A GENERICITY THEOREM FOR ALGEBRAIC STACKS AND ESSENTIAL DIMENSION OF HYPERSURFACES

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ABSTRACT. We compute the essential dimension of the functors $\mathrm{Forms}_{n,d}$ and $\mathrm{Hypersurf}_{n,d}$ of equivalence classes of homogeneous polynomials in n variables and hypersurfaces in $\mathbb{P}^{n-1}$, respectively, over any base field k of characteristic 0. Here two polynomials (or hypersurfaces) over K are considered equivalent if they are related by a linear change of coordinates with coefficients in K. Our proof is based on a new Genericity Theorem for algebraic stacks, which is of independent interest. As another application of the Genericity Theorem, we prove a new result on the essential dimension of the stack of (not necessarily smooth) local complete intersection curves.

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1. INTRODUCTION

Let k be a base field of characteristic 0, K/k be a field extension, and $F(\underline{x})$ be a homogeneous polynomial (which we call a *form*) of degree d in the n variables $\underline{x} = (x_1, \dots, x_n)$, with coefficients in K. We say that F *descends to an intermediate field* $k \subset K_0 \subset K$ if there exists a linear change of coordinates $g \in \mathrm{GL}_n(K)$ such that every coefficient of $F(g \cdot \underline{x})$ lies in K_0.

It is natural to look for a "smallest" subfield K_0 to which a given form $F(\underline{x})$ descends. A minimal such field K_0 with respect to inclusion may not exist, so we ask instead for the minimal transcendence degree $\operatorname{tr}\deg_k K_0$. This number, called

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the *essential dimension* $\operatorname{ed}_k F$ of F, may be thought of as measuring the "complexity" of F. A major goal of this paper is to compute the maximum of $\operatorname{ed}_k F$, taken over all fields K/k and all forms $F(x_1, \dots, x_n)$ of degree d. This integer, usually called the essential dimension $\operatorname{ed}_k \operatorname{Forms}_{n,d}$ of the functor of forms $\operatorname{Forms}_{n,d}$ depends only on n and d; it may be viewed as a measure of complexity of all forms of degree d in n variables.

We will also be interested in a variant of this problem, where the form $F(\underline{x}) \in K[x_1, \dots, x_n]$ of degree d is replaced by the hypersurface

$$H \stackrel{\text{def}}{=} \{(a_1 : \cdots : a_n) \mid F(a_1, \dots, a_n) = 0\} \tag{1.1}$$

in $\mathbb{P}^{n-1}$. Here we say that H descends to K_0 if there exists a linear change of coordinates $g \in \mathrm{GL}_n(K)$ and a scalar $c \in K^*$ such that every coefficient of $cF(g \cdot \underline{x})$ lies in K_0. Once again, the essential dimension $\operatorname{ed}_k(H)$ of H is defined as the minimal value of $\operatorname{tr\,deg}_k K_0$, with the minimum taken over all fields K_0/k such that H descends to K_0. We will be interested in the essential dimension $\operatorname{ed}_k(\operatorname{Hypersurf}_{n,d})$, defined as the maximal value of $\operatorname{ed}_k(H)$, where the maximum is taken over all K/k and all forms $F(\underline{x}) \in K[x_1, \dots, x_n]$ of degree d. Here H is the zero locus of F, as in (1.1).

The study of forms played a central role in 19th century algebra. The problems of computing $\operatorname{ed}_k \operatorname{Forms}_{n,d}$ and $\operatorname{ed}_k \operatorname{Hypersurf}_{n,d}$ are quite natural in this context. However, to the best of our knowledge, these questions did not appear in the literature prior to the (relatively recent) work of G. Berhuy and G. Favi, who showed that $\operatorname{ed}_k \operatorname{Hypersurf}_{3,3} = 3$; see [BF04].

In this paper we compute $\operatorname{ed}_k \operatorname{Forms}_{n,d}$ and $\operatorname{ed}_k \operatorname{Hypersurf}_{n,d}$ for all $n, d \geq 1$. Our main result is as follows.

Theorem 1.1. *Assume that $n \geq 2$ and $d \geq 3$ are integers and $(n,d) \neq (2,3)$, $(2,4)$ or $(3,3)$. Then*

(a) $\operatorname{ed}_k \operatorname{Forms}_{n,d} = \binom{n+d-1}{d} - n^2 + \operatorname{cd}(\mathrm{GL}_n/\mu_d) + 1$.
(b) $\operatorname{ed}_k \operatorname{Hypersurf}_{n,d} = \binom{n+d-1}{d} - n^2 + \operatorname{cd}(\mathrm{GL}_n/\mu_d)$.

The values of $\operatorname{ed}_k \operatorname{Forms}_{n,d}$ and $\operatorname{ed}_k \operatorname{Hypersurf}_{n,d}$ for $n, d \geq 1$ not covered by Theorem 1.1 are computed in Section 8; the results are summarized in the following table.

n	d	$\operatorname{ed}_k \operatorname{Forms}_{n,d}$	$\operatorname{ed}_k \operatorname{Hypersurf}_{n,d}$
arbitrary	1	0	0
1	≥ 2	1	0
arbitrary	2	n	$n-1$
2	3	2	1
2	4	3	2
3	3	4	3

The quantity $\operatorname{cd}(\mathrm{GL}_n/\mu_d)$ which appears in the statement of Theorem 1.1 is the canonical dimension of the algebraic group GL_n/μ_d. For the definition and basic properties of canonical dimension we refer the reader to Section 2.2; see also [BR05, KM06] for a more extensive treatment of this notion. The exact value of $\operatorname{cd}(\mathrm{GL}_n/\mu_d)$ is known in the case where $e \stackrel{\text{def}}{=} \gcd(n,d)$ is a prime power p^j. In

this case

$$\operatorname{cd}(\mathrm{GL}_n/\mu_d) = \begin{cases} p^i - 1, \text{ if } j > 0, \\ 0, \text{ otherwise,} \end{cases}$$

where p^i is the highest power of p dividing n; see [BR05, Section 11]. More generally, suppose $e = p_1^{j_1} \dots p_r^{j_r}$ is the prime decomposition of e (with $j_1, \dots, j_r \geq 1$) and $p_s^{i_s}$ is the highest power of p_s dividing n. A conjecture of J.-L. Colliot-Thélène, N. A. Karpenko, and A. S. Merkurjev [CTKM07, (2)] implies that

$$\operatorname{cd}(\mathrm{GL}_n/\mu_d) = \sum_{s=1}^{r} (p_s^{i_s} - 1). \tag{1.2}$$

This has only been proved if e is a prime power (as above) or $n = 6$ [CTKM07, Theorem 1.3]. In these two cases Theorem 1.1 gives the exact value of $\operatorname{ed}_k \operatorname{Forms}_{n,d}$ and $\operatorname{ed}_k \operatorname{Hypersurf}_{n,d}$. For other n and d Theorem 1.1 reduces the problems of computing $\operatorname{ed}_k \operatorname{Forms}_{n,d}$ and $\operatorname{ed}_k \operatorname{Hypersurf}_{n,d}$ to the problem of computing the canonical dimension $\operatorname{cd}(\mathrm{GL}_n/\mu_d)$. For partial results on the latter problem, see [BR05, Section 11].

The notions of essential dimension for forms and hypersurfaces are particular cases of Merkurjev's general definition of essential dimension of a functor [BF03]. A special case of this, upon which our approach is based, is the essential dimension of an algebraic stack. For background material on this notion we refer the reader to [BRV09]. In particular, $\operatorname{ed}_k \operatorname{Forms}_{n,d} = \operatorname{ed}_k[A_{n,d}/\mathrm{GL}_n]$ and $\operatorname{ed}_k \operatorname{Hypersurf}_{n,d} = \operatorname{ed}_k[\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$, where $A_{n,d}$ is the $\binom{n+d-1}{d}$-dimensional affine space of forms of degree d in n variables and $\mathbb{P}(A_{n,d})$ is the associated $\binom{n+d-1}{d} - 1$ dimensional projective space of degree d hypersurfaces in $\mathbb{P}^{n-1}$. (Here, as in the rest of the paper, we will follow the classical convention of defining the projectivization $\mathbb{P}(V)$ of a vector space V over k as the projective space of lines in V, that is, as $\operatorname{Proj} \operatorname{Sym}_k V^\vee$. In the present context, this seems more natural than Grothendieck's convention of defining $\mathbb{P}(V)$ as $\operatorname{Proj} \operatorname{Sym}_k V$.) The group GL_n naturally acts on these spaces, and $[A_{n,d}/\mathrm{GL}_n]$ and $[\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$ denote the quotient stacks for these actions; see [BRV09, Example 2.6].

The essential dimension of the "generic hypersurface" of degree d is $\mathbb{P}^{n-1}$, i.e., of the hypersurface H_{gen} cut out by the "generic form"

$$F_{\mathrm{gen}}(x_1, \dots, x_n) = \sum_{i_1 + \dots + i_n = d} a_{i_1, \dots, i_n} x_1^{i_1} \dots x_n^{i_n} = 0\,, \tag{1.3}$$

where $a_{i_1,\dots,i_n}$ are independent variables and K is the field generated by these variables over k, was computed in [BR05, Sections 14-15]. The question of computing the essential dimension of the generic form F_{gen} itself was left open in [BR05]. For n and d as in Theorem 1.1 we will show that $\operatorname{ed}_k F_{\mathrm{gen}} = \operatorname{ed}_k H_{\mathrm{gen}} + 1$; see Proposition 3.4.

The key new ingredient in the proof of Theorem 1.1 is the following "Genericity Theorem". Let $\mathcal{X}$ be a connected algebraic stack with quasi-affine diagonal that is smooth of finite type over k, in which the automorphism groups are generically finite (for sake of brevity, we say that $\mathcal{X}$ is *amenable*). Then we can define the *generic essential dimension* of $\mathcal{X}$, denoted by $\operatorname{ged}_k \mathcal{X}$, as the supremum of the essential dimensions of the *dominant* points $\operatorname{Spec} K \to \mathcal{X}$. If $\mathcal{X}$ is Deligne–Mumford, that is, if all stabilizers are finite, then $\operatorname{ed}_k \mathcal{X} = \operatorname{ged}_k \mathcal{X}$; see [BRV09, Theorem 6.1].

This result, which we called the Genericity Theorem for Deligne–Mumford stacks in [BRV09], is not sufficient for the applications in the present paper. Here we prove the following stronger theorem conjectured in [BRV09, Question 6.6].

Theorem 1.2. *Let $\mathcal{X}$ be an amenable stack over k. Let L be a field extension of k, and ξ be an object of $\mathcal{X}(\operatorname{Spec} L)$, such that the automorphism group scheme $\underline{\operatorname{Aut}}_L\, \xi$ is reductive. Then*

$$\operatorname{ed}_k \xi \leq \operatorname{ged}_k \mathcal{X}\,.$$

In particular, if the automorphism group of any object of $\mathcal{X}$ defined over a field is reductive, then $\operatorname{ed}_k \mathcal{X} = \operatorname{ged}_k \mathcal{X}$.

Note that Theorem 1.2 fails if the stabilizers are not required to be reductive (see [BRV09, Example 6.5(b)]), even though a weaker statement may be true in this setting (see Conjecture 5.3). We also remark that the locus of points with reductive stabilizer is constructible but not necessarily open in $\mathcal{X}$. Thus for the purpose of proving Theorem 1.2 it does not suffice to consider the case where all stabilizers are reduced.

Theorem 1.2 implies, in particular, that if the automorphism group of a form $f(x_1, \dots, x_n)$ is reductive then $\operatorname{ed}_k f \leq \operatorname{ed}_k F_{\mathrm{gen}}$. To complete the proof of Theorem 1.1(a) we supplement this inequality with additional computations, carried on in Section 6, which show that forms $f(x_1, \dots, x_n)$ whose automorphism group is not reductive have low essential dimension; for a precise statement, see Theorem 6.3. The proof of Theorem 1.1(b) is more delicate because the quotient stack $[\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$ is not amenable, so the Genericity Theorem cannot be applied to it directly. We get around this difficulty in Section 7 by relating $\operatorname{ed}_k[\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$ to the essential dimension of the amenable stack $[\mathbb{P}(A_{n,d})/\mathrm{PGL}_n]$.

In the last section we use our Genericity Theorem 1.2 to prove a new result on the essential dimension of the stack of (not necessarily smooth) local complete intersection curves, strengthening [BRV09, Theorem 7.3].

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2. Preliminaries

2.1. **Special groups.** A linear algebraic group scheme G over k is said to be *special* if for every extension K/k we have $\mathrm{H}^1(K, G) = \{1\}$. Special groups were studied by Serre [CAS58, Exposé 1] and classified by Grothendieck [CAS58, Exposé 5] (over an algebraically closed field of characteristic 0). Note that G is special if and only if $\operatorname{ed}_k G = 0$; see [TV10, Proposition 4.3].

The group GL_n is special by Hilbert's Theorem 90, and so is the special linear group SL_n. Direct products of special groups are easily seen to be special. Moreover, in characteristic 0 the group G is special if and only if the Levi subgroup of G (which is isomorphic to $G/\mathrm{R_u}\, G$) is special; see [San81, Theorem 1.13]. Here $\mathrm{R_u}\, G$ denotes the unipotent radical of G. We record the following fact for future reference.

