Coincidence in Poincaré polynomials of projective homogeneous varieties

Justin Martel

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Abstract

These notes express some basic facts concerning the Poincaré polynomials of projective homogeneous varieties (PHVs) as arising from their Bruhat-Tits cellular decomposition.

1 Introduction

1.1. For G a semisimple linear algebraic group over an algebraically closed field and P a parabolic subgroup, one obtains a projective homogeneous G-variety $X = G/P$. The remarkable Bruhat-Tits theorem says that such a homogeneous algebraic space has a cellular decomposition into affine cells $\mathbb{A}^n$ parametrized by the corresponding subgroup W_P of the Weyl group W (where, as usual, W is the Weyl group associated to the underlying root system $\Phi(G)$). Moreover, each affine cell $C(w)$ has dimension equal to the length $\ell(w)$. So Bruhat-Tits says the cellular structure of a PHV is controlled by its Weyl group (and length function).

The Poincaré polynomial of X is that polynomial whose k-th coefficient is the number of k-dimensional affine pieces in its cellular decomposition (equivalently, the number of elements $w \in W_P$ with $\ell(w) = k$). We say projective homogeneous varieties (PHVs) X, Y are *coincident* when their Poincaré polynomials coincide and express their coincidence as $X \sim_c Y$. Coincidence is the central theme of these notes; it's how one asks two PHVs to 'be made of the same stuff'.

The main result of this paper is the determination of all coincidences among simple PHVs (ie. $X \simeq G/P$, for G simple): this classification is split into our first and second main theorems, (4.3) and (4.5).

This paper divides as follows: after this introduction, the second section on lists and cyclotomic polynomials presents some terminology and lemmas that will be basic to these notes; this section is elementary, obvious, and can be quickly read through. Our third section consists of Chevalley's Factorization, which is the main stimulus to these notes; through it alone does the arithmetic of Poincaré polynomials arising from PHVs become appreciable. In our fourth section on coincidences we establish our first and second main theorems through two separate chains of elementary and tedious reasoning. Finally, in the last section we explain how these coincidences are related to the problem of the classification of Grothendieck-Chow motives of PHVs.

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2 Lists and Cyclotomic Polynomials

We introduce the necessary definitions and lemmas to be used repeatedly throughout these notes.

2.1 Definition. A *list* is a set in which multiplicities are taken into account. ie. while the set $\{a, a\}$ may be identified with $\{a\}$, the *list* $[a]$ is to be distinguished from $[a, a]$. Some authors call our lists *multisets*.

A finite list $\mathcal{A}$ with objects a_i shall be denoted $[a_1, \ldots, a_n]$. For our purposes a list shall always refer to a finite list of integers which are > 1. The notions of *membership, containment*, *join* $(\vee)$ and *meet* $(\wedge)$ are evident, as is the *difference* $\mathcal{A} - \mathcal{B}$ of lists whenever $\mathcal{B} \subset \mathcal{A}$. In practise we shall write the join of lists $\mathcal{A}, \mathcal{A}'$ as the concatentation (or product) $\mathcal{A}\mathcal{A}'$. Given lists $\mathcal{A}, \mathcal{B}$, the formal quotient $\mathcal{A}/\mathcal{B}$ will be called a *rational list*. We identify a list $\mathcal{A}$ with the rational list $\mathcal{A}/\emptyset$, where $\emptyset$ is the *empty list*. A rational list $\mathcal{A}/\mathcal{B}$ is said to be *reduced* if $\mathcal{A} \wedge \mathcal{B}$ is the empty list. Identifying the lists $\mathcal{A}/\mathcal{B}$ and $\mathcal{A} \vee \mathcal{A}'/\mathcal{B} \vee \mathcal{A}'$, we find that any rational list $\mathcal{A}/\mathcal{B}$ can be identified with a reduced rational list. The following definitions are convenient:

2.2 Definition. A list consisting of consecutive (even) integers shall be called *(even) contiguous*.

2.3 Definition. For an integer $a > 1$, its *divisors list* $Div^+[a]$ is that list of integers $d > 1$ for which $d|a$. An arbitrary list $\mathcal{A} = [a_1, \ldots, a_k]$ has *divisors list* $Div^+(\mathcal{A}) = \bigvee_i Div^+[a_i]$. We set $Div^+\emptyset = \emptyset$.

