COHOMOLOGICAL INVARIANTS OF ALGEBRAIC TORI

S. BLINSTEIN AND A. MERKURJEV

ABSTRACT. Let G be an algebraic group over a field F. As defined by Serre, a cohomological invariant of G of degree n with values in $\mathbb{Q}/\mathbb{Z}(j)$ is a functorial in K collection of maps of sets $H^1(K,G) \longrightarrow H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ for all field extensions K/F. We study the group of degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$. In particular, we compute the group $H^3_{\mathrm{nr}}\big(F(S), \mathbb{Q}/\mathbb{Z}(2)\big)$ of unramified cohomology of an algebraic torus S.

1. Introduction

Let G be a (linear) algebraic group. The notion of an *invariant* of G was defined in [16] as follows. Fix a field F (of arbitrary characteristic) and consider the category Fields_F of field extensions of F. Consider the functor

$$\mathrm{Tors}_G : \mathit{Fields}_F \longrightarrow \mathit{Sets}$$

taking a field K to the set $H^1(K,G)$ of isomorphism classes of G-torsors over $\operatorname{Spec} K$. Let

$$H : \mathit{Fields}_F \longrightarrow \mathit{Abelian\ Groups}$$

be another functor. An *H-invariant* of G is then a morphism of functors

$$i : \mathrm{Tors}_G \longrightarrow H,$$

viewing H with values in Sets, i.e., a functorial in K collection of maps of sets $H^1(K,G) \longrightarrow H(K)$ for all field extensions K/F. We denote the group of H-invariants of G by $\mathrm{Inv}(G,H)$.

An invariant $i \in \mathrm{Inv}(G,H)$ is called *normalized* if $i(I) = 0$ for the trivial G-torsor I. The normalized invariants form a subgroup $\mathrm{Inv}(G,H)_{\mathrm{norm}}$ of $\mathrm{Inv}(G,H)$ and there is a natural isomorphism

$$\mathrm{Inv}(G,H) \simeq H(F) \oplus \mathrm{Inv}(G,H)_{\mathrm{norm}},$$

so it is sufficient to study normalized invariants.

Typically, H is a cohomological functor given by Galois cohomology groups with values in a fixed Galois module. Of particular interest to us is the functor H which takes a field K/F to the Galois cohomology group $H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$, where the coefficients $\mathbb{Q}/\mathbb{Z}(j)$ are defined as follows. For a prime integer p different from $\mathrm{char}(F)$, the p-component $\mathbb{Q}_p/\mathbb{Z}_p(j)$ of $\mathbb{Q}/\mathbb{Z}(j)$ is the colimit over n of the étale sheaves $\mu_{p^n}^{\otimes j}$, where μ_m is the sheaf of m^{th} roots of unity. In

Key words and phrases. Algebraic tori, cohomological invariants, Galois cohomology *2000 Mathematical Subject Classifications: 12G05, 11E72* .

The work of the second author has been supported by the NSF grant DMS #0652316.

the case $p = \mathrm{char}(F) > 0$, $\mathbb{Q}_p/\mathbb{Z}_p(j)$ is defined via logarithmic de Rham-Witt differentials (see §3b).

We write $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the group of *cohomological invariants of G of degree n with values in $\mathbb{Q}/\mathbb{Z}(j)$*.

The second cohomology group $H^2\big(K, \mathbb{Q}/\mathbb{Z}(1)\big)$ is canonically isomorphic to the Brauer group $\mathrm{Br}(K)$ of the field K. In §2c we prove (Theorem 2.4) that if G is a connected group (reductive if F is not perfect), then $\mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} \simeq \mathrm{Pic}(G)$. The group $\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}$ for a semisimple simply connected group G has been studied by Rost (see [16]).

An essential object in the study of cohomological invariants is the notion of a *classifying torsor*: a G-torsor $E \longrightarrow X$ for some variety X over F such that every G-torsor over an infinite field K/F is isomorphic to the pull-back of $E \longrightarrow X$ along a K-point of X. If V is generically free representation of G with a nonempty open subset $U \subset V$ such that there is a G-torsor $\pi : U \longrightarrow X$, then π is classifying. The generic fiber of π is the *generic torsor* over $\mathrm{Spec}\, F(X)$. Evaluation at the generic torsor yields a homomorphism

$$\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big),$$

and in §3 we show that the image of this map is contained in the subgroup $H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ of $H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big)$, where $\mathcal{H}^n\big(\mathbb{Q}/\mathbb{Z}(j)\big)$ is the Zariski sheaf associated to the presheaf $W \mapsto H^n\big(W, \mathbb{Q}/\mathbb{Z}(j)\big)$ of the étale cohomology groups. In fact, the image is contained in the subgroup $H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}}$ of *balanced* elements, i.e., elements that have the same images under the pull-back homomorphisms with respect to the two projections $(U \times U)/G \longrightarrow X$. Moreover, the balanced elements precisely describe the image and we prove (Theorem 3.3):

Theorem. *Let G be an algebraic group over a field F. We assume that G is connected if F is a finite field. Then there is a natural isomorphism* $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}}$.