Let A be a non-zero nilpotent $n \times n$-matrix with entries in k and G_A be the image of the map $\mathbb{G}_{\mathrm{a}} \to \mathrm{GL}_n$ given by $t \to \exp(tA)$. Note that this map is algebraic, since only finitely many terms in the power series expansion of $\exp(tA)$ are non-zero.

Lemma 2.1.

(a) *The centralizer C of A (or equivalently, of G_A) in GL_n is special.*
(b) *The normalizer N of G_A in GL_n is special.*

Proof. (a) By [Jan04, Propositions 3.10 and 3.8.1] C is a semidirect product $U \rtimes H$, where $U \triangleleft C$ is unipotent and H is the direct product of general linear groups GL_r for various $r \geq 0$; cf., also [McN06, Section 2]. Thus H = Levi subgroup of C is special, and part (a) follows.

(b) The normalizer N acts on $G_A \simeq \mathbb{G}_a$ by conjugation. This gives rise to a homomorphism $\pi\colon N \to \mathbb{G}_m = \underline{\mathrm{Aut}}_k \mathbb{G}_a$ whose kernel is the centralizer $C = C_{\mathrm{GL}_n}(A)$. If π is trivial then $N = C$ is special by part (a). If π is non-trivial then it is surjective, and we have an exact sequence

$$1 \longrightarrow C \longrightarrow N \longrightarrow \mathbb{G}_m \longrightarrow 1\,.$$

The long non-abelian cohomology sequence for H^0 and H^1 associated with this short exact sequence shows that $\mathrm{H}^1(K, N) = \{1\}$ for every field K/k, as desired. ♠

2.2. **Canonical dimension.** Let K be a field and X be either a geometrically integral smooth complete K-scheme of finite type or a G-torsor for some connected linear algebraic K-group G. The canonical dimension $\operatorname{cd} X$ of X is the minimal value of $\dim Y$, where Y ranges over all integral closed K-subschemes of X admitting a rational map $X \dashrightarrow Y$ defined over K. Equivalent definitions via generic splitting fields and determination functions can be found in [BR05, KM06].

If we fix a base field k and an algebraic k-group G, the maximal value of $\operatorname{cd} X$ as K ranges over all field extensions K/k and $X \to \operatorname{Spec} K$ ranges over all G_K-torsors, is denoted by $\operatorname{cd} G$. Moreover, $\operatorname{cd} G = \operatorname{cd} X_{\mathrm{ver}}$, where $X_{\mathrm{ver}} \to \operatorname{Spec} K_{\mathrm{ver}}$ is a versal G-torsor. In particular, we can construct a versal G-torsor by starting with a generically free linear representation V of G defined over k and setting $K_{\mathrm{ver}} \stackrel{\mathrm{def}}{=} k(V)^G$. Then V has a G-invariant open subset U which is the total space of a G-torsor $U \to B$, where $k(B) = k(V)^G$. Restricting to the generic point η of B, we obtain a versal torsor $X_{\mathrm{ver}} \stackrel{\mathrm{def}}{=} U_\eta \to \operatorname{Spec} K_{\mathrm{ver}}$. For details of this construction we refer the reader to [GMS03, I.5].

Lemma 2.2.

(a) *Let X_1 and X_2 be Brauer-equivalent Brauer–Severi varieties over a field K/k. Then*

$$\operatorname{cd} X_1 = \operatorname{cd} X_2\,.$$

In other words, the canonical dimension $\operatorname{cd} \alpha$ of a Brauer class $\alpha \in \mathrm{H}^2(K, \mathbb{G}_m)$ is well defined.

(b) *Let $G = \mathrm{GL}_n$ or SL_n and let C be a central subgroup scheme of G. Then for any field K/k and any (G/C)-torsor $X \to \operatorname{Spec} K$ we have $\operatorname{cd} X = \operatorname{cd} \alpha$, where α is the image of the class of X under the coboundary map $\partial_K\colon \mathrm{H}^1(K, G/C) \to \mathrm{H}^2(K, C) \subseteq \mathrm{H}^2(K, \mathbb{G}_m)$ induced by the exact sequence $1 \to C \to G \to G/C \to 1$.*

(c) *Let K/k be a field extension and $\alpha \in \mathrm{H}^2(K, \mathbb{G}_m)$ be a Brauer class of index dividing n and exponent dividing d. Then $\operatorname{cd} \alpha \leq \operatorname{cd}(\mathrm{GL}_n/\mu_d)$.*

Proof. (a) follows from the fact that X_1 and X_2 have the same splitting fields L/K; see [BR05, Section 10] or [KM06, Section 2].

(b) By Hilbert's Theorem 90, G is special, i.e., $\mathrm{H}^1(L, G) = \{1\}$ for any field L. Hence, the coboundary map $\partial_L\colon \mathrm{H}^1(L, G/C) \to \mathrm{H}^2(K, C)$ has trivial kernel for any L, and the desired conclusion follows from [BR05, Lemma 10.2].

(c) By our assumption, α lies in the image of the coboundary map

$$\partial_K\colon \mathrm{H}^1(K, \mathrm{GL}_n/\mu_d) \longrightarrow \mathrm{H}^2(K, C)\,;$$

cf., e.g., [BR05, Lemma 2.6]. Part (c) now follows from part (b). ♠

The following result will be used repeatedly in the sequel.

Proposition 2.3.

(a) *Let $\mathcal{X} \to \operatorname{Spec} K$ be a $\mathbb{G}_{\mathrm{m}}$-gerbe over a field K. Denote the class of this gerbe in $\mathrm{H}^2(K, \mathbb{G}_{\mathrm{m}})$ by α. Then $\operatorname{ed}_K \mathcal{X} = \operatorname{cd} \alpha$.*

(b) *Let $e \geq 1$ be an integer and $\mathcal{X} \to \operatorname{Spec} K$ be a μ_e-gerbe over a field K. Denote the class of this gerbe in $\mathrm{H}^2(K, \mu_e)$ by β. Then $\operatorname{ed}_K \mathcal{X} = \operatorname{cd} \beta + 1$.*

Proof. See [BRV09, Theorem 4.1]. ♠

2.3. **Gerbes and Brauer classes.** Let $\phi\colon \mathcal{X} \to \overline{\mathcal{X}}$ be a $\mathbb{G}_{\mathrm{m}}$-gerbe over a stack $\overline{\mathcal{X}}$. If L is a field and $\xi \in \overline{\mathcal{X}}(L)$ then, pulling back $\mathcal{X}$ to $\operatorname{Spec} L$ we obtain a $\mathbb{G}_{\mathrm{m}}$-gerbe $\mathcal{X}_\xi$ over L. We will denote by $\operatorname{ind}(\mathcal{X}_\xi)$ and $\exp(\mathcal{X}_\xi)$ the index and exponent of the Brauer class of $\mathcal{X}_\xi$. The following lower semi-continuity properties of ind and exp (as functions of ξ) will be used in the proof of Theorem 1.1(b).

Lemma 2.4. *Let $\phi\colon \mathcal{X} \to \overline{\mathcal{X}}$ be a $\mathbb{G}_{\mathrm{m}}$-gerbe over an integral regular algebraic stack $\overline{\mathcal{X}}$, as above. Assume further that $\overline{\mathcal{X}}$ is generically a scheme, with generic point $\eta\colon \operatorname{Spec} K \to \overline{\mathcal{X}}$. Then for any field L/k and any $\xi \in \overline{\mathcal{X}}(L)$,*

(a) $\operatorname{ind}(\mathcal{X}_\xi)$ *divides* $\operatorname{ind}(\mathcal{X}_\eta)$, *and*

(b) $\exp(\mathcal{X}_\xi)$ *divides* $\exp(\mathcal{X}_\eta)$.

Proof. (a) The key fact we will use is that if B is a Brauer–Severi variety over a field L then $\operatorname{ind}(B)$ divides d if and only if B has a linear subspace of dimension $d-1$ defined over L; see [Art82, Proposition 3.4].

By [LMB00a, Theorem 6.3] there exists a smooth map $T \to X$ such that ξ lifts to a point $\operatorname{Spec} L \to T$; we may assume that T is affine and integral. The index of the pullback of $\mathcal{X}_\eta$ to the function field $k(T)$ divides $\operatorname{ind}(\mathcal{X}_\eta)$; hence we can substitute $\overline{\mathcal{X}}$ with T, and assume that $\overline{\mathcal{X}} = T$ is an affine regular integral variety. The étale cohomology group $\mathrm{H}^2(T, \mathbb{G}_{\mathrm{m}})$ is torsion, because T is regular; hence, by a well known result of O. Gabber [Gab81] the class of $\mathcal{X}$ is represented by a Brauer–Severi scheme $P \to T$.

Let d be the index $\operatorname{ind}(\overline{\mathcal{X}}_\eta) \stackrel{\text{def}}{=} \operatorname{ind}(P_\eta)$ and $\operatorname{Gr}(P, d-1) \to T$ be the Grassmannian bundle of linear subspaces of dimension $d-1$ in P. The generic fiber $\operatorname{Gr}(P, n-1)_\eta$ has a K-rational point; this gives rise to a section $U \to \operatorname{Gr}(P, n-1)$ over some open substack U of T. Let Y be the complement of U in T. If our point $\xi\colon \operatorname{Spec} L \to \overline{\mathcal{X}}$ lands in U, then the pullback P_ξ has a linear subspace of dimension $d-1$ defined over L, and we are done. Thus we may assume that $\xi \in Y(L)$. The morphism $\xi\colon \operatorname{Spec} L \to T$ extends to a morphism $\operatorname{Spec} R \to T$, where R is a DVR with residue field L, such that the generic point of $\operatorname{Spec} R$ lands in the complement of Y in T. The pullback $\operatorname{Gr}(P, d-1)_R$ of $\operatorname{Gr}(P, d-1)$ to $\operatorname{Spec} R$ then has a section over the generic point. By the valuative criterion of properness this section

extends to a section $\operatorname{Spec} R \to \operatorname{Gr}(P, d-1)$. Specializing to the closed point of $\operatorname{Spec} R$, we obtain a desired section $\operatorname{Spec} L \to \operatorname{Gr}(P, d-1)$. This shows that $P_{\bar{\xi}}$ has degree dividing d, as claimed.

(b) Set $e \stackrel{\text{def}}{=} \exp(\mathcal{X}_{\bar{\xi}})$ and apply part (a) to the eth power $\mathcal{Y}$ of the gerbe $\mathcal{X}$. Since $\mathcal{Y}_\eta$ is trivial (i.e., has index 1), so is $\mathcal{Y}_{\bar{\xi}}$. But $\mathcal{Y}_{\bar{\xi}}$ is the eth power of the class of $\mathcal{X}_{\bar{\xi}}$, and we are done.

An alternative proof of part (b) is based on the fact that a Brauer-Severi variety $B \to \operatorname{Spec} L$ over a field L has index dividing e if and only if P contains a hypersurface of degree e defined over L; see [Art82, (5.2)]. We may thus proceed exactly as in the proof of part (a), with the same T and $P \to T$, but using the Hilbert scheme $H(P, e) \to T$ of hypersurfaces of degree e in P instead of the Grassmannian. ♠

3. Amenable stacks and generic essential dimension

Definition 3.1. Let $\mathcal{X}$ be an algebraic stack over k. We say that $\mathcal{X}$ is *amenable* if the following conditions hold.

(a) $\mathcal{X}$ is integral with quasi-affine diagonal.
(b) $\mathcal{X}$ is locally of finite type and smooth over k.
(c) There exists a non-empty open substack of $\mathcal{X}$ that is a Deligne–Mumford stack.

Any irreducible algebraic stack has a *generic gerbe*, the residual gerbe at any dominant point $\operatorname{Spec} K \to \mathcal{X}$ [LMB00a, § 11]. For amenable stacks, there is an alternate description. Let $\mathcal{X}$ be an amenable stack over k, and $\mathcal{U}$ a non-empty open substack which is Deligne–Mumford. After shrinking $\mathcal{U}$, we may assume that the inertia stack $\mathcal{I}_{\mathcal{U}}$ is finite over $\mathcal{U}$. Let $\mathbf{U}$ be the moduli space of $\mathcal{U}$, whose existence is proved in [KM97], and let $k(\mathbf{X})$ be its residue field. The generic gerbe $\mathcal{X}_{k(\mathbf{X})} \to \operatorname{Spec} k(\mathbf{X})$ is then the fiber product $\operatorname{Spec} k(\mathbf{X}) \times_{\mathbf{U}} \mathcal{U}$. The dimension $\dim \mathcal{X}$ is the dimension of $\mathcal{U}$, or, equivalently, the dimension of $\mathbf{U}$.

Example 3.2. Consider the action of a linear algebraic group defined over k on a smooth integral k-scheme X, locally of finite type. Then the quotient stack $[X/G]$ is amenable if and only if the stabilizer $\operatorname{Stab}_G(x)$ of a general point $x \in X$ is finite.

Of particular interest to us will be the GL_n-actions on $A_{n,d}$, the $\binom{n+d-1}{d}$-dimensional affine space of forms of degree d in n variables, and $\mathbb{P}(A_{n,d}) = \binom{n+d-1}{d} - 1$ dimensional projective space of degree d hypersurfaces in $\mathbb{P}^{n-1}$, as well as the PGL_n-action on $\mathbb{P}(A_{n,d})$.

