THE UNRAMIFIED BRAUER GROUP AND THE PYTHAGORAS NUMBER OF HYPERELLIPTIC CURVES WITH GOOD REDUCTION DEFINED OVER HENSELIAN VALUED FIELDS.

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Abstract. Let F be a henselian valued field with real closed residue field, C a hyperelliptic curve over F with good reduction. A set of independend generators for the two component of the Brauer group of a curve C with good reduction defined by an affine equation $y^2 = f(x)$, $\deg f(x)$ is odd, is calculated. As an application it is shown that the Pythagoras number of the function field $F(C)$ of such curves is 2 if $F(C)$ is a real field and 3 if $F(C)$ is a non-real field.

1. Introduction and preliminary statements

A theorem of E. Becker states that a field F is hereditarily pythagorean if and only if the Pythagoras number of the rational function field in one variable $F(x)$ is equal to 2. (The Pythagoras number of a field being, if it exists, the smallest integer p such that every sum of squares in the field is a sum of p squares in the field.) It is therefore natural to study the Pythagoras number of functions fields of curves over hereditarily pythagorean fields. In previous papers [11], [12], the authors considered this problem. In [11] the first and the third author considered function fields of conics over any hereditarily pythagorean field, and in [12] the Pythagoras numbers of function fields of hyperelliptic curves with good reduction over the real formal power series field $\mathbb{R}((t))$ were determined. In fact (as mentioned in [12]) the proofs work in completely the same way if $\mathbb{R}((t))$ is replaced by a henselian discrete valued field with real closed residue field.

The results in [12] were based on the observation that the Pythagoras number of a function field $F(C)$, of a smooth projective curve C over a hereditarily pythagorean field F, is strictly bigger than 2 if and only if there is a quaternion division algebra of the form $(-1, \sum f_i^2)$, $f_i \in F(C)$. Moreover in the case F is a henselian valued field with real closed residue field, R, such a quaternion division algebra must represent an element in the Brauer group of the curve C, $Br(C)$. An essential part of the paper [12] consisted in determining the two component of $Br(C)$, for hyperelliptic curves C defined by an affine equation $y^2 = f(x)$ with good reduction, i.e. with f such that its reduction $\overline{f}$ has the same degree as f and no multiple roots in the algebraic closure $R(\sqrt{-1})$ of R.

In [2] similar results on the Pythagoras number were obtained by different methods, the condition that the hyperelliptic curves were defined by an equation with good reduction could be replaced by weaker assumptions on the defining equation. Also partial results for

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more general hereditarily pythagorean fields are obtained in that paper. In [2] the Brauer group is not used, the underlying idea however is similar namely that the existence of sums of squares that are not equal to sums of two squares yields the existence of unramified 2-fold Pfister forms over the function field $F(C)$.
Recently K. Becher, D. Grimm and the second author observed ([1]) that using the local-global results obtained by Harbatter, Hartmann and Krashen, [7] (see also [5]), it is possible to prove that if $F = R((t))$ with R a real closed field and $F(C)$ is the function field of any smooth projective curve over F then Pythagoras number of $F(C)$ is either 2 or 3. The approach is entirely different, the Brauer group of the curve does enter in any way. This answer to the question is of course very complete. However the fact that $F = R((t))$ (i.e., F a field complete with respect to a discrete valuation with real closed residue field) seems to be essential for the method. In this note we return to the original method. We consider hyperelliptic curves with good reduction over, not necessarily discrete, henselian valued fields F with real closed residue field and extend the results obtained in [12] to this more general situation. Apart from extending the results on the Pythagoras number we think that the determination of the two component of the Brauer group of the curve in this more general situation is important in its own right.

In section 2 some preliminary results are given and notation is fixed. Let F be a henselian valued field with real closed residue field, let C be a hyperelliptic curve over F with good reduction. In section 3 a set of generators for the two component of the Brauer group of C is given. It is shown that for every non-trivial class $\mathcal{D}$ in ${}_2\mathrm{Br}(C)$ either $\mathcal{D} \otimes_{F(C)} F(C)(\sqrt{-1})$ is non trivial in ${}_2\mathrm{Br}(F(C)(\sqrt{-1}))$ or for some real point P on C the local algebra $\mathcal{D}_P$ (the completion of $\mathcal{D}$ with respect to the discrete F-valuation on $F(C)$ corresponding to the point P) is non trivial. In section 4 it is shown that the Pythagoras number of $F(C)$ is 2. This result is completed by noting that the Pythagoras number is also 2 in the case $F(C)$ is a real field, and that the Pythagoras number is 3 in the case $F(C)$ is a non real field.

