DIFFERENTIAL FORMS AND BILINEAR FORMS UNDER FIELD EXTENSIONS

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Abstract. Let F be a field of characteristic $p > 0$. Let $\Omega^n(F)$ be the F-vector space of n-differentials of F over F^p. Let $K = F(g)$ be the function field of an irreducible polynomial g in $m \geqslant 1$ variables over F. We derive an explicit description of the kernel of the restriction map $\Omega^n(F) \to \Omega^n(K)$. As an application in the case $p = 2$, we determine the kernel of the restriction map when passing from the Witt ring (resp. graded Witt ring) of symmetric bilinear forms over F to that over such a function field extension K.

1. Introduction

When studying algebraic objects defined over some base field, such as quadratic forms, central simple algebras or Milnor K-groups, it is quite natural to ask how these objects behave when extending scalars to a field extension. In particular, one would like to be able to determine the kernel of the restriction homomorphism between the Witt rings, the Brauer groups, and the Milnor K-groups of the base field and of the extension field.

One of our main objectives is to derive an explicit description of the Witt kernel for symmetric bilinear forms in characteristic 2 for a large class of field extensions, namely extensions given by function fields of arbitrary hypersurfaces over the base field. This includes all finite simple extensions of the base field (the one-variable case). Our proof is based on a study of how the space of absolute Kähler differential forms $\Omega^n(F)$ behaves under field extensions. These results on differential forms are of considerable interest in their own right and they are proved for fields of arbitrary positive characteristic $p > 0$.

The strategy of the proof for determining the Witt kernels is as follows. The crucial ingredient is the determination of the kernel $\Omega^n(E/F)$ of the map $\Omega^n(F) \to \Omega^n(E)$ where E is the quotient field of $F[X]/(f(X))$ for an irreducible polynomial $f(X) \in F[X]$ where $X = (X_1, \ldots, X_n)$, and $\mathrm{char}(F) = p > 0$. This is done in several steps. We start by noticing that $\Omega^n(E/F) = 0$ if E/F is purely transcendental or separable algebraic (§7). This allows us to discard all irreducible polynomials that are not in $F[X^p] = F[X_1^p, \ldots, X_n^p]$. Next, we treat the case $n = 1$ (i.e. the case of a simple algebraic extension). The kernel can then be expressed as the subspace of elements in $\Omega^n(F)$ that are annihilated by all differentials da where $a \in F^*$ runs through all nonzero coefficients of $f(X)$ (which we may assume to be monic). This is done in §8. We then use an induction on the number of variables n. If $n \geqslant 2$, let $X' = (X_1, \ldots, X_{n-1})$. By a linear change of variables and suitable scaling, we

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then may assume that $f(X) \in F[X^p]$ is monic in X_n with coefficients in $F[X']$, so we use the 1-variable case to conclude that over $F(X')$, the kernel consists of those elements in $\Omega^n(F)$ that, over $F(X')$, are annihilated by differentials $da(X')$ where $a(X') \in F[X'^p]$ runs through the coefficients of $f(X) \in F(X')[X_n]$. By an induction on the number of variables, we can then show that the elements in $\Omega^n(F)$ annihilated by $da(X')$ are already annihilated by the $d\alpha$ where $\alpha \in F^*$ runs through all nonzero F-coefficients of $a(X')$. This induction is done in §10 where we also need some results on the kernel $\Omega^n(E/F)$ in the special case where E is the function field of a quasilinear p-form (§9).

The step from differential forms to bilinear forms in the case of characteristic $p = 2$ is done in §11 using a famous theorem by Kato (Theorem 5.2) and by invoking results by Aravire-Baeza on the kernel of the restriction map on the graded Witt ring when passing to the function field of a bilinear Pfister form (Theorem 5.6). This allows us to determine the kernel of the restriction map for the graded Witt ring for function fields of hypersurfaces as defined above (Theorem 11.3) and finally, using a fairly standard argument to reduce the situation to base fields that are finitely generated over their prime field, we obtain the Witt kernel for such extensions (Corollaries 11.4, 11.6).

To put this result into perspective, we give a short account of some previously known results on Witt kernels (in characteristic not 2 and in characteristic 2) in the next section.

To keep the paper as self contained as possible, we introduce in sections 3–5 the main objects of our study and some basic properties and theorems concerning them: quasilinear p-forms (essentially diagonal homogeneous forms of degree p in characteristic $p > 0$), bilinear forms in characteristic 2, and differential forms over field of positive characteristic. In §6, we prove a result belonging to the realm of basic Galois theory but which turns out to be extremely useful in the study of differential forms under simple algebraic extensions.

2. A short history of Witt kernels

Let F be a field and denote by $W(F)$ the Witt ring of F, i.e. the Witt ring of quadratic forms if $\mathrm{char}(F) \neq 2$ resp. of symmetric bilinear forms if $\mathrm{char}(F) = 2$. If $\mathrm{char}(F) = 2$, we denote the Witt group of quadratic forms by $W_q(F)$ which is a $W(F)$-module in a natural way. We define the Witt kernel $W(E/F)$ for a field extension E/F to be the kernel of the restriction homomorphism $W(F) \to W(E)$ (similarly, $W_q(E/F)$).

It is easy to see that $W(E/F) = 0$ if E/F is purely transcendental. One also has $W(E/F) = 0$ for odd degree extensions, a result often referred to as Springer's theorem [Sp] but that has been proved earlier by E. Artin in a communication to E. Witt (1937).

Let us now assume that $\mathrm{char}(F) \neq 2$. A well known result states that if $E = F(\sqrt{d})$ is a quadratic extension, then $W(E/F)$ is generated by the norm form $\langle 1, -d \rangle$ of that extension. While this is easy to show, it is considerably harder to determine $W(E/F)$ for $[E : F] = 4$. The case of biquadratic extensions $E = F(\sqrt{a}, \sqrt{b})$ has been solved by Elman-Lam-Wadsworth [ELW1] where it is shown that $W(E/F)$ is generated by $\langle 1, -a \rangle$ and $\langle 1, -b \rangle$, and the case of degree 4 extensions containing a quadratic subextension was treated by Lam-Leep-Tignol [LLT]. The Witt kernel of arbitrary degree 4 extensions has only been determined rather recently by Sivatski [Si]. Little is known for Witt kernels of higher even degree extensions. For triquadratic extensions $E = F(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$ it is shown by Elman-Lam-Tignol-Wadsworth [ELTW] that generally $\sum_{i=1}^{3} \langle 1, -a_i\rangle W(F) \subsetneq W(E/F)$, but an explicit description of generators of $W(E/F)$ is not known. Of course, more can be said if one makes strong assumptions on the base field. For example, for local or global fields, it is known that $W(F(\sqrt{a_1}, \ldots, \sqrt{a_n})/F) = \sum_{i=1}^{n} \langle 1, -a_i\rangle W(F)$, see Elman-Lam-Wadsworth [ELW2].

Most of the work on Witt kernels of function field extensions E/F concerns extensions of type $E = F(\varphi) := F(X_\varphi)$, where X_φ is the quadric defined by a quadratic form φ. A famous result states that if φ is an anisotropic Pfister form then $W(F(\varphi)/F) = \varphi W(F)$, see Arason [Ar] (but already implicit in Arason-Pfister [AP]). Generators for Witt kernels for function fields of other types of quadratic forms, in particular forms of small dimension, have been determined by Fitzgerald [F]. Witt kernels for function fields of hyperelliptic curves have been computed by Shick [Sh], but very little else is known.

Witt kernels in characteristic 2 have only been studied more recently. First, consider Witt kernels $W_q(E/F)$ for quadratic forms. Denote by $[1, a]$ the nondegenerate quadratic form $x^2 + xy + ax^2$. Consider first the case of a quadratic extension $E = F(\alpha)$. One can readily show that if E/F is inseparable, say $\alpha^2 = a \in F^*$, then $W_q(E/F) = \langle 1, a\rangle_b W_q(F)$ (Baeza [B1], Ahmad [A1]), and if E/F is separable, say, $\alpha^2 + \alpha + a = 0$ with $a \in F^*$, then $W_q(E/F) = W_q(F)[1, a]$ (Baeza [B2, 4.11]). Witt kernels for purely inseparable quartic extensions have been determined by Mammone-Moresi [MM] (biquadratic extension) and Ahmad [A3] (simple extension). The case of biquadratic separable extensions is due to Baeza [B2, 4.16], and that of biquadratic non-purely inseparable extensions to Ahmad [A2]. Laghribi [L2] generalized the result by Mammone-Moresi to arbitrary purely inseparable multiquadratic extensions. In all these cases, generators of the Witt kernels can be expressed in terms of the norm forms of the quadratic (inseparable or separable) subextensions. Again, not much else is known for other types of finite algebraic extensions.

As for function field extensions, the only case that has been studied thoroughly is that of function fields of quadrics where the case of singular quadrics is also of particular interest. If π is a *quadratic* Pfister form and $E = F(\pi)$ is the function field of π, then $W_q(E/F) = W_q(F)\pi$ (see Laghribi [L1], but already implicit in Baeza [B1]). If B is a *bilinear* Pfister form and E is the function field of the quadric defined by $B(x, x) = 0$ (which is the same as the function field of the quasilinear Pfister 2-form $B(x) := B(x, x)$, see §3), then $W_q(E/F) = BW_q(F)$, see Laghribi [L1] where Witt kernels $W_q(F(\varphi)/F)$ for various other types of quadratic forms φ have been determined, especially for quadratic forms in small dimension (including singular ones) in analogy to Fitzgerald's results in characteristic 2.

Finally, consider Witt kernels $W(E/F)$ for Witt rings of bilinear forms in characteristic 2. It is not difficult to show that if E/F is separable, then $W(E/F) = 0$ (Knebusch [Kn1]). In [H2], Witt kernels for a large class of purely inseparable algebraic extensions have been determined. In particular, it was shown that if E is purely inseparable of exponent 1 over F (i.e. $E^2 \subset F \subset E$), then $W(E/F)$ is generated by bilinear forms $\langle 1, t\rangle_b$ where $t \in E^2 \setminus \{0\}$. In the case of function field extensions, Laghribi [L1] determined the Witt kernel for any function field $E = F(\varphi)$ of an (anisotropic) quadratic form φ. If φ is not totally singular, then the function field can be realized as a transcendental extension followed by a separable quadratic extension, which in view of the above results implies that $W(E/F) = 0$ in that case. If φ is totally singular (which means that φ is a quasilinear 2-form in the sense of §3) then the Witt kernel is generated by certain bilinear Pfister forms that can be expressed in terms of the coefficients of φ. We prove a far more general result, namely we determine the Witt kernel for bilinear forms in characteristic 2 for *arbitrary* function fields of hypersurfaces, i.e. for extensions E/F where E is the quotient field of $F[X]/(f(X))$ for an irreducible polynomial $f(X) = f(X_1, \ldots, X_n) \in F[X_1, \ldots, X_n]$, see Corollary 11.4. This result readily implies Laghribi's, but our proof is completely different. It includes the case of arbitrary simple algebraic extensions ($n = 1$).