Since the center of GL_n acts trivially on $\mathbb{P}(A_{n,d})$, the stack $[\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$ is not amenable. On the other hand, it is classically known that the stabilizer of any smooth hypersurface in $\mathbb{P}^{n-1}$ of degree $d \geq 3$ is finite; see, e.g., [OS78, Theorem 2.1] or [MM64]. From this we deduce that the stacks $[\mathbb{P}(A_{n,d})/\mathrm{PGL}_n]$ and $[A_{n,d}/\mathrm{GL}_n]$ are both amenable for any $n \geq 2$ and $d \geq 3$.

Moreover, if $n \geq 2$, $d \geq 3$ and $(n, d) \neq (2, 3)$, $(2, 4)$ or $(3, 3)$ then the stabilizer of a general hypersurface in $\mathbb{P}^{n-1}$ of degree d is trivial; see [MM64]. For these values of n and d the quotient stack $[\mathbb{P}(A_{n,d})/\mathrm{PGL}_n]$ is generically a scheme of dimension

$$\dim \mathbb{P}(A_{n,d}) - \dim \mathrm{PGL}_n = \binom{n+d-1}{d} - n^2 .$$

Definition 3.3. The *generic essential dimension* of an amenable stack $\mathcal{X}$ is

$$\operatorname{ged}_k \mathcal{X} \stackrel{\text{def}}{=} \operatorname{ed}_{k(\mathbf{X})} \mathcal{X}_{k(\mathbf{X})} + \dim \mathcal{X} .$$

Alternatively, $\operatorname{ged}_k \mathcal{X}$ is the supremum of the essential dimension of $\zeta \in \mathcal{X}(K)$, taken over all field extensions K/k and all dominant $\zeta\colon \operatorname{Spec} K \to \mathcal{X}$. By the Genericity Theorem for Deligne–Mumford stacks [BRV09, Theorem 6.1], we see that $\operatorname{ged}_k \mathcal{X}$ is the essential dimension of any open substack of $\mathcal{X}$ that is a Deligne–Mumford stack.

We will now compute the generic essential dimension of the quotient stacks $[A_{n,d}/\mathrm{GL}_n]$ and $[\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$ for n and d as in the statement of Theorem 1.1. Recall that $\operatorname{ged}_k[A_{n,d}/\mathrm{GL}_n] = \operatorname{ed}_k F_{\mathrm{gen}}$ and $\operatorname{ged}_k[\mathbb{P}(A_{n,d})/\mathrm{GL}_n] = \operatorname{ed}_k H_{\mathrm{gen}}$, where F_{gen} is the generic forms of degree d in n variables and H_{gen} is the generic hypersurfaces, as in (1.3).

Proposition 3.4. *Let $n \geq 2$ and $d \geq 3$ be integers. Assume further that $(n,d) \neq (2,3)$, $(2,4)$ or $(3,3)$. Then*

(a) $\operatorname{ged}_k[\mathbb{P}(A_{n,d})/\mathrm{GL}_n] = \binom{n+d-1}{d} - n^2 + \operatorname{cd}(\mathrm{GL}_n/\mu_d)$.
(b) $\operatorname{ged}_k[A_{n,d}/\mathrm{GL}_n] = \binom{n+d-1}{d} - n^2 + \operatorname{cd}(\mathrm{GL}_n/\mu_d) + 1$.

Part (a) was previously known; see [BR05, Theorem 15.1]. Part (b) answers an open question from [BR05, Remark 14.8].

Proof. Let $\mathcal{X} = [\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$, $\mathcal{Y} = [A_{n,d}/\mathrm{GL}_n]$, and $\overline{\mathcal{X}} = [\mathbb{P}(A_{n,d})/\mathrm{PGL}_n]$. Consider the diagram

$$\begin{tikzcd}
 & \mathcal{Y}_\eta \arrow[dl] \arrow[ddr, "\mathbb{G}_{\mathrm{m}}\text{-gerbe}"] & & \mathcal{Y} \arrow[dl] \arrow[ddl, "\mu_d\text{-gerbe}"] \\
\mathcal{X}_\eta \arrow[d] & & \mathcal{X} \arrow[d] & \\
\eta \arrow[rr] & & \overline{\mathcal{X}} &
\end{tikzcd}$$

For n and d as in the statement of the proposition, $[\mathbb{P}(A_{n,d})/\mathrm{PGL}_n]$ is generically a scheme (see Example 3.2). Denote the generic point of this scheme by η and its function field by $k(\eta)$. The pull-backs $\mathcal{Y}_{k(\eta)}$ and $\mathcal{X}_{k(\eta)}$ are, respectively, a μ_d-gerbe and a $\mathbb{G}_{\mathrm{m}}$-gerbe over $k(\eta)$; these two gerbes give rise to the same class $\alpha \in \mathrm{H}^2(k(\eta), \mu_d) \subset \mathrm{H}^2(k(\eta), \mathbb{G}_{\mathrm{m}})$. By Proposition 2.3

$$\operatorname{ed}_{k(\eta)} \mathcal{Y}_{k(\eta)} = \operatorname{cd} \alpha \quad \text{and} \quad \operatorname{ed}_{k(\eta)} \mathcal{X}_{k(\eta)} = \operatorname{cd} \alpha + 1.$$

Since

$$\operatorname{tr\,deg}_k k(\eta) = \binom{n+d-1}{d} - n^2,$$

it remains to show that

$$\operatorname{cd} \alpha = \operatorname{cd}(\mathrm{GL}_n/\mu_d). \tag{3.1}$$

The action of $G = \mathrm{GL}_n/\mu_d$ on $A_{n,d}$ is linear and generically free. Thus it gives rise to a versal G-torsor $t \in \mathrm{H}^1(k(\eta), G)$, and α is the image of t under the natural coboundary map $\mathrm{H}^1(k(\eta), G) \to \mathrm{H}^2(k(\eta), \mu_d)$ associated with the exact sequence $1 \to \mu_d \to \mathrm{GL}_n \to G \to 1$. As we explained in Section 2.2, $\operatorname{cd} t = \operatorname{cd}(\mathrm{GL}_n/\mu_d)$. On the other hand, by Lemma 2.2(b), $\operatorname{cd} \alpha = \operatorname{cd} t$, and (3.1) follows. ♠

4. Gerbe-like stacks

The purpose of the next two sections is to prove the Genericity Theorem 1.2. The proof of the genericity theorem for Deligne–Mumford stacks in [BRV09] relied on a stronger form of genericity for gerbes; see [BRV09, Theorem 5.13]. Our proof of Theorem 1.2 will follow a similar pattern, except that instead of working with gerbes we will need to work in the more general setting of gerbe-like stacks, defined below. The main result of this section, Theorem 4.6, is a strong form of genericity for gerbe-like stacks.

Definition 4.1. A Deligne–Mumford stack $\mathcal{X}$ is *gerbe-like* if its inertia stack $\mathcal{I}_{\mathcal{X}}$ is étale over $\mathcal{X}$.

If $\mathcal{X}$ is an algebraic stack, the *gerbe-like part* $\mathcal{X}^0$ of $\mathcal{X}$ is the largest open substack of $\mathcal{X}$ that is Deligne–Mumford and gerbe-like.

Remark 4.2. If an algebraic stack $\mathcal{X}$ is Deligne–Mumford, then the inertia stack $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is unramified. Hence, if $\mathcal{X}$ is also reduced then by generic flatness the gerbe-like part $\mathcal{X}^0$ of $\mathcal{X}$ is dense in $\mathcal{X}$.

Lemma 4.3. *Let $\mathcal{X}$ be a reduced Deligne–Mumford stack. Suppose that the inertia stack $\mathcal{I}_{\mathcal{X}}$ is finite and étale over $\mathcal{X}$. Then $\mathcal{X}$ is a proper étale gerbe over an algebraic space.*

Remark 4.4. The condition that $\mathcal{X}$ be reduced can be eliminated. However, it makes the proof marginally simpler, and will be satisfied in all cases of interest to us in this paper.

Proof. Let $\mathbf{X}$ be the moduli space of $\mathcal{X}$; we claim that $\mathcal{X}$ is a proper étale gerbe over $\mathbf{X}$. This is a local problem in the étale topology of $\mathbf{X}$. Hence, after passing to an étale covering of $\mathbf{X}$, we may assume that $\mathbf{X}$ is a connected scheme, and there exists a finite reduced connected scheme U, with a finite group G acting on U, such that $\mathcal{X} = [U/G]$. The pullback of $\mathcal{I}_{\mathcal{X}}$ to U is the closed subscheme of $G \times U$ defined as representing the functor of pairs (g, u) with $gu = u$. The fact that this pullback is étale over U translates into the condition that the order of the stabilizer of a geometric point is locally constant on U. Since U is connected, this means that there exists a subgroup H of G that is the stabilizer of all the geometric points of G; this subgroup is necessarily normal. The induced action of (G/H) is free, and $U/(G/H) = \mathbf{X}$; hence U is étale over $\mathbf{X}$, and $\mathcal{X} = [U/G]$ is a gerbe banded by H over $\mathbf{X}$. ♠

Lemma 4.5. *Suppose that $\mathcal{X}$ is a gerbe-like Deligne–Mumford stack, $\mathcal{Y} \to \mathcal{X}$ a representable unramified morphism. Then $\mathcal{Y}$ is also gerbe-like.*

Proof. The inertia stack $\mathcal{I}_{\mathcal{X}}$ of a stack $\mathcal{X}$ is the fiber product $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$. We have a diagram

$$\begin{tikzcd}
\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \arrow[dl] \arrow[dd] \\
& \mathcal{I}_{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
& \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} \\
\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \arrow[rr] \arrow[ur] & & \mathcal{Y} \times \mathcal{Y} \arrow[ul]
\end{tikzcd}$$

in which all the squares are cartesian. This implies the equality

$$\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}} = (\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}) \times_{\mathcal{X}} (\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}),$$

which in turn tells us that $(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}) \times_{\mathcal{X}} (\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y})$ is étale over $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$. The hypotheses on $\mathcal{Y} \to \mathcal{X}$ imply that the diagonal $\mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is an open embedding. Thus $\mathcal{I}_{\mathcal{Y}} = \mathcal{Y}_{\mathcal{Y}\times\mathcal{Y}}$ is an open substack of $(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}) \times_{\mathcal{X}} (\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y})$, so it is étale over $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ hence it is étale over $\mathcal{Y}$, as claimed. ♠

From this and the results in [BRV09], it is easy to deduce the following. Given a field L/k and $\xi \in \mathcal{X}(\operatorname{Spec} L)$, we denote by $\operatorname{codim}_{\mathcal{X}} \xi$ the codimension of the closure of the image of the corresponding morphism $\operatorname{Spec} L \to \mathcal{X}$.

Theorem 4.6. *Let $\mathcal{X}$ be an integral gerbe-like Deligne–Mumford stack which is smooth of finite type over a field k. Let L be an extension of k and $\xi \in \mathcal{X}(\operatorname{Spec} L)$. Then*

$$\operatorname{ed}_{\mathcal{X}} \xi \leq \operatorname{ed}_{k(\mathbf{X})} \mathcal{X}_{k(\mathbf{X})} + \dim \mathbf{X} - \operatorname{codim}_{\mathcal{X}} \xi.$$

Proof. If the inertia stack $\mathcal{I}_{\mathcal{X}}$ is finite over $\mathcal{X}$, then, by Lemma 4.3, $\mathcal{X}$ is an étale proper gerbe over a smooth k-scheme, and the statement reduces to [BRV09, Theorem 5.13]. In the general case, from [BRV09, Lemma 6.4] we deduce the existence of an étale representable morphism $\mathcal{Y} \to \mathcal{X}$, such that $\mathcal{Y}$ is an integral Deligne–Mumford stack with finite inertia, and the morphism $\operatorname{Spec} L \to \mathcal{Y}$ factors through $\mathcal{Y}$. By Lemmas 4.3 and 4.5, the stack $\mathcal{Y}$ is a proper étale gerbe over a smooth algebraic space, hence [BRV09, Theorem 5.13] can be applied to it. Let $\eta \in \operatorname{Spec} \mathcal{Y}$ be a point in $\mathcal{Y}(L)$ mapping to ξ. Then we have the relations

$$\begin{aligned} \operatorname{ed}_k \xi &\leq \operatorname{ed}_k \eta, \\ \operatorname{ed}_{k(\mathcal{X})} \mathcal{X}_{k(\mathbf{X})} &\geq \operatorname{ed}_{k(\mathbf{Y})} \text{ and} \\ \operatorname{codim}_{\mathcal{X}} \xi &= \operatorname{codim}_{\mathcal{Y}} \eta; \end{aligned}$$

hence the general case of the Theorem follows from [BRV09, Theorem 5.13]. ♠

5. The genericity theorem

We now proceed with the proof of Theorem 1.2. As before, let $\mathcal{Y}$ be the closure of the image of $\xi\colon \operatorname{Spec} L \to \mathcal{X}$. The stack $\mathcal{Y}$ is integral, and since $\operatorname{char} k = 0$, $\mathcal{Y}$ is generically smooth. Let $\pi\colon \mathcal{M} \to \mathbb{A}^1_k$ the deformation to the normal bundle of $\mathcal{Y}$ inside $\mathcal{X}$; then $\pi^{-1}(\mathbb{A}^1_k \smallsetminus \{0\}) = \mathcal{X} \times_{\operatorname{Spec} k} (\mathbb{A}^1_k \smallsetminus \{0\})$, while $\pi^{-1}(0)$ is isomorphic to the normal bundle $\mathcal{N}$ of $\mathcal{Y}$ in $\mathcal{X}$.