2. Notation, terminology and preliminary results.

Let F be a field of characteristic 0. With ΣF^2 we denote the set of all sums of squares of elements of F. For $a \in \Sigma F^2$ the minimal $n \in \mathbb{N}$ such that a is a sum of n squares in F is called the length of a and denoted by $l(a)$. A field F is called a real field (also called a formally real field) if $-1 \notin \Sigma F^2$. If F is a nonreal field then $-1 \in \Sigma F^2$, and $s(F) = l(-1)$ is called the level of F.
The number $p(F) = \sup\{l(a) \mid a \in \Sigma F^2\}$ is called the Pythagoras number of F. If F is nonreal, then it is known that $s(F) \leq p(F) \leq s(F)+1$. A real field F is called pythagorean if $p(F) = 1$, and hereditarily pythagorean (abbreviated h. p.) if any real algebraic extension of F is pythagorean and any nonreal extension contains $\sqrt{-1}$. We recall that a field F is a h. p. field iff the Pythagoras number of the rational function field in one variable over F is 2, ([3, Chap. III, theorem 4]). For further properties of h. p. fields we refer the reader to [3]. We refer to [12], the paper to which this note is a sequel, for further information on hyperelliptic curves and the Brauer group of such curves.
We give one further characterization of h. p. fields,

Proposition 2.1. *The following conditions are equivalent*
a) F is hereditarily pythagorean.
b) F admits a henselian valuation such that the residue field k is hereditarily pythagorean with at most two orderings.

Proof. See Prop. 3.5 in [4]. □

The work presented in [12] and [2], and the observation [1] that the local-global result of [7] implies that the Pythagoras number of function fields of hyperelliptic curves over complete discrete valued fields with real closed residue field is either 2 or 3, leads to the natural question wether or not the Pythagoras number of a function of a curve over a h. p. field k is less than or equal to 3. In view of the above proposition considering henselian valued fields in general is a step towards answering this question. In the next section we extend the results obtained in [12] on the Brauer group of curves with good reduction over henselian discrete valued fields with real closed residue field to the case of henselian valued fields (we keep the condition that the residue field is real closed). In section 4 we apply this to determine the Pythagoras number of the function fields of such curves.

We will use the following fact,

Lemma 2.2. *Let F be a pythagorean field. Then any square class of $F(\sqrt{-1})$ is represented by an element from F^*.*

Proof. We need to prove that for any $u + v\sqrt{-1} \in F(\sqrt{-1})$ there exists $a, b, c \in F$ such that $c(u + v\sqrt{-1}) = (a + b\sqrt{-1})^2$. If $v = 0$, then we set $c = 1/u$, $a = 1$, $b = 0$. Thus without loss of generality we can assume that $v = 1$. Hence we have the following system of equations

$$\begin{cases} uc = (a^2 - b^2); \\ c = 2ab. \end{cases}$$

Set $b = 1$. Since F is pythagorean, then $u^2 + 1 \in F^{*2}$. Then there exists $a \in F$ such that $a^2 - 1 = 2ua$. □

Throughout the rest of the paper F is a henselian valued field with real closed residue field R, so F is a h. p. field. The valuation ring is denoted by O_v, m_v is its maximal ideal. The residues of elements in $u \in O_v$ are denoted by $\overline{u}$. As the following lemma states all the square classes of F, except $-1 \bmod F^{*2}$ are of positive value,

Lemma 2.3. *Let F be a henselian valued field with real closed residue field R. Let m_v be the maximal ideal of the valuation ring O_v. Let $\{-1\} \cup \{\alpha_j\}_{j\in J}$, with $\{\alpha_j\}_{j\in J} \subset O_v$, be a basis of the $\mathbb{F}_2$-vector space F^*/F^{*2}. Then for all $j \in J$, $\alpha_j \in m_v$.*

Proof. Assume that there exists an $\alpha_j, j \in J$ such that $\alpha_j \notin m_v$, then $\overline{\alpha}_j \notin R^{*2}$ and $-\overline{\alpha}_j \notin R^{*2}$. Contradicting the fact that R is real closed. □

3. The two component of the Brauer group of hyperelliptic curves with good reduction over F.

Let C be a smooth hyperelliptic curve over F given by an affine equation of the form

$$y^2 = f(x), \quad f(x) \in O_v[x] \tag{3.1}$$

with

$$f(x) = (x - a_1)...(x - a_{2n+1})g_1(x)...g_m(x), \tag{3.2}$$

or

$$f(x) = \pm g_1(x)...g_m(x), \tag{3.3}$$

where the residues $\overline{a}_i \neq \overline{a}_j$ if $i \neq j$, and $\overline{g}_l$, $l = 1, \dots, m$ are different quadratic irreducible polynomials over the residue field R with splitting field $R(\sqrt{-1})$. Without loss of generality we also assume that the coefficients a_i, $i = 1, \dots, 2n + 1$ are enumerated in such a way that $\overline{a}_i < \overline{a}_j$ if $i < j$.
These conditions on the lineair and the quadratic factors of f imply that the reduced polynomial $\overline{f}$ has no multiple roots and that it is of the same degree as f. This means that C is a curve with good reduction with respect to the valuation v. Actually if an equation $y^2 = f(x)$, $f(x) \in O_v[x]$, has good reduction up to a birational transformation of C the polynomial $f(x)$ can be chosen in the form (3.2) or (3.3). We refer to [12] for details, it is easy to see that the arguments given there remain valid if the henselian valuation on F is not necessarily a discrete valuation.
If f is of the form (3.2) and of the form (3.3) with the plus sign, $F(C)$ is a real field and C contains an F-rational point ($(a_1, 0)$ in the first case and $(0, \sqrt{f(0)})$ in the second case, where $\sqrt{f(0)} \in F$ since every sum of two squares is a square in F). If the equation is of the form (3.3) with the minus sign then $F(C)$ is a nonreal field of level 2 since $-f$ is a sum of two squares.

