\sqrt{b})}$ is isotropic and $\left(\varphi_{k(\sqrt{a},\sqrt{b})}\right)_{an}$ is defined over k.*

2) *The form $\varphi\langle\langle x\rangle\rangle_{k(x)(C_1\times C_2)}$ is isotropic.*

Proof. 1) $\Longrightarrow$ 2). Let $\psi_{k(\sqrt{a},\sqrt{b})} \simeq (\varphi_{k(\sqrt{a},\sqrt{b})})_{an}$ for some form ψ over k. We have $\varphi-\psi \in W(k(\sqrt{a},\sqrt{b})/k)$, hence $\varphi-\psi = \langle\langle a\rangle\rangle\varphi_1 + \langle\langle b\rangle\rangle\varphi_2$ for some forms φ_1, φ_2 over k. Thus $\varphi\langle\langle x\rangle\rangle_{k(x)(C_1\times C_2)} = \psi\langle\langle x\rangle\rangle_{k(x)(C_1\times C_2)}$, and so the form $\varphi\langle\langle x\rangle\rangle_{k(x)(C_1\times C_2)}$ is isotropic.

2) $\Longrightarrow$ 1). Let $F = k(t)$. We may assume that $\varphi \simeq \langle 1,-u,-v,uvd\rangle$. Let further C be a conic corresponding to the quaternion algebra (b, t^2-a). In view of the equations determining the conics C_1, C_2 it is easy to see that the form $\varphi\langle\langle t^2-a\rangle\rangle_{F(C)}$ is isotropic. Since the extension $F(C)/F$ is excellent and the Clifford algebra of $\varphi\langle\langle t^2-a\rangle\rangle$ is similar to (d, t^2-a), we have

$$\varphi\langle\langle t^2-a\rangle\rangle = f\langle\langle d, t^2-a\rangle\rangle + \langle\langle b, t^2-a\rangle\rangle\tau,$$

where $f \in k(t)^*$ and τ is an even-dimensional form over k. It is easy to see that consequently

$$\langle\langle u,v,t^2-a\rangle\rangle \equiv \langle\langle d,t^2-a,Q\rangle\rangle + \langle\langle b,t^2-a,R\rangle\rangle \ (mod\ I^4(F)) \qquad (*)$$

for some squarefree nonzero polynomials $Q, R \in k[t]$. We may assume that $\deg Q + \deg R$ is as small as possible. In particular, t^2-a divides neither Q, nor R. We are going to prove that there exists $c \in k^*$ such that $R = cQ$. Indeed, suppose that there exists a prime monic polynomial p such that $p|Q$, but $p \not| R$. Put $\pi \simeq \langle\langle d, t^2-a, p\rangle\rangle$. Then

$$\partial_p(\pi) = \partial_p(\langle\langle d, t^2-a, Q\rangle\rangle) \in \partial_p(\langle\langle u, v, t^2-a\rangle\rangle) + I^3(k_p) = I^3(k_p),$$

which implies $\partial_p(\pi) = 0$. Also $\partial_\infty(\pi) = 0$, where ∂_∞ is the residue map associated with the local parameter t^{-1} at the infinity point. By the Scharlau reciprocity law for $W(F)$ ([Sch], Ch.6, Th. 3.5) we have $s_{t^2-a}(\partial_{t^2-a}(\pi)) = 0$, where s_{t^2-a} is the transfer determined by the k-linear map $k_{t^2-a} \to k$, taking 1 to 0, and t to 1. Thus, $\langle\langle d, p\rangle\rangle \in \mathrm{res}_{k_{t^2-a}/k} W(k)$. Hence $\langle\langle d, p\rangle\rangle = \langle\langle d, e\rangle\rangle \in W(k_{t^2-a})$ for some $e \in k^*$, which implies that

$$\langle\langle d, t^2-a, p\rangle\rangle = \langle\langle d, t^2-a, e\rangle\rangle,$$

and we can replace Q by $e\frac{Q}{p}$ in $(*)$. But this is a contradiction to minimality of $\deg Q + \deg R$. Quite similarly one can prove that the case where $p \not| Q$, but $p|R$ is impossible as well. Therefore, we conclude that $R = cQ$ for some $c \in k^*$. It follows that

$$\langle\langle u, v, t^2-a\rangle\rangle \equiv \langle\langle d, t^2-a, Q\rangle\rangle + \langle\langle b, t^2-a, cQ\rangle\rangle = \langle\langle bd, t^2-a, Q\rangle\rangle + \langle\langle b, c, t^2-a\rangle\rangle \ (mod\ (I^4(F)).$$

Hence $\partial_p\langle\langle bd, t^2-a, Q\rangle\rangle = 0$ if $p \neq t^2-a$. Comparing residues we get that

$$\langle\langle bd, t^2-a, Q\rangle\rangle = \langle\langle bd, t^2-a, e\rangle\rangle$$

for some $e \in k^*$. Thus

$$\langle\langle u, v, t^2-a\rangle\rangle \equiv \langle\langle bd, e, t^2-a\rangle\rangle + \langle\langle b, c, t^2-a\rangle\rangle \ mod(I^4(F)).$$