Lemma 5.1. $\mathcal{M}^0 \cap \mathcal{N} \neq \emptyset$.

Theorem 1.2 follows from Lemma 5.1 and Theorem 4.6 by the same argument as in [BRV09, Theorem 6.1]. This argument is quite short, and we reproduce it here for sake of completeness.

Let L be an extension of k and let ξ be an object of $\mathcal{X}(L)$. Call $\mathcal{Y}$ the closure of the image of the morphism $\xi\colon \operatorname{Spec} L \to \mathcal{X}$, with its reduced stack structure. Set $\mathcal{N}^0 \stackrel{\text{def}}{=} \mathcal{M}^0 \cap \mathcal{N}$. Then the fiber product $\operatorname{Spec} L \times_{\mathcal{Y}} \mathcal{N}$ is a vector bundle over $\operatorname{Spec} L$, and $\operatorname{Spec} L \times_{\mathcal{X}} \mathcal{M}^0$ is a non-empty open subscheme. Hence $\xi\colon \operatorname{Spec} L \to \mathcal{Y}$ can be lifted to $\mathcal{N}^0$; this gives an object η of $\mathcal{N}^0(\operatorname{Spec} L)$ mapping to ξ in $\mathcal{Y}$. Clearly the essential dimension of ξ as an object of $\mathcal{X}$ is the same as its essential dimension as an object of $\mathcal{Y}$, and $\operatorname{ed}_k \xi \leq \operatorname{ed}_k \eta$. Let us apply Theorem 4.6 to the gerbe $\mathcal{M}^0$. The function field of the moduli space $\mathbf{M}$ of $\mathcal{M}$ is $k(\mathbf{X})(t)$, and its generic gerbe is

$\mathcal{X}_{k(\mathbf{X})(t)}$; by [BRV09, Proposition 2.8], we have $\operatorname{ed}_{k(\mathbf{X})(t)} \mathcal{X}_{k(\mathbf{X})(t)} \leq \operatorname{ed}_{k(\mathbf{X})} \mathcal{X}_{k(\mathbf{X})}$. The composite $\operatorname{Spec} L \to \mathcal{N}^0 \subseteq \mathcal{M}^0$ has codimension at least 1, hence we obtain

$$\begin{aligned} \operatorname{ed}_k \xi &< \operatorname{ed}_{k(\mathbf{X})(t)} \mathcal{X}_{k(\mathbf{X})(t)} + \dim \mathcal{M} \\ &= \operatorname{ed}_{k(\mathbf{X})} \mathcal{X}_{k(\mathbf{X})} + \dim \mathcal{X} + 1. \end{aligned}$$

This concludes the proof.

Proof of Lemma 5.1. If X is a finite dimensional representation of a group scheme G over a field k, we will identify X with the affine space $\operatorname{Spec}(\operatorname{Sym}_k^\bullet X^\vee)$.

Let us suppose that X is a finite dimensional representation of a linearly reductive algebraic group G and $Y \subseteq X$ is a subrepresentation. Since G is reductive, we have a G-equivariant splitting $X \simeq Y \oplus Y'$. Set $\mathcal{X} \stackrel{\text{def}}{=} [X/G]$ and $\mathcal{Y} \stackrel{\text{def}}{=} [Y/G]$. Assume that the generic stabilizer of the action of G on X is finite. Then $\mathcal{X}$ is amenable, and $\mathcal{Y} \subseteq \mathcal{X}$ is a closed integral substack.

It is easy to see that the deformation to the normal bundle M of Y in X is G-equivariantly isomorphic to $X \times_k \mathbb{A}^1_k$ (where the group G acts trivially on $\mathbb{A}^1_k$); the projection

$$Y \times Y' \times_k \mathbb{A}^1_k = X \times_k \mathbb{A}^1_k \simeq M \longrightarrow X \times_k \mathbb{A}^1_k$$

is given by the formula $(y, y', t) \mapsto (ty, y', t)$. The deformation to the normal bundle $\mathcal{M}$ of $\mathcal{Y}$ is $[M/G] = [X/G] \times_k \mathbb{A}^1_k$; hence $\mathcal{M}^0 = \mathcal{X}^0 \times_k \mathbb{A}^1_k$, and it is obvious that $\mathcal{M}^0 \cap \mathcal{N} \neq \emptyset$.

The proof in the general case will be reduced to this by a formal slice argument.

We may base-change to the algebraic closure of k; so we may assume that k is algebraically closed. By deleting the singular locus of $\mathcal{Y}$, we may assume that $\mathcal{Y}$ is smooth; by further restricting, we may assume that the inertia stack $\mathcal{I}_{\mathcal{Y}}$ is flat over $\mathcal{Y}$, and that all geometric fibers are reductive, and have the same numbers of connected components.

Let $y_0 \to \operatorname{Spec} k \to \mathcal{Y}$ be a general closed point. The residual gerbe $\mathcal{G}_{y_0} \to \operatorname{Spec} k$ [LMB00a, § 11] admits a section $\operatorname{Spec} k \to \mathcal{G}_{y_0}$, since k is algebraically closed; hence, if $G \stackrel{\text{def}}{=} \underline{\operatorname{Aut}}_k\, y_0$ it the automorphism group scheme of y_0, we have $\mathcal{G}_{y_0} \simeq \mathcal{B}_k G$. The embedding of stacks $\mathcal{B}_k G \to \mathcal{Y}$ is of finite type; hence it is easy to see that it is a locally closed embedding, from Zariski's main Theorem for stacks [LMB00a, Théorème 16.5].

Let $U \to \mathcal{X}$ be a smooth morphism, where U is a scheme, together with a lifting $u_0 \colon \operatorname{Spec} k \to U$ of y_0. Call $\mathcal{X}_n$ the nth infinitesimal neighborhood of $\mathcal{B}_k G$ inside $\mathcal{X}$: in other words, if $\mathcal{U}$ is an open substack of $\mathcal{X}$ containing $\mathcal{B}_k G$ as a closed substack, and we denote by I the sheaf of ideals of $\mathcal{B}_k G$ inside $\mathcal{U}$, then $\mathcal{X}_n$ is the closed substack of $\mathcal{X}$ defined by the sheaf of ideals I^{n+1}. In particular, $\mathcal{X}_0 = \mathcal{B}_k G$.

Lemma 5.2. *There exists a finite dimensional representation X of G with finite generic stabilizers and a trivial subrepresentation $Y \subseteq X$, with the following property. If we denote by X_n and $\mathcal{Y}_n$ the nth infinitesimal neighborhoods of the origin, there is a sequence of isomorphisms $\mathcal{X}_n \simeq [X_n/G]$, compatible with the embeddings $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$ and $[X_n/G] \subseteq [X_{n+1}/G]$, that induce isomorphisms of $\mathcal{Y}_n$ with $[Y_n/G]$.*

Furthermore, denote by $\widehat{X}$ the spectrum of the completion of the local ring of X at the origin. Then there exists a smooth morphism $U \to \mathcal{X}$ with a closed point $u_0 \in U$ mapping to y_0 in $\mathcal{Y}$, and an isomorphism of $\widehat{X}$ with the spectrum $\widehat{U}$ of the completion of the local ring of U at u_0, such that

(a) *the sequence of composites* $X_n \to [X_n/G] \simeq \mathcal{X}_n \subseteq \mathcal{X}$ *is obtained by restriction from the composite morphism* $\widehat{X} \simeq \widehat{U} \to U \to \mathcal{X}$, *and*
(b) *the inverse image of* Y *in* $\widehat{X}$ *corresponds to the inverse image of* $\mathcal{Y}$ *in* $\widehat{U}$.

Proof. The tautological G-torsor $P_0 \stackrel{\text{def}}{=} \operatorname{Spec} k \to \mathcal{B}_k G$ extends to a G-torsor $P_n \to \mathcal{X}_n$, in such a way that the restriction of P_{n+1} to $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$ is isomorphic to P_n, by [Alp09, Propositions 4.1 and 4.2]. Each of the stacks P_n is in fact a scheme, because its reduced substack is; in fact, P_n must be the spectrum of a local artinian k-algebra R_n. Clearly, $\mathcal{X}_n = [P_n/G]$.

If we denote by V the maximal ideal of R_1, then $R_1 = k \oplus V$; the action of G on R_1 induces a linear action of G on V. The space V is isomorphic to I/I^2, which is a coherent sheaf on $\mathcal{B}_k G$, i.e., a representation of G. In turn, I/I^2 is the cotangent space of deformations of $\operatorname{Spec} k \to \mathcal{X}$, that is, the dual to the space of isomorphism classes of liftings $\operatorname{Spec} k[\epsilon] \to \mathcal{X}$ of $\operatorname{Spec} k \to \mathcal{X}$ (here $k[\epsilon]$ denotes, as usual, the ring of dual numbers $k[x]/(x^2)$).

The homomorphism $R_{n+1} \to R_n$ induced by the embedding $P_n \simeq P_{n+1}|_{\mathcal{X}_n} \subseteq P_{n+1}$ is surjective; its kernel is the ideal $I^n R_{n+1}$. Denote by R the projective limit $\varprojlim_n R_n$. (Notice that, while G acts, by definition, on each of R_n, this does not, unfortunately, induced an action of G on R, as an algebraic group, unless G is finite; it if did, this would make the proof conceptually much simpler.) If $x_1, \dots, x_n$ is a set of elements of R that project to a basis for V in R_1, the ring R is a quotient of the power series ring $k[\![x_1, \dots, x_n]\!]$ by an ideal J contained in $\mathfrak{m}_R^2$. We claim that $J = 0$, i.e., R is a power series ring. For this, it is enough to check that R is formally smooth over k, or, in other words, that if A is a local artinian k-algebra with reside field k and B is a quotient of A, any homomorphism of k-algebras $R \to B$ lifts to a homomorphism $R \to A$. Take $n \gg 0$; then $R \to B$ factors through R_n. Consider the composite

$$\operatorname{Spec} B \longrightarrow \operatorname{Spec} R_n \longrightarrow \mathcal{X}_n \subseteq \mathcal{X}\,;$$

since $\mathcal{X}$ is smooth, deformations are unobstructed, i.e., this morphism extends to $\operatorname{Spec} A \to \mathcal{X}$. If $n \gg 0$, this factors as $\operatorname{Spec} A \to \mathcal{X}_n \subseteq \mathcal{X}$; and since $\operatorname{Spec} R_n$ is smooth over $\mathcal{X}_n$, as it is a G-torsor, the section $\operatorname{Spec} B \to \operatorname{Spec} R_n$ of $\operatorname{Spec} B \to \mathcal{X}_n$ lifts to a section $\operatorname{Spec} A \to R_n$, giving the desired extension $R \to R_n \to A$.

Suppose that $U \to \mathcal{X}$ a smooth morphism, where U is a scheme, with a lifting $u_0\colon \operatorname{Spec} k \to U$ of y_0. Let us assume that U is minimal at u_0, or, in other words, that the tangent space of U at u_0 maps isomorphically onto the deformation space of $\mathcal{X}$ at y_0. Since U is smooth over $\mathcal{X}$, the morphisms $\operatorname{Spec} R_n \to \mathcal{X}$ lift to a compatible system of morphism $\operatorname{Spec} R_n \to U$, sending $\operatorname{Spec} k$ into u_0; these yield a morphism $\operatorname{Spec} R \to \mathcal{O}_{U,u_0}$, inducing a homomorphism of k-algebras $\widehat{\mathcal{O}}_{U,u_0} \to R$. This is a homomorphism of power series algebras over k which gives an isomorphism of tangent spaces; hence it is an isomorphism. This shows that the morphisms $\operatorname{Spec} R_n \to \mathcal{X}$ yield a flat morphism $\operatorname{Spec} R \to \mathcal{X}$.

Call d the codimension of $\mathcal{Y}$ in $\mathcal{X}$ at the point y_0; after a base change in $R = k[\![x_1, \dots, x_n]\!]$, we may assume that the inverse image of the ideal of $\mathcal{Y}$ in $\mathcal{X}$ is $(x_1, \dots, x_d)$. Call X the scheme corresponding to the dual of the vector space $V = \langle x_1, \dots, x_n \rangle$, that is, $X \stackrel{\text{def}}{=} \operatorname{Spec} \operatorname{Sym}_k^\bullet V$; call Y the linear subscheme defined by the ideal $(x_1, \dots, x_d)$. Then $\widehat{X} = \operatorname{Spec} R$; the representation X has all the required properties, except that we have not yet proved that the action of G on X has finite generic stabilizers, and the representation Y is trivial.