Remark 3.1. Let C be a curve over F. Our definition of good reduction refers to the existence of an affine equation of type $y^2 = f(x)$ such that the reduced equation $y^2 = \overline{f}(x)$ defines an irreducible affine curve of the same degree.
In general the notion of good reduction refers to the existence of "good" models of the curve over the valuation ring O_v. In the case the valuation ring is discrete it is known that for hyperelliptic curves both notions coincide, we refer to [8] for more details.

Throughout the remaining part of this section we assume the hyperelliptic curve C over F is defined by an affine equation of the form (3.2). We determine the two component of the Brauer group of such curves C by presenting a set of generators for the two-torsion part of the unramified Brauer group, ${}_2\mathrm{Br}(C)$ (cf. also [12]).
Consider the following list of classes in ${}_2\mathrm{Br}(F(C))$.

$$\mathcal{A}_i = (-1, (x - a_1)(x - a_i))_{F(C)}, \ i = 2, \dots, 2n + 1; \tag{3.4}$$

$$\mathcal{B}_i^k = (\alpha_k, (x - a_1)(x - a_i))_{F(C)}, \ i = 2, \dots, 2n + 1, k \in J; \tag{3.5}$$

$$\mathcal{C}_j^l = (\alpha_l, g_j(x))_{F(C)}, \ j = 1, \dots, m, l \in J. \tag{3.6}$$

Where $\{-1\} \cup \{\alpha_j\}_{j\in J}$ is a basis of $\mathbb{F}_2$-vector space F^*/F^{*2}, with $\alpha_j, j \in J$, elements of the maximal ideal m_v of the heselian valuation ring in F (cf. lemma 2.3).
We note that all the algebras in the list are unramified and therefore represent elements in the Brauer group of the curve C. This can be seen as follows, the Brauer group of the curve coincides with the intersection of the kernels of the ramification map associated to the F-discrete valuations w on $F(C)$. For quaternion algebras the map is given by the ramification formula (for instance cf. [12]),

$$\partial_w((a,b)_{F(C)}) = \overline{(-1)^{w(a)w(b)}\left(\frac{a^{w(b)}}{b^{w(a)}}\right)} \in \kappa(w)^*/\kappa(w)^{*2}, \tag{3.7}$$

with $\kappa(w)$ the residue field of w. If we apply the formula to the algebras $\mathcal{A}_i$, $\mathcal{B}_i^k$ and $\mathcal{C}_j^l$ in the list it is clear that the ramification map is zero for all discrete F-valuations w because the valuation of $(x-a_1)(x-a_i)$ and that of $g_j(x)$ is even for all w, this follows from the equation $y^2 = f(x)$.
We claim that the classes of the algebras in the list together with the classes of the constant algebras, by which we mean the elements in the natural image of ${}_2\mathrm{Br}(F)$, generate ${}_2\mathrm{Br}(C)$. The proof is based on the following theorem ([10]).

Theorem 3.2. *Let K be a field of characteristic not 2. Let H be a hyperelliptic curve over K with affine part defined by the equation*

$$y^2 = g(x),$$

where $g(x) = (x-b)h_1(x)...h_r(x)$, deg $g(x)$ is odd and $h_1(x),...,h_r(x)$ are irreducible monic polynomials in $K[x]$. Let for $i = 1,\dots,r$, b_i be a root of $h_i(x)$ in the algebraic closure of K. Then any element of ${}_2\mathrm{Br}(H)$ can be represented as a tensor product of a constant algebra (i.e. algebra defined over K) and an algebra of the form

$$\otimes_{i=1}^r \mathrm{cor}_{K(b_i)(H)/K(H)}((c_i,(x-b)(x-b_i))_{K(b_i)(H)}),$$

with $c_i \in K(b_i)$.
Conversely, any algebra of the above form is an element of ${}_2\mathrm{Br}(H)$.
Let $\mathcal{D}$ be such an algebra then the Brauer class of $\mathcal{D}$ is trivial if and only if the algebra is Brauer equivalent to a tensor product

$$D_1 \otimes \cdots \otimes D_r,$$

with $D_i \in \mathrm{cor}_{K(b_i)(H)/K(H)}((s_i,(x-b)(x-b_i))_{K(b_i)(H)})$, $s_i = \prod_j (x_j - b_i)^{n_j}$ such that $\sum_j n_j(x_j,y_j)$ is a K-divisor of degree 0 on H whose support does not contain any Weierstrass points of H.
(The Weierstrass points of a hyperelliptic curve $y^2 = h(x)$ are the points $(\theta, 0)$ with θ a root of $h(x)$ in case $\deg h(x)$ is even and the same points plus the point P_∞ at infinity in case $\deg h(x)$ is odd.)