3. Quasilinear p-forms

Quasilinear p-forms have been studied in quite some detail in [H1] where they are called p-forms for short. In the sequel, we introduce only all those basic notions and results about quasilinear p-forms which we will require for our purposes. For proofs and more details we refer to [H1].

Let F be a field of characteristic $p > 0$ and let V be an F-vector space of finite dimension n. A quasilinear p-form φ on V is a map $\varphi : V \to F$ satisfying $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(\lambda x) = \lambda^p \varphi(x)$. In the case $p = 2$, quasilinear 2-forms are nothing else but totally singular quadratic forms.

If $\{e_1, \ldots, e_n\}$ is an F-basis of V and $\varphi(e_i) = a_i \in F$, then we write $\varphi = \langle a_1, \ldots, a_n \rangle$. We adopt the usual definitions from the context of quadratic forms. Two quasilinear p-forms (φ, V) and (ψ, W) are isometric, $\varphi \cong \psi$, if there exists a linear isomorphism $t : V \to W$ such that $\psi(tx) = \varphi(x)$ for all $x \in V$.

The value sets are denoted by $D_F(\varphi) = \{\varphi(x) \mid x \in V \setminus \{0\}\}$ and $D_F^0(\varphi) = D_F(\varphi) \cup \{0\}$. Note that if $\varphi = \langle a_1, \ldots, a_n \rangle$, then $D_F^0(\varphi) = \mathrm{span}_{F^p}\{a_1, \ldots, a_n\}$ which is a finite dimensional F^p-subvector space of the F^p-vector space F, and $\varphi \cong \psi$ iff $\dim \varphi = \dim \psi$ and $D_F^0(\varphi) = D_F^0(\psi)$.

One defines in the obvious way an "orthogonal" sum $\varphi \perp \psi$ and a product $\varphi \otimes \psi$ of quasilinear p-forms. In particular,

$$\langle a_1, \ldots, a_n \rangle \otimes \langle b_1, \ldots, b_m \rangle \cong \langle a_1 b_1, \ldots, a_1 b_m, a_2 b_1 \ldots, a_n b_m \rangle \ .$$

$\varphi = \langle a_1, \ldots, a_n \rangle$ is called anisotropic if $0 \notin D_F(\varphi)$, which is equivalent to saying that $\dim \varphi = \dim_{F^p} D_F^0(\varphi)$, i.e. the elements $a_1, \ldots a_n$ are F^p-linearly independent. Every quasilinear p-form φ decomposes in a unique way (up to isometry) as $\varphi \cong \varphi_{\mathrm{an}} \perp \langle 0, \ldots, 0 \rangle$ with φ_{an} anisotropic. One then has $\dim \varphi_{\mathrm{an}} = \dim_{F^p} D_F^0(\varphi)$. φ is called p-split if $\dim \varphi_{\mathrm{an}} \leqslant \frac{1}{p} \dim \varphi$.

For $a \in F$, we define $\langle\langle a \rangle\rangle = \langle 1, a, a^2, \ldots, a^{p-1} \rangle$, and a form $\pi = \langle\langle a_1, \ldots, a_n \rangle\rangle = \langle\langle a_1 \rangle\rangle \otimes \ldots \langle\langle a_n \rangle\rangle$ will be called an n-fold quasilinear Pfister p-form. Note that $D_F^0(\pi) = F^p(a_1, \ldots, a_n)$, and that $\dim \pi_{\mathrm{an}} = [F^p(a_1, \ldots, a_n) : F^p] = p^m$ for some $m \leqslant n$. So n-fold quasilinear Pfister p-forms are either anisotropic or p-split.

For each quasilinear p-form φ we define the norm field $N_F(\varphi)$ and the norm degree $\mathrm{ndeg}_F(\varphi)$ as follows:

$$\begin{aligned} N_F(\varphi) &= F^p(\tfrac{a}{b} \mid a, b \in D_F(\varphi),\ b \neq 0\} \ , \\ \mathrm{ndeg}_F(\varphi) &= [N_F(\varphi) : F^p] \ . \end{aligned}$$

Note that for an n-fold quasilinear Pfister p-form π, one has $N_F(\pi) = D_F^0(\pi)$.

By the above, using value sets, anisotropic quasilinear p-forms can be identified with finite dimensional F^p-subvector spaces of F, and anisotropic quasilinear Pfister p-forms with finite extensions of F^p inside F.

We define the set of similarity factors of φ by $G_F^*(\varphi) = \{x \in F^* \mid x\varphi \cong \varphi\}$ and put $G_F(\varphi) = G_F^*(\varphi) \cup \{0\}$. It is not difficult to prove that $G_F(\varphi)$ is a finite extension of F^p inside $N_F(\varphi)$ ([H1, Proposition 6.4]). If π is a quasilinear Pfister p-form then $N_F(\pi) = G_F(\pi)$.

Let φ be a quasilinear p-form over F and let E/F be a field extension. Then φ_E denotes the quasilinear p-form over E obtained from φ by scalar extension.

We will need the following result on the p-splitting of p-forms under function field extensions (see [H1, Theorem 6.10]).

Theorem 3.1. *Let $X = (X_1, \dots, X_n)$ be an n-tuple of variables and put $X^p = (X_1^p, \dots, X_n^p)$. Let $f(X) \in F[X]$ be an irreducible polynomial and $E = F(f)$ be the function field of $f(X)$ over F, i.e. the quotient field of $F[X]/(f(X))$. Let $a^* \in F^*$ denote the leading coefficient of $f(X)$ (with respect to the lexicographical ordering of monomials). Let φ be an anisotropic quasilinear p-form over F. Then the following are equivalent.*

(i) *$f(X) \in G_{F(X)}(\varphi)$;*
(ii) *$f(X) \in G_F(\varphi)[X^p]$ (i.e., $f(X) \in F[X^p]$ and each coefficient of f is in $G_F(\varphi)$);*
(iii) *$a^* \in G_F(\varphi)$ and φ_E is p-split.*

4. Bilinear forms and the Witt ring in characteristic 2

In this section, all fields are assumed to be of characteristic 2. For all undefined notations and statements mentioned below without proof, we refer to [EKM, Ch. I], [M].

By a bilinear form over a field F we will always mean a finite-dimensional symmetric nondegenerate bilinear form over F. A bilinear form $B : V \times V \to F$ on an f-vector space V is called isotropic if there exists $x \in V \setminus \{0\}$ such that $B(x) := B(x,x) = 0$, i.e., 0 is contained in the value set

$$D_F(B) = \{B(x) \mid x \in V \setminus \{0\}\} .$$

We put $D_F^0(B) = D_F(B) \cup \{0\}$ and $D_F^*(B) = D_F(B) \setminus \{0\}$. Note that $D_F^0(B)$ is an F^2-subvector space of F. The group of similarity factors of B is defined by $G_F(B) = \{x \in F^* \mid xB \cong B\}$.

2-dimensional isotropic bilinear forms are called metabolic planes. Such a metabolic plane is always isometric to a form of type

$$\mathbb{M}_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \quad a \in F,$$

and $M_a \cong M_{a'}$ iff $aF^{*2} = a'F^{*2}$. The form $\mathbb{H} := \mathbb{M}_0$ is called a hyperbolic plane. A metabolic (hyperbolic) form is just an orthogonal sum of metabolic (hyperbolic) planes.

By Witt decomposition, every bilinear form B decomposes as $B \cong B_{\mathrm{an}} \perp B_m$ with B_{an} anisotropic and B_m metabolic. In this decomposition, B_{an} is uniquely determined up to isometry, and $\dim B_m$ is also uniquely determined, but generally, B_m is not uniquely determined up to isometry. However, the isometry class of B is uniquely determined by the triple $(B_{\mathrm{an}},\ \dim B_m,\ D_F^0(B))$.

A bilinear form B can be diagonalized iff $D_F(B) \neq \{0\}$ iff B is not hyperbolic. If B is diagonalizable and $\{e_1, \dots, e_n\}$ is an orthogonal basis of the underlying vector space of B, and $a_i = \delta_{ij} B(e_i, e_j)$, then we write $B \cong \langle a_1, \dots, a_n\rangle_b$.

A form of type $\langle\langle c_1, \dots, c_n\rangle\rangle_b := \langle 1, c_1\rangle_b \otimes \dots \otimes \langle 1, c_m\rangle_b$ (where $\otimes$ denotes the usual tensor product of bilinear forms) is called an m-fold bilinear Pfister form. Any Pfister form π is always anisotropic or metabolic and one has $D_F^*(\pi) = G_F(\pi)$. The set of isometry classes of m-fold bilinear Pfister forms over F is denoted by $BP_n(F)$.

Two bilinear forms B, B' are called Witt-equivalent if $B_{\mathrm{an}} \cong B'_{\mathrm{an}}$, or, equivalently, if $B \perp -B'$ is metabolic. The Witt classes of bilinear forms make up the Witt ring $W(F)$ of F, with addition induced by the orthogonal sum and multiplication induced by the tensor product.

The Witt classes of even-dimensional bilinear forms form the fundamental ideal $I(F)$, and we put $I^n(F) := I(F)^n$, the n-th power of $I(F)$. $I^n(F)$ is additively generated by $BP_n(F)$. The quotients $\overline{I^n}(F) := I^n(F)/I^{n+1}(F)$ give rise to the graded Witt ring. The Arason-Pfister Hauptsatz states that if $0 \neq B \in I^n(F)$ is anisotropic then $\dim B \geqslant 2^n$, and if $\dim B = 2^n$ then $B \cong \lambda\pi$ for some $\lambda \in F^*$ and some $\pi \in BP_n(F)$, see, e.g., [L3, Lemma 4.8]. (This Hauptsatz for bilinear forms in characteristic 2 is already contained in the original article by Arason-Pfister [AP], but the proof there contains an error.)

We will later on need the following lemma which is well known (see, e.g., [M]), but we will give a quick proof for the reader's convenience.

Lemma 4.1. *Let F be a field of characteristic* 2 *that is finitely generated, say, by n elements, over its subfield F^2. Then every bilinear form over F of dimension $> 2^n$ is isotropic and $I^{n+1}(F) = 0$.*

This holds in particular if F can be generated by $\leqslant n$ elements over its prime field $\mathbb{F} = \mathbb{F}_2$.

Proof. Say, $F = F^2(x_1, \dots, x_n)$. Since $x_i^2 \in F^2$, we clearly have $[F : F^2] \leqslant 2^n$. Let $B = \langle a_1, \dots, a_m\rangle_b$ be any nonhyperbolic bilinear form over F. Then $D_F^0(B)$ is the F^2-vector space spanned by $\{a_1, \dots, a_m\}$ in F. Suppose $m > 2^n$, then $a_1, \dots, a_m \in F$ are necessarily F^2-linearly dependent, implying that B is isotropic. In particular, if $\pi \in BP_{n+1}(F)$, then π is isotropic, hence metabolic, i.e. $\pi = 0 \in W(F)$, and thus $I^{n+1}(F) = 0$.