Recall the nth cyclotomic polynomials $\Phi_n(t)$ are those polynomials irreducible in $\mathbb{Z}[t]$ and recursively satisfying $\Phi_0 = \Phi_1 = 1$ and $\Pi_{d|n}\Phi_d(t) = t^n - 1$ for $n \geq 1$.

2.4 Lemma. *Let $\mathcal{A}, \mathcal{B}$ be lists of integers > 1 such that*

$$\prod_{a \in \mathcal{A}} \frac{1 - t^a}{1 - t} = \prod_{b \in \mathcal{B}} \frac{1 - t^b}{1 - t}.$$

Then $\mathcal{A} = \mathcal{B}$.

Proof. By definition the above equality becomes $\Pi_{Div^+(\mathcal{A})}\Phi_d = \Pi_{Div^+(\mathcal{B})}\Phi_d$. The irreducibility of Φ_d then says $Div^+(\mathcal{A}) = Div^+(\mathcal{B})$. Considering maximal elements in $Div^+(\mathcal{A})$, one can retrieve $\mathcal{A}$. Similarly, one retrieves $\mathcal{B}$. Being retrieved from the same divisors list, $\mathcal{A} = \mathcal{B}$. □

We easily find the following useful

2.5 Lemma. *Let $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$ be lists of integers > 1. Then*

$$\prod_{Div^+(\mathcal{A})} \Phi_d / \prod_{Div^+(\mathcal{B})} \Phi_d = \prod_{Div^+(\mathcal{A}')} \Phi_d / \prod_{Div^+(\mathcal{B}')} \Phi_d$$

iff $\mathcal{A}/\mathcal{B}, \mathcal{A}'/\mathcal{B}'$ reduce to identical rational lists.

Proof. One has $Div^+(\mathcal{A})Div^+(\mathcal{B}') = Div^+(\mathcal{A}\mathcal{B}') = Div^+(\mathcal{A}'\mathcal{B})$ iff $\mathcal{A}\mathcal{B}' = \mathcal{A}'\mathcal{B}$ (refer to proof of (2.4). Having cancelled common factors between $\mathcal{A}, \mathcal{B}$ and $\mathcal{A}', \mathcal{B}'$, one then sees that the above equality occurs iff $\mathcal{A} = \mathcal{A}'$ and $\mathcal{B} = \mathcal{B}'$, qed. □

3 Chevalley's Factorization

3.1. To be concerned with G only up to isogeny, we shall without qualms consider G as determined by its underlying root system $\Phi(G)$.

To a given PHV $X \simeq G/P$, we associate the formal quotient

$$\frac{\mathscr{D}_1 \cdots \mathscr{D}_r}{\mathscr{D}'_1 \cdots \mathscr{D}'_s},$$

where $\mathscr{D}_1 \cdots \mathscr{D}_r$ is the decomposition of $\Phi(G)$ into irreducible root systems, and $\mathscr{D}'_1 \cdots \mathscr{D}'_s$ likewise the decomposition of $\Phi(P)$. Such decompositions will be called *root system words.* Through the Chevalley Factorization we associate a list $\mathcal{A}_i$ of so-called *basic invariant degrees* to each irreducible root system type $\mathscr{D}_i$ (cf.[5]).

3.2. Let (W, S) be a Coxeter system (cf.[1], [5]). Then a length function $\ell(\cdot)$ is defined relative to S on W. We define the *Poincaré polynomial $P(W,t)$ of W* to be

$$P(W,t) = \sum_{w \in W} t^{\ell(w)}.$$

This definition is independent of S cf.[5].

The following proposition–due to Chevalley–is for us crucial.