Corollary 3.3. *Let C be an hyperelliptic curve over F defined by the affine equation (3.2) with odd $\deg f(x)$.*

(a) *Then every element of* ${}_2\mathrm{Br}\, C$ *is represented as a tensor product of a constant algebra and an algebra from*

$$\begin{aligned}&\mathrm{cor}_{F(\sqrt{-1})(C)/F(C)}((c_1,(x-a_1)(x-\theta_1))_{F(\sqrt{-1})(C)})\otimes\dots\\&\quad\otimes\mathrm{cor}_{F(\sqrt{-1})(C)/F(C)}((c_m,(x-a_1)(x-\theta_m))_{F(\sqrt{-1})(C)})\\&\quad\otimes(d_2,(x-a_1)(x-a_2))_{F(C)}\otimes\dots\otimes(d_{2n+1},(x-a_1)(x-a_{2n+1}))_{F(C)},\end{aligned}$$

with θ_i *a root of* g_i *and* $d_i \in F$, $c_i \in F(\sqrt{-1})$.
(b) *Every element of* ${}_2\mathrm{Br}(C)$ *is equivalent to a tensor product of a constant algebra and algebras taken from the set* $\mathcal{A}_i$, $i = 2,\dots,2n+1$, $\mathcal{B}_j^k$, $j = 2,\dots,2n+1$, $k \in J$, $\mathcal{C}_j^l$, $j = 1,\dots,m$, $l \in J$.

Proof. The first point (a) follows directly from theorem 3.2.
(b) Since the splitting field of all the polynomials $g_j(x)$ is $F(\sqrt{-1})$ it follows form lemma 2.2 that all the elements c_i, $i = 1,\dots,m$, can be taken in F. The corestriction can then be calculated using the formula

$$\mathrm{cor}_{F(\sqrt{-1})(C)/F(C)}((a,(x-a_1)(x-\theta_i))_{F(\sqrt{-1})(C)}) = (a, g_i(x))_{F(C)}$$

for $a \in F$. The statement now follows from part (a) of the corollary. □

Lemma 3.4. *Let* $\mathcal{E}$ *be a non-trivial constant algebra over* $F(C)$, *then for every* F*-rational point* P *on* C *the algebra* $\mathcal{E}\otimes_{F(C)} F(C)_P$ *is non trivial in the Brauer group of the completion* $F(C)_P$.

Proof. The completion of $F(C)$ at an F-rational point is isomorphic to the field of formal power series $F((\pi))$ over F. A central simple algebra is a division algebra iff its reduced norm polynomial has no non-trivial zero. Since the reduced norm polynomial of a central division F-algebra remains without non-trivial zeros over $F((\pi))$, the restriction map $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(F((\pi)))$ is injective. □

Lemma 3.5. *For every* $j = 1,\dots,n$, *the algebras* $\mathcal{A}_{2j}$ *and* $\mathcal{A}_{2j+1}$ *are Brauer equivalent.*

Proof. For $i = 1,\dots,m$, let $\theta_i = u_i + v_i\sqrt{-1}$, with $u_i, v_i \in F$, be a root of the quadratic polynomial g_i occurring in the factorisation of f, cf. equation 3.2. Let j be any element in $\{1,\dots,n\}$. Choose $d_1, d_2 \in F$ such that $\overline{d}_1 \neq \overline{u}_i$, $\overline{d}_2 \neq \overline{u}_i$, for all $i = 1,\dots,m$, $\overline{a}_{2j-1} < \overline{d}_1 < \overline{a}_{2j}$, and $\overline{a}_{2j+1} < \overline{d}_2 < \overline{a}_{2j+2}$ if $1 \leq j < n$, $\overline{a}_{2j+1} < \overline{d}_2$ if $j = n$. It follows from this choice of d_1 and d_2 that $f(d_1)$ and $f(d_2)$ are squares in F. Hence $(d_1, \sqrt{f(d_1)})$ and $(d_2, \sqrt{f(d_2)})$ are F-rational points of the affine part of C. The divisor $(d_1, \sqrt{f(d_1)}) - (d_2, \sqrt{f(d_2)})$ is an F-rational divisor of degree 0 and its support does not contain any Weierstrass points. Applying theorem 3.2 we see that the following algebra

$$\begin{aligned}\mathcal{D} \;=\; &\otimes_{i=1}^m \mathrm{cor}_{F(\sqrt{-1})(C)/F(C)}(((d_1-\theta_i)(d_2-\theta_i),(x-a_1)(x-\theta_i))_{F(\sqrt{-1})(C)}\otimes\\&\otimes_{l=2}^{2n+1}((d_1-a_l)(d_2-a_l),(x-a_1)(x-a_l))_{F(C)}\end{aligned}$$

is trivial in ${}_2\mathrm{Br}(C)$.
The choice of d_1 and d_2 also yields that for $s \neq 2j, 2j+1$, we have that $(\overline{d}_1 - \overline{a}_s)(\overline{d}_2 - \overline{a}_s) > 0$, implying that $(d_1 - a_s)(d_2 - a_s) \equiv 1 \bmod F^{*2}$. For $s = 2j, 2j+1$ we have $(\overline{d}_1 - \overline{a}_s)(\overline{d}_2 -$