If $F = \mathbb{F}(x_1, \dots, x_n)$, then clearly $F^2 = \mathbb{F}(x_1^2, \dots, x_n^2)$ and thus $F = F^2(x_1, \dots, x_n)$ and we can apply the above. □

If E/F is a field extension, then for a bilinear form B over F, scalar extension yields the form $B_E = B \otimes E$ over E, and we have the natural restriction homomorphisms $WF \to WE$, $I^n(F) \to I^n(E)$, and $\overline{I^n}(F) \to \overline{I^n}(E)$. The study of the kernels of these maps for function field extensions will be the subject of Section 11.

5. Differential forms

Let F be a field of characteristic $p > 0$. The space $\Omega^1(F)$ of absolute *differential* 1*-forms* or 1*-differentials* over F is defined to be the F-vector space generated by symbols da, $a \in F$, subject to the relations given by additivity, $d(a + b) = da + db$, and the product rule, $d(ab) = adb + bda$. In particular, one has $d(F^p) = 0$ for $F^p = \{a^p \mid a \in F\}$, and $d : F \to \Omega^1(F)$ is an F^p-derivation.

The space of *n-differentials* $\Omega^n(F)$ $(n \geqslant 1)$ is then defined by the n-fold exterior power, $\Omega^n(F) := \bigwedge^n(\Omega^1(F))$, which is therefore an F-vector space generated by *symbols* $da_1 \wedge \ldots \wedge da_n$, $a_i \in F$. The derivation d extends to an operator $d : \Omega^n(F) \to \Omega^{n+1}(F)$ by $d(a_0 da_1 \wedge \ldots \wedge da_n) = da_0 \wedge da_1 \wedge \ldots \wedge da_n$. We put $\Omega^0(F) = F$, $\Omega^n(F) = 0$ for $n < 0$, and $\Omega^*(F) = \bigoplus_{n\geqslant 0} \Omega^n(F)$, the algebra of differential forms over F with multiplication naturally defined by $(a_0 da_1 \wedge \ldots \wedge da_n)(b_0 db_1 \wedge \ldots \wedge db_m) = a_0 b_0 da_1 \wedge \ldots \wedge da_n \wedge db_1 \wedge \ldots \wedge db_m$.

A subset S of F is called *p-independent* if for every finite subset $\{s_1, \ldots, s_n\} \subset S$, $s_i \neq s_j$ for all $i \neq j$, one has $[F^p(s_1, \ldots, s_n) : F^p] = p^n$. A *p-basis* $\mathfrak{B}$ of F is a p-independent subset of F such that $F^p(\mathfrak{B}) = F$. In other words, a p-basis is a minimal generating set for the extension F/F^p.

Let $\mathfrak{B} = \{b_i \,|\, i \in I\}$ be a p-basis of F for some well-ordered index set $I = (I, <)$. For any subset $S \subset \mathfrak{B}$, we define

$$\bigwedge\nolimits_S^n = \{db_{i_1} \wedge \ldots \wedge db_{i_n} \,|\, b_{i_j} \in S,\ i_1 < i_2 < \ldots < i_n\} \ .$$

In particular, it is then well known that $\bigwedge_{\mathfrak{B}}^n$ is a basis of the F-vector space $\Omega^n(F)$. Furthermore, we define the sub-vector space

$$\Omega_S^n(F) = \mathrm{span}_F\{\bigwedge\nolimits_S^n\} \ .$$

So $\Omega_{\mathfrak{B}}^n(F) = \Omega^n(F)$.

There exists a well-defined group homomorphism $\Omega^n(F) \to \Omega^n(F)/d\Omega^{n-1}F$, the Artin-Schreier map $\wp$, which acts on *logarithmic* differentials as follows:

$$\wp : \Omega^n(F) \to \Omega^n(F)/d\Omega^{n-1}F : a\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} \longmapsto (a^p - a)\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n}$$

We define $\nu_n(F) := \ker(\wp)$. Kato [K1] has shown the following:

Theorem 5.1. *$\nu_n(F)$ is additively generated by the logarithmic differentials of the form $\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n}$, $a_i \in F^*$.*

The groups $\nu_n(F)$ are intimately related to bilinear forms as shown by Kato [K1]:

Theorem 5.2. *Let F be a field of characteristic 2. Then there is an isomorphism $\beta_{n,F} : \nu_n(F) \xrightarrow{\sim} I^n(F)/I^{n+1}F$ defined on generators as follows:*

$$\beta_{n,F} : \nu_n(F) \to I^n(F)/I^{n+1}F : \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} \longmapsto \langle\langle a_1, \ldots, a_n \rangle\rangle \bmod I^{n+1}F \ .$$

Remark 5.3. (i) Our notations differ slightly from those used by Kato and others who write $\nu_F(n)$, Ω_F^n. We chose our notations to give our functors a more uniform appearance in line with what one commonly uses, for example, for the Witt ring $W(F)$ and the higher powers of its fundamental ideals $I^n(F)$.

(ii) The cokernel $\mathrm{coker}(\wp)$ is denoted by $H_p^{n+1}(F)$. We will not consider this group here but only remark that it is of importance in the study of *quadratic* forms in the case $p = 2$. More precisely, Kato [K1] has shown that $H_2^{n+1}(F) \cong I^n(F)W_q(F)/I^{n+1}(F)W_q(F)$, where $W_q(F)$ denotes the Witt group of quadratic forms over F considered as module over the Witt ring $W(F)$ of symmetric bilinear forms over F.

Symbols in $\Omega^*(F)$ and quasilinear Pfister p-forms are related by the following observations (see [H1, Lemma 8.1]).

Lemma 5.4. (a) *Let F be a field of characteristic $p > 0$ and let $a_1, \ldots, a_n \in F$. The following are equivalent:*

(i) *The a_i are p-independent, i.e. $[F^p(a_1, \ldots, a_n) : F^p] = p^n$;*
(ii) *$da_1 \wedge \ldots \wedge da_n \neq 0 \in \Omega^n(F)$;*
(iii) *The quasilinear Pfister p-form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ is anisotropic.*

(b) *If the equivalent conditions in (a) hold then for $b_1, \ldots, b_n \in F$ the following are equivalent:*

(i) *$F^p(a_1, \ldots, a_n) = F^p(b_1, \ldots, b_n)$;*
(ii) *$F da_1 \wedge \ldots \wedge da_n = F db_1 \wedge \ldots \wedge db_n$;*
(iii) *$\langle\langle a_1, \ldots, a_n\rangle\rangle \cong \langle\langle b_1, \ldots, b_n\rangle\rangle$.*

Let $W \subset \Omega^m(F)$. The *annihilator subspace* of W in $\Omega^n(F)$ is defined to be

$$\mathrm{ann}_{\Omega^n(F)}(W) = \{\eta \in \Omega^n(F) \mid \eta \wedge \omega = 0 \text{ for all } \omega \in W\}\ .$$

Similarly, one defines the subspace $\mathrm{ann}_{\Omega^*(F)}(W)$ of $\Omega^*(F)$.

Proposition 5.5. *Consider a subset $S \subset F$ such that $[F^p(S) : F^p] = p^n$ for some $n \geqslant 1$, and let $dS = \{da \mid a \in S\}$. Let $a_1, \ldots, a_n \in F$ be such that $F^p(S) = F^p(a_1, \ldots, a_n)$. Extend $a_1, \ldots, a_n$ to a p-basis $\mathfrak{B} = \{a_1, \ldots, a_n\} \cup T$ of F. Then*

$$\mathrm{ann}_{\Omega^m(F)}(dS) = \begin{cases} 0 & \text{if } m < n \\ \Omega_T^{m-n}(F) \wedge da_1 \wedge \ldots \wedge da_n & \text{if } m \geqslant n \end{cases}$$

In particular, $\mathrm{ann}_{\Omega^*(F)}(dS) = \Omega_T^*(F) \wedge da_1 \wedge \ldots \wedge da_n = \Omega^*(F) \wedge da_1 \wedge \ldots \wedge da_n$.

Proof. First, let us remark that for S as above, the existence of such a_i with $F^p(S) = F^p(a_1, \ldots, a_n)$ is evident. Indeed, the a_i can be chosen among the elements of S, and in view of Lemma 5.4(b), it suffices to show the proposition in this case.

Working with the particular p-basis $\mathfrak{B}$ and the corresponding F-basis $\bigwedge_{\mathfrak{B}}^n$ of $\Omega^n(F)$, it is easy to see that $\Omega_T^k(F) \wedge da_1 \wedge \ldots \wedge da_n = \Omega^k(F) \wedge da_1 \wedge \ldots \wedge da_n$ for all $k \geqslant 0$.

If $s \in S$ then by the choice of the a_i, the elements $a_1, \ldots, a_n, s$ are p-dependent, so by Lemma 5.4(a), $da_1 \wedge \ldots \wedge da_n \wedge ds = 0$ and it follows readily that $\Omega_T^k(F) \wedge da_1 \wedge \ldots \wedge da_n \subset \mathrm{ann}_{\Omega^{k+n}(F)}(dS)$.

For the converse inclusion, let $1 \leqslant i \leqslant n$ and let $T_i = \{a_1, \ldots, a_{n-i}\} \cup T$, so $T_0 = \mathfrak{B}$ and $T_n = T$. Let $\omega \in \mathrm{ann}_{\Omega^m(F)}(dS)$. We use induction and assume that ω can be written as $\omega = \omega' \wedge da_{n-i+1} \wedge \ldots \wedge da_n$ with $\omega' \in \Omega_{T_i}^{m-i}(F)$ (the case $i = 0$ is trivial).

ω' then decomposes as $\eta + \rho \wedge da_{n-i}$ with uniquely determined $\eta \in \Omega_{T_{i+1}}^{m-i}(F)$, $\rho \in \Omega_{T_{i+1}}^{m-i-1}(F)$. But then $0 = da_{n-i} \wedge \omega$ implies $0 = da_{n-i} \wedge \eta$ which clearly implies $\eta = 0$ and thus $\omega = \rho \wedge da_{n-i} \wedge \ldots \wedge da_n$.

Induction then shows that $\omega \in \Omega_T^{m-n}(F) \wedge da_1 \wedge \ldots \wedge da_n$ (with $\Omega_T^{m-n}(F) = 0$ for $m - n < 0$ by convention). □

To be able to apply our results on the behaviour of differential forms under field extensions to the determination of Witt kernels, we have to consider those elements in $\mathrm{ann}_{\Omega^m(F)}(dS)$ as above that are in $\nu_m(F)$. The crucial result that we need and which is essentially due to Aravire-Baeza [AB2, §3] is the following:

Theorem 5.6. *Let F be a field of characteristic 2 and let $a_1, \dots, a_n \in F^*$ be 2-independent. Then*

$$\beta_{m,F}\big(\nu_m(F) \cap \Omega^{m-n}(F) \wedge da_1 \wedge \ldots \wedge da_n\big)$$
$$= \big\{\ \overline{\langle\langle c_1, \dots, c_n \rangle\rangle_b \otimes \varphi}\ \big|\ \varphi \in I^{m-n}(F),\ c_1, \dots, c_n \in F^2(a_1, \dots, a_n)^* \big\}$$

(with the usual convention that $\Omega^{m-n}(F) = 0$ and $I^{m-n}(F) = 0$ whenever $m < n$).