3.3 Proposition (Chevalley's factorization)**.** *Let (W, S) be an irreducible Coxeter system. Then*

$$P(W,t) = \prod_{i=1}^{n} \frac{1 - t^{d_i}}{1 - t},$$

where $[d_1, \ldots, d_n]$ is a list of basic invariant degrees assigned according to the following table:

Type	$d_1, \ldots, d_n$
A_n	$2, 3, \ldots, n+1$
B_n, C_n	$2, 4, 6, \ldots, 2n$
D_n	$2, 4, 6, \ldots, 2n-2, n$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
G_2	$2, 6$

3.4 Remark. (i) For our purposes the Coxeter system (W, S) will arise as $W = W(\Phi(G)$, for G a semisimple linear algebraic group, and S the simple reflections corresponding to a base Δ of $\Phi(G)$ (or equivalently, a choice B of Borel subgroup of G).
(ii) Henceforth we shall denote by a_n the degree list $[2, 3, \ldots, n]$ of the root system A_{n-1}.

3.5. Now suppose $\mathscr{D}_1, \ldots, \mathscr{D}_r, \mathscr{D}'_1, \ldots, \mathscr{D}'_s$ are irreducible root system types with associated basic invariant degree lists $\mathcal{A}_1, \ldots, \mathcal{A}_r, \mathcal{A}'_1, \ldots, \mathcal{A}'_s$. Set $\mathcal{A} = \vee_i \mathcal{A}_i$ and $\mathcal{A}' = \vee_j \mathcal{A}'_j$. For a (PHV) $X \simeq \mathscr{D}_1 \cdots \mathscr{D}_r / \mathscr{D}'_1 \cdots \mathscr{D}'_s$, we set

$$P(X, t) := P(\Pi \mathscr{D}_i / \Pi \mathscr{D}'_j, t) = \prod_{Div^+(\mathcal{A}) - Div^+(\mathcal{A}')} \Phi_d.$$

As immediate corollary to (2.5) we obtain

3.6 Proposition. *PHVs are coincident iff their reduced rational degree lists coincide.*

4 Coincidences

The question of 'coincidence' is the question of "to what extent do Poincaré polynomials determine their PHVs?' We use the following terminology to state our main results:

4.1 Definition. Let $\mathscr{D}$ be a root systems and $\mathscr{D}' \subset \mathscr{D}$ a sub-root system. Then the *$\mathscr{D}'$-reduction of $\mathscr{D}$* is the formal quotient $\mathscr{D}/\mathscr{D}'$. If $\mathscr{D}, \mathscr{D}'$ have rational degree lists $\mathcal{A}, \mathcal{A}'$, then $\mathscr{D}/\mathscr{D}'$ has degree list $\mathcal{A}/\mathcal{A}'$. Finally, if $\mathscr{D}''$ is another sub-root system $\subset \mathscr{D}$ and disjoint from $\mathscr{D}'$, then the *$\mathscr{D}''$-reduction of $\mathscr{D}/\mathscr{D}'$* is the formal quotient $\mathscr{D}/\mathscr{D}'\mathscr{D}''$ of $\mathscr{D}$ by the sum $\mathscr{D}'\mathscr{D}''$.

4.2 Definition. We say simple PHVs $\mathscr{D}/\mathscr{D}'$ and $\mathscr{D}''/\mathscr{D}'''$ are *type-distinct* if *type*$(\mathscr{D}) \neq$ *type*$(\mathscr{D}'')$, and we say they are *type-similar* otherwise.

Our first main result classifies all type-distinct coincidences among simple PHVs:

4.3 Theorem (First Main Theorem)**.** *The only type-distinct coincidences between simple PHVs are those between:*

- $A_{2\ell-1}/A_{2\ell-2}$ *and* $B_\ell/B_{\ell-1}$ $(\ell > 1)$*;*
- $D_{\ell+1}/A_\ell$ *and* $B_\ell/A_{\ell-1}$ $(\ell > 1)$*;*
- G_2/A_1 *and* A_5/A_4*,* B_3/B_2*.*

For the expression of our second main result we introduce the following device:

4.4 Definition. A *swindle* is the substitution of the coincidence $B_m A_{2m-2} \sim_c B_{m-1}A_{2m-1}$ (for $m \geq 1$) within a root system word $\mathscr{D}\mathscr{D}' \cdots$.