$\overline{a}_s) < 0$ implying that $(d_1 - a_s)(d_2 - a_s) \equiv -1 \bmod F^{*2}$. For $i = 1, \ldots, m$, the elements $(d_1 - \theta_i)(d_2 - \theta_i)$ have nonzero residues in $R(\sqrt{-1})$, so the elements are in $F(\sqrt{-1})^{*2}$ by the henselian property. It follows that

$$\begin{aligned} \mathcal{D} &\sim ((d_1 - a_{2j})(d_2 - a_{2j}), (x - a_1)(x - a_{2j}))_{F(C)} \otimes \\ &\quad ((d_1 - a_{2j+1})(d_2 - a_{2j+1}), (x - a_1)(x - a_{2j+1}))_{F(C)} \\ &\sim \mathcal{A}_{2j} \otimes \mathcal{A}_{2j+1} \end{aligned}$$

So the latter algebra is trivial, which proves that $\mathcal{A}_{2j} \sim \mathcal{A}_{2j+1}$ in ${}_2\mathrm{Br}(C)$. □

Lemma 3.6. *Let $\mathcal{A} = \otimes_{i \in I} \mathcal{A}_i$, with $I \subset \{2, 4, \ldots, 2n\}$, and $\mathcal{E}$ a non-trivial constant algebra.*
(1) For the algebras $\mathcal{A} \otimes \mathcal{E}^\epsilon$, $\epsilon \in \{0, 1\}$, there exists an F-rational point P on C such that $\mathcal{A} \otimes \mathcal{E}^\epsilon \otimes F(C)_P$ is non-trivial, where $F(C)_P$ is the completion of $F(C)$ at the discrete valuation associated to the point P.
(2) For any non-trivial constant algebra $\mathcal{E}$, the algebras $\mathcal{A}$, and $\mathcal{A} \otimes \mathcal{E}$ represent non-trivial elements in ${}_2\mathrm{Br}(F(C))$.

Proof. It is clear that (2) follows from (1).
Write $\mathcal{A} = \mathcal{A}_{i_1} \otimes \cdots \otimes \mathcal{A}_{i_r}$ in such a way that $i_j < i_k$ if $j < k$. Choose $c \in F$ such that $\overline{a}_{i_r - 1} < \overline{c} < \overline{a}_{i_r}$. Then $\overline{f}(\overline{c}) > 0$ and therefore $f(c) \in F^{*2}$. Hence $P = (c, \sqrt{f(c)})$ is an F-rational point of C. It is well known that the completion of $F(C)$ at P is isomorphic to $F((x - c))$, so it is a real field. We obtain (since for a unit $a \in O_v[[x - c]]$, $(x - a) \equiv (c - a) \bmod F((x - c))^{*2}$),

$$\begin{aligned} \mathcal{A}_P &:= \mathcal{A} \otimes_{F(C)} F(C)_P \\ &\sim (-1, (c - a_1)(c - a_{i_r}))_{F(C)_P} \\ &\sim (-1, -1)_{F(C)_P} \not\sim 1. \end{aligned}$$

Let $\mathcal{E}$ be a constant algebra over $F(C)$ not in the Brauer class of the algebra $(-1, -1)_{F(C)}$, we have (using lemma 3.4) that

$$(\mathcal{A}_P \otimes \mathcal{E}_{F(C)_P}) \sim ((-1, -1)_{F(C)_P} \otimes \mathcal{E}_{F(C)_P}) \not\sim 1.$$

Finally for the rational point $Q = (d, \sqrt{f(d)})$, where d is chosen such that $\overline{d} > \overline{a}_{2n+1}$, we have

$$\mathcal{A}_Q \otimes (-1, -1)_{F(C)_Q} \sim (-1, -1)_{F(C)_Q} \not\sim 1.$$

This completes the proof of the lemma. □

Lemma 3.7. *The algebras*

$$\mathcal{D} = \left(\bigotimes_{(i,k) \in M_1} \mathcal{B}_i^k \right) \otimes \left(\bigotimes_{(j,l) \in M_2} \mathcal{C}_j^l \right) \otimes \mathcal{E} \otimes_{F(C)} F(\sqrt{-1})(C),$$

where M_1 and M_2 are finite subsets respectively of $\{2, \ldots, 2n+1\} \times J$ and $\{1, \ldots, m\} \times J$ and $\mathcal{E}$ is any constant F-algebra, represent non-trivial elements in ${}_2\mathrm{Br}(F(\sqrt{-1})(C))$.

Proof. At first assume that $\mathcal{E} \otimes F(\sqrt{-1})(C)$ is trivial. We can rewrite $\mathcal{D}$ as

$$\bigotimes_k \left(\alpha_k, \prod_{i,(i,k)\in M_1} (x-a_1)(x-a_i) \cdot \prod_{j,(j,k)\in M_2} g_j(x) \right)_{F(C)} \otimes_{F(C)} F(\sqrt{-1})(C),$$

We will use the induction on the number of α_k's appearing in the latter algebra. Assume that there is only one α_k, i.e. $\mathcal{D} = (\alpha_k, h(x))_{F(C)}$, where

$$h(x) = \prod_{i,(i,k)\in M_1} (x-a_1)(x-a_i) \cdot \prod_{j,(j,k)\in M_2} g_j(x).$$