To do this, let us call I the pullback of the inertia stack of $[X/G]$ to X; in other words, I is the subscheme of $G \times_{\operatorname{Spec} k} X$ defined by the equation $gx = x$. We need to show that I is generically finite over X. Since I is a group scheme over X, it has equidimensional fibers, hence it is enough that there is an étale neighborhood $I' \to I$ of the pair $(1, u_0)$ in I which is generically finite over X. The inverse image of $\mathcal{X}_n$ in U is the nth infinitesimal neighborhood U_n of u_0 in U. Denote by J the pullback of the inertia stack of $\mathcal{X}$ to U; we have isomorphisms $X_n \simeq U_n$, and compatible isomorphisms of the pullbacks of I and J to X_n and U_n respectively. These induce an isomorphism of the completions of I and J at $(1, x_0)$ and $(1, u_0)$ respectively; by Artin approximation, the morphisms $I \to X$ and $J \to U$ are étale-locally equivalent at $(1, x_0)$ and $(1, u_0)$. Since J is generically finite over U it follows that the action of G on X has generically finite stabilizers. Also, this implies that the stabilizer of a general closed point of Y is isomorphic to the isomorphism group scheme of a general point of $\operatorname{Spec} k \to \mathcal{Y}$, hence it has the same dimension and the same number of connected components of G; hence it equals G, and so the action of G on Y is trivial, as claimed. ♠

Consider the complement $\mathcal{F}$ of $(\mathcal{M} \smallsetminus \mathcal{N})^0$ in $\mathcal{M} \smallsetminus \mathcal{N}$. Since $\mathcal{N}$ is a divisor on $\mathcal{M}$, the closure of $\mathcal{F}$ in $\mathcal{M}$ will not contain it; hence, if v is a general k-rational point of $\mathcal{N}$, this has neighborhood in which the only point of the inertia stack $\mathcal{I}_{\mathcal{M}}$ where this can fail to be étale are over $\mathcal{N}$; so it is enough to show that $\mathcal{I}_{\mathcal{M}}$ is étale over $\mathcal{M}$ at a general point of $\mathcal{N}$. For this it suffices to show that the locus of point of the inverse image $\mathcal{I}_{\mathcal{N}}$ of $\mathcal{N}$ in $\mathcal{I}_{\mathcal{M}}$ at which $\mathcal{I}_{\mathcal{M}}$ is étale over $\mathcal{M}$ surjects onto $\mathcal{Y}$.

Denote by N the normal bundle of Y in X, and by M the deformation to the normal bundle. If n_0 is a general closed point of N, then the pullback I_M of the inertia stack of $[M/G]$ to M is étale at n_0. Notice that I_M is étale at a general closed point of the fiber of N over any $y_0 \in Y(k)$, since the action of G on Y is trivial, so translation by any closed point of Y is G-equivariant. Denote by $\mathcal{M}_n$ the inverse image of $\mathcal{X}_n \times_{\operatorname{Spec} k} \mathbb{A}^1_k$ in $\mathcal{M}$, and by M_n the inverse image of $X_n \times_{\operatorname{Spec} k} \mathbb{A}^1_k$ in M.

Claim. There is sequence of isomorphisms $\mathcal{M}_n \simeq [M_n/G]$ compatible with the isomorphisms $\mathcal{X}_n \simeq [X_n/G]$, and with the identity on $\mathbb{A}^1_k$.

Let us assume the claim. We know that $\mathcal{I}_{[M/G]}$ is étale over $\mathcal{M}$ at a general k-rational point of $\mathcal{I}_{[M/G]}$ lying over the image of the origin in $[N/G] \subseteq [M/G]$; let $v \in \mathcal{I}_{\mathcal{M}}(k)$ be a general k-rational point lying over $\mathcal{N}$, and let us show that $\mathcal{I}_{\mathcal{M}} \to \mathcal{M}$ is étale at v. For this we use the infinitesimal criterion for étaleness. Let A be a finite k-algebra with residue field equal to k, let I be a proper ideal in A, and consider a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(A/I) \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{M}} \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{M} \end{tikzcd}$$

in which composite $\operatorname{Spec} k \subseteq \operatorname{Spec} A \to \mathcal{I}_{\mathcal{M}}$ is isomorphic to v; we need to show that we can fill in the dashed arrow in a unique way. For $n \gg 0$, the morphism $\operatorname{Spec} A \to \mathcal{M}$ factor through $\mathcal{M}_n$. Since $\mathcal{I}_{\mathcal{M}_n} = \mathcal{M}_n \times_{\mathcal{M}} \mathcal{I}_{\mathcal{M}}$, the square above

factors through a square

$$\begin{tikzcd} \operatorname{Spec}(A/I) \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{M}_n} \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{M}_n \end{tikzcd}$$

in which again we have to show the existence and uniqueness of the lifting. However, the isomorphism $\mathcal{M}_n \simeq [M_n/G]$ induces an isomorphism of the morphism $\mathcal{I}_{\mathcal{M}_n} \to \mathcal{M}_n$ with $\mathcal{I}_{[M_n/G]} \to [M_n/G]$; we know that $\mathcal{I}_{[M_n/G]} = [M_n/G] \times_{[M/G]} \mathcal{I}_{[M/G]}$ is étale at the point corresponding to v. Hence the lifting exists and is unique.

Now let us prove the claim. Set

$$\begin{aligned} R &\stackrel{\text{def}}{=} U \times_{\mathcal{X}} U, \\ R_n &\stackrel{\text{def}}{=} U_n \times_{\mathcal{X}_n} U_n, \\ S &\stackrel{\text{def}}{=} X \times_{[X/G]} X = G \times_{\operatorname{Spec} k} X, \\ S_n &\stackrel{\text{def}}{=} X_n \times_{[X_n/G]} X_n = G \times_{\operatorname{Spec} k} X_n\,. \end{aligned}$$

The compatible isomorphisms $\phi_n \colon X_n \simeq U_n$ and $\mathcal{X}_n \simeq [X_n/G]$ yield yield isomorphisms of schemes in groupoids of $R_n \rightrightarrows U_n$ with $S_n \rightrightarrows X_n$, for each $n \geq 0$.

Denote by I_U the sheaf of ideals of the inverse image of $\mathcal{Y} \times \{0\} \subseteq \mathcal{X} \times \mathbb{A}^1$ in $U \times \mathbb{A}^1$, and by I_R the sheaf of ideals of its inverse image in $R \times \mathbb{A}^1$. Also denote with J_U the sheaf of ideals of $\mathcal{Y} \times \{0\} \subseteq \mathcal{X} \times \{0\}$ in $U \times \{0\}$, pushed forward to $U \times \mathbb{A}^1$, and by J_R the sheaf of ideals of its inverse image in $R \times \{0\}$, pushed forward to $R \times \mathbb{A}^1$. There are natural surjection $I_U \twoheadrightarrow J_U$ and $I_R \twoheadrightarrow J_R$. Set

$$\begin{aligned} U' &\stackrel{\text{def}}{=} \operatorname{Proj}_{U \times \mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} I_U^m\Big), \quad R' \stackrel{\text{def}}{=} \operatorname{Proj}_{R \times \mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} I_R^m\Big), \\ U'' &\stackrel{\text{def}}{=} \operatorname{Proj}_{U \times \mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} J_U^m\Big), \quad R'' \stackrel{\text{def}}{=} \operatorname{Proj}_{R \times \mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} J_R^m\Big). \end{aligned}$$

Then $R' \rightrightarrows U'$ is a scheme in groupoids, $R'' \rightrightarrows U''$ is a closed subgroupoid, and the difference groupoid $R' \smallsetminus R'' \rightrightarrows U' \smallsetminus U''$ gives a smooth presentation of $\mathcal{M}$. Let us denote by U'_n and U''_n the inverse images of U_n in U' and U'', and by R'_n and R''_n the inverse images of $U_n \times U_n$ in R' and R''. Then the groupoid $R'_n \smallsetminus R''_n \rightrightarrows U'_n \smallsetminus U''_n$ gives a smooth presentation of $\mathcal{M}_n$; furthermore, we have

$$\begin{aligned} U'_n &= \operatorname{Proj}_{U_n \times \mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} I_U^m \otimes_{\mathcal{O}_{U \times \mathbb{A}^1}} \mathcal{O}_{U_n \times \mathbb{A}^1}\Big), \\ R'_n &= \operatorname{Proj}_{R_n \times \mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} I_R^m \otimes_{\mathcal{O}_{U \times \mathbb{A}^1}} \mathcal{O}_{R_n \times \mathbb{A}^1}\Big), \\ U''_n &= \operatorname{Proj}_{U_n \times \mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} J_U^m \otimes_{\mathcal{O}_{R \times \mathbb{A}^1}} \mathcal{O}_{U_n \times \mathbb{A}^1}\Big), \end{aligned}$$

$$R''_n = \mathrm{Proj}_{R_n\times\mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} J_R^m \otimes_{\mathcal{O}_{R\times\mathbb{A}^1}} \mathcal{O}_{R_n\times\mathbb{A}^1}\Big).$$

In a completely analogous manner, denote by I_X the sheaf of ideals of $Y\times\{0\}$ in $U\times\mathbb{A}^1$, and by I_S the sheaf of ideals of its inverse image in $S\times\mathbb{A}^1$. Also denote with J_X the sheaf of ideals of $Y\times\{0\}$ in $X\times\{0\}$, pushed forward to $X\times\mathbb{A}^1$, and by J_S the sheaf of ideals of its inverse image in $S\times\{0\}$, pushed forward to $S\times\mathbb{A}^1$. Set

$$X'_n \stackrel{\text{def}}{=} \mathrm{Proj}_{X_n\times\mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} I_X^m \otimes_{\mathcal{O}_{X\times\mathbb{A}^1}} \mathcal{O}_{X_n\times\mathbb{A}^1}\Big),$$

$$S'_n \stackrel{\text{def}}{=} \mathrm{Proj}_{S_n\times\mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} I_S^m \otimes_{\mathcal{O}_{S\times\mathbb{A}^1}} \mathcal{O}_{S_n\times\mathbb{A}^1}\Big),$$

$$X''_n \stackrel{\text{def}}{=} \mathrm{Proj}_{U\times\mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} J_U^m \otimes_{\mathcal{O}_{R\times\mathbb{A}^1}} \mathcal{O}_{R_n\times\mathbb{A}^1}\Big),$$

$$S''_n \stackrel{\text{def}}{=} \mathrm{Proj}_{R\times\mathbb{A}^1}\Big(\bigoplus_{m=0}^{\infty} J_R^m \otimes_{\mathcal{O}_{R\times\mathbb{A}^1}} \mathcal{O}_{R_n\times\mathbb{A}^1}\Big).$$

By the same argument as before, we see that $[M_n/G]$ has a smooth presentation $S'_n \smallsetminus S''_n \rightrightarrows X'_n \smallsetminus X''_n$; hence, to complete the proof we need to establish the existence of isomorphisms $U'_n \simeq X'_n$ and $R'_n \simeq S'_n$, compatible with the groupoid structures, the isomorphisms $\phi_n\colon U_n \simeq X_n$ and $\psi_n\colon R_n \simeq S_n$, and the embeddings $?_{n-1} \subseteq ?_n$.

Let us denote by $\widetilde{R}$ and $\widetilde{U}$ the formal schemes obtained by completing $R\times\mathbb{A}^1$ and $U\times\mathbb{A}^1$ respectively along the inverse images of $u_0\in U$, and by $\widetilde{S}$ and $\widetilde{X}$ the formal schemes obtained by completing $S\times\mathbb{A}^1$ and $X\times\mathbb{A}^1$ respectively along the inverse images of the origin in X. The structure maps of the schemes in groupoids $R_n \rightrightarrows U_n$ and $S_n \rightrightarrows X_n$ pass to the limit, yielding formal schemes in groupoids $\widetilde{R}\rightrightarrows\widetilde{U}$ and $\widetilde{S}\rightrightarrows\widetilde{X}$. The isomorphisms $\phi_n\colon U_n \simeq X_n$ and $\psi_n\colon R_n \simeq S_n$ give isomorphisms of formal schemes $\widetilde{\phi}\colon \widetilde{U}\simeq\widetilde{X}$ and $\widetilde{\psi}\colon \widetilde{R}\simeq\widetilde{S}$, yielding an isomorphism of formal schemes in groupoids of $\widetilde{R}\rightrightarrows\widetilde{U}$ with $\widetilde{S}\rightrightarrows\widetilde{X}$.

Denote by $I_{\widetilde{U}}$ and $I_{\widetilde{R}}$ the sheaves of ideals of the inverse images of $\mathcal{Y}\times\{0\}\subseteq\mathcal{X}\times\mathbb{A}^1$ in $\widetilde{U}$ and $\widetilde{R}$ respectively, and by $J_{\widetilde{U}}$ and $J_{\widetilde{R}}$ the pushforwards to U and $\widetilde{R}$ of the sheaves of ideals of the pullbacks of the inverse images of $\mathcal{Y}\times\{0\}$ in the inverse images of $\mathcal{X}\times\{0\}$. Analogously, denote by $I_{\widetilde{X}}$ and $I_{\widetilde{S}}$ the sheaves of ideals of the inverse images of $[Y/G]\times\{0\}\subseteq[X/G]\times\mathbb{A}^1$ in $\widetilde{X}$ and $\widetilde{S}$ respectively, and by $J_{\widetilde{X}}$ and $J_{\widetilde{S}}$ the pushforwards to $\widetilde{X}$ and $\widetilde{S}$ of the sheaves of ideals of the pullbacks of the inverse images of $[Y/G]\times\{0\}$ in the inverse images of $[X/G]\times\{0\}$.

