METABOLIC INVOLUTIONS AND QUADRATIC RADICAL EXTENSIONS

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Abstract. In this paper we characterise involutions that become metabolic over a quadratic field extension attained by adjoining a square root.

1. Introduction

Let K be a field. In [3] we characterised those central simple algebras with involution that become metabolic over a given quadratic separable extension. We showed that a K–algebra with involution (A,σ) becomes metabolic over a quadratic separable extension L/K with non-trivial K–automorphism ι if and only if the algebra with involution (L,ι) maps injectively into the anisotropic part of (A,σ).

In this follow up we investigate the case of an extension L/K of the form $L = K(\sqrt{d})$ with $d \in K\backslash K^2$. These extensions are only distinct from those studied in [3] in characteristic 2, where these extensions are inseparable, and hence not covered by the results of [3]. We show that a K–algebra with involution (A,σ) over K becomes metabolic over $L = K(\sqrt{d})$ if and only if the algebra with involution (L,ι) maps injectively into the anisotropic part of (A,σ), where here ι is the K–linear involution on L defined by $\iota(\sqrt{d}) = -\sqrt{d}$. This is equivalent to the previous characterisation when $\mathrm{char}(K) \neq 2$, otherwise ι is the identity map on L.

Results on this question have already been found for fields of characteristic different from 2 in [1] for involutions of the first kind, and then extended in [2] to involutions of the second kind. For fields of characteristic 2, only the case of symplectic involutions has been investigated, in [8]. Our main results here do not involve any assumption on the characteristic of the field or the type of involution. In particular our result holds when the involution is orthogonal or unitary and the field is of characteristic 2, which are cases that have not been considered before.

2. Algebras with involution and hermitian forms

In this section we recall the basic definitions and results we use on central simple algebras with involution. We refer to [9] for a general reference on central simple algebras.

Throughout, let K be a field and let $\mathrm{char}(K)$ denote its characteristic. Let A be a finite-dimensional K–algebra. If A is simple (i.e. it has no non-trivial two sided ideals) and K' is the centre of A, we can view A as a K'–algebra and, by Wedderburn's Theorem, we have that $A \simeq \mathrm{End}_D(V)$ for a K–division algebra D with centre K' and a right D–vector space V. In this case $\dim_{K'}(A)$ is a square,

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and the positive root of this integer is called the *degree of* A and is denoted $\deg(A)$. The degree of D is called the *index of* A and is denoted $\operatorname{ind}(A)$. We call A *split* if $\operatorname{ind}(A) = 1$. For any field extension L/K we will use the notation $A_L = A \otimes_K L$. We call a field extension L/K a *splitting field of* A if A_L is split. If $K = K'$, then we call the K–algebra A *central simple*.

A K*–involution* on A is a K–linear map $\sigma : A \to A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$ and $\sigma^2 = \mathrm{id}_A$. A K*–algebra with involution* is a pair (A, σ) of a finite-dimensional K–algebra A and a K–involution σ of A such that, with K' being the centre of A, one has $K = \{x \in K' \mid \sigma(x) = x\}$, and such that either A is simple or A is a product of two simple K–algebras that are mapped to each other by σ. In this situation, there are two possibilities: either $K = K'$, so that A is a central simple K–algebra, or K'/K is a quadratic étale extension with σ restricting to the nontrivial K–automorphism of K'. To distinguish these two situations, we speak of involutions of the first or second kind; more precisely, we say that the K–algebra with involution (A, σ) is of the *first kind* if $K' = K$ and of the *second kind* otherwise. For more information on involutions of the second kind, also known as *unitary involutions*, we refer to [7, Section 2.B]. For any field extension L/K we will use the notations $\sigma_L = \sigma \otimes \mathrm{id}_L$ and $(A, \sigma)_L = (A_L, \sigma_L)$.

A *homomorphism of* K*–algebras with involution* is a map $\varphi : (A, \sigma) \to (B, \tau)$, where (A, σ) are (B, τ) K–algebras with involution, such that $\varphi : A \to B$ is an K–algebra homomorphism and $\varphi \circ \sigma = \tau \circ \varphi$: if φ is injective then this is an *embedding*.