Let v be the unique valuation on $F(\sqrt{-1})$ which extends the valuation of F (uniqueness follows from the fact that the residue field of any extension is $R(\sqrt{-1})$). Defining, for $g(x) = \sum a_l x^l \in F(\sqrt{-1})[x]$, $w(g(x)) := \min_l\{v(a_l)\}$ we obtain a valuation w on $F(\sqrt{-1})(x)$ extending the valuation v.
We have $v(\alpha_k) = w(\alpha_k) \notin 2\Gamma_v$, where Γ_v is the valuation group of v. Indeed, if $v(\alpha_k) \in 2\Gamma_v$, then there exists $\beta \in F(\sqrt{-1})$ such that $v(\alpha_k) = v(\beta^2)$. Hence $v(\alpha_k/\beta^2) = 0$. Since $R(\sqrt{-1})$ is algebraically closed, one has that $\alpha_k/\beta^2 \in F(\sqrt{-1})^{*^2}$. Contradiction.
The residue field of w is $R(\sqrt{-1})(x)$. Since the polynomial $\overline{f}(x)$ has no multiple roots, $\overline{f}(x)$ is not a square in the residue field $R(\sqrt{-1})(x)$. Hence, the field $F(\sqrt{-1})(C) = F(\sqrt{-1})(x)(\sqrt{f(x)})$ is an unramified extension of $F(\sqrt{-1})(x)$. So the valuation w on $F(\sqrt{-1})(x)$ can be extended to a valuation w' on $F(\sqrt{-1})(C)$ in such a way that $w'(\alpha_k) \notin 2\Gamma_{w'}$. The residue field $R(\sqrt{-1})(x)(\sqrt{\overline{f}(x)})$ of w' is a quadratic extension of $R(\sqrt{-1})(x)$.
For the sake of contradiction we assume that the algebra $\mathcal{D} = (\alpha_k, h(x))_{F(C)}$ is trivial in ${}_2\mathrm{Br}(F(\sqrt{-1})(C))$. Then there exist x_1,$x_2 \in F(\sqrt{-1})(C)$ such that $h(x) = x_1^2 - \alpha_k x_2^2$. Since $h(x)$ is monic and since $w'(x_1^2) \in 2\Gamma_{w'}$ and $w'(\alpha_k x_2^2) \notin 2\Gamma_{w'}$, it follows that $0 = w'(h(x)) = w'(x_1^2 - \alpha_k x_2^2) = \min(w'(x_1^2), w'(\alpha_k x_2^2)) = w'(x_1^2)$. So $\overline{h}(x) = \overline{x_1^2}$ in the residue field $R(\sqrt{-1})(x)(\sqrt{\overline{f}(x)})$. Equivalently we find that $\overline{h}(x) \in R(\sqrt{-1})(x)^{*2}$ or $\overline{f}(x)\overline{h}(x) \in R(\sqrt{-1})(x)^{*2}$. Since $\overline{h}(x), \overline{f}(x)\overline{h}(x)$ are elements if $R(x)$, either $\overline{h}(x) \equiv -1 \bmod R(x)^{*2}$ or $\overline{f}(x)\overline{h}(x) \equiv -1 \bmod R(x)^{*2}$. This is impossible since $\overline{f(x)}$ has no multiple roots which implies that $\overline{h}(x)$ and $\overline{f}(x)\overline{h}(x)$ contain irreducible factors to an odd power. We obtained a contradiction implying that the algebra $\mathcal{D}$ is non-trivial in ${}_2\mathrm{Br}(F(\sqrt{-1})(C))$.
Assume that we have proved the non-triviality of $\mathcal{D}$ in the case where there are n elements α_k occurring in the representation of the algebra as a tensor product of quaternions. Let $\mathcal{D}$ be such that there are $n+1$ elements α_k occurring in its representation.
Let also α_s appears in $\mathcal{D}$. Since $\alpha_s \notin -F^{*2}$, then the field $F(\sqrt{\alpha_s})$ is real and hence it is a h.p. field. Note that the elements α_k, $k \neq s$, are linear independent in $\mathbb{F}_2$-vector space $F(\sqrt{\alpha_s})^*/F(\sqrt{\alpha_s})^{*2}$. So we can apply the induction hypothesis for the algebra $\mathcal{D} \otimes F(C)(\sqrt{\alpha_s})$. It follows that the latter algebra is non-trivial. Hence $\mathcal{D}$ is non-trivial. This finishes the case where $\mathcal{E} \otimes F(\sqrt{-1})$ is trivial.

Now assume that $\mathcal{E} \otimes F(\sqrt{-1})$ represents a non-trivial element in $\mathrm{Br}(F(\sqrt{-1}))$. Let P be the F-rational point $(x_0, \sqrt{f(x_0)})$ on C, with the first coordinate $x_0 \in F^*$ such that $\overline{x_0} > \overline{a_i}$, $i = 1, \ldots, 2n+1$, in R. Then $(\mathcal{D} \otimes \mathcal{E}) \otimes_{F(C)} F(\sqrt{-1})(C)_P \sim \mathcal{E} \otimes_{F(C)} F(\sqrt{-1})(C)_P \not\sim 1$, (lemma 3.4). □

Theorem 3.8. *The algebras $\mathcal{A}_{2j}$, $j = 1, \ldots, n$, $\mathcal{B}_i^k$, $i = 2, \ldots, 2n+1$, $k \in J$, and $\mathcal{C}_j^l$, $j = 1, \ldots, m$, $l \in J$, together with the constant algebras form a set of generators for the two component of the Brauer group of the curve C.*

Proof. This follows from corollary 3.3, lemma 3.5 and lemma 3.6. □

4. The Pythagoras number of $F(C)$.

The next lemma links the Brauer group of a curve over F to the Pythagoras number of the function field $F(C)$.