Remark 5.7. (i) Aravire and Baeza didn't state the result in the above form, but it can very easily be extracted from Lemma 3.1, Corollaries 3.2 and 3.3 in [AB2].

(ii) In the above statement, we certainly have $[F^2(c_1, \dots, c_n) : F^2] \leqslant 2^n$, with *strict* inequality iff $F^2(c_1, \dots, c_n) \subsetneq F^2(a_1, \dots, a_n)$ iff $c_1, \dots, c_n$ are 2-dependent, in which case $\langle\langle c_1, \dots, c_n \rangle\rangle_b$ is in fact metabolic (Lemma 5.4). So without loss of generality, the condition $c_1, \dots, c_n \in F^2(a_1, \dots, a_n)^*$ can be replaced by the condition $F^2(c_1, \dots, c_n) = F^2(a_1, \dots, a_n)$.

6. Roots and coefficients of polynomials in positive characteristic

Lemma 6.1. *Let F be a field of characteristic p with prime field $\mathbb{F}$. Let $g(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0 \in F[X]$ be a separable polynomial with roots $\alpha_1, \dots, \alpha_n$ in a separable closure F^{sep} of F. Let $m_1, \dots, m_n \in \mathbb{N}$. Then*

$$\mathbb{F}(\alpha_1, \dots, \alpha_n) = \mathbb{F}(\alpha_1^{p^{m_1}}, \dots, \alpha_n^{p^{m_n}}, a_0, \dots, a_{n-1}) \ .$$

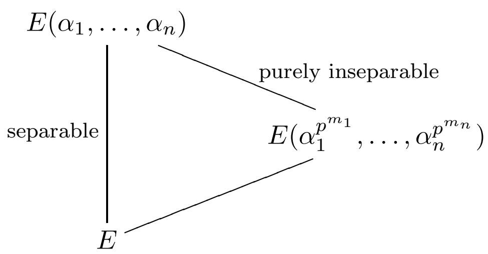
Proof. Let $E = \mathbb{F}(a_0, \dots, a_{n-1})$. Then

$$\mathbb{F}(\alpha_1, \dots, \alpha_n) = E(\alpha_1, \dots, \alpha_n) \ ,$$

which is a splitting field of the separable polynomial $g(X) \in E[X]$, and also

$$\mathbb{F}(\alpha_1^{p^{m_1}}, \dots, \alpha_n^{p^{m_n}}, a_0, \dots, a_{n-1}) = E(\alpha_1^{p^{m_1}}, \dots, \alpha_n^{p^{m_n}}) \ ,$$

and we have the following diagram:



Since an algebraic extension that is separable over its base field will also be separable over any intermediate field, we necessarily have $E(\alpha_1^{p^{m_1}}, \dots, \alpha_n^{p^{m_n}}) = E(\alpha_1, \dots, \alpha_n)$. □

Corollary 6.2. *With the same notations and hypotheses as in Lemma 6.1:*

(i) *If $L/\mathbb{F}$ is any field extension with $\alpha_1, \dots, \alpha_n \in L$ and if $m \in \mathbb{N}$, then*

$$L^{p^m}(\alpha_1, \dots, \alpha_n) = L^{p^m}(a_0, \dots, a_{n-1})$$

.

(ii) *Let* $K = F(\alpha_1,\dots,\alpha_n)$ *and let* $\{a_{i_1},\dots,a_{i_r}\}$, $0 \leqslant i_1 < \ldots < i_r \leqslant n-1$, *be a maximal* p*-independent subset of the set of coefficients* $\mathcal{C}(g) = \{a_0,\dots,a_{n-1},a_n = 1\}$ *of* g *over* F*. Then* $\{a_{i_1},\dots,a_{i_r}\}$ *is* p*-independent over* K*, and there exist* $1 \leqslant j_1 < \ldots < j_r \leqslant n$ *such that*

$$\begin{aligned} K^p(\alpha_{j_1},\dots,\alpha_{j_r}) &= K^p(\alpha_1,\dots,\alpha_n) \\ &= K^p(\mathcal{C}(g)) \\ &= K^p(a_{i_1},\dots,a_{i_r})\ . \end{aligned}$$

In particular, $K d\alpha_{j_1} \wedge \ldots \wedge d\alpha_{j_r} = K da_{i_1} \wedge \ldots \wedge da_{i_r}$ *in* $\Omega^r(K)$*.*

Proof. (i) Since $\mathbb{F} = \mathbb{F}^{p^m}$, we clearly have $\mathbb{F}(\alpha_1^{p^m},\dots,\alpha_n^{p^m}) \subset L^{p^m}$ and the claim follows readily from Lemma 6.1.

(ii) Using (i), we have $K^p(\mathcal{C}(g)) = K^p(\alpha_1,\dots,\alpha_n)$. Now K/F is separable, so any maximal p-independent subset of $\mathcal{C}(g)$ over F will also be a maximal p-independent subset of $\mathcal{C}(g)$ over K. Hence, $[K^p(\mathcal{C}(g)) : K^p] = [F^p(\mathcal{C}(g)) : F^p] = p^r$, and by part (i), any maximal p-independent subset of $\{\alpha_1,\dots,\alpha_n\}$ over K will have r elements, so any such p-independent subset, say, $\{\alpha_{j_1},\dots,\alpha_{j_r}\}$, $1 \leqslant j_1 < \ldots < j_r \leqslant n$ will satisfy $K^p(\alpha_{j_1},\dots,\alpha_{j_r}) = K^p(\alpha_1,\dots,\alpha_n) = K^p(\mathcal{C}(g))$. The statement about the wedge products then follows from Lemma 5.4(b). □

Remark 6.3. In a certain sense, Lemma 6.1 says that the roots of a separable equation in characteristic $p > 0$ can be expressed rationally in terms of the coefficients of the polynomial and the p^{m_i}-th powers of the respective roots α_i

To illustrate this in the case $p = 2$, let $g(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0$ be a separable polynomial as above, and let α be one of its roots. Write $g(X) = h_0(X^2) + Xh_1(X^2)$. Taking derivatives and using $p = 2$, we get $g'(X) = h_1(X^2)$. Since α is a root of the separable polynomial $g(X)$, we have $g'(\alpha) = h_1(\alpha^2) \neq 0$. But $g(\alpha) = h_0(\alpha^2) + \alpha h_1(\alpha^2) = 0$ and thus

$$\alpha = \frac{h_0(\alpha^2)}{h_1(\alpha^2)}\ ,$$

showing that α is a rational expression (over $\mathbb{F}$) in $a_0,\dots,a_{n-1}$ and α^2. Squaring this equation and feeding it back into itself, we can express α as a rational expression in $a_0,\dots,a_{n-1}$ and α^4 and so on. □

We finish this section with another elementary observation about roots and coefficients of polynomials in positive characteristic.

Corollary 6.4. *Let* F *be a field of characteristic* p*. Let* $g(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0 \in F[X]$ *be a separable and irreducible polynomial with roots* $\alpha_1,\dots,\alpha_n$ *in a separable closure* F^{sep} *of* F*. Let* $K = F(\alpha_1,\dots,\alpha_n)$*. Then the following are equivalent:*

(i) $\alpha_i \in K^p$ *for some* $i \in \{1,\dots,n\}$*.*
(ii) $\alpha_i \in K^p$ *for all* $i \in \{1,\dots,n\}$*.*
(iii) $g(X) \in F^p[X]$*.*

Proof. (ii)$\Longleftrightarrow$(iii). By Corollary 6.2, we have $K^p(\alpha_1,\dots,\alpha_n) = K^p(a_0,\dots,a_{n-1})$. Also, since K/F is separable, one has $K^p \cap F = F^p$. The equivalence follows readily.

(ii)$\Longrightarrow$(i) is trivial, so it remains to show (i)$\Longrightarrow$(ii). K is the splitting field of $g(X)$ over F, hence K/F is Galois, and its Galois group $G = \mathrm{Gal}(K/F)$ acts transitively on the roots of $g(X)$ as it is irreducible over F. Suppose $\alpha_i = \beta^p$

with $\beta \in K$. Let α_j be any root of $g(X)$, and let $\sigma \in G$ with $\sigma(\alpha_i) = \alpha_j$, then $\alpha_j = \sigma(\beta)^p \in K^p$. □

7. Differential forms under field extensions: first results

Throughout this section, all fields are assumed to be of arbitrary positive characteristic $p > 0$. We may also assume that all fields considered are non perfect (and so in particular infinite) as for perfect fields, $\Omega^n(F) = 0$ for $n \geqslant 1$ and our results are either trivially true or don't apply.

The next two lemmas are folklore but we include a proof for the reader's convenience.

Lemma 7.1. *Let E/F be a separable algebraic extension. Then $\Omega^n(E/F) = 0$ for all n.*

Proof. If $\mathfrak{B}$ is a p-basis of F and E/F is separable algebraic, then it is well known that $\mathfrak{B}$ is also a p-basis of E (see, e.g., [H1, Lemma 8.6]). Therefore, the basis $\bigwedge_{\mathfrak{B}}^n$ of $\Omega^n(F)$ (over F) is also a basis of $\Omega^n(E)$ (over E), and the result follows readily. □

Lemma 7.2. *Let E/F be a purely transcendental extension. Then $\Omega^n(E/F) = 0$ for all n.*

Proof. Clearly, it suffices to show this for the case $E = F(X)$, the rational function field in one variable X over F. Let $\mathfrak{B}$ be a p-basis of F. If $0 \neq \omega \in \Omega^n(F)$, then there exists $a_i \in F^*$, $\omega_i \in \bigwedge_{\mathfrak{B}}^n$, $1 \leqslant i \leqslant n$, such that $\omega = \sum_i a_i\omega_i$. But we have that $\mathfrak{B}' = \{X\} \cup \mathfrak{B}$ is a p-basis of $F(X)$, hence the ω_i are in $\bigwedge_{\mathfrak{B}'}^n$ and stay linearly independent over E and thus $0 \neq \omega_E \in \Omega^n(E)$. □

Now every field extension E/F has a transcendence basis T. So we have $F \subset F(T) \subset E$ with $F(T)/F$ purely transcendental and $E/F(T)$ algebraic. If T can be chosen such that $E/F(T)$ is separable, then we call E/F a separable extension. The previous two lemmas now imply

Corollary 7.3. *Let E/F be a separable extension. Then $\Omega^n(E/F) = 0$ for all n.*

For simple purely inseparable extensions we have the following result (cf. [AB1, Lemma 2.4] in the case $p = 2$):

Lemma 7.4. *Let $a \in F \setminus F^p$, $n \geqslant 1$ and let $E = F(\sqrt[p^n]{a})$. Then $\Omega^m(E/F) = \Omega^{m-1}(F) \wedge da = \mathrm{ann}_{\Omega^m(F)}(da)$.*

Proof. Let $\alpha = \sqrt[p^n]{a}$. Clearly $da = d\alpha^{p^n} = 0 \in \Omega^1(E)$, hence $\Omega^{m-1}(F) \wedge da \subset \Omega^m(E/F)$.