We call a K–algebra with involution (A, σ) *isotropic* if there exists an $a \in A \backslash \{0\}$ such that $\sigma(a)a = 0$, and *anisotropic* otherwise. We call an idempotent $e \in A$ *hyperbolic with respect to* σ if $\sigma(e) = 1 - e$. An idempotent $e \in A$ is called *metabolic with respect to* σ if $\sigma(e)e = 0$ and $\dim_{K'} eA = \frac{1}{2}\dim_{K'} A$. Note that, by [3, Corollary 4.3], we may substitute the condition that $\dim_{K'} eA = \frac{1}{2}\dim_{K'} A$ for the condition that $(1 - e)(1 - \sigma(e)) = 0$ in the definition of metabolic. An algebra with involution (A, σ) is called *hyperbolic* (resp. *metabolic*) if A contains a hyperbolic (resp. metabolic) idempotent with respect to σ.

Let (A, σ) be an K–algebra with involution and K' be the centre of A. For $\lambda \in K'$, let $\mathrm{Sym}_\lambda(A, \sigma) = \{a \in A \mid \lambda\sigma(a) = a\}$ and $\mathrm{Alt}(A, \sigma) = \{\sigma(a) - a \mid a \in A\}$.

Throughout the rest of this section, let (D, θ) be a K–division algebra with involution and K' be the centre of D. Further, fix $\lambda \in K'$ such that $\lambda\theta(\lambda) = 1$. A λ*–hermitian form* over (D, θ) is a pair (V, h) where V is a finite-dimensional right D–vector space and h is a non-degenerate bi-additive map $h : V \times V \to D$ such that

$$h(x, yd) = h(x, y)d \quad \text{ and } \quad h(y, x) = \lambda\theta(h(x, y))$$

holds for all $x, y \in V$ and $d \in D$.

A *isometry of* λ*–hermitian forms over* (D, θ) is a bijective map $\phi : (V, h) \to (W, h')$, where (V, h) and (W, h') are λ–hermitian forms, such that $\phi : V \to W$ is D–linear and $h(v, w) = h'(\phi(v), \phi(w))$ for all $v, w \in V$. If such a isometry exists, we say (V, h) are (W, h') are *isometric as* λ*–hermitian forms* and we write $(V, h) \simeq (W, h')$. We denote the *orthogonal sum* of λ–hermitian forms (V, h) and (W, h') over (D, θ) by $(V, h) \perp (W, h')$. If $(D, \theta) = (F, \mathrm{id}_F)$ then we must have that $\lambda = \pm 1$ and in this case we call a λ–hermitian form over (D, θ) a λ*–bilinear form*, which we call *symmetric* if $\lambda = 1$.

Let V be a finite dimensional right D–vector space and let $V^* = \mathrm{End}_D(V, D)$, the dual of V. We define a K–bilinear map $h_\lambda : (V^* \oplus V) \times (V^* \oplus V) \to D$ by

$$h_\lambda(\varphi + x, \psi + y) = \varphi(y) + \lambda\theta(\psi(x)) \quad \text{for } \varphi, \psi \in V^* \text{ and } x, y \in V.$$

Then $\mathbb{H}_\lambda(V) = (V^* \oplus V, h_\lambda)$ is a regular λ–hermitian form over (D, θ). We call a λ-hermitian form over (D, θ) *hyperbolic* if it is isometric to $\mathbb{H}_\lambda(V)$ for some right D–vector space V.

Let $S \subset V$. We define the *orthogonal complement* $S^\perp$ of S with respect to (V, h) as $S^\perp = \{x \in V \mid h(x, s) = 0 \text{ for all } s \in S\}$. A hermitian space (V, h) is called *metabolic* if there exists a subspace $S \subset V$ such that $S = S^\perp$.

Proposition 2.1. *Let (V, h) be a λ–hermitian form over (D, θ). There exists a K–involution σ on $\mathrm{End}_D(V)$ such that*

$$h(f(x), y) = h(x, \sigma(f)(y)) \text{ for all } x, y \in V \text{ and } f \in A.$$

This involution σ is uniquely determined by h. Further, $(\mathrm{End}_D(V), \sigma)$ is a K–algebra with involution of the same kind as (D, θ).

Proof. See, for example, [7, (4.1)]. □

In the situation of (2.1), we call σ the *adjoint involution to h* and denote it by ad_h, and we further write $\mathrm{Ad}(V, h)$ for the K-algebra with involution $(\mathrm{End}_D(V), \mathrm{ad}_h)$.

A K–algebra with involution (A, σ) of the first kind is said to be *symplectic* if $(A, \sigma)_L \simeq \mathrm{Ad}(V, b)$, where (V, b) is a λ–bilinear form such that $b(x, x) = 0$ for all $x \in V$, and *orthogonal* otherwise. This definition is independent of the choice of the splitting field L (see [7, Section 2.A]).