The *swindle* formalizes the fact that all type-similar coincidences are consequences of the same 'trick'. On degree lists the *swindle* is just the relation $na_{n-1} = a_n$

We may now present our second main theorem classifying all type-similar coincidences among simple PHVs:

4.5 Theorem (Second Main Theorem)**.** *A type-similar coincidence between simple PHVs is either a coincidence* $B_k/\mathscr{D} \sim_c B_k/\mathscr{D}'$*, where* $\mathscr{D} \sim_c \mathscr{D}'$ *with* $\mathscr{D}, \mathscr{D}'$ *of the form* $B \prod A$*; or a coincidence* $D_k/\mathscr{D} \sim_c D_k/\mathscr{D}'$*, where* $\mathscr{D} \sim_c \mathscr{D}'$ *with* $\mathscr{D}, \mathscr{D}'$ *of the form* $D \prod A$*. Moreover, all semisimple coincidences* $B \prod A \sim_c B \prod A$ *and* $D \prod A \sim_c D \prod A$ *are generated by swindles.*

In (4.6) is expressed an observation which organizes our casewise proofs of (4.3), (4.5):

4.6 Lemma. *Let* $\mathscr{D}$ *be a connected Dynkin diagram and* $\mathscr{D}'$ *a connected subdiagram. Then either* $\mathscr{D}'$ *has type* A_k*, with* $k \leq rank(\mathscr{D})$*, or* $\mathscr{D}'$ *is the unique subdiagram having rank* $\leq rank(\mathscr{D})$ *and* $type(\mathscr{D}') = type(\mathscr{D})$*.*

Proof. By inspection. □

Using (4.1) we may rephrase (4.6): a connected Dynkin diagram $\mathscr{D}$ admits only A.-type reductions after at most one $\mathscr{D}$-type reduction.

Proof of First Main Theorem. A coincidence between irreducible types comes in three 'flavours': classical-classical, classical-exceptional, and exceptional-exceptional. Correspondingly we split the proofs of our first main theorem (4.3) 'flavourwise' into some lemmas and sublemmas.

4.7 Sublemma. *The only coincidences produced by A-type reductions between pairwise type-distinct PHVs among* A*,* B/B_k*, and* $D/D_{k'}$ *(with* $k \geq 2$*,* $k' \geq 4$*) is that between* $A_{2\ell-1}/A_{2\ell-2}$ *and* $B_\ell/B_{\ell-1}, (\ell > 1)$*.*

Proof. Any quotient A_ℓ/A_k has a degree list containing an odd integer unless ℓ is odd and $k = \ell - 1$. Since a nontrivial quotient B/B has a degree list consisting of even integers, coincidence only occurs when $k = \ell - 1$. This produces the coincident pair $A_{2\ell-1}/A_{2\ell-2}$, $B_\ell/B_{\ell-1}$.

A coincidence between reductions of B/B and D_ℓ/D_k must see ℓ even. If $2k \leq \ell$, then the even ℓ occurs twice in the numerator of D/D's reduced rational degree list. By (3.6) D/D cannot coincide with any A-type reductions of B/B (since numerator of reduced rational degree list is without any repetitions). When $2k > \ell$, B/B is coincident with D_ℓ/D_k through A-type reductions only if $B/B = B_{\ell-1}/B_{k-1}$ (since we need maximal elements of the numerators in the reduced rational degree lists to coincide). But no further A-type reduction can produce the necessary k on the denominator since $B_{\ell-1}/B_{k-1}$ does not have an A_{k-1}-type connected component (because $\ell < 2k$). Consequently no A-type reductions of $B/B,\ D/D$ are coincident.

