DECOMPOSITION OF ALGEBRAS WITH INVOLUTION IN CHARACTERISTIC 2

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Abstract. In this paper we present a decomposition theorem for hermitian forms over fields of characteristic 2 refining the usual Witt decomposition in this case. We apply this decomposition to algebras with involution over fields of characteristic 2 to give a complete description of the effect of passing to a generic splitting field of the algebra on the isotropy of the involution.

1. Introduction

The decomposition theorem of Witt (see [15]) states that any regular symmetric quadratic form over a field of characteristic different from 2 uniquely decomposes into an orthogonal sum of an anisotropic quadratic form and a hyperbolic quadratic form. This theorem has since been generalised and extended in many ways and to many different contexts. Our interest will be in its generalisation to the theory of hermitian forms, where it says that every regular hermitian form over a division algebra with involution can be decomposed into an orthogonal sum of an anisotropic hermitian form and a metabolic hermitian form, but where the latter need not be uniquely determined if the characteristic of the underlying field is 2.

In this paper we will show that in the case of a division algebra over a base field F of characteristic 2 with an F–linear involution, an anisotropic hermitian form has a natural decomposition into what we will call its *direct* and *alternating parts*. As we shall see, the notion of a direct hermitian form allows the generalisation to hermitian form theory of several results on anisotropic bilinear forms, which do not generally hold for anisotropic hermitian forms.

In particular, in (6.7) we shall show that in characteristic 2 direct hermitian forms remain direct, and in particular anisotropic, over the separable closure of the base field. This is a generalisation of [9, Satz 10.2.1], a theorem on bilinear forms over fields of characteristic 2. We shall also see, in (6.3), that in characteristic 2, the direct hermitian forms are precisely those forms that remain anisotropic when the algebra is generically split.

Most of our final results will be stated in terms of algebras with involution, rather than hermitian forms. Any hermitian form over a division algebra is associated with an algebra with involution. In Section 2 we will briefly cover this association, as well as cover the basic definitions and results we need from the theory of algebras with involution and hermitian forms. In Section 3 we introduce the concept of direct hermitian forms and show that any anisotropic hermitian form decomposes into direct and alternating parts in (3.9).

2000 *Mathematics Subject Classification.* Primary 11E81; Secondary 11E39, 12F10. *Key words and phrases.* Central simple algebras; involutions; hermitian forms; characteristic two, generic splitting.

In Section 4 we translate the concepts of direct and alternating from the theory of hermitian forms to the theory of algebras with involution. We shall show that every involution over a base field of characteristic 2 has well-defined direct and alternating parts in (4.5).

Section 5 will be a brief digression into the study of the notion of the orthogonal sum of algebras with involution. This concept was introduced by Dejaiffe in [3] for fields of characteristic different from 2. We follow the more general presentation of the concept found in [2].

Finally, in Section 6 we will apply (3.9) in order to study the effect of generically splitting the algebra on the isotropy of an involution over that algebra. This is a topic of much recent study of fields of characteristic different from 2, see e.g. [8]. In particular, in characteristic different from 2, it is an open problem whether any anisotropic involution on a central simple algebra remains anisotropic after extension to some splitting field of A. As we shall see in (6.4), over fields of characteristic 2, there exists such a field precisely when the involution is direct. Moreover, we shall show that the anisotropic part of an algebra with involution after generically splitting the algebra is simply its direct part extended to the splitting field.

2. Algebras with involution

In this section we recall the basic definitions and results we will use on central simple algebras with involution and hermitian forms. We refer to [13] for a general reference on central simple algebras.

Throughout, let F be a field and A be a finite-dimensional simple F–algebra. The centre of A is a field E, and A can be viewed as a E–algebra. In this case $\dim_E(A)$ is a square, and the positive root of this integer is called the *degree of* A and is denoted $\deg_E(A)$. By Wedderburn's theorem, $A \simeq \mathrm{End}_D(V)$ for F–division algebra D with centre E and a right D–vector space V. The degree of D is called the *index of* A and is denoted $\mathrm{ind}_E(A)$. We call any A with $\mathrm{ind}_E(A) = 1$ *split.* We call a field extension L/E a *splitting field of* A if $A \otimes_F L$ is split. If $E = F$, then we call the F–algebra A *central simple.*

An F*–involution* on A is an F–linear map $\sigma : A \to A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$ and $\sigma^2 = \mathrm{id}_A$. An F*–algebra with involution* is a pair (A, σ) of a finite-dimensional F–algebra A and an F–involution σ of A such that, with E being the centre of A, one has $F = \{x \in E \mid \sigma(x) = x\}$, and such that either A is simple or a product of two simple F–algebras that are mapped to each other by σ. In this situation, there are two possibilities: either $E = F$, so that A a central simple F–algebra, or E/F is a quadratic étale extension with σ restricting to the nontrivial F–automorphism of E. To distinguish these two situations, we speak of involutions of the first or second kind; more precisely, we say that the F–algebra with involution (A, σ) is of the *first kind* if $E = F$ and of the *second kind* otherwise. Involutions of the second kind are also known as *unitary involutions.* For any field extension K/F we will use the notations $A_K = A \otimes_F K$, $\sigma_K = \sigma \otimes \mathrm{id}_K$ and $(A, \sigma)_K = (A_K, \sigma_K)$.

Let (A, σ) be an F–algebra with involution E be the centre of A. For $\lambda \in E$, let $\mathrm{Sym}_\lambda(A, \sigma) = \{a \in A \mid \lambda\sigma(a) = a\}$ and $\mathrm{Alt}(A, \sigma) = \{\sigma(a) - a \,|\, a \in A\}$. These are F–linear subspaces of A.

We call an idempotent $e \in A$ *hyperbolic with respect to* σ if $\sigma(e) = 1 - e$. Moreover, we call an F–algebra with involution (A, σ) *hyperbolic* if A contains a hyperbolic idempotent with respect to σ.

An idempotent $e \in A$ is called *metabolic with respect to* σ if $\sigma(e)e = 0$ and $\dim_F eA = \frac{1}{2}\dim_F A$. Note that, by [4, Corollary 4.3], we may substitute the condition that $\dim_F eA = \frac{1}{2}\dim_F A$ for the condition that $(1-e)(1-\sigma(e)) = 0$ in this definition. An algebra with involution (A, σ) is called *metabolic* if A contains a metabolic idempotent with respect to σ.