For the converse inclusion, note that a can be chosen as part of a p-basis $\mathfrak{B} = \{a\} \cup T$ of F, in which case $\{\alpha\} \cup T$ is a p-basis of E. Let $\omega \in \Omega^m(E/F)$ and write $\omega = \eta + \mu \wedge da$ with uniquely determined $\eta \in \Omega_T^m(F)$, $\mu \in \Omega_T^{m-1}(F)$. Then $0 = \omega_E = \eta_E$. But since T is part of a p-basis of E, we must have $\eta = 0 \in \Omega_T^m(F)$, so $\omega = \mu \wedge da \in \Omega^{m-1}(F) \wedge da$. □

8. Differential forms under field extensions: simple algebraic extensions

In this section, we determine the kernel $\Omega^m(E/F)$ for a simple algebraic extension E/F of a field F of characteristic $p > 0$.

Proposition 8.1. *Let $\overline{F}$ be an algebraic closure of F and $\xi \in \overline{F}$. Let $f(X) \in F[X]$ be the monic irreducible polynomial with $f(\xi) = 0$ and let $E = F(\xi)$. Let $\mathcal{C}(f) \subset F^*$ be the set of nonzero coefficients of f.*

(i) *If $f(X) \notin F[X^p]$, then $\Omega^m(E/F) = 0$.*
(ii) *If $f(X) \in F[X^p]$ then $[F^p(\mathcal{C}(f)) : F^p] = p^r$ with $r \geqslant 1$. If $b_i \in F^*$, $1 \leqslant i \leqslant r$ are such that $F^p(\mathcal{C}(f)) = F^p(b_1, \dots, b_r)$ then*

$$\Omega^m(E/F) = \Omega^{m-r}(F) \wedge db_1 \wedge \dots \wedge db_r = \mathrm{ann}_{\Omega^m(F)}(d\mathcal{C}(f))$$

(with the usual convention $\Omega^{m-r}(F) = 0$ if $m < r$).

Proof. (i) If $f(X) \notin F[X^p]$ then $f'(X) \neq 0$ and E/F is separable. The result then follows from Lemma 7.1.

(ii) Suppose $f(X) \in F[X^p]$. Since $\mathrm{char}(F) = p$, we cannot have $\mathcal{C}(f) \subset F^p$ or else $f(X) = h(X)^p$ for some $h(X) \in F[X]$. Hence, $[F^p(\mathcal{C}(f)) : F^p] = p^r$ with $r \geqslant 1$.

We make the following observations and fix the following notations for the remainder of the proof:

- By the general theory of polynomials over fields, there exist uniquely determined $\ell \geqslant 1$ and $g(X) \in F[X]$ such that $g(X) \in F[X]$ is monic, irreducible and separable, and $f(X) = g(X^{p^\ell})$. We write
$$g(X) = X^n + a_{n-1}X^{n-1} + \dots + a_1X + a_0 \ .$$
- $\alpha_1, \dots, \alpha_n$ are the n different roots of $g(X)$ in a separable closure F^{sep} of F (inside $\overline{F}$). We put $K = F(\alpha_1, \dots, \alpha_n)$, the splitting field of $g(X)$ over F.
- $f(X)$ has n different roots in $\overline{F}$, say, $\beta_1, \dots, \beta_n$ with $\beta_i^{p^\ell} = \alpha_i$.
- As already remarked, $\mathcal{C}(f) = \mathcal{C}(g) \not\subset F^p$, i.e. $g(X) \notin F^p[X]$, so in particular $\alpha_i \notin K^p$ for all i and $[K^p(\alpha_1, \dots, \alpha_n) : K^p] = [K^p(\mathcal{C}(g)) : K^p] = p^r$ (where $n \geqslant r \geqslant 1$) by Corollaries 6.2 and 6.4. Without loss of generality, $K^p(\alpha_1, \dots, \alpha_n) = K^p(\alpha_1, \dots, \alpha_r)$.
- Again by Corollary 6.2, any maximal p-independent subset of $\mathcal{C}(g) = \{a_0, \dots, a_{n-1}, 1\} \setminus \{0\}$ (over F) will have r elements. If $\{a_{i_1}, \dots, a_{i_r},\ 0 \leqslant i_1 < \dots < i_r \leqslant n-1\}$ is such a maximal p-independent subset, then $K^p(\alpha_1, \dots, \alpha_r) = K^p(a_{i_1}, \dots, a_{i_r})$.
- Clearly, $F^p(\mathcal{C}(f)) = F^p(\mathcal{C}(g)) = F^p(a_{i_1}, \dots, a_{i_r})$, and by Lemma 5.4(b), it suffices to show the result with $b_1, \dots, b_r$ replaced by $a_{i_1}, \dots, a_{i_r}$.

We first show that $\Omega^{m-r}(F) \wedge da_{i_1} \wedge \dots \wedge da_{i_r} \subset \Omega^m(E/F)$. By Lemma 5.4(a), it suffices to show that the quasilinear Pfister p-form $\pi = \langle\langle a_{i_1}, \dots, a_{i_r} \rangle\rangle$ (which is anisotropic over F) becomes isotropic (and hence p-split) over E.

But $E \cong_F F[X]/(f(X))$, the function field of f over F, and all the nonzero coefficients of the monic polynomial $f(X) \in F[X^p]$ are in $F^p(a_0, \dots, a_{n-1})^* = F^p(a_{i_1}, \dots, a_{i_r})^* = D_F(\pi) = G_F(\pi)$. Hence, by Theorem 3.1, $f(X) \in G_{F(X)}(\pi)$ and π_E is p-split.

For the converse inclusion, let $\omega \in \Omega^m(E/F)$. Since $E \cong_F F(\beta_i)$ for $1 \leqslant i \leqslant r$, we clearly have $\omega \in \Omega^m(F(\beta_i)/F)$ as well, hence $\omega_K \in \Omega^m(K(\beta_i)/K)$. But $\beta_i^{p^\ell} = \alpha_i \notin K^p$. Therefore, by Lemma 7.4,

$$\omega_K \in \Omega^{m-1}(K) \wedge d\alpha_i \quad \text{for all } 1 \leqslant i \leqslant r.$$

Hence, by Proposition 5.5,

$$\omega_K \in \mathrm{ann}_{\Omega^m(K)}(d\alpha_1, \ldots, d\alpha_r) = \Omega^{m-r}(K) \wedge d\alpha_1 \wedge \ldots \wedge d\alpha_r$$

(with the usual convention $\Omega^{m-r}(K) = 0$ if $r > m$).

Since $K^p(\alpha_1, \ldots, \alpha_r) = K^p(a_{i_1}, \ldots, a_{i_r})$, it then follows from Lemma 5.4(b) that

$$\omega_K \in \Omega^{m-r}(K) \wedge da_{i_1} \wedge \ldots \wedge da_{i_r} \ .$$

Now extend the p-independent elements $a_{i_1}, \ldots, a_{i_r}$ to a p-basis $\mathfrak{B} = \{a_{i_1}, \ldots, a_{i_r}\} \cup T$ of F. Since K/F is separable, $\mathfrak{B}$ is also a p-basis of K and we have that $\omega_K \in \Omega_T^{m-r}(K) \wedge da_{i_1} \wedge \ldots \wedge da_{i_r}$. So if we write ω as a linear combination over F of the basis $\bigwedge_{\mathfrak{B}}^m$ of $\Omega^m(F)$, then ω_K is of course the same linear combination with respect to the same basis $\bigwedge_{\mathfrak{B}}^m$ now considered as basis of $\Omega^m(K)$. It clearly follows that we must have $\omega \in \Omega_T^{m-r}(F) \wedge da_{i_1} \wedge \ldots \wedge da_{i_r} = \Omega^{m-r}(F) \wedge da_{i_1} \wedge \ldots \wedge da_{i_r}$.

The fact that $\Omega^{m-r}(F) \wedge da_{i_1} \wedge \ldots \wedge da_{i_r} = \mathrm{ann}_{\Omega^m(F)}(d\mathcal{C}(f))$ follows also from Proposition 5.5. □

Corollary 8.2. *Let* $f(X) = X^{pn} + a_{n-1}X^{p(n-1)} + \ldots + a_1X^p + a_0 \in F[X]$ *be irreducible and let* ζ *be a root of* $f(X)$ *in an algebraic closure* $\overline{F}$ *of* F. *Then* $\Omega^m(F(\zeta)/F) = \mathrm{ann}_{\Omega^m(F)}(da_0, \ldots, da_n)$, *i.e.* $\omega \in \Omega^m(F(\zeta)/F)$ *iff* $\omega \in \Omega^{m-1}(F) \wedge da_i = \mathrm{ann}_{\Omega^m(F)}(da_i)$ *for all* $a_i \notin F^p$.

Proof. This follows readily from Propositions 5.5 and 8.1(ii). □

9. Differential forms under field extensions: function fields of quasilinear p-forms

Function fields of quasilinear p-forms form a special case of function fields of hypersurfaces, and the behaviour of differential forms over function fields of quasilinear p-forms will play an important role in the investigation of the more general situation. For further reference on function fields of quasilinear p-forms and some basic facts we mention without further proof or comment, see [H1, §§ 7.1, 7.2].

Let $\varphi = \langle a_0, \cdots, a_n \rangle$ be a nonzero quasilinear p-form over F, let $X = (X_0, \cdots, X_n)$ be an $(n+1)$-tuple of variables. The function field $F(\varphi)$ of φ over F is defined as follows (cf. [H1, Definition 7.3]) :

- If $\mathrm{ndeg}_F(\varphi) = 1$, then $F(\varphi) = F(X_1, \cdots, X_n)$, the rational function field in n variables over F.
- If $\mathrm{ndeg}_F(\varphi) > 1$, then $\varphi(X) = \varphi(X_0, \cdots, X_n)$ is irreducible, and one defines

$$F(\varphi) = \mathrm{Quot}\left(\frac{F[X_0, \cdots, X_n]}{(\varphi(X))}\right) ,$$

the quotient field of the integral domain $F[X_0, \cdots, X_n]/(\varphi(X))$.

Clearly, if $\dim \varphi \geqslant 2$, then φ is isotropic over $F(\varphi)$, and if φ and ψ are similar, then $F(\varphi) \cong F(\psi)$.

Note also that if $\psi = \varphi \oplus (t \times \langle 0 \rangle)$, then $F(\psi) \cong F(\varphi)(T_1, \cdots, T_t)$, where the T_i are variables. In particular, $F(\varphi)$ is F-isomorphic to a purely transcendental extension of transcendence degree $i_d(\varphi)$ over $F(\varphi_{\mathrm{an}})$.

Suppose (after possibly scaling) that $\varphi = \langle 1, a_1, \cdots, a_n \rangle$ and assume that $\mathrm{ndeg}_F(\varphi) > 1$, then $\sum_{i=1}^n a_i X_i^p \notin F(X_1, \cdots, X_n)^p$ and

$$F(\varphi) \cong F(X_1, \cdots, X_n)(\sqrt[p]{\textstyle\sum_{i=1}^n a_i X_i^p}) = F(X_1, \cdots, X_n)(\textstyle\sum_{i=1}^n \sqrt[p]{a_i} X_i) \ ,$$

i.e. $F(\varphi)$ can be realized as a purely transcendental extension of transcendence degree $n = \dim \varphi - 1$, followed by a purely inseparable extension of degree p (note that $-1 = (-1)^p$).