At this point it is convenient to make use of a construction due to Totaro [33]: because the Chow groups are homotopy invariant, the groups $\mathrm{CH}^n(X)$ do not depend on the choice of the representation V and the open set $U \subset V$ provided the codimension of $V \setminus U$ in V is large enough. This leads to the notation $\mathrm{CH}^n(BG)$, the Chow groups of the so-called *classifying space* BG, although BG itself is not defined in this paper.

Unfortunately, the étale cohomology groups with values in $\mathbb{Q}_p/\mathbb{Z}_p(j)$, where $p = \mathrm{char}(F) > 0$, are not homotopy invariant. In particular, we cannot use Rost's theory of cycle modules of Rost [31].

The main result of this paper is the exact sequence in Theorem 4.4 describing degree 3 cohomological invariants of an algebraic torus T. Writing $\widehat{T}_{\mathrm{sep}}$ for the character lattice of T over a separable closure of F and T° for the dual torus, we prove:

Theorem. *Let T be an algebraic torus a field F. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(BT)_{\mathrm{tors}} \longrightarrow H^1(F, T^0) \xrightarrow{\alpha} \mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \longrightarrow H^0\big(F, S^2(\widehat{T}_{\mathrm{sep}})\big)/\,\mathrm{Dec} \longrightarrow H^2(F, T^0).$$

The homomorphism α is given by $\alpha(a)(b) = a_K \cup b$ for every $a \in H^1(F, T^0)$ and $b \in H^1(K, T)$ and every field extension K/F, where the cup-product is defined in (4.5), and Dec *is the subgroup of decomposable elements in the symmetric square $S^2(\widehat{T}_{\mathrm{sep}})$ defined in Appendix A-II.*

We prove that the torsion group $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ is finite of exponent 2 and the last homomorphism in the sequence is also of exponent 2. Moreover, if p is an odd prime, the group $\mathrm{Inv}^3\big(T, \mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{norm}}$, which is the p-primary component of $\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}$, splits canonically into the direct sum of *linear* invariants (those that induce group homomorphisms from Tors_T to H^3) and *quadratic* invariants, i.e., the invariants i such that the function $h(a, b) := i(a+b)-i(a)-i(b)$ is bilinear and $h(a, a) = 2i(a)$ for all a and b. Furthermore, the groups of linear and quadratic invariants with values in $\mathbb{Q}_p/\mathbb{Z}_p(2)$ are canonically isomorphic to $H^1(F, T^\circ)\{p\}$ and $H^0\big(F, S^2(\widehat{T}_{\mathrm{sep}})\big)/\,\mathrm{Dec}\big)\{p\}$, respectively.

We also prove (Theorem 4.9) that the degree 3 invariants have control over the structure of all invariants. In particular, the group $\mathrm{Inv}^3(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is trivial for all K/F if and only if T is *universally special*, i.e., T has no nontrivial torsors over any field K/F, which in particular means T has no nonconstant H-invariants for every functor H.

Our motivation for considering invariants of tori comes from their connection with unramified cohomology (defined in §5). Specifically, this work began as an investigation of a problem posed by Colliot-Thélène in [5, p. 39]: for n prime to $\mathrm{char}(F)$ and $i \geq 0$, determine the unramified cohomology group $H^i_{\mathrm{nr}}\big(F(S), \mu_n^{\otimes(i-1)}\big)$, where $F(S)$ is the function field of a torus S over F. The connection is provided by Theorem 5.6 where we show that the unramified cohomology of a torus S is calculated by the invariants of an auxiliary torus:

Theorem. *Let S be a torus over F and let $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ be a flasque resolution of S, i.e., T is flasque and P is quasi-split. Then there is a natural isomorphism*

$$H^n_{\mathrm{nr}}\big(F(S), \mathbb{Q}/\mathbb{Z}(j)\big) \simeq \mathrm{Inv}^n\big(T, \mathbb{Q}/\mathbb{Z}(j)\big).$$

In the present paper, F denotes a field of arbitrary characteristic, F_{sep} a separable closure of F, and Γ the absolute Galois group $\mathrm{Gal}(F_{\mathrm{sep}}/F)$ of F. The word "scheme" over a field F means a separated scheme over F and a "variety" over F is an integral scheme of finite type over F. If X is a scheme over F and L/F is a field extension then we write X_L for $X \times_F \operatorname{Spec} L$. When $L = F_{\mathrm{sep}}$ we write simply X_{sep}.