An F*–quaternion algebra* is a central simple F-algebra of degree 2. The description of quaternion algebras over fields of any characteristic is give in [14, Section 8.11]. We recall it for fields of characteristic 2. If $\mathrm{char}(F) = 2$, then given any $\alpha \in F$ and $\beta \in F^\times$, there exists an F–quaternion algebra with an F–basis $(1, i, j, k)$ subject to the relations that $i^2 + i = \alpha$, $j^2 = \beta$ and $k = ij = ji + j$; we denote this F–quaternion algebra by $[\alpha, \beta)_F$. If $\mathrm{char}(F) = 2$, by [14, Section 8.11], every quaternion algebra is isomorphic to $[\alpha, \beta)_F$ for some $\alpha \in F$ and $\beta \in F^\times$. Note that $[\alpha, \beta)_F$ is split if $\alpha = u^2 + u$ for some $u \in F$. In particular, any quaternion division algebra splits over a quadratic separable extension.

Every quaternion algebra Q has a so-called *canonical involution* (see [11, Proposition 2.21]). When the characteristic of F is 2 and $(1, i, j, k)$ is an F–basis of Q with relations as above, this involution is given by

$$x_0 + x_1 i + x_2 j + x_3 k \longmapsto x_0 + x_1(i+1) + x_2 j + x_3 k$$

for $x_0, x_1, x_2, x_3 \in F$. This involution is symplectic, and is in fact the unique symplectic involution on Q.

Throughout the rest of this section, let (D, θ) be an F–division algebra with involution and E be the centre of D. Further, fix $\lambda \in E$ such that $\lambda\theta(\lambda) = 1$.

A λ*–hermitian form* over (D, θ) is a pair (V, h) where V is a finite-dimensional right D-vector space and h is a non-degenerate bi-additive map $h : V \times V \to D$ such that

$$h(x, yd) = h(x, y)d \quad \text{and} \quad h(y, x) = \lambda\theta(h(x, y))$$

holds for all $x, y \in V$ and $d \in D$. Let $\simeq$ denote *isometry* between hermitian forms and we denote the *orthogonal sum* of two hermitian forms (V, h) and (W, h') as $(V, h)\perp(W, h')$.

Throughout the rest of this section we fix (V, h) to be a λ–hermitian form over an F–division algebra with involution (D, θ). We say that (V, h) is *hermitian* when $\lambda = 1$ and *skew* when $\lambda = -1$. Note that if (D, θ) is of the first kind, then (V, h) is always either hermitian or skew. If D is split and $\theta = \mathrm{Id}_D$, then (V, h) is a *symmetric bilinear form.* We say (V, h) *represents* $a \in D$ if there exists some $x \in V$ such that $h(x, x) = a$, and we say (V, h) *represents* $a \in D$ *non-trivially* if there exists some $x \in V \backslash \{0\}$ such that $h(x, x) = a$. We say (V, h) is *isotropic* if it represents 0 non-trivially, and *anisotropic* otherwise. We call (V, h) *alternating* if $h(x, x) \in \mathrm{Alt}(D, \theta)$ for all $x \in V$.

Let V be a finite-dimensional right D–vector space V and $V^* = \mathrm{End}_D(V, D)$, the dual of V. Then there exists a regular λ–hermitian form $\mathbb{H}_\lambda(V) = (V^* \oplus V, h_\lambda)$ over (D, θ), where

$$h_\lambda : (V^* \oplus V) \times (V^* \oplus V) \to D$$

is given by

$$h_\lambda(\varphi + x, \psi + y) = \varphi(y) + \lambda\theta(\psi(x)) \text{ for } \varphi, \psi \in V^* \text{ and } x, y \in V.$$

We call a λ-hermitian form over (D,θ) *hyperbolic* if it is isometric to $\mathbb{H}_\lambda(V)$ for some right D–vector space V.

Let $S \subset V$. We define the *orthogonal complement* $S^\perp$ of S with respect to a fixed λ–hermitian form (V,h) as

$$S^\perp = \{x \in V \mid h(x,s) = 0 \text{ for all } s \in S\}.$$

The λ–hermitian form (V,h) is called *metabolic* if there exists a subspace $S \subset V$ such that $S = S^\perp$.

For $a_1,\dots,a_n \in D^\times$ such that $a_i \in \mathrm{Sym}_\lambda(A,\sigma)$, for $i = 1\dots,n$, we denote by $\langle a_1,\dots,a_n\rangle_{(\mathrm{D},\theta,\lambda)}$ the λ-hermitian form (D^n,h) where

$$h : D^n \times D^n \to D, \qquad (x,y) \mapsto \sum_{i=1}^n \theta(x_i)a_i y_i.$$

We call such a form a *diagonal form.* We call a hermitian form *diagonalisable* if it is isometric to a diagonal form.

Proposition 2.1. *The λ–hermitian form (V,h) is diagonalisable, except if $D = F$ and either* $\mathrm{char}(F) \neq 2$ *and (V,h) is a skew-symmetric bilinear space, or* $\mathrm{char}(F) = 2$ *and (V,h) is a hyperbolic symmetric bilinear space.*

Proof. See [10, Proposition 6.2.4]. □

Proposition 2.2. *Let (V,h) be a λ–hermitian form over (D,θ). There is a unique F–involution σ on $\mathrm{End}_D(V)$ such that*

$$h(f(x),y) = h(x,\sigma(f)(y)) \textit{ for all } x,y \in V \textit{ and } f \in A.$$

This involution σ is uniquely determined by h. Further, $(\mathrm{End}_D(V),\mathrm{ad}_h)$ is an F–algebra with involution of the same kind as (D,θ) and A is Brauer equivalent to D.

Proof. See, for example, [11, Theorem 4.1]. □

In the situation of (2.2), we call σ the *adjoint involution to h* and denote it by ad_h, and we further write $\mathrm{Ad}(V,h) = (\mathrm{End}_D(V),\mathrm{ad}_h)$.

Let L be a splitting field of the F–algebra A. An involution (A,σ) of the first kind is said to be *symplectic* if $(A,\sigma)_L \simeq \mathrm{Ad}(V,b)$, where (V,b) is an alternating bilinear form, and *orthogonal* otherwise. This definition is independent of the choice of the splitting field L (see [11, Section 2.A]).