Lemma 4.1. *Let C be any irreducible smooth projective curve over h.p. field F. Let $f_i \in F(C)$, $i = 1, \ldots, r$. Then the quaternion algebra $A = (-1, \sum_{i=1}^r f_i^2)_{F(C)}$ over $F(C)$ is trivial over all completions at F-discrete valuations on $F(C)$, and therefore it is an unramified algebra.*

Proof. Let w be an F-discrete valuation correspondig to a non-real point. Then the residue field of w contains $\sqrt{-1}$ since F is hereditarily pythagorian. It follows that $F(C)_w$ contains $\sqrt{-1}$ and therefore the algebra of the form $(-1, \sum_{i=1}^r f_i^2)_{F(C)}$ is trivial over $F(C)_w$.
Let w be a valuation corresponding to a real-point of C, then the residue field of w is a real extension of F and therefore hereditarily pythagorian. It follows that $F(C)_w$ is hereditarily pythagorian and so $\sum_{i=1}^r f_i^2$ is equal a square, so the quaternion algebra $(-1, \sum_{i=1}^r f_i^2)_{F(C)_w}$ must be trivial. □

Theorem 4.2. *Let F be a field with henselian valuation such that the residue field is real closed. Let also C be a smooth hyperelliptic curve over F with good reduction such that the function field $F(C)$ of C is a real field. Then the Pythagoras number of $F(C))$ is equal to 2.*

Proof. We first deal with the case that C is given by an equation of the form (3.2).
To prove that the Pythagoras number $p(F(C)) = 2$, it is enough to prove that for any sum of squares $\sum f_i^2 \in F(C)$, the quaternion algebra $\mathcal{A} = (-1, \sum f_i^2)_{F(C)}$ is trivial in ${}_2\mathrm{Br}(F(C))$. From lemma 4.1 it follows that $\mathcal{A} \in {}_2\mathrm{Br}(C)$.
We can make use of lemma 3.6 and lemma 3.7.
Since $\mathcal{A} \in {}_2\mathrm{Br}(C)$ we have

$$\mathcal{A} \sim \otimes_{i\in I}\mathcal{A}_i \left(\bigotimes_{(i,k)\in M_1} \mathcal{B}_i^k \right) \otimes \left(\bigotimes_{(j,l)\in M_2} \mathcal{C}_j^l \right) \otimes \mathcal{E}^\epsilon,$$

with $I \subset \{2, 4, \ldots, 2n\}$, M_1 and M_2 finite subsets of respectively, $\{2, \ldots, 2n+1\} \times J$ and $\{1, \ldots, m\} \times J$, $\mathcal{E}$ any constant F-algebra, and $\epsilon \in \{0, 1\}$. If $M_1 \neq \varnothing$ or $M_2 \neq \varnothing$, then $\mathcal{A} \otimes F(\sqrt{-1})(C) \not\sim 1$ by lemma 3.7. But $\mathcal{A} \otimes F(\sqrt{-1})(C) = (-1, \sum f_i^2)_{F(C)} \otimes F(\sqrt{-1})(C) \sim$

1 so it follows that $\mathcal{A} \sim \otimes_{i\in I}\mathcal{A}_i \otimes \mathcal{E}^\epsilon$. Either $\mathcal{A} \sim 1$ or by lemma 3.6 there is an F-rational point P such that $\mathcal{A} \otimes F(C)_P \not\sim 1$. The latter contradicts lemma 4.1 so we obtain that $\mathcal{A} = (-1, \sum f_i^2)_{F(C)} \sim 1$ and therefore $p(F(C)) = 2$.
In the case C is given by the affine equation (3.3) with positive sign we have $f = g^2 + h^2$, $g, h \in F[x]$. In this case $F(C)$ is the totally positive quadratic extension $F(x)(\sqrt{g^2+h^2})$ of $F(x)$. The result now follows from the more general fact that if K is a field with $p(K) = 2$ and if $K(\sqrt{\alpha})$ is a totally positive quadratic extension of K then $p(K(\sqrt{\alpha})) = 2$ (see [6, proposition 3.2]) (note that the reduced height as defined in [6, page 22] is exactly the Pythagoras number). This finishes the proof of our theorem. □

Theorem 4.3. *Let F be a field with non-trivial henselian valuation such that the residue field R is real closed. Let C be a smooth hyperelliptic curve over F with good reduction and such that $F(C)$ is non-real. Then $p(F(C)) = 3$.*