Proposition 2.2. *Let (A, σ) be a K-algebra with symplectic or unitary involution. Then (A, σ) is metabolic if and only if it is hyperbolic.*

Proof. See [2, (A.3)]. □

Proposition 2.3. *A λ–hermitian form (V, h) over (D, θ) is hyperbolic (resp. metabolic) if and only if $\mathrm{Ad}(V, h)$ is hyperbolic (resp. metabolic).*

Proof. See [7, (6.7)] for the statement on hyperbolicity, [3, (4.8)] for the statement on metabolicity in the case of involutions of the first kind and [4] for involutions of the second kind. □

Proposition 2.4. *A λ–hermitian form (V, h) over (D, θ) has a a decomposition $(V, h) \simeq (U, h') \perp (W, b)$, where (U, h') is an anisotropic λ–hermitian form and (W, b) is a metabolic λ–hermitian form, both over (D, θ). Moreover (U, h') is unique up to isometry.*

Proof. See [6, Chap. 1 (6.1.1)] and [6, Chap. 1 (6.1.4)]. □

In the situation of (2.4) we call (U, h') the *anisotropic part of* (V, h), denoted $(V, h)_{\mathrm{an}}$.

Proposition 2.5. *Let (V, h) be a λ–hermitian form over (D, θ), and let (W, b) be a λ'–hermitian form over (D, θ'), for some involution θ' on D and $\lambda' \in K'$. If $\mathrm{Ad}(V, h) \simeq \mathrm{Ad}(W, b)$ then $\mathrm{Ad}((V, h)_{\mathrm{an}}) \simeq \mathrm{Ad}((W, b)_{\mathrm{an}})$.*

Proof. See [3, (3.6)] for the statement for the case of involutions of the first kind. It is explained in [4] that the same argument holds for involutions of the second kind. □

Hence, we call a K–algebra with involution (B,τ) the *anisotropic part of a K–algebra with involution* (A,σ), if $(B,\tau)\simeq \mathrm{Ad}((V,h)_{\mathrm{an}})$ for a λ–hermitian space (V,h) over a K–division algebra with involution such that $\mathrm{Ad}(V,h)\simeq(A,\sigma)$.

3. Extension by a square root

We now investigate the effect on a K–algebra with involution of passing to a quadratic field extension given by adjoining a square root to the ground field. Note that this is an inseparable quadratic extension when the field is of characteristic 2, and a separable quadratic extension otherwise. While it would suffice to work in characteristic 2 (since otherwise the behaviour is known), we give this result in arbitrary characteristic to show the parallelism.

First, we fix some notation for this section. Let K be a field of arbitrary characteristic. Let (A,σ) be a K–algebra with involution of any kind, and let L/K be a field extension given by $L=K(\sqrt{d})$ for some $d\in K\backslash K^2$. Let ι be the involution on L such that $\iota|_K=\mathrm{id}_K$ and $\iota(\sqrt{d})=-\sqrt{d}$. The pair (L,ι) is an L–algebra with trivial involution if $\mathrm{char}\,K=2$, and a K–algebra with involution otherwise.

Proposition 3.1. *Assume there exists a K–linear embedding $(L,\iota)\hookrightarrow(A,\sigma)$. Then $(A,\sigma)_L$ is metabolic.*

Proof. The embedding $(L,\iota)\hookrightarrow(A,\sigma)$ gives an element $r\in\mathrm{Sym}_{-1}(A,\sigma)$ such that $r^2\in K^\times$. By [8, (6.3)] every element in the centraliser of r in A is of the form $ry+yr$ for some $y\in A$. In particular, $r=rx+xr$ for some $x\in A$.

We have that $A_L=A\oplus A\sqrt{d}$. Let $e=1-x+\frac{xr}{d}\sqrt{d}\in A_L$ with $r,x\in A$ as above. Then, since $rx=r-xr$, we have

$$\begin{aligned} e^2 &= \left(1-x+\frac{xr}{d}\sqrt{d}\right)\left(1-x+\frac{xr}{d}\sqrt{d}\right)\\ &= 1-2x+2\frac{xr}{d}\sqrt{d}+x^2-\frac{x^2r}{d}\sqrt{d}-\frac{xrx}{d}\sqrt{d}+\frac{xrxr}{d}. \end{aligned}$$