Proposition 2.3. *Assume* $\mathrm{char}(F) = 2$ *and (A,σ) is of the first kind. Then (V,h) is alternating if and only if* $\mathrm{Ad}(V,h)$ *is symplectic.*

Proof. See [11, Theorem 4.2] □

Proposition 2.4. (V,h) *is hyperbolic (resp. metabolic) if and only if* $\mathrm{Ad}(V,h)$ *is hyperbolic (resp. metabolic).*

Proof. See [11, Proposition 6.7] for the statement on hyperbolicity and [4, Theorem 4.8] for the statement on metabolicity in the case of involutions of the first kind. The argument there remains valid for unitary involutions. □

We will often use a single letter to denote a hermitian form.

Proposition 2.5. *Any λ–hermitian form has a decomposition $\eta \simeq \varphi \perp \rho$, where φ is an anisotropic λ–hermitian form and ρ is a metabolic λ–hermitian form, both over (D, θ). Moreover φ is unique up to isometry.*

Proof. See [10, Proposition 6.1.1] and [10, Proposition 6.1.4]. □

In the situation of (2.5) we call φ the *anisotropic part of* η, denoted η_{an}.

Let (A, σ) be an F–algebra with involution. We call an algebra (B, τ) with involution an *anisotropic part* of (A, σ), if $(B, \tau) \simeq \mathrm{Ad}(\eta_{\mathrm{an}})$ for some hermitian space η such that $\mathrm{Ad}(\eta) \simeq (A, \sigma)$.

Proposition 2.6. *Let (A, σ) be an F–algebra with involution. Then the anisotropic part of (A, σ) is uniquely determined up to isomorphism by (A, σ).*

Proof. See [4, Proposition 3.6] for the case of involutions of the first kind. The argument there remains valid for unitary involutions. □

We may therefore speak of the anisotropic part of (A, σ), which we denote by $(A, \sigma)_{\mathrm{an}}$.

3. Alternating and direct Hermitian forms

In this section we shall show a decomposition theorem for hermitian forms. This result will be only of interest of fields of characteristic 2, and for involutions of the first kind. In all other cases, the decomposition theorem will be trivial (see (3.12)). The decomposition will be in terms of alternating forms, and forms that we will call direct hermitian forms. Firstly we will show a couple of results on alternating forms.

We now fix some notation for the rest of this section. Let (D, θ) be an F–division algebra with involution and E be the centre of D. Further, fix $\lambda \in E$ such that $\lambda\theta(\lambda) = 1$.

Proposition 3.1. *Assume* $\mathrm{char}(F) \neq 2$. *Then a λ–hermitian form over (D, θ) is alternating if and only if it is skew.*

Proof. Since $\mathrm{char}(F) \neq 2$, by [11, Section 2.A.] and [11, Proposition 2.17] we have $\mathrm{Sym}_{-1}(D, \theta) = \mathrm{Alt}(A, \theta)$ and the result is clear. □

Proposition 3.2. *Let η be an alternating hermitian form over (D, θ). If $(D, \theta) = (F, \mathrm{id}_F)$ then η is a hyperbolic form. Otherwise, $\eta \simeq \langle a_1, \dots, a_n \rangle_{(\mathrm{D}, \theta, \lambda)}$ for some $a_1, \dots, a_n \in \mathrm{Alt}(D, \theta)$.*

Proof. See [4, Proposition 3.6] for the case of involutions of the first kind. The argument there is easily adapted to the case of unitary involutions. □

We will call an hermitian form (V, h) *direct* if $h(x, x) \notin \mathrm{Alt}(D, \theta)$ for all $x \in V \setminus \{0\}$. Direct hermitian forms are necessarily anisotropic and non-alternating.

Proposition 3.3. *Let φ be a symmetric bilinear form over a field F. Then φ is direct if and only if φ is anisotropic.*

Proof. This follows easily from the fact that $\mathrm{Alt}(F, \mathrm{id}_F) = \{0\}$. □

Proposition 3.4. *Assume that* $\mathrm{char}(F) \neq 2$. *Then an anisotropic λ–hermitian form over (D, θ) is direct if and only if $\lambda \neq -1$.*

Proof. Let $\eta = (V, h)$ be an anisotropic λ–hermitian form over (D, θ). If η is skew, then η is not direct by (3.1). Conversely, suppose η is not direct. Then

$$\lambda h(x, x) = \theta(h(x, x)) = -h(x, x) \neq 0,$$

as η is anisotropic, so $\lambda = -1$. $\square$

Proposition 3.5. *Assume $\eta \simeq \langle a \rangle_{(D,\theta,\lambda)}$ for some $a \in \mathrm{Sym}_\lambda(D, \theta)$. If $a \in \mathrm{Alt}(D, \theta)$ then η is alternating. Otherwise η is direct.*

Proof. If $a \in \mathrm{Alt}(D, \theta)$ then η is alternating by (3.2).

Otherwise, η represents elements of the form $\theta(b)ab$ for some $b \in D$. We have that $\theta(b)ab \in \mathrm{Alt}(D, \theta)$ if and only if $a \in \mathrm{Alt}(D, \theta)$, hence η is direct. $\square$

Proposition 3.6. *Let η be an anisotropic λ–hermitian form over (D, θ) and suppose that there exist λ–hermitian forms φ and ψ over (D, θ) such that φ is direct, ψ is alternating and $\eta \simeq \varphi \perp \psi$. Let $a \in \mathrm{Alt}(D, \theta)$ be an element represented non-trivially by η. Then a is represented non-trivially by ψ.*

Proof. Let $\varphi = (U, h')$ and $\psi = (W, b)$ for right D–vector spaces U, W and let $x \in V \backslash \{0\}$ be such that $h(x, x) = a$. Then $x = y + w$ for $y \in U$ and $w \in W$, and

$$h(x, x) = h'(y, y) + b(w, w) = a.$$

However, $a - b(w, w) \in \mathrm{Alt}(D, \theta)$, hence $y = 0$ as φ is direct. Since $x \neq 0$, we must have that a is represented non-trivally by ψ. $\square$

Corollary 3.7. *The sum of a direct hermitian form and an anisotropic alternating form over the same F–division algebra with involution is anisotropic.*

Lemma 3.8. *Let φ, ψ and ρ be λ-hermitian forms over (D, θ). If φ and ψ are anisotropic, then $\varphi \perp \rho \simeq \psi \perp \rho$ implies that $\varphi \simeq \psi$.*

Proof. We have that $\rho \perp \rho$ is metabolic by [10, Proposition 3.7.8], and $\varphi \perp \rho \perp \rho \simeq \psi \perp \rho \perp \rho$. Hence the result follows from [10, Proposition 6.1.4]. $\square$

Theorem 3.9. *Let η be an anisotropic λ–hermitian form over (D, θ). Then $\eta \simeq \varphi \perp \psi$ for some direct λ-hermitian form φ and an alternating λ-hermitian ψ form, over (D, θ). Moreover φ and ψ are unique up to isometry.*

Proof. First we show existence of such a decomposition. If η does not represent any non-zero element in $\mathrm{Alt}(D, \theta)$, then η is direct, so we are done.