Acknowledgements: The authors would like to thank Jean-Louis Colliot-Thélène, Skip Garibaldi, Thomas Geisser, David Harari, Bruno Kahn, Kazuya Kato, and Alena Pirutka for helpful discussions.

2. Invariants of algebraic groups

2a. **Definitions and basic properties.** Let G be a (smooth linear) algebraic group over a field F. Consider the functor

$$\mathrm{Tors}_G : \textsf{\textit{Fields}}_F \longrightarrow \textsf{\textit{Sets}}$$

from the category of field extensions of F to the category of sets taking a field K to the set $H^1(K,G) := H^1\big(\mathrm{Gal}(K_{\mathrm{sep}}/K), G(K_{\mathrm{sep}})\big)$ of isomorphism classes of G-torsors over $\operatorname{Spec} K$.

Let $H : \textsf{\textit{Fields}}_F \longrightarrow \textsf{\textit{Abelian Groups}}$ be a functor. We also view H as a functor with values in $\textsf{\textit{Sets}}$. Following [16], we define an *H-invariant of G* as a morphism of functors $\mathrm{Tors}_G \longrightarrow H$ from the category $\textsf{\textit{Fields}}_F$ to $\textsf{\textit{Sets}}$. All the H-invariants of G form the abelian group of invariants $\mathrm{Inv}(G,H)$.

An invariant $i \in \mathrm{Inv}(G,H)$ is called *constant* if there is an element $h \in H(F)$ such that $i(I) = h_K$ for every G-torsor $I \longrightarrow \operatorname{Spec} K$, where h_K is the image of h under natural map $H(F) \longrightarrow H(K)$. The constant invariants form a subgroup $\mathrm{Inv}(G,H)_{\mathrm{const}}$ of $\mathrm{Inv}(G,H)$ isomorphic to $H(F)$. An invariant $i \in \mathrm{Inv}(G,H)$ is called *normalized*, if $i(I) = 0$ for the trivial G-torsor I. The normalized invariants form a subgroup $\mathrm{Inv}(G,H)_{\mathrm{norm}}$ of $\mathrm{Inv}(G,H)$ and we have the decomposition

$$\mathrm{Inv}(G,H) = \mathrm{Inv}(G,H)_{\mathrm{const}} \oplus \mathrm{Inv}(G,H)_{\mathrm{norm}} \simeq H(F) \oplus \mathrm{Inv}(G,H)_{\mathrm{norm}},$$

so it suffices to determine the normalized invariants.

2b. **Classifying torsors.** Let G be an algebraic group over a field F. A G-torsor $E \longrightarrow X$ over a variety X over F is called *classifying* if for every field extension K/F, with K infinite, and for every G-torsor $I \longrightarrow \operatorname{Spec} K$, there is a point $x : \operatorname{Spec} K \longrightarrow X$ such that the torsor I is isomorphic to the fiber $E(x)$ of $E \longrightarrow X$ over x, i.e., $I \simeq E(x) := x^*(E) = \operatorname{Spec}(K) \times_X E$. The generic fiber $E_{\mathrm{gen}} \longrightarrow \operatorname{Spec} F(X)$ of a classifying torsor is called a *generic G-torsor* (see [16, Part 1, §5.3]).

If V is generically free representation of G with a nonempty open subset $U \subset V$ such that there is a G-torsor $\pi : U \longrightarrow X$, then π is classifying (see [16, Part 1, §5.4]). We will write U/G for X and call π a *standard classifying G-torsor*. Standard classifying G-torsors exist: we can embed G into $U := \mathbf{GL}_{n,F}$ for some n as a closed subgroup. Then U is an open subset in the affine space $M_n(F)$ on which G acts linearly. Note that $U(F) \neq \emptyset$.

We say that a G-variety Y is *G-rational* if there is an affine space V with a linear G-action such that Y and V have G-isomorphic nonempty open G-invariant subvarieties. Note that if $U \longrightarrow U/G$ is a standard classifying G-torsor, then U is a G-rational variety.