Applying $xrx=xr-x^2r$ and $xrxr=xd-x^2d$ gives

$$\begin{aligned} e^2 &= 1-2x+2\frac{xr}{d}\sqrt{d}+x^2-\frac{x^2r}{d}\sqrt{d}+\left(\frac{x^2r}{d}\sqrt{d}-\frac{xr}{d}\sqrt{d}\right)+(x-x^2)\\ &= 1-x+\frac{xr}{d}\sqrt{d}=e. \end{aligned}$$

So e is an idempotent in A_L. We also have

$$\begin{aligned} \sigma_L(e)e &= \left(1-\sigma(x)-\frac{r\sigma(x)}{d}\sqrt{d}\right)\left(1-x+\frac{xr}{d}\sqrt{d}\right)\\ &= 1-x+\frac{xr}{d}\sqrt{d}-\sigma(x)+\sigma(x)x-\frac{\sigma(x)xr}{d}\sqrt{d}-\frac{r\sigma(x)}{d}\sqrt{d}\\ &\quad +\frac{r\sigma(x)x}{d}\sqrt{d}-\frac{r\sigma(x)xr}{d}. \end{aligned}$$

From $rx=r-xr$ we see that $r\sigma(x)=r-\sigma(x)r$. Therefore

$$r\sigma(x)x=r-xr-\sigma(x)r+\sigma(x)xr \quad\text{and}\quad r\sigma(x)xr=d-xd-\sigma(x)d+\sigma(x)xd.$$

Substituting these into the above equation gives

$$\sigma_L(e)e=\frac{r}{d}\sqrt{d}-\frac{r\sigma(x)}{d}\sqrt{d}-\frac{\sigma(x)r}{d}\sqrt{d}=0.$$

Finally we have

$$\begin{aligned}(1-e)(1-\sigma_L(e)) &= \left(x-\frac{xr}{d}\sqrt{d}\right)\left(\sigma(x)+\frac{r\sigma(x)}{d}\sqrt{d}\right)\\ &= x\sigma(x)+\frac{xr\sigma(x)}{d}\sqrt{d}-\frac{xr\sigma(x)}{d}\sqrt{d}-\frac{xr^2\sigma(x)}{d}=0.\end{aligned}$$

Hence $e \in A_L$ is a metabolic idempotent with respect to σ_L. □

Note that over fields of characteristic different from 2, one can take $x = \frac{1}{2}$ in the above proof, which simplifies the calculation a great deal.

For a right (resp. left) ideal $I \subset A$ we denote by I^0 the left (resp. right) annihilator ideal, that is,

$$I^0 = \{a \in A \mid ax = 0 \text{ for all } x \in I\}$$

$$(\text{resp. } I^0 = \{a \in A \mid xa = 0 \text{ for all } x \in I\}).$$

Note that we have that $(I^0)^0 = I$ for all ideals $I \subseteq A$ (see [7, (1.14)]).

Lemma 3.2. *Let (A, σ) be a K–algebra with involution and $e \in A$ be a metabolic idempotent. Then $A\sigma(e) = (eA)^0$ and $eA = (A\sigma(e))^0$.*

Proof. That $A\sigma(e) = (eA)^0$ is shown in [3, (4.12)] for the case of involutions of the first kind and it is noted in [4] that the argument holds for involutions of the second kind. That $eA = (A\sigma(e))^0$ then follows immediately. □

Lemma 3.3. *Assume (A, σ) is anisotropic and $(A, \sigma)_L$ is metabolic and let $e \in A_L$ be a metabolic idempotent with respect to σ_L. Then there exists a K–linear map $\epsilon : A_L \to A$ defined by*

$$\sigma_L(e)(x - \epsilon(x) \otimes 1) = 0$$

for all $x \in A_L$. Moreover the map $\epsilon|_L : L \to A$ is an injective K–algebra homomorphism.

Proof. See [3, (5.3)] for the case of involutions of the first kind. The argument for unitary involutions can be found in the proof of [2, (A.9)]. □

Proposition 3.4. *Assume (A, σ) is anisotropic. Then $(A, \sigma)_L$ is metabolic if and only if there exists an embedding $(L, \iota) \hookrightarrow (A, \sigma)$.*

Proof. If such an embedding exists, then $(A, \sigma)_L$ is metabolic by (3.1).

Let $e \in A_L$ be a metabolic idempotent with respect to σ_L, and let $\epsilon|_L$ be the embedding $L \hookrightarrow A$ associated with e given in (3.3). Let $r = \epsilon(\sqrt{d})$. Clearly we have that $r^2 = d$. We now show that $r \in \mathrm{Sym}_{-1}(A, \sigma)$.