Furthermore, F is algebraically closed in $F(\varphi)$ iff $\mathrm{ndeg}_F(\varphi) \geqslant p^2$, and if $\mathrm{ndeg}_F(\varphi) = p$, say, $N_F(\varphi) = F^p(d)$ with $d \in F \setminus F^p$, then $F(\varphi) \cong F(\sqrt[p]{d})(X_1, \dots, X_n)$, so $F(\sqrt[p]{d})$ is the algebraic closure of F inside $F(\varphi)$ (see [H1, Proposition 7.6])

The function field of the *projective* variety defined by $\varphi = 0$ will be denoted by $\widehat{F}(\varphi)$ and (with $\varphi = \langle 1, a_1, \cdots, a_n \rangle$) we have

$$\widehat{F}(\varphi) \cong F(X_1, \cdots, X_{n-1})\big(\sqrt[p]{a_1 X_1^p + \ldots + a_{n-1} X_{n-1}^p + a_n}\big) .$$

One readily checks that $F(\varphi) \cong \widehat{F}(\varphi)(T)$, the rational function field in one variable T over $\widehat{F}(\varphi)$.

The vanishing of differential forms when passing to the function field of a quasilinear p-form is described by the following.

Proposition 9.1. *Let φ be a quasilinear p-form over F with $\mathrm{ndeg}_F(\varphi) = p^n > 1$. Let $b_1, \cdots, b_n \in F^*$ be such that $N_F(\varphi) = F^p(b_1, \cdots, b_n)$. Then*

$$\Omega^m(F(\varphi)/F) = \Omega^m(\widehat{F}(\varphi)/F) = \begin{cases} 0 & \text{if } m < n; \\ \Omega_F^{m-n} \wedge db_1 \wedge \cdots \wedge db_n & \text{if } m \geqslant n. \end{cases}$$

Proof. The statement for $\Omega^m(F(\varphi)/F)$ is nothing but [H1, Proposition 8.4] and we refer to the proof there. The statement for $\Omega^m(\widehat{F}(\varphi)/F)$ then follows from Lemma 7.2, using the fact that $F(\varphi)/\widehat{F}(\varphi)$ can be realized as a purely transcendental extension as remarked above. □

The following corollary will be an ingredient in the proof of Theorem 10.3.

Corollary 9.2. *Let $a, b \in F$ and $\omega \in \Omega^m(F)$. Suppose that $[F^p(a,b) : F^p] = p^r$ with $r \in \{1, 2\}$ (so $a \notin F^p$ or $b \notin F^p$). Consider $aX^p + b \in F[X]$ and suppose that $\omega_{F(X)} \in \Omega^{m-1}(F(X)) \wedge d(aX^p + b)$.*

If $r = 1$ then $F^p(a,b) = F^p(a)$ if $a \notin F^p$ resp. $F^p(a,b) = F^p(b)$ if $b \notin F^p$, and $\omega \in \Omega^{m-1}(F) \wedge da$ resp. $\omega \in \Omega^{m-1}(F) \wedge db$.

If $r = 2$ then $\omega \in \Omega^{m-2}(F) \wedge da \wedge db$

Proof. By assumption, $aX^p + b \notin F(X)^p$. Let $E = F(X)\big(\sqrt[p]{aX^p + b}\big)$. Now $\omega_{F(X)} \in \Omega^{m-1}(F(X)) \wedge d(aX^p + b)$ means that $\omega_{F(X)} \in \Omega^m(E/F(X))$ by Lemma 7.4, hence $\omega \in \Omega^m(E/F)$. But $E \cong \widehat{F}(\varphi)$ for the quasilinear p-form $\varphi = \langle 1, a, b \rangle$ and $N_F(\varphi) = F^p(a,b)$. The result now follows from Proposition 9.1. □

10. Differential forms under field extensions: function fields of hypersurfaces

The next lemma is rather technical but it will provide a crucial ingredient in the induction on the number of variables when treating function fields of hypersurfaces.

Lemma 10.1. *Let F be a field of characteristic $p > 0$ and let $q(X) = c_n X^{pn} + c_{n-1} X^{p(n-1)} + \ldots + c_1 X^p + c_0 \in F[X^p]$ with $c_n \neq 0$. Let $\mathcal{C}(q) = \{c_0, \dots, c_n\} \setminus \{0\}$ be the set of nonzero coefficients of $q(X)$. Suppose that $\mathcal{C}(q) \not\subset F^p$, so $q(X) \notin F^p[X^p]$ and $[F^p(\mathcal{C}(q)) : F^p] = p^s$ for some $s \geqslant 1$. Let $b_1, \dots, b_s \in F$ such that $F^p(\mathcal{C}(q)) = F^p(b_1, \dots, b_s)$.*

Let $\omega \in \Omega^m(F)$ *and suppose that* $\omega_{F(X)} \in \Omega^{m-1}(F(X)) \wedge dq$. *Then* $\omega \in \Omega^{m-s}(F) \wedge db_1 \wedge \ldots \wedge db_s = \mathrm{ann}_{\Omega^m(F)}(dc_0, \ldots, dc_n)$.

Proof. Since not all $c_i \in F^p$, it follows that $q(X)$ is *not* a p-th power in $K = F(X)$. In particular, the polynomial $h(X,Y) = Y^p - q(X) \in F[X,Y]$ is irreducible over $K[Y]$ and by Gauss' Lemma also over $F[X,Y]$ and over $L[X]$ where $L = F(Y)$.

Now $\omega_K \in \Omega^{m-1}(K) \wedge dq$ then just means that $\omega_K \in \Omega^m(K(\sqrt[p]{q})/K)$ by Lemma 7.4 and hence $\omega \in \Omega^m(K(\sqrt[p]{q})/F)$.

Consider the function field $E = F(h)$, i.e. the quotient field of the integral domain $F[X,Y]/(h)$. Clearly, $E \cong_F K(\sqrt[p]{q})$. Now consider the monic irreducible polynomial

$$\tilde{h}(X) = -\frac{1}{c_n} h(X,Y) = X^{pn} + \frac{c_{n-1}}{c_n} X^{p(n-1)} + \ldots + \frac{c_1}{c_n} X^p + \frac{c_0 - Y^p}{c_n} \in L[X]$$

and let ζ be a root of $\tilde{h}(X)$ in an algebraic closure $\overline{L}$ of L. We then also have $E \cong_F L(\zeta)$.

It follows that $\omega_L \in \Omega^m(L(\zeta)/L)$ and we can invoke Corollary 8.2 to conclude that

$$\omega_L \in \mathrm{ann}_{\Omega^m(L)}\big(d\tfrac{c_1}{c_n}, \ldots, d\tfrac{c_{n-1}}{c_n}, d\tfrac{c_0 - Y^p}{c_n}\big) \ .$$

In particular,

$$\begin{aligned} &\omega_L \in \Omega^{m-1}(L) \wedge d\tfrac{c_i}{c_n} && \text{for all } i \in \{1, \ldots, n-1\} \text{ with } \tfrac{c_i}{c_n} \notin L^p, \\ &\omega_L \in \Omega^{m-1}(L) \wedge d\big(\tfrac{c_0 - Y^p}{c_n}\big) && \text{if } \tfrac{c_0 - Y^p}{c_n} \notin L^p. \end{aligned}$$

Now $\frac{c_i}{c_n} \notin L^p$ iff $\frac{c_i}{c_n} \notin F^p$ in which case this element can be chosen as part of a p-basis of F. Since $L = F(Y)/F$ is purely transcendental, it readily follows that $\omega \in \Omega^{m-1}(F) \wedge d\frac{c_i}{c_n}$.

Case 1. $[F^p(c_0, c_n) : F^p] = 1$, i.e. $c_n, c_0 \in F^p$. Then $\Omega^{m-1}(F) \wedge d\frac{c_i}{c_n} = \Omega^{m-1}(F) \wedge dc_i$ and $F^p(c_0, \ldots, c_n) = F^p(c_1, \ldots, c_{n-1})$. It follows readily that $\omega \in \mathrm{ann}_{\Omega^m(F)}(dc_0, \ldots, dc_n)$ and we are done by Proposition 5.5.

Case 2. $[F^p(c_0, c_n) : F^p] = p$ and $c_n \in F^p$, so in particular $c_0, \frac{c_0}{c_n} \notin F^p$ and $F^p(c_0, c_n) = F^p(c_0)$. By Corollary 9.2 we have $\omega \in \Omega^{m-1}(F) \wedge d\frac{c_0}{c_n} = \Omega^{m-1}(F) \wedge dc_0$ and hence

$$\omega \in \mathrm{ann}_{\Omega^m(F)}\big(d\tfrac{c_1}{c_n}, \ldots, d\tfrac{c_{n-1}}{c_n}, dc_0\big) \ .$$

But clearly $F^p\big(\frac{c_1}{c_n}, \ldots, \frac{c_{n-1}}{c_n}, c_0\big) = F^p(c_0, c_1, \ldots, c_n)$, thus $\omega \in \mathrm{ann}_{\Omega^m(F)}(dc_0, \ldots, dc_n)$ and the result follows again from Proposition 5.5.

Case 3. $[F^p(c_0, c_n) : F^p] = p$ and $c_n \notin F^p$. Here, $F^p(c_0, c_n) = F^p(c_n)$. By Corollary 9.2 we have $\omega \in \Omega^{m-1}(F) \wedge d\big(\frac{-1}{c_n}\big) = \Omega^{m-1}(F) \wedge dc_n$ since $F^p\big(\frac{-1}{c_n}\big) = F^p(c_n)$. This time, we have

$$\omega \in \mathrm{ann}_{\Omega^m(F)}\big(d\tfrac{c_1}{c_n}, \ldots, d\tfrac{c_{n-1}}{c_n}, dc_n\big)$$

but also $F^p\big(\frac{c_1}{c_n}, \ldots, \frac{c_{n-1}}{c_n}, c_n\big) = F^p(c_0, c_1, \ldots, c_n)$ and we conclude as in Case 2.