Proof. The hypothesis implies that C is defined by an affine equation of the form $y^2 = -f(x)$, with f a sum of two squares in $F[x]$. It follows that -1 is a sum of two squares in $F(C)$. Thus $s(F(C)) = 2$ and every element of $F(C)$ is a sum of squares. As mentioned in the introduction it is well known that since the level is finite we have $s(F(C)) \le p(F(C)) \le s(F(C))+1$, this because for every element $h \in F(C)$ one has $4h = (h+1)^2 - (h-1)^2$ (see also cf. [9, chap. 7, lemma 1.3]). So to prove the theorem it is enough to find an element in $F(C)$ which is not a sum of two squares.
Consider the algebra $\mathcal{A} = (-1, \alpha)_{F(C)}$, where $\alpha \neq -1$ is an element of the basis of $\mathbb{F}_2$-vector space F^*/F^{*2} (such α exists since F is not real closed). We claim that $\mathcal{A}$ is non-trivial in ${}_2\mathrm{Br}(F(C))$.
Let us, as before, define $w(g(x)) := \min_l\{v(a_l)\}$ for $g(x) = \sum a_l x^l \in F[x]$. We obtain a valuation w on $F(x)$ extending the valuation v. We further extend w to a valuation w' on $F(C)$.
We have $v(\alpha) = w'(\alpha) \notin 2\Gamma_{w'}$, where $\Gamma_{w'}$ is the valuation group of w', cf. page 8. The residue field of w' is equal to $R(x)(\sqrt{-\overline{f}(x)})$.
Now suppose for the sake of contradiction that $\mathcal{A}$ is trivial. Then $-1 = x_1^2 - \alpha x_2^2$, for some $x_1, x_2 \in F(C)$. We have $0 = w'(-1) = w'(x_1^2 - \alpha x_2^2) = \min(w'(x_1^2), w'(\alpha x_2^2)) = w'(x_1^2)$ since $w'(x_1^2) \in 2\Gamma_{w'}$ and $w'(\alpha x_2^2) \notin 2\Gamma_{w'}$. Hence $-1 = \overline{x}_1^2$ in the residue field $R(x)(\sqrt{-\overline{f}(x)})$, or equivalently $-1 \in R(x)^{*2}$ or $\overline{f}(x) \in R(x)^{*2}$. The first is clearly not true and the second is impossible since $\overline{f}(x)$ has no multiple roots. So we obtained the desired contradiction. Thus the algebra $\mathcal{A}$ is non-trivial. Hence α is not a sum of two squares. □

Remark 4.4. If F is a real closed field (i.e., a henselian valued field as in the theorem but with trivial valuation), then it is known by a theorem of Witt (cf. [PD, Theorem 3.4.11]) that for any K, function field in one variable over F, the Pythagoras number $p(K) = 2$.

References

[1] K. Becher, D. Grimm, and J. Van Geel, *Quadratic Forms over Function Fields of Curves over a Complete Field,* preprint 2010.

[2] K. Becher, J. Van Geel, *Sums of squares in function fields of hyperelliptic curves*, Math.Z. **261**, (2009), 829-844.

[3] E. Becker, *Hereditarily-Pythagorean fields and orderings of higher level*, Monografias de Mathematica 29, Rio de Janeiro, 1978.

[4] L. Bröcker, *Über eine Klasse pythagoreischer Körper,* Arch. Math. **23**, (1970), 405-407.

[5] J.-L. Colliot-Thélène, R. Parimala, and V. Suresh, *Patching and Local-Global Principles for Homogeous Spaces over Function Fields of p-adic Curves.* To appear in Commentarii Mathematici Helvetic in 2012.

[6] R. Elman and T.Y. Lam, *Quadratic Forms Under Algebraic Extensions*, Math. Ann. **219**, (1976), 21-42.

[7] D. Harbater, J. Hartmann, and D. Krashen, *Applications of Patching to Quadratic forms and Central Simple Algebras.* Invent. math., **178** (2009), 231-263.

[8] Qing Liu, *Algebraic Geometry and Arithmetic Curves.* Oxford Graduate Texts in Mathematics, 6, Oxford University Press, Oxford, 2002.

[9] A. Pfister, *Quadratic Forms with Applications to Algebraic Geometry and Topology*, LMS Lecture Note Series 217, Cambridge University Press, 1995.

[PD] A. Prestel, C. Delzell, *Positive polynomials. From Hilbert's 17th problem to real algebra*, Springer Monographs in Mathematics, Berlin, Springer 2001.

[10] U. Rehmann, S.V. Tikhonov, and V.I. Yanchevskii, *Two-torsion of the Brauer groups of hyperelliptic curves and unramified algebras over their function fields*, Comm. in Algebra **29** (2001), $n°$ 9, 3971-3987.

[11] S. V. Tikhonov and V. I. Yanchevskiĭ, *Pythagoras numbers of function fields of conics over hereditarily pythagorean fields*, Dokl. Nats. Akad. Nauk Belarusi **47** (2003), $n°2$, 5–8.

[12] S.V. Tikhonov, J. Van Geel, and V.I. Yanchevskii, *Pythagoras numbers of function fields of hyperelliptic curves with good reduction*, Manuscripta Math. **119** (2006), 305-322.

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