By our choice of r, we have that

$$\sigma_L(e)(1 \otimes \sqrt{d} - r \otimes 1) = 0,$$

hence

$$(1 \otimes \sqrt{d} - r \otimes 1) \in (A_L\sigma_L(e))^0 = eA_L$$

by (3.2). Applying σ to the above expression we see that

$$(1 \otimes \sqrt{d} - \sigma(r) \otimes 1)e = 0,$$

and hence

$$(1 \otimes \sqrt{d} - \sigma(r) \otimes 1) \in (eA_L)^0 = A_L\sigma_L(e)$$

by (3.2). Since $(A_L\sigma_L(e))^0 = eA_L$, we have that

$$\begin{aligned} 0 &= (1\otimes\sqrt{d} - \sigma(r)\otimes 1)(1\otimes\sqrt{d} - r\otimes 1)\\ &= (d+\sigma(r)r)\otimes 1 - (\sigma(r)+r)\otimes\sqrt{d}. \end{aligned}$$

Hence, $\sigma(r)r = -d$ and $\sigma(r) = -r$. □

Theorem 3.5. *Let (A,σ) be a K-algebra with involution. Then $(A,\sigma)_L$ is metabolic if and only if either (A,σ) is metabolic or there exists an embedding $(L,\iota)\hookrightarrow (A,\sigma)_{\mathrm{an}}$.*

Proof. It is clear that $(A,\sigma)_L$ is metabolic if (A,σ) is metabolic. If there exists an embedding $(L,\iota)\hookrightarrow (A,\sigma)_{\mathrm{an}}$, then $(A,\sigma)_L$ is metabolic by (3.1).

Assume now that $(A,\sigma)_L$ is metabolic. Then $(V,h)_L$ is metabolic for any (V,h) such that $\mathrm{Ad}(V,h)\simeq (A,\sigma)$ by (2.3). Therefore $((V,h)_{\mathrm{an}})_L$ is metabolic for any such (V,h), and hence $((A,\sigma)_{\mathrm{an}})_L$ is metabolic by (2.3). Hence there exists an embedding $(L,\iota)\hookrightarrow (A,\sigma)_{\mathrm{an}}$ by (3.4). □

The following corollary, well known over fields of characteristic different from 2 (see, for example, [1, (3.1)]), seems to be a new result for fields of characteristic 2.

Corollary 3.6. *Let $L = K(\sqrt{d})$ for some $d\in K^2\backslash K$ and let (V,b) be an anisotropic symmetric bilinear form over K. Then $(V,b)_K$ is metabolic if and only if there exists an $f\in \mathrm{End}_K(V)$ such that $f^2 = d$ and $b(f(x),y) = -b(x,f(y))$ for all $x,y\in V$.*

Another characterisation of symmetric bilinear forms that become metabolic over such an extension, in terms of an ideal in the Witt ring, can be found in [5].

4. Hyperbolic Involutions

In this section we discuss a strengthening of our result that can be achieved in the case of a symplectic or unitary involution. Recall that in these cases, an algebra with involution is metabolic if and only if it is hyperbolic. These results are known for fields of characteristic different from 2. The approach here is characteristic free. As in the last section, let L/K be a field extension given by $L = K(\sqrt{d})$ for some $d\in K\backslash K^2$ and let ι be the involution on L with $\iota|_K = \mathrm{id}_K$ and $\iota(\sqrt{d}) = -\sqrt{d}$ throughout. We denote the K–algebra of $n\times n$ matrices over K by $M_n(K)$.

Lemma 4.1. *Let γ be a symplectic K–involution on $M_2(K)$. Then there is an embedding $(L,\iota)\hookrightarrow (M_2(K),\gamma)$*

Proof. By [7, (2.21)] there exists only one symplectic involution on $M_2(K)$, therefore we may assume that γ is given by

$$\gamma\begin{pmatrix} x_1 & x_2\\ x_3 & x_4\end{pmatrix} = \begin{pmatrix} x_4 & -x_2\\ -x_3 & x_1\end{pmatrix}$$

for $x_1,x_2,x_3,x_4\in K$. Consider the element $r = \begin{pmatrix} 0 & d\\ 1 & 0\end{pmatrix}\in M_2(K)$. We have that $r^2 = d$ and $\gamma(r) = -r$, hence we have the required embedding. □

Let K be a field and V a K–vector space. Let γ be an involution on K and $\lambda = \pm 1$. We denote by $\Theta_{\lambda,\gamma}$ the involution on $\mathrm{End}_K(V)\simeq M_2(K)$ given by

$$\Theta_{\lambda,\gamma}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} \gamma(d) & \lambda\gamma(b)\\ \lambda\gamma(c) & \gamma(a)\end{pmatrix}.$$