Finally, consider A-type reductions of A and D_ℓ/D_k ($\ell \geq 4$). If ℓ is even, then coincidence would see no odd elements in the numerator of any reductions of A, which means $A/A = A_{2\ell-3}/A_{2\ell-4}$. But then A/A has numerator [2ℓ-2] and admits no further A-type reduction, and hence not coincident with any A-type reduction of D/D. If ℓ is odd, a coincidence between A-type reductions of $A,\ D/D$ would require the (reduced) numerator of D/D to be contiguous. However for $\ell \geq 4$ this does not occur. □

4.8 Sublemma. *The only type-distinct coincidences produced by A-type reductions between A, B, and D is that between $D_{\ell+1}/A_\ell$ and $B_\ell/A_{\ell-1}$.*

Proof. The reduced rational list of any A-type reduced B has an even-contiguous numerator. So by (3.6) it's clear that no A-type reductions generates a coincidence between A and B. Likewise we see that no A-type reductions of A and D_ℓ are coincident for $\ell \geq 4$. Furthermore it's clear by (3.6) that for $k \leq k'$ no A-type reductions render either B_k and $B_{k'}$ or D_k and $D_{k'}$ coincident. Finally any A-type reduction of $D_{\ell+1}$ is coincident with an A-type reduction of B only if we first reduce $D_{\ell+1}$ to $D_{\ell+1}/A_\ell$ by (3.6) and after which we may reduce no more; here we find the coincidence with $B_\ell/A_{\ell-1}$. □

4.9 Lemma (Classical-Classical (4.3))**.**

Proof. All together (4.7)–(4.8) yield (4.3) in the classical-classical case. □

4.10 Lemma (Classical-Exceptional (4.3))**.**

Proof. The exceptional types have reduced rational degree lists with numerators which are neither contiguous nor even-contiguous. Their reductions to quotients $\mathscr{D}/\mathscr{D}$ have contiguous or even-contiguous numerator degree lists only in the case G_2/A_1. □

4.11 Lemma (Exceptional-Exceptional (4.3))**.**

Proof. The basic invariant degree lists of the exceptional irreducible types have distinct max elements, except in the case of E_6 and F_4. But E_6 has an odd basic invariant degree, while F_4 cannot. □

Together (4.7)-(4.12) establish our First Main Theorem (4.3).

Proof of Second Main Theorem. We begin with some observations:

4.12 Sublemma. *The simple PHVs $\mathscr{D}/\mathscr{D}'$ and $\mathscr{D}''/\mathscr{D}'''$ are type-similar coincident only if $\mathscr{D}=\mathscr{D}''$ and $\mathscr{D}'\sim_c\mathscr{D}'''$.*

Proof. For type-similar irreducible root systems $\mathscr{D},\mathscr{D}''$, one observes the numerators of their reduced rational degree lists have distinct maximal elements unless $\mathscr{D}=\mathscr{D}''$. Now by (3.6) they must have been reduced by coincident PHVs. □

4.13 Sublemma. *There are no type-similar coincidence among reductions of the simple PHVs of type $A,E,F,$ or G.*

Proof. By (4.12) this follows by realizing there are no type-similar semisimple coincidences between root systems of the form $\prod A$, $E\prod A$, $F\prod A$, or $G\prod A$. □

It is convenient now to introduce the following

4.14 Definition. We define the *sup.rank* of a product $\mathscr{D}_1\mathscr{D}_2\mathscr{D}_3\cdots$ of irreducible root systems $\mathscr{D}_i$ to be $sup_i rank(\mathscr{D}_i)$.

Together with (4.12) and (4.13) the following lemma establishes (4.5).

4.15 Lemma. *All semisimple coincidences $B\prod A\sim_c B\prod A$ and $D\prod A\sim_c D\prod A$ are generated by swindles.*

Proof. We proceed recursively by reducing via swindles the *sup.rank*s of coincident semisimple pairs. This establishes that indeed all such coincidences are generated by swindles.