Case 4. $[F^p(c_0, c_n) : F^p] = p^2$. This time, by Corollary 9.2 we have $\omega \in \Omega^{m-2}(F) \wedge d\frac{c_0}{c_n} \wedge d\big(\frac{-1}{c_n}\big) = \Omega^{m-2}(F) \wedge dc_0 \wedge dc_n$ since $F^p\big(\frac{c_0}{c_n}, \frac{-1}{c_n}\big) = F^p(c_0, c_n)$. Hence

$$\omega \in \mathrm{ann}_{\Omega^m(F)}\big(d\tfrac{c_1}{c_n}, \ldots, d\tfrac{c_{n-1}}{c_n}, dc_0, dc_n\big) \ .$$

Since $F^p\big(\frac{c_1}{c_n}, \ldots, \frac{c_{n-1}}{c_n}, c_0, c_n\big) = F^p(c_0, \ldots, c_n)$, the result follows once more from Proposition 5.5. $\square$

Proposition 10.2. *Let F be a field of characteristic $p > 0$, let $X = (X_1, \dots, X_n)$ be an n-tuple of variables, $X' = (X_1, \dots, X_{n-1})$, and $X^p = (X_1^p, \dots, X_n^p)$, $X'^p = (X_1^p, \dots, X_{n-1}^p)$. Let*

$$h(X) = X_n^{p\ell} + g_{\ell-1}(X')X_n^{p(\ell-1)} + \dots + g_1(X')X_n^p + g_0(X') \in F[X^p]$$

be irreducible, where $g_i(X') \in F[X'^p]$. Let $E = F(h)$ be the function field of $h(X)$ over F, i.e., $E = \mathrm{Quot}\big(F[X]/(h)\big)$. Let $\mathcal{C}(h) \subset F^*$ be the set of nonzero F-coefficients of h. Then $[F^p(\mathcal{C}(h)) : F^p] = p^s$ with $s \geqslant 1$. Let $b_1, \dots, b_s \in F$ such that $F^p(\mathcal{C}(h)) = F^p(b_1, \dots, b_s)$. Then

$$\Omega^m(E/F) = \Omega^{m-s}(F) \wedge db_1 \wedge \dots \wedge db_s = \mathrm{ann}_{\Omega^m(F)}(d\mathcal{C}(h))$$

(with the usual convention $\Omega^{m-s}(F) = 0$ for $m < s$).

Proof. Note that the irreducibility of $h(X)$ implies that we cannot have $\mathcal{C}(h) \subset F^p$ for otherwise $h(X) = f(X)^p$ for some $f(X) \in F[X]$. Thus, we indeed have $s \geqslant 1$.

First, we show that $\Omega^{m-s}(F) \wedge db_1 \wedge \dots \wedge db_s \subset \Omega^m(E/F)$. Consider the quasilinear Pfister p-form $\pi = \langle\langle b_1, \dots b_s \rangle\rangle$. We then have $D_F(\pi) = G_F(\pi) = F^p(b_1, \dots b_s)^* = F^p(\mathcal{C}(h))^*$, hence $h(X) \in G_F(\pi)[X^p]$ and by Theorem 3.1, π is p-split over E, so in particular π is isotropic over E and thus $db_1 \wedge \dots \wedge db_s = 0 \in \Omega^s(E)$ by Lemma 5.4(a), so indeed $\Omega^{m-s}(F) \wedge db_1 \wedge \dots \wedge db_s \subset \Omega^m(E/F)$.

For the reverse inclusion, consider $\omega \in \Omega^m(E/F)$. By Proposition 5.5, it suffices to show that for every coefficient $c \in \mathcal{C}(h)$ with $c \notin F^p$ we have that $\omega \in \mathrm{ann}_{\Omega^m(F)}(dc)$. The one variable case $n = 1$ follows from Proposition 8.1(ii).

So assume $n \geqslant 2$. Since $c \in \mathcal{C}(h)$, we must have $c \in \mathcal{C}(g_i)$ for some $i \in \{0, \dots, \ell - 1\}$, and since $c \notin F^p$ we have $g_i(X') \notin F(X')^p$.

Note also that $E \cong F(X')(\zeta)$ with ζ being a root of $h(X)$ (considered as polynomial in X_n over $F(X')$) in an algebraic closure $\overline{F(X')}$. By Proposition 8.1(ii),

$$\omega_{F(X')} \in \mathrm{ann}_{\Omega^m(F(X'))}(dg_i(X')) = \Omega^{m-1}(F(X')) \wedge dg_i(X') \ .$$

Let $X'' = (X_1, \dots X_{n-2})$ and write

$$g_i(X') = q_k(X'')X_{n-1}^{pk} + q_{k-1}(X'')X_{n-1}^{p(k-1)} + \dots + q_1(X'')X_{n-1}^p + q_0(X'')$$

with $q_i(X'') \in F[X''^p]$. Then there exists $j \in \{0, \dots, k\}$ such that $c \in \mathcal{C}(q_j)$, so in particular $q_j \notin F(X'')^p$, and we can apply Lemma 10.1 to conclude that

$$\omega_{F(X'')} \in \mathrm{ann}_{\Omega^m(F(X''))}(dq_j(X'')) = \Omega^{m-1}(F(X'')) \wedge dq_j(X'') \ .$$

Continuing like this by eliminating one variable at a time using Lemma 10.1 but retaining each time a polynomial containing c as a coefficient, we end up with $\omega \in ann_{\Omega^m(F)}(dc)$ as desired. □

Theorem 10.3. *Let F be a field of characteristic $p > 0$, let $X = (X_1, \dots, X_n)$ be an n-tuple of variables, $X' = (X_1, \dots, X_{n-1})$, and $X^p = (X_1^p, \dots, X_n^p)$, $X'^p = (X_1^p, \dots, X_{n-1}^p)$. Let $f(X) \in F[X]$ be an irreducible polynomial and let $E = F(f)$, the function field of f over F.*

(i) *If $f(X) \notin F[X^p]$, then $\Omega^m(E/F) = 0$.*

(ii) *If $f(X) \in F[X^p]$, write*

$$f(X) = h_{p\ell}(X) + h_{p(\ell-1)}(X) + \dots + h_p(X) + h_0(X)$$

where $h_{pi}(X)$ is homogeneous of total degree pi and $h_{p\ell}(X) \neq 0$. Let $a \in F^*$ be any nonzero element represented by $h_{p\ell}(X)$, and let $\widehat{\mathcal{C}} = \mathcal{C}\big(\frac{1}{a}f\big)$ be the set *of nonzero F-coefficients of the scaled polynomial $\frac{1}{a}f(X) \in F[X^p]$. Then $[F^p(\widehat{\mathcal{C}}) : F^p] = p^s$ with $s \geqslant 1$. Let $b_1, \ldots, b_s \in F$ such that $F^p(\widehat{\mathcal{C}}) = F^p(b_1, \ldots, b_s)$. Then*

$$\Omega^m(E/F) = \Omega^{m-s}(F) \wedge db_1 \wedge \ldots \wedge db_s = \mathrm{ann}_{\Omega^m(F)}(d\widehat{\mathcal{C}})$$

(with the usual convention $\Omega^{m-s}(F) = 0$ for $m < s$).

Proof. (i) In this situation, $f(X)$ contains a monomial in which one of the variables, say, X_n, has an exponent not divisible by p. $f(X)$ is then also an irreducible polynomial in the variable X_n over the rational function field $F(X')$, and we have $E \cong F(X')(\zeta)$ where ζ is a root of $f(X) \in F(X')[X_n]$ in some algebraic closure $\overline{F(X')}$. But since $f(X) \notin F(X')[X_n^p]$, this irreducible polynomial is separable and hence E/F is a separable extension (in the general sense) and we have $\Omega^m(E/F) = 0$ by Corollary 7.3.

(ii) First note that the irreducibility of $f(X) \in F[X^p]$ implies that not all coefficients of $f(X)$ can be in F^p (for otherwise $f(X) = g(X)^p$, a contradiction). In particular, $F \neq F^p$ and F is nonperfect, hence infinite. Thus, since $h_{p\ell}(X) \neq 0$, there exist $c = (c_1, \ldots, c_n) \in F^{(n)}$ such that $a := h_{p\ell}(c) \neq 0$. After relabeling the variables if necessary, we may assume $c_n \neq 0$. Consider the following invertible linear change A of variables:

$$A : \begin{cases} X_i \mapsto X_i + c_i X_n & \text{for } 1 \leqslant i < n, \\ X_n \mapsto c_n X_n . \end{cases}$$

Denote $f_A(X) := f(AX)$. Since $f(X) \in F[X^p]$, we clearly have $f_A(X) \in F[X^p]$, and for the sets of nonzero coefficients, one readily sees that $\mathcal{C}(f_A) \subset F^p(\mathcal{C}(f))$ and (since the linear change of variables is invertible) also $\mathcal{C}(f) \subset F^p(\mathcal{C}(f_A))$ and therefore $F^p(\mathcal{C}(f_A)) = F^p(\mathcal{C}(f))$. Let $\widehat{f}(X) = \frac{1}{a}f_A(X) \in F[X^p]$. By the above, we have

$$F^p(\mathcal{C}(\widehat{f})) = F^p(\widehat{\mathcal{C}}) \ . \tag{10.1}$$

Also, one readily sees that

$$\widehat{f}(X) = X_n^{p\ell} + g_{\ell-1}(X')X_n^{p(\ell-1)} + \ldots + g_1(X')X_n^p + g_0(X') \in F[X^p] \tag{10.2}$$

with $g_i(X') \in F[X'^p]$. Since $\widehat{f}$ is obtained from f through an invertible linear change of variables and subsequent scaling, we clearly have $F(\widehat{f}) \cong E = F(f)$, hence

$$\Omega^m(E/F) = \Omega^m(F(\widehat{f})/F) \ . \tag{10.3}$$

Using Eqs. 10.1, 10.2, 10.3, the result then follows by applying Proposition 10.2. □

11. Witt kernels for bilinear forms for function field extensions

Consider the Witt ring $W(F)$ of nondegenerate symmetric bilinear forms over a field F of characteristic 2.

For any field extension E/F, we want to study the kernel of the restriction maps

$$W(E/F) = \ker(W(F) \to W(E))$$
$$I^n(E/F) = \ker(I^n(F) \to I^n(E))$$
$$\overline{I^n}(E/F) = \ker(\overline{I^n}(F) \to \overline{I^n}(E))$$

for function field extensions.

For the remainder of this section, we will fix the following notations:

Notation.

- F is a field of characteristic 2.
- $X = (X_1, \ldots, X_n)$ is an n-tuple of variables, and $X^2 = (X_1^2, \ldots, X_n^2)$.
- $f(X) \in F[X]$ is an irreducible polynomial.
- $E = F(f)$ is the function field of f over F.
- If $f(X) \in F[X^2]$, write
$$f(X) = h_{2\ell}(X) + h_{2(\ell-1)}(X) + \ldots + h_2(X) + h_0(X)$$
where $h_{2i}(X)$ is homogeneous of total degree $2i$ and $h_{2\ell}(X) \neq 0$. Let $a \in F^*$ be any nonzero element represented by $h_{2\ell}(X)$, and let $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}(f) = \mathcal{C}\big(\frac{1}{a}f\big)$ be the set of nonzero F-coefficients of the scaled polynomial $\frac{1}{a}f(X) \in F[X^2]$. Then $[F^2(\widehat{\mathcal{C}}) : F^2] = 2^s$ with $s \geqslant 1$, and there are elements $b_1, \ldots, b_s \in F$ such that $F^2(\widehat{\mathcal{C}}) = F^2(b_1, \ldots, b_s)$. Note that the b_i are necessarily 2-independent.
- $\mathfrak{Pf}(f) = \mathfrak{Pf}(b_1, \ldots, b_s)$ is the set of s-fold Pfister forms defined as follows:
$$\mathfrak{Pf}(f) = \mathfrak{Pf}(b_1, \ldots, b_s) = \{\langle\langle c_1, \ldots, c_s\rangle\rangle_b \,|\, F^2(c_1, \ldots, c_s) = F^2(b_1, \ldots, b_s)\}\ .$$
Note that such Pfister forms will be anisotropic since $[F^2(c_1, \ldots, c_s) : F^2] = [F^2(b_1, \ldots, b_s) : F^2] = 2^s$ and by Lemma 5.4(i).