The natural morphisms $\widetilde{u}\colon\widetilde{U}\to U\times\mathbb{A}^1$, $\widetilde{r}\colon\widetilde{R}\to R\times\mathbb{A}^1$, $\widetilde{x}\colon\widetilde{X}\to X\times\mathbb{A}^1$ and $\widetilde{s}\colon\widetilde{S}\to S\times\mathbb{A}^1$ are flat (this is actually the key point of this part of the proof). Furthermore, the inverse images of $\mathcal{Y}\times\{0\}\subseteq\mathcal{X}\times\mathbb{A}^1$ and of $\mathcal{Y}\times\{0\}\subseteq\mathcal{X}\times\mathbb{A}^1$ in $\widetilde{U}$ and $\widetilde{R}$ and the inverse images of $[Y/G]\times\{0\}\subseteq[X/G]\times\mathbb{A}^1$ and of $[X/G]\times\{0\}\subseteq[X/G]\times\mathbb{A}^1$ in $\widetilde{U}$ and $\widetilde{R}$ correspond under $\widetilde{\phi}$ and $\widetilde{\psi}$; hence for each $m\geq 0$

we obtain canonical isomorphisms of coherent sheaves

$$\widetilde{u}^* I_U^m \simeq \widetilde{\phi}^* \widetilde{x}^* I_X^m, \quad \widetilde{r}^* I_R^m \simeq \widetilde{\psi}^* \widetilde{S}^* I_S^m,$$
$$\widetilde{u}^* J_U^m \simeq \widetilde{\phi}^* \widetilde{x}^* J_X^m, \quad \widetilde{r}^* J_R^m \simeq \widetilde{\psi}^* \widetilde{S}^* J_S^m.$$

By restricting to U_n and R_n we obtain isomorphism of coherent sheaves

$$I_U^m \otimes_{\mathcal{O}_{U\times\mathbb{A}^1}} \mathcal{O}_{U_n\times\mathbb{A}^1} \simeq \phi_n^*(I_X^m \otimes_{\mathcal{O}_{X\times\mathbb{A}^1}} \mathcal{O}_{X_n\times\mathbb{A}^1}),$$
$$I_R^m \otimes_{\mathcal{O}_{R\times\mathbb{A}^1}} \mathcal{O}_{R_n\times\mathbb{A}^1} \simeq \phi_n^*(I_S^m \otimes_{\mathcal{O}_{X\times\mathbb{A}^1}} \mathcal{O}_{S_n\times\mathbb{A}^1}),$$
$$J_U^m \otimes_{\mathcal{O}_{U\times\mathbb{A}^1}} \mathcal{O}_{U_n\times\mathbb{A}^1} \simeq \phi_n^*(J_X^m \otimes_{\mathcal{O}_{X\times\mathbb{A}^1}} \mathcal{O}_{X_n\times\mathbb{A}^1}),$$
$$J_R^m \otimes_{\mathcal{O}_{R\times\mathbb{A}^1}} \mathcal{O}_{R_n\times\mathbb{A}^1} \simeq \phi_n^*(J_S^m \otimes_{\mathcal{O}_{X\times\mathbb{A}^1}} \mathcal{O}_{S_n\times\mathbb{A}^1}).$$

By summing up over all m we obtain isomorphism of the corresponding Rees algebras, which yield the desired isomorphisms $U_n' \simeq X_n'$ and $R_n' \simeq S_n'$.

This ends the proof of Lemma 5.1, and of the Theorem. ♠

On the basis of examples, the following generalization of Theorem 1.2 seems plausible.

Conjecture 5.3. *Let $\mathcal{X}$ be an amenable stack over k. Let L be an extension of k, and let ξ be an object of $\mathcal{X}(\operatorname{Spec} L)$. Then* $\operatorname{ed}_k \xi \leq \operatorname{ged}_k \mathcal{X} + \dim \mathrm{R_u}(\underline{\mathrm{Aut}}_L\, \xi)$.

Here $\mathrm{R_u}\, G$ denotes the unipotent radical of G, as in Section 2.1. Unfortunately, the approach used in this section breaks down in the more general setting of the above conjecture: if the stabilizer is not reductive, the slice theorem does not apply.

6. Essential dimension of GL_n-quotients

Suppose that G is a special affine algebraic group over k acting on a scheme X locally of finite type over k. For each field L/k we have an equivalence between $[X/G](L)$ and the quotient category for the action of the discrete group $G(L)$ on the set $X(L)$; hence the essential dimension of $[X/G]$ equals the essential dimension of the *functor of orbits*

$$\mathrm{Orb}_{G,X}\colon (\mathrm{Field/k}) \longrightarrow (\mathrm{Set})$$

from the category $(\mathrm{Field/k})$ of extensions of k to the category of sets, sending L to the set of orbits $\mathrm{Orb}_{G,X}(L) \stackrel{\text{def}}{=} X(L)/G(L)$; see [BRV09, Example 2.6].

For the rest of this section we will assume that X is an integral scheme, locally of finite type and smooth over k, and GL_n acts on X with generically finite stabilizers. Then the quotient stack $[X/\mathrm{GL}_n]$ is amenable; however the Genericity Theorem 1.2 does not tell us that $\operatorname{ed}_k[X/\mathrm{GL}_n] = \operatorname{ged}_k[X/\mathrm{GL}_n]$ because we are not assuming that the stabilizer of every point of X is reductive. Nevertheless, in some cases one can still establish this equality by estimating $\operatorname{ed}_k \xi$ from above and proving, in an ad-hoc fashion, that $\operatorname{ed}_k(\xi) \leq \operatorname{ged}_k[X/\mathrm{GL}_n]$ for every $\xi \in [X/\mathrm{GL}_n](L)$ whose automorphism group is not reductive. The rest of this section will be devoted to such estimates. These estimates will ultimately allow us to deduce Theorem 1.1 from Proposition 3.4.

For each positive integer λ, denote by J_λ the $\lambda \times \lambda$ Jordan block with eigenvalue 0, that is, the $\lambda \times \lambda$ matrix, that is, the linear transformation $k^\lambda \to k^\lambda$ defined by $e_1 \mapsto 0$ and $e_i \mapsto e_{i-1}$ for $i = 2, \dots, \lambda$, where $e_1, \dots, e_\lambda$ is the canonical basis of k^λ. Let $\underline{\lambda}$ be a partition of n; that is, $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ is a non-increasing sequence

of positive integers with $\lambda_1 + \cdots + \lambda_r = n$. We denote by $A_{\underline{\lambda}}$ the $n \times n$ nilpotent matrix which is written in block form as

$$A_{\underline{\lambda}} \stackrel{\text{def}}{=} \begin{pmatrix} J_{\lambda_1} & 0 & \dots & 0 \\ 0 & J_{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\lambda_r} \end{pmatrix}$$

Every nilpotent $n \times n$ matrix is conjugate to a unique $A_{\underline{\lambda}}$. Consider the 1-parameter subgroup $\omega_{\underline{\lambda}} \colon \mathrm{G_a} \to \mathrm{GL}_n$ defined by $\omega_{\underline{\lambda}}(t) = \exp(tA_{\underline{\lambda}})$. We will usually assume that $\underline{\lambda} \neq (1^n)$; under this assumption $\omega_{\underline{\lambda}}$ is injective. Denote by $N_{\underline{\lambda}}$ the normalizer of the image of $\omega_{\underline{\lambda}}$; the group $N_{\underline{\lambda}}$ acts on the fixed point locus $X^{\omega_{\underline{\lambda}}}$.

Lemma 6.1. *Let L be a field extension of k, and ξ be an object of $[X/\mathrm{GL}_n](L)$ whose automorphism group scheme $\underline{\mathrm{Aut}}_L\, \xi$ is not reductive. Then*

$$\mathrm{ed}_k\, \xi \leq \max_{\underline{\lambda}} \mathrm{ed}_k [X^{\omega_{\underline{\lambda}}}/N_{\underline{\lambda}}]\,,$$

where the maximum is taken over all partitions $\underline{\lambda}$ of n different from (1^n).

Proof. Suppose ξ corresponds to the GL_n-orbit of a point $p \in X(L)$. The automorphism group scheme $\underline{\mathrm{Aut}}_L\, \xi$ is isomorphic to the stabilizer G_p of p in GL_n. Since we are assuming that this group is not reductive, G_p will contain a copy of $\mathbb{G}_\mathrm{a}$, which is conjugate to the image of $\omega_{\underline{\lambda}}$ for some $\underline{\lambda} \neq (1^n)$. After changing p to a suitable GL_n-translate, we may assume that the image of $\omega_{\underline{\lambda}}$ is contained in G_p; hence $p \in X^{\omega_{\underline{\lambda}}}$. The composite $X^{\omega_{\underline{\lambda}}} \hookrightarrow X \to [X/\mathrm{GL}_n]$ factors through $[X^{\omega_{\underline{\lambda}}}/N_{\underline{\lambda}}]$; hence ξ is in the essential image of $[X^{\omega_{\underline{\lambda}}}/N_{\underline{\lambda}}](L)$ in $[X/\mathrm{GL}_n](L)$, and $\mathrm{ed}_k\, \xi \leq \mathrm{ed}_k [X^{\omega_{\underline{\lambda}}}/N_{\underline{\lambda}}]$. ♠

Lemma 6.2. $\mathrm{ed}_k [X^{\omega_{\underline{\lambda}}}/N_{\underline{\lambda}}] \leq \dim X^{\omega_{\underline{\lambda}}}$ *for any $\underline{\lambda} \neq (1^n)$.*

Proof. By Lemma 2.1, $N_{\underline{\lambda}}$ is special. Hence,

$$\mathrm{ed}_k [X^{\omega_{\underline{\lambda}}}/N_{\underline{\lambda}}] = \mathrm{ed}_k\, \mathrm{Orb}_{N_{\underline{\lambda}}, X^{\omega_{\underline{\lambda}}}} \leq \dim X^{\omega_{\underline{\lambda}}}\,,$$

as claimed. ♠

We now further specialize X to the affine space $A_{n,d}$ of forms of degree d in the n variables $\underline{x} = (x_1, \dots, x_n)$ over k. The general linear group GL_n acts on $A_{n,d}$ in the usual way, via $(Af)(\underline{x}) \stackrel{\text{def}}{=} f(\underline{x} \cdot A^{-1})$ for any $A \in \mathrm{GL}_n$. We are now ready for the main result of this section.

Theorem 6.3. *Let L be a field extension of k, and ξ be an object of $[A_{n,d}/\mathrm{GL}_n](L)$ whose automorphism group scheme $\underline{\mathrm{Aut}}_L\, \xi$ is not reductive. Assume that either $d \geq 4$ and $n \geq 2$ or $d = 3$ and $n \geq 3$. Then $\mathrm{ed}_k\, \xi \leq \binom{n+d-1}{d} - n^2$.*

Proof. By Lemma 6.1 it suffices to show that

$$\mathrm{ed}_k [X^{\omega_{\underline{\lambda}}}/N_{\underline{\lambda}}] \leq \binom{n+d-1}{d} - n^2\,. \tag{6.1}$$

for any $\underline{\lambda} \neq (1^n)$. The space $A_{n,d}^{\omega_{\underline{\lambda}}}$ consists of the forms $f(\underline{x})$ such that

$$f\big(\exp(-tA_{\underline{\lambda}} x)\big) = f(\underline{x})\,.$$

By differentiating and applying the chain rule, this is equivalent to

$$\bigtriangledown f(\underline{x}) \cdot A_{\underline{\lambda}} = 0\,, \tag{6.2}$$

where $\bigtriangledown f = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)$ is the gradient of f. We now proceed with the proof of (6.1) in three steps.

Case 1: Assume $d \geq 4$. By Lemma 6.2 it suffices to show that

$$\dim A^{\omega_{\underline{\lambda}}}_{n,d} \leq \binom{n+d-1}{d} - n^2\,. \tag{6.3}$$

for any $\underline{\lambda} \neq (1^n)$. For $\underline{\lambda} \neq (1^n)$ formula (6.2) tells us that $\partial f/\partial x_1$, is identically zero. In other words, $f(\underline{x})$ is a form in $x_2, \dots, x_n$. Such forms lie in an affine subspace of $A_{n,d}$ isomorphic to $A_{n-1,d}$. Hence,

$$\dim A^{\omega_{\underline{\lambda}}}_{n,d} \leq \dim A_{n-1,d} = \binom{n+d-2}{d}\,,$$

and it suffices to prove the inequality

$$\binom{n+d-1}{d} - \binom{n+d-2}{d} \geq n^2$$

or equivalently,

$$\binom{n+d-2}{d-1} \geq n^2\,. \tag{6.4}$$

Since $\binom{n+d-2}{d-1} = \binom{n+d-2}{n-1}$ is an increasing function of d for any given $n \geq 1$, it suffices to prove (6.4) for $d = 4$. In this case

$$\binom{n+d-2}{d-1} - n^2 = \binom{n+2}{3} - n^2 = \frac{n(n-1)(n-2)}{6} \geq 0$$

for any $n \geq 2$, as desired.