Assume now that η represents $0 \neq a \in \mathrm{Alt}(D, \theta)$. Then

$$\eta \simeq \eta' \perp \langle a \rangle_{(D,\theta)}$$

for some λ-hermitian form η' by [10, Lemma 3.6.2]. The existence the required a decomposition then follows by induction on the dimension of V and the fact that the sum of two alternating forms is an alternating form.

We now show the uniqueness of the decomposition. Let φ_1 and φ_2 be direct λ-hermitian forms and ψ_1 and ψ_2 be alternating anisotropic λ-hermitian forms, all over (D, θ) such that $\varphi_1 \perp \psi_1 \simeq \varphi_2 \perp \psi_2$.

If $\varphi_1 \perp \psi_1$ represents no alternating elements, then both ψ_1 and ψ_2 must be trivial and we are done.

Otherwise, let $a \in \mathrm{Alt}(D, \theta)$ be an element represented by $\varphi_1 \perp \psi_1$, and hence also by $\varphi_2 \perp \psi_2$. By (3.6), a is represented by ψ_1 and ψ_2. Hence

$$\psi_1 \simeq \psi_1' \perp \langle a \rangle_{(D,\theta,\lambda)} \text{ and } \psi_2 \simeq \psi_2' \perp \langle a \rangle_{(D,\theta,\lambda)}$$

where ψ_1' and ψ_2' are alternating λ–hermitian forms of smaller dimension. Therefore

$$\varphi_1 \perp \psi_1' \perp \langle a \rangle_{(D,\theta,\lambda)} \simeq \varphi_2 \perp \psi_2' \perp \langle a \rangle_{(D,\theta,\lambda)},$$

and hence by (3.8) we have

$$\varphi_1 \perp \psi_1' \simeq \varphi_2 \perp \psi_2'.$$

Repeating this process, we see that we eventually arrive at

$$\varphi_1 \perp \psi_1'' \simeq \varphi_2 \perp \psi_2''$$

where ψ_1'' and ψ_2'' are alternating λ–hermitian forms and such that $\varphi_1 \perp \psi_1''$ does not represent any alternating elements. In which case both ψ_1'' and ψ_2'' must be trivial and $\varphi_1 \simeq \varphi_2$. That $\psi_1 \simeq \psi_2$ now follows from (3.8). □

Corollary 3.10. *Let η be a λ-hermitian form over an F–division algebra (D, θ). Then there exists a decomposition*

$$\eta \simeq \varphi \perp \psi \perp \rho,$$

for some direct λ-hermitian form φ, an anisotropic alternating λ-hermitian form ψ and ρ a metabolic hermitian form, over (D, θ). Moreover, φ and ψ are unique up to isometry.

Proof. By (2.5) we have a decomposition $\eta \simeq \eta_{\mathrm{an}} \perp \rho$ where ρ is a metabolic hermitian form and η_{an} is a λ–hermitian form that is uniquely defined. We can then apply (3.9) to η_{an} in order to give the full decomposition. □

In the situation of (3.10), we call the direct form φ the *direct part* of η, denoted η_{dir}, and we call the alternating form ψ the *alternating part* of η, denoted η_{alt}.

Note that a subform of a direct form is always direct, and that a subform of an alternating form is always alternating. However, while a sum of alternating hermitian forms is always alternating, a sum of two direct forms need not be direct, even in the case where the sum is anisotropic, as the following example shows.

Example 3.11. Let Q be an F–quaternion division algebra and γ is the canonical involution on Q. There exists $j \in Q \setminus F$ such that $j^2 \in F^\times$ and $\gamma(j) = j$, and $\langle j, j+1 \rangle_{(Q,\gamma)}$ is an anisotropic sum of two direct hermitian forms that is not direct.

Proof. Since $j, j+1 \notin \mathrm{Alt}(Q, \gamma) = F$, the forms $\langle j \rangle_{(Q,\gamma)}$ and $\langle j+1 \rangle_{(Q,\gamma)}$ are direct by (3.5).

It is clear that h is not direct as it represents $1 \in \mathrm{Alt}(Q, \gamma)$. Finally, note that the change of basis matrix $\begin{pmatrix} 1 & 1+\beta^{-1}j \\ 1 & 1 \end{pmatrix}$ gives the isometry $h \simeq \langle 1, 1+\beta^{-1}j \rangle_{(Q,\gamma)}$. Since $\langle 1 \rangle_{(Q,\gamma)}$ is alternating and $\langle 1+\beta^{-1}j \rangle_{(Q,\gamma)}$ is direct, it follows from (3.7) that h is anisotropic. □

While the results in this section have been given free from any assumption on the characteristic of the underlying field and on the kind of involution, the following propositions show that these results are trivial when the characteristic is different from 2 or the involution is of unitary type.

Proposition 3.12. *Assume* $\mathrm{char}(F) \neq 2$ *or that (D, θ) is unitary. Let η be a λ-hermitian form over (D, θ). Then $\eta_{\mathrm{alt}} \simeq \eta_{\mathrm{an}}$ and η_{dir} is trivial if and only if $\lambda = -1$. Otherwise, $\eta_{\mathrm{dir}} \simeq \eta_{\mathrm{an}}$ and η_{alt} is trivial.*

Proof. For $\mathrm{char}(F) \neq 2$, this follows directly from (3.1) and (3.4).

Assume $\mathrm{char}(F) = 2$ and that (A, σ) is unitary. Then we have that $\mathrm{Sym}_1(A, \sigma) = \mathrm{Alt}(A, \sigma)$ by [11, Proposition 2.17]. Let a be any element represented by η, then $\theta(a) = \lambda a$. So $a \in \mathrm{Sym}_1(A, \sigma) = \mathrm{Alt}(A, \sigma)$ for all elements $a \in D$ represented by η if and only if $\lambda = 1$. □

4. Decomposition of algebras with involution

In this section we will use the results of Section 3 to decompose algebras with involution into direct and alternating parts. We call an F–algebra with involution *direct* if it is isomorphic to the adjoint involution of some direct λ–hermitian form.