We will first compute $\overline{I^m}(E/F)$.

Theorem 11.1. (i) *If $f(X) \notin F[X^2]$, then $\overline{I^m}(E/F) = 0$.*

(ii) *If $f(X) \in F[X^2]$, then*
$$\overline{I^m}(E/F) = \big\{\ \overline{\pi \otimes \varphi}\ \big|\ \varphi \in I^{m-s}(F),\ \pi \in \mathfrak{Pf}(f)\big\}$$
(with the usual convention $I^{m-s}(F) = 0$ for $m < s$).

Proof. We have the following commutative diagram (the upward arrows are the usual restriction maps):

$$\begin{array}{ccc} \nu_m(E) & \xrightarrow[\sim]{\beta_{m,E}} & \overline{I^m}(E) \\ \uparrow & & \uparrow \\ \nu_m(F) & \xrightarrow[\beta_{m,F}]{\sim} & \overline{I^m}(F) \end{array}$$

For the kernel of the restriction maps, it thus follows that $\beta_{m,F}(\nu_m(E/F)) = \overline{I^m}(E/F)$. However, $\nu_m(E/F) = \nu_m(F) \cap \Omega^m(E/F)$ and, by Theorem 10.3, we have that $\Omega^m(E/F) = 0$ in case (i), and $\Omega^m(E/F) = \Omega^{m-s}(F) \wedge db_1 \wedge \ldots \wedge db_s$ in case (ii).

Hence, in case (i), $\overline{I^m}(E/F) = 0$. In case (ii), we have
$$\overline{I^m}(E/F) = \beta_{m,F}\big(\nu_m(F) \cap \Omega^{m-s}(F) \wedge db_1 \wedge \ldots \wedge db_s\big)$$
and the result now follows from Theorem 5.6 and Remark 5.7(ii). □

Lemma 11.2. $\mathfrak{Pf}(f) \subseteq W(E/F)$.

Proof. Let $\pi = \langle\langle c_1, \ldots, c_s\rangle\rangle_b \in \mathfrak{Pf}(f)$. By the definition of $\mathfrak{Pf}(f)$, Lemma 5.4 and Theorem 10.3, we have that $dc_1 \wedge \ldots \wedge dc_s = db_1 \wedge \ldots \wedge db_s = 0 \in \Omega^s(E)$, hence the quasilinear 2-form $\langle\langle c_1, \ldots, c_s\rangle\rangle$ is isotropic over E, thus also the bilinear Pfister form $\pi = \langle\langle c_1, \ldots, c_s\rangle\rangle_b$. Therefore π_E is in fact metabolic and the result follows. □

For $I^m(E/F)$, we obtain the following.

Theorem 11.3. (i) *If $f(X) \notin F[X^2]$, then $I^m(E/F) = 0$.*

(ii) *If $f(X) \in F[X^2]$ and $s \leqslant m$, then $I^m(E/F)$ is additively generated by Pfister forms of type $\pi \otimes \varphi$ with $\pi \in \mathfrak{P}\mathfrak{f}(f)$ and $\varphi \in P_{m-s}(F)$.*

(iii) *If $f(X) \in F[X^2]$ and $s > m$, then $I^m(E/F)$ is additively generated by $\mathfrak{P}\mathfrak{f}(f)$, so in particular $I^m(E/F) = I^s(E/F)$.*

Proof. Let $0 \neq \psi$ be an anisotropic bilinear form in $I^m(F)$. By the Arason-Pfister Hauptsatz for bilinear forms in characteristic 2, there exists $\ell \geqslant m$ be such that $\psi \in I^\ell(F) \setminus I^{\ell+1}(F)$. Let $\overline{\psi} := \psi \bmod I^{\ell+1}(F) \in \overline{I^\ell}(F)$.

If $f(X) \notin F[X^2]$ or if $\ell < s$ in the case $f(X) \in F[X^2]$, then by Theorem 11.1, $\overline{\psi}_E \neq 0 \in \overline{I^\ell}(E)$ and thus $\psi_E \in I^\ell(E) \setminus I^{\ell+1}(E)$. In particular ψ_E is not metabolic. This already shows (i).

Now suppose that $f(X) \in F[X^2]$. By Lemma 11.2, bilinear forms in $I^m(F)$ that can be additively generated by forms of type $\pi \otimes \varphi$ with $\pi \in \mathfrak{P}\mathfrak{f}(f)$ and $\varphi \in P_{m-s}(F)$ if $m \geqslant s$, and by forms in $\mathfrak{P}\mathfrak{f}(f)$ if $s > m$ are clearly in $I^m(E/F)$.

So let $0 \neq \psi$ be an anisotropic bilinear form in $I^m(E/F)$. With ℓ as above and by what was said before, we may assume without loss of generality that $\ell = m \geqslant s$.

To express ψ as an element in $I^m(F)$, the field extension E/F and the metabolicity of ψ over E, only finitely many coefficients from F are needed, so that without loss of generality, we may assume that F is finitely generated over its prime field $\mathbb{F}$, and so by Lemma 4.1 we have $I^n(F) = 0$ for some $n \geqslant m + 1$.

Since $\overline{\psi} \in \overline{I^m}(E/F)$, it follows from Theorem 11.1 that there exists $\pi_1 \in \mathfrak{P}\mathfrak{f}(f)$ and $\varphi_1 \in I^{m-s}(F)$ such that

$$\psi \equiv \pi_1 \otimes \varphi_1 \bmod I^{m+1}(F) \ .$$

Consider $\psi_1 := \psi \perp -\pi_1 \otimes \varphi_1$. Note that $\psi_1 \in I^{m+1}(F)$ and that $(\psi_1)_E = 0 \in W(E)$ since $\psi_E = (\pi_1)_E = 0$. Repeating the argument, we see that there exists $\pi_2 \in \mathfrak{P}\mathfrak{f}(f)$ and $\varphi_2 \in I^{m-s+1}(F)$ such that

$$\psi_1 \equiv \pi_2 \otimes \varphi_2 \bmod I^{m+2}(F) \ ,$$

i.e.

$$\psi \equiv \pi_1 \otimes \varphi_1 + \pi_2 \otimes \varphi_2 \bmod I^{m+2}(F) \ .$$

Continuing like this, we conclude that there exist $\pi_i \in \mathfrak{P}\mathfrak{f}(f)$ and $\varphi_i \in I^{m-s+i-1}(F)$, $1 \leqslant i \leqslant n - m$ such that

$$\psi \equiv \pi_1 \otimes \varphi_1 + \pi_2 \otimes \varphi_2 + \ldots + \pi_{n-m} \otimes \varphi_{n-m} \bmod I^n(F) \ .$$

But $I^n(F) = 0$ and thus $\psi = \sum_{i=1}^{n-m} \pi_i \otimes \varphi_i \in W(F)$. Since $\varphi_i \in I^{m-s+i-1}(F) \subset I^{m-s}(F)$, it follows readily that ψ can be written as a sum of Pfister forms of type $\pi \otimes \varphi$ with $\pi \in \mathfrak{P}\mathfrak{f}(f)$, $\varphi \in BP_{m-s}(F)$. This completes the proof. □

Corollary 11.4. (i) *If $f(X) \notin F[X^2]$, then $W(E/F) = 0$.*

(ii) *If $f(X) \in F[X^2]$, then $W(E/F)$ is additively generated by $\mathfrak{P}\mathfrak{f}(f)$. More precisely, if $0 \neq \psi \in W(E/F)$ is anisotropic, then there exist $\ell \in \mathbb{N}$, $\lambda_i \in F^*$, $\pi_i \in \mathfrak{P}\mathfrak{f}(f)$, $1 \leqslant i \leqslant \ell$, such that*

$$\psi \cong \lambda_1\pi_1 \perp \ldots \perp \lambda_\ell\pi_\ell \ .$$

Proof. In view of 11.3, the only thing that remains to be shown is in part (ii) the fact that any anisotropic $\psi \in W(E/F)$ can be written as orthogonal sum of forms of type $\lambda\pi$ with $\lambda \in F^*$, $\pi \in \mathfrak{P}\mathfrak{f}(f)$.

We keep the notations from the beginning of this section, in particular $\mathfrak{Pf}(f) = \mathfrak{Pf}(b_1, \dots, b_s)$. Our proofs show that for any irreducible polynomial $g \in F[T^2]$ (where T a finite tuple of variables) with $F^2(\widehat{\mathcal{C}}(f)) = F^2(\widehat{\mathcal{C}}(g))$ we have $W(E/F) = W(F(g)/F)$.

Consider the s-fold quasilinear Pfister 2-form $g = \langle\langle b_1, \dots, b_s\rangle\rangle$ which is anisotropic since $[F^2(b_1, \dots, b_s) : F^2] = 2^s$ (Lemma 5.4(a)). Then, by the very definition of g, we have $F^2(\widehat{\mathcal{C}}(g)) = F^2(b_1, \dots, b_s) = F^2(\widehat{\mathcal{C}}(f))$, and thus $W(E/F) = W(F(g)/F)$. But in [L1, Theorem 1.2], it was shown that any anisotropic form $0 \neq \psi \in W(F(g)/F)$ can be written as $\psi \cong \lambda_1\pi_1 \perp \ldots \perp \lambda_\ell\pi_\ell$ with $\lambda_i \in F^*$, $\pi_i \in \mathfrak{Pf}(g) = \mathfrak{Pf}(f)$, which completes the proof. □

Remark 11.5. The determination of $W(F(\varphi)/F)$ for quasilinear 2-forms (i.e., totally singular quadratic forms) in [L1] is very different from our proof for function fields of arbitrary irreducible polynomials in $F[X^2]$. It doesn't use any differential forms but is based on very specific and nice properties of totally singular quadratic forms that do not generalize to arbitrary polynomials.

Combining the above with Knebusch's norm principle [Kn2, Theorem 4.2], we can summarize (still keeping the notations from above):

Corollary 11.6. *Let $f(X) \in F[X]$ be an irreducible polynomial with leading coefficient $a^* \in F^*$ (with respect to the lexicographical ordering of monomials) and let $\psi \neq 0$ be an anisotropic bilinear form over F. Then the following are equivalent.*

(i) *ψ_E is metabolic, i.e. $\psi \in W(E/F)$;*
(ii) *$f(X) \in F[X^2]$ and $\psi \cong \lambda_1\pi_1 \perp \ldots \perp \lambda_\ell\pi_\ell$ for some $\ell \in \mathbb{N}$, $\lambda_i \in F^*$, $\pi_i \in \mathfrak{Pf}(f)$;*
(iii) *$a^* f(X) \in G_{F(X)}(\psi)$.*

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