Proposition 4.2. *Let (A,σ) be a symplectic or unitary K–algebra with involution such that $\sigma(x)=\gamma(x)$ for all $x\in K$. Then (A,σ) is hyperbolic if and only if there exists a σ-invariant subalgebra M of A such that*

$$(M,\sigma|_M)\simeq(M_2(K),\Theta_{-1,\gamma}).$$

Proof. The case of $\mathrm{char}(K)\neq 2$ is covered in [1, (2.2)], and the proof there works for the case $\mathrm{char}(K)=2$ with no change in the argument. □

Proposition 4.3. *Let (A,σ) be a symplectic or unitary K–algebra with involution. If (A,σ) is hyperbolic, then there exists an embedding $(L,\iota)\hookrightarrow(A,\sigma)$.*

Proof. We have by (4.2) and the Double Centralizer Theorem (see [9, Section 12.7]) that $A\simeq M_2(K)\otimes_K A'$, for some central simple K–algebra A' that has involutions of the same kind as (A,σ). Let (A',σ') be a K–algebra with involution of orthogonal type if (A,σ) is symplectic, and of unitary type if (A,σ) is of unitary type, with $\sigma|_K=\sigma'|_K$.

Let γ be a symplectic involution on $M_2(K)$. This involution is the unique symplectic involution on $M_2(K)$ by [7, (2.21)]. Since $(M_2(F),\gamma)$ is split and symplectic, it is hyperbolic by [3, (4.7)]. It follows that the involution $(M_2(K)\otimes_K A',\gamma\otimes\sigma')$ is hyperbolic. Since all hyperbolic hermitian forms of a given dimension are isometric, it follows that $(A,\sigma)\simeq(M_2(K)\otimes_K A',\gamma\otimes\sigma')$ by [7, (4.2)]. By (4.1), there exists an embedding $(L,\iota)\hookrightarrow(M_2(F),\gamma)$, and hence also an embedding $(L,\iota)\hookrightarrow(A,\sigma)$. □

Theorem 4.4. *Let (A,σ) be a symplectic or unitary K–algebra with involution. Then $(A,\sigma)_L$ is hyperbolic if and only if there exists an embedding $(L,\iota)\hookrightarrow(A,\sigma)$.*

Proof. If such an embedding exists, then $(A,\sigma)_L$ is metabolic by (3.1). That $(A,\sigma)_L$ is hyperbolic follows from (2.2). The result then follows from (3.5), (4.3) and an argument as in [2, (1.7)]. □

A version of this theorem for fields of characteristic different from 2 was already shown in [1, (3.3)], where the case of orthogonal involutions was also covered. The above approach does not work generally in the case of an orthogonal involution over a field of characteristic 2 since (L,ι) does not inject into every metabolic orthogonal algebra with involution, as the following example shows.

Example 4.5. Assume that $\mathrm{char}(K)=2$ and for $a\in K^\times$ take $(V,b)\simeq(K^2,\langle a,a\rangle)$, that is,

$$b:K^2\times K^2\to K,\qquad (x,y)\mapsto x^t\begin{pmatrix}a&0\\0&a\end{pmatrix}y.$$

Then there does not exist an $f\in\mathrm{End}_K(V)$ such that $f^2=d$ for some $d\in K\backslash K^2$ and $b(f(x),y)=b(x,f(y))$ for all $x,y\in V$, as we now show.

The adjoint involution to (V,b) is $(\mathrm{End}_K(V),\sigma)\simeq(M_2(K),t)$, where t is the transposition involution. Assume there exists an $f\in M_2(K)$ such that $f^2=\begin{pmatrix}d&0\\0&d\end{pmatrix}$ for $d\in K\backslash K^2$ and $b(f(x),y)=b(x,f(y))$ for all $x,y\in V$. Then, as $b(f(x),y)=b(x,f(y))$, we have $t(f)=f$, so $f=\begin{pmatrix}a_{11}&a_{12}\\a_{12}&a_{22}\end{pmatrix}$ for some $a_{ij}\in K$. Consider now the top right hand entry of f^2, that is $d=a_{11}^2+a_{12}^2\in K^2$. This contradicts the assumption that $d\notin K^2$.

Note that one easily adapt this argument to show that (L,ι) does not embedded into $\mathrm{Ad}(K^{2n},n\times\langle a,a\rangle)$ for $a\in K^\times$.

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