Proposition 4.1. *Let (A, σ) be an F–algebra with involution and assume that* $\mathrm{char}(F) = 2$ *or (A, σ) is unitary. Then (A, σ) is direct if and only if (A, σ) is anisotropic.*

Proof. First note that if (A, σ) is isotropic, then any involution that it is the adjoint to is isotropic, and hence not direct.

Assume that $\mathrm{char}(F) \neq 2$ and (A, σ) of the first kind. Then (A, σ) isomorphic to the adjoint involution of an anisotropic 1–hermitian form by [11, Theorem 4.2], and hence is direct by (3.4).

Now assume that (A, σ) is unitary and over a field of arbitary characteristic. Let (V, h) be an anisotropic λ–hermitian form over an F–division algebra with involution (D, θ), such that $\mathrm{Ad}(V, h) \simeq (A, \sigma)$. Then by [14, Lemma 7.6.6] and the remarks that follow, there exists a $u \in E^\times$, where E is the centre of D, such that (V, uh) is a 1-hermitian form over (D, θ). Note that for all $f \in \mathrm{End}_D(V)$ we have

$$uh(\mathrm{ad}_h(x), y) = uh(x, \mathrm{ad}_h(f)(y)),$$

as u is invertible, and hence $\mathrm{Ad}(V, h) = \mathrm{Ad}(V, uh)$ by (2.2). We have that (V, uh) is direct by (3.12), and therefore (A, σ) is direct. □

Hence, the concept of a direct involution is trivial if the characteristic of the underlying field is not 2, or the involution is of the second kind. Therefore, we shall assume throughout the rest of this section that $\mathrm{char}(F) = 2$, and that (A, σ) is an F–algebra with involution of the first kind. In this setting we must have that $\lambda = 1$ for any λ–hermitian form over an F–division algebra of the first kind, so we will drop λ from all notation.

We shall characterise direct involutions of the first kind without reference to hermitian forms. First we show the following characterisation of symplectic involutions for comparison, as the alternating part of an involution is always symplectic.

Proposition 4.2. *(A, σ) is symplectic if and only if $\sigma(a)a \in \mathrm{Alt}(A, \sigma)$ for all $a \in A$.*

Proof. Since $\mathrm{char}(F) = 2$, (A, σ) is symplectic if and only if $1 \in \mathrm{Alt}(A, \sigma)$ by [11, Proposition 2.6]. If $\sigma(a)a \in \mathrm{Alt}(A, \sigma)$ for all $a \in A$ then in particular $1 = \sigma(1) \cdot 1 \in \mathrm{Alt}(A, \sigma)$, and hence (A, σ) is symplectic.

If $1 \in \mathrm{Alt}(A, \sigma)$ then $1 = \sigma(b) - b$ for some $b \in A$. Then, for all $a \in A$,

$$\sigma(a)a = \sigma(a)(\sigma(b) - b)a = \sigma(\sigma(a)ba) - \sigma(a)ba \in \mathrm{Alt}(A, \sigma).$$

□

Proposition 4.3. *(A, σ) is direct if and only if $\sigma(a)a \notin \mathrm{Alt}(A, \sigma)$ for all $a \in A\backslash\{0\}$.*

Proof. Let (V,h) be a hermitian space over an F–division algebra with involution (D,θ) such that $\mathrm{Ad}(V,h)\simeq(A,\sigma)$.

Assume that there exist $a\in A\backslash\{0\}$ and $b\in A$ such that $\sigma(a)a=\sigma(b)-b$. Then for all $x\in V$

$$\begin{aligned} h(a(x),a(x)) &= h((\sigma(a)a)(x),x)\\ &= h(\sigma(b)(x),x)-h(b(x),x)=h(x,b(x))-h(b(x),x)\\ &= \theta(h(b(x),x))-h(b(x),x)\in\mathrm{Alt}(D,\theta).\end{aligned}$$

In particular, since $a\neq 0$, we must have that $a(x)\neq 0$ for some $x\in V$, therefore (V,h) is not direct, and (A,σ) is not direct.

Now assume that (V,h) is not direct. Then there exists some $x\in V\backslash\{0\}$ such that $h(x,x)=d\in\mathrm{Alt}(D,\theta)$. Let $e\in\mathrm{End}_D(V)$ be the restriction map $e:V\to dD$, and let σ' be the restriction of σ to the F–algebra $A'=eAe$. Then (A',σ') is a symplectic involution, as it is adjoint to $\langle d\rangle_{(D,\theta)}$. Hence for all $a'\in A'\backslash\{0\}$ we have $\sigma'(a')a'\in\mathrm{Alt}(A',\sigma')$ by (4.2). In particular $\sigma'(e)e=\sigma'(b')-b'$ for some $b'\in A'$.

Then for all $e\in A\backslash\{0\}$ we have that

$$\sigma(e)=\sigma'(b')-b'=\sigma(ebe)-ebe$$

for some $b\in A$. That is $\sigma(e)e\in\mathrm{Alt}(A,\sigma)$. □

We call an F–algebra with involution (B,τ) a *direct part* of (A,σ) if $(B,\tau)\simeq\mathrm{Ad}(\eta_{\mathrm{dir}})$ for some hermitian form η such that $\mathrm{Ad}(\eta)\simeq(A,\sigma)$. Similarly we call an F–algebra with involution (C,ρ) an *alternating part* of (A,σ) if $(C,\rho)\simeq\mathrm{Ad}(\eta_{\mathrm{alt}})$.

For $u\in A$, let $\mathrm{Int}(u)\in\mathrm{End}_F(A)$ denote the *inner automorphism*, that is the map $A\to A$ given by $a\mapsto uau^{-1}$ for all $a\in A$.

Lemma 4.4. *For every involution σ' of the first kind on A, there exists some $u\in A^{\times}$, uniquely determined up to a factor in $F^{\times}$, such that*

$$\sigma'=\mathrm{Int}(u)\circ\sigma\quad\text{and}\quad\sigma(u)=u.$$

Further, $\mathrm{Alt}(A,\sigma')=u\cdot\mathrm{Alt}(A,\sigma)$.