Case 2: $d = 3$ and $\underline{\lambda} \neq (1^n)$ or $(2, 1^{n-1})$. Once again, it suffices to prove (6.3). If $\underline{\lambda} \neq (1^n)$ or $(2, 1^{n-1})$ then (6.2) shows that for every $f(\underline{x})$ in $A^{\omega_{\underline{\lambda}}}_{n,d}$ at least two of the partial derivatives $\partial f/\partial x_1$, and $\partial f/\partial x_i$ are identically zero. For notational simplicity we will assume that $i = 2$. Then $f(\underline{x})$ is a form in the variables $x_3, \dots, x_n$. Hence,

$$\dim A^{\omega_{\underline{\lambda}}}_{n,3} \leq \dim A_{n-2,3} = \binom{n}{3} = \binom{n+2}{3} - n^2\,,$$

as desired.

Case 3: Finally assume $d = 3$ and $\underline{\lambda} = (2, 1^{n-1})$. Set $\omega \stackrel{\text{def}}{=} \omega_{(2,1^{n-1})}$ and $N \stackrel{\text{def}}{=} N_{(2,1^{n-1})}$. By Lemma 2.1(b) N is a special group. Hence, we may identify the set of isomorphism classes in $[A^{\omega}_{n,3}/N](K)$ with the set of $N(K)$-orbits in $[A^{\omega}_{n,3}](K)$, for every field K/k.

By (6.2) $A^{\omega}_{n,d}$ consists of degree d forms $f(x_1, \dots, x_n)$ such that $\partial f/\partial x_1 = 0$. That is, $f(x_1, \dots, x_n) \in A^{\omega}_{n,d}$ if and only if f is a form in the variables $x_2, \dots, x_n$. Thus $A^{\omega}_{n,d}$ is an affine subspace of $A_{n,d}$ isomorphic to $A_{n-1,d}$. Our goal is to show that

$$\operatorname{ed}_k f \leq \binom{n+2}{3} - n^2 \tag{6.5}$$

for any $f(\underline{x}) \in A^{\omega}_{n,d}(K)$.

The normalizer N contains a subgroup

$$\Gamma \simeq \mathrm{GL}_{n-2} \ltimes \mathbb{G}_{\mathrm{a}}^{n-2}$$

consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & a_3 & & & \\ \vdots & \vdots & & A & \\ 0 & a_n & & & \end{pmatrix}$$

where $(a_3, \dots, a_n) \in \mathbb{G}_{\mathrm{a}}^{n-2}$ and $A \in \mathrm{GL}_{n-2}$. We may assume without loss of generality that the stabilizer of f in Γ does not contain a non-trivial unipotent subgroup. Indeed, if it does then f is a Γ-translate of an element of $A_{n,3}^{\omega_{\underline{\lambda}}}$ for some $\underline{\lambda} \neq (1^n)$ or $(2, 1^{n-1})$. For such f the inequality (6.5) was established in Case 2.

Since both Γ and N are special, it is obvious that the essential dimension of f, viewed as an element of $[A_{n,3}^{\omega}/N](K) = \mathrm{Orb}_{N, A_{n,3}^{\omega}}(K)$ is no greater than the essential dimension of f, viewed as an element of $[A_{n,3}^{\omega}/\Gamma](K) = \mathrm{Orb}_{\Gamma, A_{n,3}^{\omega}}(K)$. Since we are assuming that the stabilizer of f in Γ does not contain any non-trivial unipotent subgroups, the Genericity Theorem 1.2 tells us that

$$\mathrm{ed}_k f \leq \mathrm{ged}_k [A_{n,3}^{\omega}/\Gamma] \,.$$

By Lemma 6.4 below for $n \geq 4$ the action of $\Gamma \subset \mathrm{GL}_{n-1}$ on the space $A_{n-1,3}$ of forms of degree 3 in the $n-1$ indeterminates $x_2, \dots, x_n$ is generically free. Thus $[A_{n,3}/\Gamma]$ is an amenable stack and is generically a scheme. Consequently,

$$\mathrm{ged}_k [A_{n,3}^{\omega}/\Gamma_{n-2}] = \dim A_{n-1,3} - \dim \Gamma_{n-2} = \binom{n+1}{3} - (n-2)^2 - (n-2) \,.$$

A simple computation shows that

$$\binom{n+1}{3} - (n-2)^2 - (n-2) \leq \binom{n+2}{3} - n^2 \,,$$

for any $n \geq 4$; indeed,

$$\binom{n+2}{3} - n^2 - \left(\binom{n+1}{3} - (n-2)^2 - (n-2) \right) = \frac{(n-2)(n-3)}{2} - 1 \geq 0. \quad \spadesuit$$

Lemma 6.4.

(a) *Assume that the base field k is algebraically closed and G is a connected linear algebraic k-group such that $N \cap Z(G) \neq \{1\}$ for every closed normal subgroup $\{1\} \neq N \triangleleft G$. Here $Z(G)$ denotes the center of G. (For example, G could be almost simple or GL_n.) Let H_1, H_2 be closed subgroups of G such that H_1 is finite and H_2 contains no non-trivial central elements of G. Then for $g \in G(k)$ in general position, $H_1 \cap g H_2 g^{-1} = \{1\}$.*

(b) *Assume $d \geq 3$ and $n \geq 1$. Let GL_{n-1} be the subgroup of GL_n acting on the variables $x_2, \dots, x_n$. Then for $f \in A_{n,d}$ in general position, $\mathrm{Stab}_{\mathrm{GL}_{n-1}}(f) = \{1\}$.*

Proof. (a) Assume the contrary. Consider the natural (translation) action of H_1 on the homogeneous space G/H_2. By our assumption this action is not generically free. Since H_1 is finite, we conclude that this action is not faithful, i.e., some $1 \neq h \in H_1$ acts trivially on G/H_2. Then h lies in $N = \bigcap_{g \in G} g H_2 g^{-1}$. Consequently,

N is a non-trivial normal subgroup of G. By our assumption N (and hence, H_2) contains a non-trivial central element of G, a contradiction.

(b) We may assume that k is algebraically closed. By [Ric72, Theorem A], there exists a subgroup $S_{n,d} \subset \mathrm{GL}_n$ and a dense open subset $U \subset A_{n,d}$ such that $\mathrm{Stab}_{\mathrm{GL}_n}(f)$ is conjugate to S for every $f \in U$. Moreover, for $d \geq 3$ (and any $n \geq 1$) $S_{n,d}$ is a finite group; see Example 3.2.

Write $f(x_1, \dots, x_n) = \sum_{i=0}^{d} x_1^{d-i} f_i(x_2, \dots, x_n)$, where f_i is a form of degree i in $x_2, \dots, x_n$. Clearly $g \in \mathrm{GL}_{n-1}$ stabilizes f if and only if it stabilizes $f_1, f_2, \dots, f_d$. In other words,

$$\mathrm{Stab}_{\mathrm{GL}_{n-1}}(f) = \bigcap_{i=1}^{d} \mathrm{Stab}_{\mathrm{GL}_{n-1}}(f_i)\,.$$

Moreover, each $\mathrm{Stab}_{\mathrm{GL}_{n-1}}(f_i)$ is a conjugate of $S_{n-1,i}$ in GL_{n-1}. Thus it suffices to show that for $g_1, \dots, g_d$ in general position in GL_{n-1},

$$g_1 S_{n-1,1} g_1^{-1} \cap \dots \cap g_d S_{n-1,d} g_d^{-1} = \{1\}\,. \tag{6.6}$$

This is a consequence of part (a), with $G = \mathrm{GL}_n$, $H_1 = S_{n-1,1}$ and $H_2 = S_{n-1,d}$. ♠

Remark 6.5. We note that if $d \geq 4$ and $n \geq 3$ or $d = 3$ and $n \geq 5$ then (6.6) is immediate, since $S_{n-1,d} = \{1\}$; see Example (3.2). However, this argument does not cover the case where $d = 3$ and $n = 3$ or 4, which are needed for the proof of Theorem 6.3 above.

7. Proof of Theorem 1.1

Theorem 1.1(a) is an immediate consequence of what we have done so far. Indeed, by Proposition 3.4(a),

$$\mathrm{ged}_k[A_{n,d}/\mathrm{GL}_n] = \binom{n+d-1}{d} - n^2 + 1 + \mathrm{cd}(\mathrm{GL}_n/\mu_d)\,.$$

Thus it suffices to show that for any field extension K/k and any K-point ζ of $[A_{n,d}/\mathrm{GL}_n](K)$, we have

$$\mathrm{ed}_k\,\zeta \leq \mathrm{ged}_k[A_{n,d}/\mathrm{GL}_n]\,.$$

If the automorphism group scheme $\underline{\mathrm{Aut}}_K(\zeta)$ is reductive, this is a direct consequence of Theorem 1.2, and if $\underline{\mathrm{Aut}}_K(\zeta)$ is not reductive, then Theorem 6.3 tells us that

$$\mathrm{ed}_k\,\zeta \leq \binom{n+d-1}{d} - n^2 < \mathrm{ged}_k[A_{n,d}/\mathrm{GL}_n]\,.$$

This completes the proof of Theorem 1.1(a). The rest of this section will be devoted to proving Theorem 1.1(b). The main complication here is that the stack $[\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$ is not amenable (see Example 3.2) and thus our Genericity Theorem 1.2 does not apply. We will get around this difficulty by relating $[\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$ to the amenable stack $[\mathbb{P}(A_{n,d})/\mathrm{PGL}_n]$.

Proposition 7.1. *Let $\mathcal{X} = [\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$ and $\overline{\mathcal{X}} \stackrel{\mathrm{def}}{=} [\mathbb{P}(A_{n,d})/\mathrm{PGL}_n]$, with the natural projection $\phi\colon \mathcal{X} \to \overline{\mathcal{X}}$. Then for any extension L/k and any L-point $\xi\colon \operatorname{Spec} L \to \mathcal{X}$,*

$$\mathrm{ed}_k\,\xi \leq \mathrm{ed}_k\,\phi(\xi) + \mathrm{cd}(\mathrm{GL}_n/\mu_d)\,.$$

Proof. Note that $\mathcal{X}$ is a gerbe banded by $\mathbb{G}_{\mathrm{m}}$ over $\overline{\mathcal{X}}$.

By the definition of $\operatorname{ed}_k \phi(\xi)$ there exists an intermediate field $k \subset K \subset L$ such that $\phi(\xi)$ descends to $\operatorname{Spec} K$ and $\operatorname{tr\,deg}_k K = \operatorname{ed}_k \phi(\xi)$. Moreover, $\xi\colon \operatorname{Spec} L \to \mathcal{X}_K$ factors through a point $\xi_0\colon \operatorname{Spec} L \to \mathcal{X}_K$, as in the diagram below.

$$\begin{tikzcd}
 & \mathcal{X}_K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\phi"] \\
\overline{\xi}\colon \operatorname{Spec} L \arrow[r] \arrow[ur, "\xi_0"] & \operatorname{Spec} K \arrow[r] & \overline{\mathcal{X}},
\end{tikzcd}$$

Note that $\mathcal{X}_K$ is a $\mathbb{G}_{\mathrm{m}}$-gerbe over K. So ξ_0 (and hence, ξ) descends to some intermediate subfield of $K \subset K_0 \subset L$ such that $\operatorname{tr\,deg}_K K_0 \le \operatorname{ed}_k(\mathcal{X}_K) = \operatorname{cd}(\mathcal{X}_K)$, where the last equality is Proposition 2.3(a). Let η be the generic point of $\overline{\mathcal{X}}$. We know that the Brauer class of $\mathcal{X}_\eta$ has index dividing n and exponent dividing d; see the proof of Proposition 3.4. By Lemma 2.4 the same is true of the Brauer class of $\mathcal{X}_K$. Therefore, by Lemma 2.2(c), $\operatorname{cd}(\mathcal{X}_K) \le \operatorname{cd}(\mathrm{GL}_n/\mu_d)$. In summary,

$$\operatorname{ed}_k \xi \le \operatorname{tr\,deg}_k K_0 = \operatorname{tr\,deg}_k K + \operatorname{tr\,deg}_K K_0 \le \operatorname{ed}_k(\phi(\xi)) + \operatorname{cd}(\mathcal{X}_K),$$

as claimed. ♠

Proof of Theorem 1.1(b). Let $\mathcal{X} \stackrel{\text{def}}{=} [\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$, $\overline{\mathcal{X}} \stackrel{\text{def}}{=} [\mathbb{P}(A_{n,d})/\mathrm{PGL}_n]$, and $\mathcal{Y} \stackrel{\text{def}}{=} [A_{n,d}/\mathrm{GL}_n]$ and consider the following diagram of natural maps.

$$\begin{tikzcd}
 & \mathcal{Y} \arrow[dl] \arrow[ddl, "\psi,\ \text{a } \mu_d\text{-gerbe}"] \\
\mathcal{X} \arrow[d, "\phi,\ \text{a } \mathbb{G}_{\mathrm{m}}\text{-gerbe}"'] & \\
\overline{\mathcal{X}} &
\end{tikzcd}$$

In view of Proposition 7.1, it suffices to show that for every field extension L/k and every L-point $\xi\colon \operatorname{Spec} L \to \overline{\mathcal{X}}$,

$$\operatorname{ed}_k \phi(\xi) \le \binom{n+d-1}{d} - n^2. \tag{7.1}$$

Recall from Example 3.2 that under our assumptions on n and d, we have that $\overline{\mathcal{X}} = [\mathbb{P}(A_{n,d})/\mathrm{PGL}_n]$ is amenable and is generically a scheme of dimension

$$\dim \mathbb{P}(A_{n,d}) - \dim \mathrm{PGL}_n = \binom{n+d-1}{d} - n^2.$$

If the automorphism group scheme $\underline{\mathrm{Aut}}_L(\phi(\zeta))$ is reductive then (7.1) holds by the Genericity Theorem 1.2 applied to $\overline{\mathcal{X}}$.