Proceeding first with the B-type case, we consider a coincidence

$$B_{k-1}\prod_I A_{i-1}\sim_c B_m\prod_J A_{j-1}. \qquad (*_0)$$

With respect to degree lists $(*_0)$ becomes

$$\prod_I a_i=[2k,\ldots,2m]_{even}\prod_J a_j. \qquad (*_0)$$

After cancelling common factors, the maximal degree M occurring in $(*_0)$ is seen to be $2m$ (hence the *sup.rank* is finite). Consequently there is a unique factor of $a_M=a_{2m}$ on the left-hand side of $(*_0)$, all of which may be rewritten as

$$a_M\prod_{\leq M-1}a_i=[2k,\ldots,M]_{even}\prod_{\leq M-1}a_j \qquad (*_1)$$

There are now two cases: If (a) $2k=M$, then after the swindle $a_M=Ma_{M-1}$ $(*_1)$ becomes

$$a_{M_1}\prod_{\leq M-1}a_i=\prod_{\leq M-1}a_j.$$

By (4.12), then $I_{\leq M-1}.M = J_{\leq M-1}$.

Otherwise, if (b) $2k < M$, then $2k < M-1$. As left-hand side of $(*_1)$ contains a $M-1$ factor and $[2k, \dots, M]_{even}$ does not, we find that after having cancelled common factors there is a unique a_{M-1} factor on the right-hand side. The swindle $a_M = Ma_{M-1}$ then reduces $(*_1)$ to a coincidence with *sup.rank* $< M$.

Clearly, in either of the cases we reduce the *sup.rank*s of the coincidence via a swindle.

We proceed similarly in the D-type case, by reducing the *sup.rank* through the successive use of swindles. Except for these details (which the reader may provide for themselves), this completes our proof. □

As corollary we've established our second main theorem.

Coincidence in Semisimple Case The question of coincidence among semisimple PHVs is not entirely resolved by Lemma 3.6. In addition to the coincidences generated by our main theorems there are many *non-interesting* coincidences which arise from products of A_k/A_{k-1} (a PHV with rational degree list $[k+1]$). For instance, it is immediate that the exceptional root system types E, F, G are all coincident with products of the classical types A/A. One also has the obvious coincidence between E_7A_{29}/A_{28} and $E_8A_5A_9/A_4A_8$. Supposing one calls a coincidence *interesting* if not generated by PHVs of the form A_k/A_{k-1}, there is the obvious question: are all interesting coincidences generated by those coincidences of (4.3) and (4.5)?

While (3.6) does not resolve the question of semisimple coincidence, it does permit the following obvious conclusions in the exceptional-exceptional case:

4.16 Proposition. *There are no coincidences between PHVs generated by the root system types E_6, E_7, E_8 and those generated by the root system types F_4, G_2.*

4.17 Proposition. *There are no coincidences between PHVs generated by the root system types E_i, F_4, G_2 and E_j, F_4, G_2 for $i \neq j$ (unless of course there is no E-type factor, ie. the PHV was generated by F_4, G_2).*

5 Application to motives

In the present section we relate our main theorems (4.3) and (4.5) with the problem of classification of Grothendieck-Chow motives of PHV. [1] Note that this question has been intensively studied during the last few years (see [3],[4], [7],[9],[10]).

Following (2.2) of [10], the Chow motive $\mathscr{M}(X)$ of the PHV $X = G/P$ satisfies an isomorphism

$$\mathscr{M}(X) \simeq \bigoplus_{i=0}^{dim X} \mathbb{Z}(i)^{\oplus a_i(X)},$$

[1] For an introduction to Chow motives, see [8]

where $Z(i)$ denotes the Tate motive shifted by i and

$$\sum_{i \geq 0} a_i(X) t^i$$

is the Poincare polynomial of X. By (2.3) of [10], motives are the same iff they have the same Poincaré polynomials. In view of our main theorems (4.3) and (4.5) we immediately obtain the

5.1 Corollary. *Let X, Y be simple PHVs. Then X, Y have isomorphic motives iff X, Y belong to the coincidence classes given in our main theorems.*

5.2 Remark. Observe that the coincidence of Poincare polynomials (and hence, their motives) doesn't imply that the respective varieties are isomorphic. For instance, the variety from (4.3) of type A_{2l-1}/A_{2l-2} is a projective space of dimension $2l-1$ and the variety of type B_l/B_{l-1} is the projective quadric of the same dimension. Both varieties have the same Poincare polynomial $\frac{1-t^{2l}}{1-t}$ but are not isomorphic.

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