Proof. See [11, Proposition 2.7]. □

Proposition 4.5. *Let (V,h) be a hermitian form over an F–division algebra (D,θ) and let (W,b) be a hermitian form with over an F–division algebra with involution (D,θ'). If $\mathrm{Ad}(V,h)\simeq\mathrm{Ad}(W,b)$, then $\mathrm{Ad}((V,h)_{\mathrm{dir}})\simeq\mathrm{Ad}((W,b)_{\mathrm{dir}})$ and $\mathrm{Ad}((V,h)_{\mathrm{alt}})\simeq\mathrm{Ad}((W,b)_{\mathrm{alt}})$.*

Proof. By (4.4), there exists some $u\in D^{\times}$ such that $\theta'=\mathrm{Int}(u)\circ\theta$ and $\theta(u)=u$. We shall first show that (V,uh) is a hermitian form over (D,θ'). We only need to check that $uh(y,x)=\theta'(uh(x,y))$, for all $x,y\in V$. We have

$$\begin{aligned} uh(y,x) &= u\theta(h(x,y))=u\cdot u^{-1}\theta'(h(x,y))u=\theta'(h(x,y))u\\ &= \theta'(h(x,y))\theta'(u)=\theta'(uh(x,y)),\end{aligned}$$

for all $x,y\in V$, as required.

Now note that for all $f\in\mathrm{End}_D(V)$ we have

$$uh(\mathrm{ad}_h(x),y)=uh(x,\mathrm{ad}_h(f)(y)),$$

as u is invertible, and hence $\mathrm{Ad}(V,h)=\mathrm{Ad}(V,uh)$ by (2.2). By assumption we have $\mathrm{Ad}(V,h)\simeq\mathrm{Ad}(W,b)$, and hence $\mathrm{Ad}(V,uh)\simeq\mathrm{Ad}(W,b)$.

Since (V, uh) and (W, b) are both hermitian forms over (D, θ') with the same adjoint algebra with involution, it follows from [11, Theorem 4.2] that $V = W$ and $\mu uh = b$ for some $\mu \in F^\times$. Since $a \in \mathrm{Alt}(D, \theta')$ if and only if $\mu a \in \mathrm{Alt}(D, \theta')$ for all $\mu \in F$, it follows that $\mathrm{Ad}((V, uh)_{\mathrm{dir}}) \simeq \mathrm{Ad}((W, b)_{\mathrm{dir}})$ and $\mathrm{Ad}((V, uh)_{\mathrm{alt}}) \simeq \mathrm{Ad}((W, b)_{\mathrm{alt}})$.

Finally, since $\mathrm{Alt}(D, \theta') = u \cdot \mathrm{Alt}(D, \theta)$ by (4.4), it follows that $\mathrm{Ad}((V, h)_{\mathrm{dir}}) \simeq \mathrm{Ad}((V, uh)_{\mathrm{dir}})$ and $\mathrm{Ad}((V, h)_{\mathrm{alt}}) \simeq \mathrm{Ad}((V, uh)_{\mathrm{alt}})$. □

Corollary 4.6. *The direct and alternating parts of* (A, σ) *are uniquely determined up to isomorphism by* (A, σ).

We may therefore speak of the direct part and the alternating part of (A, σ), which we denote by $(A, \sigma)_{\mathrm{dir}}$ and $(A, \sigma)_{\mathrm{alt}}$ respectively. Note that for any (A, σ), $(A, \sigma)_{\mathrm{alt}}$ is necessarily symplectic, $(A, \sigma)_{\mathrm{dir}}$ is necessarily orthogonal, and both are necessarily anisotropic.

Example 4.7. Assume $\mathrm{char}(F) = 2$. Let Q be an F–quaternion algebra and γ it's canonical involution. There exists $j \in Q \setminus F$ such that $j^2 \in F^\times$ and $\gamma(j) = j$. Let $A = M_2(Q)$ and σ be the F–involution on A given by

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \gamma(a) & \gamma(c)j^{-1} \\ j\gamma(b) & j\gamma(d)j^{-1} \end{pmatrix}.$$

Then $(A, \sigma)_{\mathrm{dir}} = (Q, \mathrm{Int}(j) \circ \gamma)$ and $(A, \sigma)_{\mathrm{alt}} = (Q, \gamma)$.

Proof. We have that $(A, \sigma) = \mathrm{Ad}(\langle 1, j \rangle_{(Q,\gamma)})$. We have $(\langle 1, j \rangle_{(Q,\gamma)})_{\mathrm{alt}} = \langle 1 \rangle_{(Q,\gamma)}$ and $(\langle 1, j \rangle_{(Q,\gamma)})_{\mathrm{dir}} = \langle j \rangle_{(Q,\gamma)}$. The result now follows by taking the adjoint involutions of these hermitian forms. □

For further discussion of this example, see [4, Section 6].

5. Orthogonal Sums

In this section we will briefly discuss the related topic of orthogonal sums of algebras with involution.

A central simple F–algebra with involution (A, σ) is an *orthogonal sum* of central simple F–algebras with involution (A_1, σ_1) and (A_2, σ_2) if there are idempotents $e_1, e_2 \in A$ such that $e_1 + e_2 = 1$ and $\sigma(e_1) = e_1, \sigma(e_2) = e_2$ and for $i = 1, 2$, there is an F–algebra isomorphism

$$\varphi_i : A_i \xrightarrow{\sim} e_i A e_i$$

such that $\varphi_i \circ \sigma_i = \sigma \circ \varphi_i$. We may use φ_1 and φ_2 to identify A_1 and A_2 to subsets of A.

Note that an orthogonal sum is not uniquely determined up to isomorphism by its summands. However, if two algebras with involution (A, σ) and (A', σ') are the direct sum of the same summands, then $A \simeq A'$ and

$$\sigma(a) = (u_1 e_1 + u_2 e_2)\sigma'(a)(u_1^{-1} e_1 + u_2^{-1} e_2)$$

for $u_1, u_2 \in F^\times$, $u_1 \neq u_2$.

We say two F–algebras with involution (A_1, σ_1) and (A_2, σ_2) are *compatible* if the following conditions all hold:

1. A_1 and A_2 are Brauer equivalent,
2. If E is the centre of A_1 and A_2, then $\sigma_1|_E = \sigma_2|_E$,

(3) If $\mathrm{char}(F) \neq 2$ and the involutions are of the first kind, then (A_1, σ_1) and (A_2, σ_2) are either both orthogonal or both symplectic.