Taking the residue at t^2-a we get

$$\langle\langle u, v\rangle\rangle_{k(\sqrt{a})} \equiv (\langle\langle bd, e\rangle\rangle + \langle\langle b, c\rangle\rangle)_{k(\sqrt{a})} \ modI^3k(\sqrt{a})), \qquad (**)$$

hence $\langle\langle u, v\rangle\rangle_{k(\sqrt{a},\sqrt{b})} = \langle\langle d, e\rangle\rangle_{k(\sqrt{a},\sqrt{b})}$. Finally, $(**)$ implies

$$\begin{aligned}\langle 1, -u, -v, uvd\rangle_{k(\sqrt{a},\sqrt{b})} &= \langle\langle u, v\rangle\rangle_{k(\sqrt{a},\sqrt{b})} + \langle uvd, -uv\rangle_{k(\sqrt{a},\sqrt{b})} \equiv (\langle\langle d, e\rangle\rangle + \langle uvd, -uv\rangle)_{k(\sqrt{a},\sqrt{b})} \\ &= \langle\langle d\rangle\rangle\langle 1, -e, -uv\rangle_{k(\sqrt{a},\sqrt{b})} \equiv -euv\langle\langle d\rangle\rangle_{k(\sqrt{a},\sqrt{b})}(modI^3(k(\sqrt{a},\sqrt{b}))),\end{aligned}$$

hence $(\varphi_{k(\sqrt{a},\sqrt{b})})_{an} \simeq -euv\langle\langle d\rangle\rangle_{k(\sqrt{a},\sqrt{b})}$. □

Theorem 2.3 can be reformulated without mentioning any conic at all. Namely, keeping the notation in this theorem we have the following

Corollary 2.4. *Let α, β, z be indeterminates. Let further*

$$p(\alpha, \beta, z) = z^4 - 2(a\alpha^2 + b\beta^2)z^2 + (a\alpha^2 - b\beta^2)^2 \in k(\alpha, \beta)[z]$$

be the minimal polynomial of $\alpha\sqrt{a} + \beta\sqrt{b}$ over the field $k(\alpha, \beta)$. The following two conditions are equivalent.

1) *The form $\varphi_{k(\sqrt{a},\sqrt{b})}$ is isotropic and the form $\varphi_{k(\sqrt{a},\sqrt{b})_{an}}$ is defined over k.*
2) *The form $\varphi\langle\langle p(\alpha, \beta, z)\rangle\rangle$ is isotropic over $k(\alpha, \beta, z)$.*

Proof. We have $\alpha^2(t^2-a) = \beta^2(u^2-b) = x$, which implies $(\alpha t - \beta u)(\alpha t + \beta u) = a\alpha^2 - b\beta^2$. Put $z = \alpha t - \beta u$. Then it is easy to see by straightforward computation that $k(x)(C_1 \times C_2) = k(\alpha, \beta, z)$ and $4xz^2 = p(\alpha, \beta, z)$. Theorem 2.3 then yields the result. □

Corollary 2.5. *Suppose the form* $\varphi\langle\langle p(\alpha,\beta,z)\rangle\rangle$ *from Corollary* 2.4 *is anisotropic over* $k(\alpha,\beta,z)$. *Let* ψ *be a form over* k, $\dim\psi \geq 5$. *Then the form* $\varphi\langle\langle p(\alpha,\beta,z)\rangle\rangle$ *remains anisotropic over the field* $k(\psi)$.

Proof. Recall that we may assume that $\varphi \simeq \langle 1, -u, -v, uvd\rangle$. Suppose the form $\varphi\langle\langle p(\alpha,\beta,z)\rangle\rangle_{k(\alpha,\beta,z)(\psi)}$ is isotropic. By Corollary 2.4 the form $\varphi_{k(\psi)(\sqrt{a},\sqrt{b})}$ is isotropic and its anisotropic part is defined over $k(\psi)$. It is easy to see that then $(u,v)_{k(\psi)} = (a,x)+(b,y)+(d,z)$ for some $x,y,z \in k(\psi)^*$. By [S2], Prop.5 $(u,v) = (a,x')+(b,y')+(d,z')$ for some $x',y',z' \in k^*$. Then the form $\varphi_{k(\sqrt{a},\sqrt{b})}$ is isotropic, and the form $(\varphi_{k(\sqrt{a},\sqrt{b})})_{an}$ is defined over k. By Corollary 2.4 the form $\varphi\langle\langle p(\alpha,\beta,z)\rangle\rangle$ is isotropic. □

Remark. In Corollary 2.5 the condition $\dim\psi \geq 5$ is essential. For example, if $\psi \simeq \varphi$, then $\varphi\langle\langle p(\alpha,\beta,z)\rangle\rangle$ is, obviously, isotropic.

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REFERENCES

[A] Arason J. Kr., *Excellence of $F(\varphi)/F$ for 2-fold Pfister forms*, Appendix II in [ELW] (1977).

[ELW] Elman R., Lam T.Y. and Wadsworth A.R., *Amenable fields and Pfister extensions*, Queen's Papers Pure Appl. Math. **46** (1977), 445-491.

[M] Merkurjev A.S., *Simple algebras and quadratic forms*, Soviet Math. Docl. **38** (1992), 215-221.

[R] Rost M., *Quadratic forms isotropic over the function field of a conic*, Math. Ann. **288** (1990), 511-513.

[RST] L.H. Rowen, A.S. Sivatski, J.-P. Tignol, *Division algebras over rational function fields in one variable*, Algebra and Number Theory, Proceedings of the Silver Jubilee Conference 2003 (2005), 158-180.

[Sch] Scharlau W., *Quadratic and Hermitian forms*, Springer, Berlin Heidelberg New York (1985).

[Si1] Sivatski A.S., *Nonexcellence of certain field extensions*, Journal of Mathematical Sciences **145** (2007), no. 1, 4811-4817.

[Si2] Sivatski A.S., *On some elements of the Brauer group of a conic*, Journal of Mathematical Sciences **145** (2007), no. 1, 4818-4822.

[T] Tignol J.P., *Algebres indecomposables d'exposant premier*, Advances in Math. **65** (1987), no. 3, 205-228.

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