We may therefore assume that $\underline{\mathrm{Aut}}_L(\phi(\zeta))$ is not reductive. Lift ξ to some $\zeta\colon \operatorname{Spec} L \to \mathcal{Y}$. (This can be done because $\mathcal{Y} \to \mathcal{X}$ is a $\mathbb{G}_{\mathrm{m}}$-torsor.) Since the automorphism group scheme $\underline{\mathrm{Aut}}_L(\zeta)$ is contained in the preimage of $\underline{\mathrm{Aut}}_L(\phi(\zeta))$

under the natural projection map $\mathrm{GL}_n \to \mathrm{PGL}_n$, we see that $\underline{\mathrm{Aut}}_L(\zeta)$ is not reductive. Now $\mathrm{ed}_k\, \phi(\xi) \leq \mathrm{ed}_k\, \zeta$ and in view of Theorem 6.3

$$\mathrm{ed}_k\, \zeta \leq \binom{n+d-1}{d} - n^2\,.$$

This completes the proof of (7.1) and thus of Theorem 1.1(b). ♠

8. Small n and d

In this section we compute $\mathrm{ed}_k\, \mathrm{Forms}_{n,d}$ and $\mathrm{ed}_k\, \mathrm{Hypersurf}_{n,d}$ in the cases not covered by Theorem 1.1, building on the results of [BF03] and [BR05, Section 16].

To handle the case where $n = 2$, we need the following variant of [BR05, Lemma 16.1]. The proof is similar; we reproduce it here, with the necessary modifications, for the sake of completeness.

Lemma 8.1. $\mathrm{ed}_k\, \mathrm{Forms}_{2,d} \leq d-1$ *and* $\mathrm{ed}_k\, \mathrm{Hypersurf}_{2,d} \leq d-2$ *for any* $d \geq 3$.

Proof. Let $f(x_1, x_2) = a_0 x_1^d + a_1 x_1^{d-1} x_2 + \cdots + a_d x_2^d$ be a non-zero binary form of degree d over a field K/k. We claim that that f is equivalent (up to a linear coordinate changes by elements of $\mathrm{GL}_2(K)$) to a binary form with (i) $a_0 = 0$ or $a_1 = 0$ and (ii) $a_{d-1} = 0$ or $a_{d-1} = a_d$. In each case f descends to the field $k(a_0, \dots, a_d)$ and the hypersurface in $\mathbb{P}^1$ cut out by f descends to the field $k(a_i/a_j \,|\, a_j \neq 0)$. If (i) and (ii) are satisfied then the transcendence degrees of these fields over k are clearly $\leq d-1$ and $d-2$, respectively. So, the lemma follows from the claim.

To prove the claim, we first reduce f to a form satisfying (i). If $a_0 = 0$, we are done. If $a_0 \neq 0$, then performing the Tschirnhaus substitution

$$x_1 \mapsto x_1 - \frac{a_1}{d a_0} x_2\,, \quad x_2 \mapsto x_2$$

we reduce f to a binary form with $a_1 = 0$.

Now assume that f satisfies (i). We want to further reduce it to a form satisfying both (i) and (ii). If $a_{d-1} = 0$, we are done. If $a_{d-1} \neq 0$, rescale x_1 as follows

$$x_1 \mapsto \frac{a_d}{a_{d-1}} x_1\,, \quad x_2 \mapsto x_2\,,$$

to reduce f to a form satisfying (i) and $a_{d-1} = a_d$. This completes the proof of the claim and the lemma. ♠

Proposition 8.2. *For any* $n \geq 1$ *and* $d \geq 2$ *we have*

(a) $\mathrm{ed}_k\, \mathrm{Forms}_{n,1} = \mathrm{ed}_k\, \mathrm{Hypersurf}_{n,1} = 0$,
(b) $\mathrm{ed}_k\, \mathrm{Forms}_{1,d} = 1$ *and* $\mathrm{ed}_k\, \mathrm{Hypersurf}_{1,d} = 0$,
(c) $\mathrm{ed}_k\, \mathrm{Forms}_{n,2} = n$ *and* $\mathrm{ed}_k\, \mathrm{Hypersurf}_{n,2} = n-1$,
(d) $\mathrm{ed}_k\, \mathrm{Forms}_{2,3} = 2$ *and* $\mathrm{ed}_k\, \mathrm{Hypersurf}_{2,3} = 1$,
(e) $\mathrm{ed}_k\, \mathrm{Forms}_{2,4} = 3$ *and* $\mathrm{ed}_k\, \mathrm{Hypersurf}_{2,4} = 2$,
(f) $\mathrm{ed}_k\, \mathrm{Forms}_{3,3} = 4$ *and* $\mathrm{ed}_k\, \mathrm{Hypersurf}_{3,3} = 3$.

Proof. First we note that

$$\mathrm{ed}_k\, \mathrm{Forms}_{n,d} \leq \mathrm{ed}_k\, \mathrm{Hypersurf}_{n,d} + 1\,. \tag{8.1}$$

This is easy to see directly from the definition or, alternatively, as a special case of the Fiber Dimension Theorem [BRV09, Theorem 3.2(b)], applied to the representable morphism of quotient stacks $[A_{n,d} \smallsetminus \{0\}/\mathrm{GL}_n] \to [\mathbb{P}(A_{n,d})/\mathrm{GL}_n]$ of relative dimension 1.

(a) Any non-zero linear form $F(x_1, \dots, x_n)$ over any field K/k is equivalent to x_1.

(b) Degree d forms $f_1(x) = ax^d$ and $f_2(x) = bx^d$, over a field K/k are equivalent if and only if $b = ac^d$ for some $c \in K^* = \mathrm{GL}_1(K)$. The assertions of part (b) follow easily from this.

(c) Any quadratic form $F(x_1, \dots, x_n)$ over K/k can be diagonalized and hence, is defined over an intermediate field $k \subset K_0 \subset K$ such that $\operatorname{tr\,deg}_k K_0 \le n$. This implies that $\mathrm{ed}_k \mathrm{Forms}_{n,2} \le n$ and $\mathrm{ed}_k \mathrm{Hypersurf}_{n,2} \le n-1$. The opposite inequalities follow from [BR05, Proposition 16.2(b)].

(d) By Lemma 8.1, $\mathrm{ed}_k \mathrm{Forms}_{2,3} \le 2$ and $\mathrm{ed}_k \mathrm{Hypersurf}_{2,3} \le 1$, respectively. On the other hand, by [BR05, Proposition 16.2(c)], for the generic binary form F_{gen} in three variables (as in (1.3)) and the hypersurface H_{gen} it cuts out in $\mathbb{P}^1$, we have $\mathrm{ed}_k F_{\mathrm{gen}} = 2$ and $\mathrm{ed}_k H_{\mathrm{gen}} = 1$.

Part (e) is proved in a similar manner, by combining Lemma 8.1 with [BR05, Proposition 16.2(d)].

(f) The identity $\mathrm{ed}_k \mathrm{Hypersurf}_{3,3} = 3$ is the main result of [BF03]. By (8.1), $\mathrm{ed}_k \mathrm{Forms}_{3,3} \le 4$.

In order to show that equality holds, it suffices to prove that the essential dimension $\mathrm{ged}_k[X_{3,3}/\mathrm{GL}_3]$ of the generic form F_{gen} of degree 3 in 3 variables is at least 4. By [Ric72, Theorem A] the GL_3-action on $X_{3,3}$ has a stabilizer in general position. Denote it by $S_{3,3}$, as in the proof of Lemma 6.4. As we mentioned there (and in Example 3.2), $S_{3,3}$ is a finite subgroup of GL_3. Since the dimension of $[X_{3,3}/\mathrm{GL}_3]$ is 1, by [BR05, Lemma 15.4 and Proposition 5.5(c)] $\mathrm{ed}_k F_{\mathrm{gen}} \ge \mathrm{ed}_{\overline{k}}(S_{3,3}) + 1$, where $\mathrm{ed}_k S_{3,3}$ denotes the essential dimension of the finite group $S_{3,3}$. (Note that in [BR05] the symbol $\phi_{n,d}$ was used in place of F_{gen}.) Thus it suffices to show that $\mathrm{ed}_{\overline{k}}(S_{3,3}) \ge 3$.

To get a better idea about the structure of $S_{3,3}$, note that a form in $X_{3,3}(\overline{k})$ in general position is a scalar multiple of $x_1^3 + x_2^3 + x_3^3 + 3ax_1x_2x_3$ for some $a \in \overline{k}$. Hence, S contains a non-abelian subgroup H of order 27, generated by diagonal permutation matrices $\mathrm{diag}(\zeta_1, \zeta_2, \zeta_3)$, where ζ_1, ζ_2 and ζ_3 are cube roots of unity satisfying $\zeta_1\zeta_2\zeta_3 = 1$, and the permutation matrices cyclically permuting x_1, x_2 and x_3. Now

$$\mathrm{ed}_{\overline{k}}(S) \ge \mathrm{ed}_{\overline{k}}(H) \ge 3\,, \tag{8.2}$$

where the second inequality is a consequence of the Karpenko-Merkurjev theorem; see [MR10, Theorem 1.3]. This completes the proof of part (f). ♠

Remark 8.3.

(i) Since S is a finite subgroup of GL_3, it has a natural faithful 3-dimensional representation. Hence, $\mathrm{ed}_k S_{3,3} \le 3$, and both inequalities in (8.2) are actually equalities.

(ii) The proof of part (e) shows that $\operatorname{ed}_k F_{\mathrm{gen}} = 4$, where F_{gen} is the generic form of degree 3 in 3 variables, as in (1.3). This answers an open question posed after the statement of Proposition 16.2 in [BR05].

9. Essential dimension of singular curves

In this section we use our new Genericity Theorem (1.2) to strengthen [BRV09, Theorem 7.3] on the essential dimension of the stack on (not necessarily smooth) local complete intersection curves with finite automorphism group presented in [BRV09, Theorem 7.3]. Let us recall the set-up. Denote by $\mathfrak{M}_{g,n}$ the stack of all reduced n-pointed local complete intersection curves of genus g, that is, the algebraic stack over $\operatorname{Spec} k$ whose objects over a k-scheme T are finitely presented proper flat morphisms $\pi\colon C \to T$, together with n sections $s_1, \dots, s_n\colon T \to C$, where C is an algebraic space, the geometric fibers of π are connected reduced local complete intersection curves of genus g, and the image of each s_i is contained in the smooth locus of $C \to T$. (We do not require the images of the sections to be disjoint.)

The stack $\mathfrak{M}_{g,n}$ contains the stack $\mathcal{M}_{g,n}$ of smooth n-pointed curves of genus g as an open substack (here the sections are supposed to be disjoint). By standard results in deformation theory, every reduced local complete intersection curve is unobstructed, and is a limit of smooth curves. Furthermore there is no obstruction to extending the sections, since these map into the smooth locus. Therefore $\mathfrak{M}_{g,n}$ is smooth and connected, and $\mathcal{M}_{g,n}$ is dense in $\mathfrak{M}_{g,n}$. However, the stack $\mathfrak{M}_{g,n}$ is very large (it is certainly not of finite type), and in fact it is very easy to see that its essential dimension is infinite. Assume that we are in the stable range, i.e., $2g - 2 + n > 0$: then in [BRV09] we show that the essential dimension of the open substack $\mathfrak{M}^{\mathrm{fin}}_{g,n}$ of $\mathfrak{M}_{g,n}$ of curves with finite automorphism group equals the essential dimension of $\mathcal{M}_{g,n}$.

Let C be an object of $\mathfrak{M}_{g,n}$ defined over an algebraically closed field K. We say that C is *reductive* if the automorphism group scheme $\underline{\mathrm{Aut}}_K C$ is reductive. The marked curve C is *not* reductive if and only the smooth part $C_{\mathrm{sm}} \subseteq C$ contains a component that is isomorphic to $\mathbb{A}^1_K$ and contains no marked points. A reductive object of $\mathfrak{M}_{g,n}$ is an object $C \to S$, whose geometric fibers over S are reductive. It is not hard to see that the reductive objects form an open substack $\mathfrak{M}^{\mathrm{red}}_{g,n}$ of $\mathfrak{M}_{g,n}$. Then our new genericity theorem applies, and allows to conclude that the essential dimensions of $\mathfrak{M}^{\mathrm{red}}_{g,n}$ and of $\mathcal{M}_{g,n}$ are the same. From [BRV09, Theorem 1.2] we obtain the following.

Theorem 9.1. *If $2g - 2 + n > 0$ and the characteristic of k is 0, then*

$$\operatorname{ed}_k \mathfrak{M}^{\mathrm{red}}_{g,n} = \begin{cases} 2 & \text{if } (g,n) = (1,1), \\ 5 & \text{if } (g,n) = (2,0), \\ 3g - 3 + n & \text{otherwise.} \end{cases}$$

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