The following propositions are corrections of [2, Proposition 1.2 and 1.3]. The results there hold in the case of $\mathrm{char}(F) \neq 2$, but they are incorrectly stated for $\mathrm{char}(F) = 2$.

Proposition 5.1. *If an F–algebra with involution (A, σ) is an orthogonal sum of (A_1, σ_1) and (A_2, σ_2), then (A_1, σ_1) and (A_2, σ_2) are compatible. Moreover* $\deg A = \deg A_1 + \deg A_2$.

Assume (A, σ) is of the first kind. If $\mathrm{char}(F) \neq 2$, then (A, σ) has the same type as (A_1, σ_1) and (A_2, σ_2). If $\mathrm{char}(F) = 2$, then (A, σ) is symplectic if and only if (A_1, σ_1) and (A_2, σ_2) are both symplectic.

Proof. The proof in [2, Proposition 1.2] holds for every statement, except for the statement on the types of (A, σ), (A_1, σ_1) and (A_2, σ_2) when $\mathrm{char}(F) = 2$. Assume we are in this case.

As in [2, Proposition 1.2] we may identify $(A, \sigma) = \mathrm{Ad}(\eta)$, $(A_1, \sigma_1) = \mathrm{Ad}(\eta_1)$ and $(A_2, \sigma_2) = \mathrm{Ad}(\eta_2)$ where η, η_1 and η_2 are λ–hermitian forms over some F–division algebra (D, θ) such that $\eta = \eta_1 + \eta_2$. The result now follows as η is alternating if and only if both η_1 and η_2 are alternating. □

Proposition 5.2. *Let $(A_1 \sigma_1)$ and (A_2, σ_2) be compatible F–algebras with involution. Then there exists a central simple F–algebra with involution (A, σ) such that (A, σ) is isomorphic to an orthogonal sum (A_1, σ_1) and (A_2, σ_2).*

Proof. When $\mathrm{char}(F) \neq 2$ the proof in [2, Proposition 1.3] holds. When $\mathrm{char}(F) = 2$ we may adapt the proof in [2, Proposition 1.3] as in (5.1) to allow for the sum of two algebras with involution of different type. □

Lemma 5.3. *An orthogonal sum of metabolic algebras with involution is metabolic.*

Proof. With notation as in the proof of (5.1) we may identify any F–algebra with involution (A, σ) that is a direct sum of two F–algebras with involution with $\mathrm{Ad}(\eta_1 + \eta_2)$ for λ–hermitian forms η_1 and η_2. Then by (2.4) η_1 and η_2 are metabolic, hence $\eta_1 \perp \eta_2$ is metabolic by [10, Corollary 3.7.7], and therefore, again by (2.4), (A, σ) is metabolic. □

Lemma 5.4. *Every F–algebra with involution (A, σ) decomposes as an orthogonal sum of $(A, \sigma)_{\mathrm{an}}$ and (A', σ') where (A', σ') is some metabolic F–algebra with involution.*

Proof. We may identify $(A, \sigma) = \mathrm{Ad}(\eta)$ for a λ–hermitian form η over an F–division algebra with involution (D, θ). The result now follows from (2.5). □

Proposition 5.5. *Every F–algebra with involution (A, σ) decomposes as an orthogonal sum of $(A, \sigma)_{\mathrm{dir}}$, $(A, \sigma)_{\mathrm{alt}}$ and (A', σ'), where (A', σ') is some metabolic F–algebra with involution. Further $(A, \sigma)_{\mathrm{an}}$ is isomorphic to an orthogonal sum of $(A, \sigma)_{\mathrm{dir}}$ and $(A, \sigma)_{\mathrm{alt}}$.*

Proof. This follows directly from (5.4) and (3.10). □

Note that while $(A, \sigma)_{\mathrm{dir}}$ and $(A, \sigma)_{\mathrm{alt}}$ are determined up to isomorophism by (4.6), the metabolic involution (A', σ') is not generally.

6. Generic splitting over fields of characteristic 2

In this section we will apply the decomposition theorems from the previous sections in order to describe the effect of passing to the generic splitting field of an algebra on the isotropy of involutions on that algebra.

Let A be a central simple F–algebra. A field extension L/F is called a *generic splitting field* if for any splitting field L' there exists an F–place of L to L'. It was shown in [1] that every central simple algebra has a generic splitting field.

In [7] the following conjecture is given.

Conjecture 6.1. *Let* $\mathrm{char}(F) \neq 2$ *and let* (A, σ) *be a central simple* F*–algebra with anisotropic orthogonal involution. Let* L *be a generic splitting field of* A*. Then* $(A, \sigma)_L$ *is anisotropic.*

We shall show, in (6.4), that the corresponding conjecture for fields of characteristic 2 is false in general, but holds precisely when the involution is direct.

Since our decomposition theorems only have weight over fields of characteristic 2 and involutions of the first kind, throughout the rest of this section we assume F is a field with $\mathrm{char}(F) = 2$ and all involutions are of the first kind. We will show that the anisotropic part of an algebra with involution over a generic splitting field is precisely its direct part extended to the same field.

Proposition 6.2. *Let* L/F *be a separable quadratic extension, and let* (A, σ) *be a direct* F*–algebra with involution. Then* $(A, \sigma)_L$ *is direct. In particular* $(A, \sigma)_L$ *is anisotropic.*

Proof. We may write $L = F(\delta)$ with $\delta \in L\backslash F$ such that $\delta^2 + \delta + \alpha = 0$ for some $\alpha \in F^\times$. Let $a \in A_L$ be such that $\sigma_L(a)a \in \mathrm{Alt}(A, \sigma)_L$. Then $a = b \otimes 1 + c \otimes \delta$ for some $b, c \in A_L$ and we get

$$\begin{aligned}\sigma_L(a)a &= \sigma(b)b \otimes 1 + (\sigma(b)c + \sigma(c)b) \otimes \delta + \sigma(c)c \otimes \delta^2 \\ &= (\sigma(b)b + \sigma(c)c) \otimes 1 + (\sigma(b)c + \sigma(c)b + \alpha\sigma(c)c) \otimes \delta.\end{aligned}$$

We also have that $\sigma_L(a)a = \sigma_L(d) - d$ for some $d \in A_L$. Then $d = e \otimes 1 + f \otimes \delta$ for some $e, f \in A_L$ and we get

$$\sigma_L(a)a = (\sigma(e) - e) \otimes 1 + (\sigma(f) - f) \otimes \delta.$$

Comparing coefficients of the basis vectors $1 \otimes 1$ and $1 \otimes \delta$ gives

$$\begin{aligned}\sigma(b)b + \sigma(c)c &= \sigma(e) - e \in \mathrm{Alt}(A, \sigma) \\ \sigma(b)c + \sigma(c)b + \alpha\sigma(c)c &= \sigma(f) - f \in \mathrm{Alt}(A, \sigma).\end{aligned}$$

However,

$$\sigma(b)c + \sigma(c)b = \sigma(\sigma(c)b) + \sigma(c)b \in \mathrm{Alt}(A, \sigma)$$

hence $\sigma(c)c \in \mathrm{Alt}(A, \sigma)$, and hence $c = 0$, as (A, σ) is direct. It follows that we must have $b = 0$ as (A, σ) is direct. Hence $a = 0$, and $(A, \sigma)_L$ is direct. □

Theorem 6.3. *Let* (A, σ) *be an* F*–algebra with involution and let* L *be a generic splitting field of* A*. Then*

$$((A, \sigma)_L)_{\mathrm{an}} \simeq ((A, \sigma)_{\mathrm{dir}})_L.$$

Proof. By [6, Theorem 9.1.8], every central simple F-algebra with involution is Brauer equivalent to a product of quaternion algebras. Since every quaternion algebra splits over a separable quadratic field extension, it follows that A splits over a field constructed from series of successive quadratic separable field extensions, which we call L'. We have that $((A,\sigma)_{\mathrm{dir}})_{L'}$ is anisotropic by (6.2).

By (5.5), we have that $(A,\sigma)_L$ is some orthogonal sum of $((A,\sigma)_{\mathrm{dir}})_L$, $((A,\sigma)_{\mathrm{alt}})_L$ and $(A',\sigma')_L$, where (A',σ') is a metabolic F–algebra with involution. Clearly $(A',\sigma')_L$ is metabolic and $((A,\sigma)_{\mathrm{alt}})_L$ is metabolic as any symplectic involution is metabolic over any splitting field by [4, Corollary 4.7]. Hence any sum of $((A,\sigma)_{\mathrm{alt}})_L$ and $(A',\sigma')_L$ is metabolic by (5.4).

Finally, we must have $((A,\sigma)_{\mathrm{dir}})_L$ is anisotropic as $((A,\sigma)_{\mathrm{dir}})_{L'}$ is anisotropic and there exists an F–place from L to L'. Hence $((A,\sigma)_L)_{\mathrm{an}} \simeq ((A,\sigma)_{\mathrm{dir}})_L$ follows from (5.5). □

Theorem 6.4. *Let (A,σ) be an anisotropic F–algebra with involution and L a generic splitting field of A.*

(1) *$(A,\sigma)_L$ is metabolic if and only if (A,σ) is symplectic.*
(2) *$(A,\sigma)_L$ is anisotropic if and only if (A,σ) is direct.*

Proof. If (A,σ) is symplectic then $(A,\sigma)_L$ is metabolic by [4, Corollary 4.7]. If (A,σ) is not symplectic then $(A,\sigma)_{\mathrm{dir}}$ is non-trivial, and hence $(A,\sigma)_L$ is not metabolic by (6.3).

Similarly, if (A,σ) is direct then $(A,\sigma)_L$ is anisotropic by (6.3). If (A,σ) is not direct then $(A,\sigma)_{\mathrm{alt}}$ is non-trivial, and hence $(A,\sigma)_L$ is not anisotropic by (6.3). □

Corollary 6.5. *Let (A,σ) be an F–algebra with involution and L a generic splitting field of A. Then $(A,\sigma)_L$ is metabolic if and only if $(A,\sigma)_{\mathrm{an}} = (A,\sigma)_{\mathrm{alt}}$.*

Proof. If $(A,\sigma)_{\mathrm{an}} = (A,\sigma)_{\mathrm{alt}}$ then $((A,\sigma)_{\mathrm{an}})_L$ is metabolic by (6.4), and hence $(A,\sigma)_L$ is metabolic by (5.3). Otherwise $(A,\sigma)_{\mathrm{dir}}$ is non-trivial, and $(A,\sigma)_L$ is not metabolic by (6.3). □

Remark 6.6. Note that (6.5) shows that non-metabolic orthogonal algebra with involution can become metabolic over a generic splitting field. Take (A,σ) to be any anisotropic symplectic involution, and (B,τ) to be a compatible metabolic orthogonal involution. Then any orthogonal sum of (A,σ) and (B,τ) will be orthogonal by (5.4), and become metabolic over any splitting field by (6.5).

Theorem 6.7. *Let E be the separable closure of F. Then $(A,\sigma)_E$ is direct if and only if (A,σ) is direct.*

Proof. Note that any algebra splits over a separable closure. Therefore if (A,σ) is not direct, then $(A,\sigma)_E$ is isotropic by (6.4).

Now assume (A,σ) is direct. We consider the effect of passing to a separable extension K/F, and show that $(A,\sigma)_K$ is anisotropic. If K/F is a quadratic separable extension, then this follows from (6.2).

Assume now that K/F is an odd degree extension. Let L be a generic splitting field of A (if A is already split, then $F = L$). Then there exists a field L'/F such that L'/L is of odd degree and K embeds into L' over F.

By (6.4), we have that $(A,\sigma)_L$ is anisotropic, and by Springer's theorem, see [5, Corollary 18.5], it follows that $(A,\sigma)_{L'}$ is anisotropic. L' is also a splitting field for

A_K, hence if $(A,\sigma)_K$ were not direct, it would become isotropic over the extension L'/K by (6.4). But $(A,\sigma)_{L'}$ is anisotropic, hence $(A,\sigma)_K$ is direct

The result now follows, as any separable extension is a series of quadratic separable extensions and odd degree extensions by basic Galois theory. □

Note that the question of whether any general anisotropic algebra with involution stays anisotropic over any odd degree extension, an analogue of Springer's theorem, is still open. Partial results have been found, such as [12], over fields of characteristic different from 2. (6.7) shows that an analogue of Springer's theorem holds for direct algebras with involution over fields of characteristic 2.

Acknowledgements

This work was supported by the Deutsche Forschungsgemeinschaft (project *Quadratic Forms and Invariants*, BE 2614/3-1) and by the Zukunftskolleg, Universität Konstanz.

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