Let $E \longrightarrow X$ be a classifying G-torsor and let $H : \mathsf{Fields}_F \longrightarrow \mathsf{Abelian\ Groups}$ be a functor. Define the map

$$\begin{aligned} \theta : \operatorname{Inv}(G, H) &\longrightarrow H\big(F(X)\big) \\ i &\longmapsto i(E_{\mathrm{gen}}), \end{aligned} \tag{2.1}$$

by sending an invariant to its value at the generic torsor E_{gen}.

Consider the following property of the functor H:

Property 2.1. *The map $H(K) \longrightarrow H\big(K(\!(t)\!)\big)$ is injective for any field extension K/F.*

The following theorem, due to M. Rost, was proved in [16, Part II, Th. 3.3]. For completeness, we give a slightly modified proof in Appendix A-I.

Theorem 2.2. *Let G be an algebraic group over F. If a functor $H : \mathsf{Fields}_F \longrightarrow \mathsf{Abelian\ Groups}$ has Property 2.1, then the map θ is injective, i.e., every H-invariant of G is determined by its value at the generic G-torsor.*

2c. **The Brauer group invariants.** Let G be a connected algebraic group over F. Every cohomological invariant of G of degree 1 is constant by [24, Prop. 31.15]. In this section we study (degree 2) Br-invariants for the Brauer group functor $K \longmapsto \operatorname{Br}(K)$. We assume that G is reductive if F is not perfect as required by the auxiliary result [32, Prop. 6.10].

Lemma 2.3. *For any field extension K/F such that F is algebraically closed in K, the natural map $\operatorname{Pic}(G) \longrightarrow \operatorname{Pic}(G_K)$ is an isomorphism.*

Proof. We may assume that G is reductive by factoring out the unipotent radical in the case that F is perfect. There is an exact sequence (see [8, Th. 1.2])

$$1 \longrightarrow C \longrightarrow G' \longrightarrow G \longrightarrow 1$$

with C a group of multiplicative type and G' a reductive group with $\operatorname{Pic}(G'_L) = 0$ for any field extension L/F. Let T be the factor group of G' by the semisimple part. The result follows from the exact sequence [32, Prop. 6.10]

$$\widehat{T}(L) \longrightarrow \widehat{C}(L) \longrightarrow \operatorname{Pic}(G_L) \longrightarrow \operatorname{Pic}(G'_L) = 0$$

with $L = F$ and K since the groups $\widehat{T}(F)$ and $\widehat{C}(F)$ don't change when F is replaced by K. $\square$

Since for any G-torsor $E \longrightarrow \operatorname{Spec}(K)$ over a field extension K/F one has [32, Prop. 6.10] the exact sequence

$$\operatorname{Pic}(E) \longrightarrow \operatorname{Pic}(G_K) \xrightarrow{\delta} \operatorname{Br}(K) \longrightarrow \operatorname{Br}(E), \tag{2.2}$$

we obtain the homomorphism

$$\nu : \operatorname{Pic}(G) \longrightarrow \operatorname{Inv}(G, \operatorname{Br})_{\mathrm{norm}}$$

which takes an element $\alpha \in \operatorname{Pic}(G)$ to the invariant that sends a G-torsor E over a field extension K/F to $\delta(\alpha_K)$.

Theorem 2.4. *Let G be a connected algebraic group over F. Assume that G is reductive if F is not perfect. Then the map $\nu : \mathrm{Pic}(G) \longrightarrow \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}}$ is an isomorphism.*

Proof. Choose a standard classifying G-torsor $U \longrightarrow U/G$. Write K for the function field $F(U/G)$ and let U_{gen} be the generic G-torsor over K. Consider the commutative diagram

$$\begin{tikzcd} & \mathrm{Pic}(G) \arrow[r, "\nu"] \arrow[d, "j"] & \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} \arrow[d, "\theta"] & \\ \mathrm{Pic}(U_{\mathrm{gen}}) \arrow[r] & \mathrm{Pic}(G_K) \arrow[r, "\delta"] & \mathrm{Br}(K) \arrow[r, "i"] & \mathrm{Br}(K(U_{\mathrm{gen}})), \end{tikzcd}$$

where the bottom sequence is (2.2) for the G-torsor $U_{\mathrm{gen}} \longrightarrow \mathrm{Spec}(K)$ followed by the injection $\mathrm{Br}(U_{\mathrm{gen}}) \longrightarrow \mathrm{Br}\big(K(U_{\mathrm{gen}})\big)$ (see [29, Ch. IV, Cor. 2.6]), and the map θ is evaluation at the generic torsor U_{gen} given in (2.1) and is injective by Theorem 2.2. Since the generic torsor is split over $K(U_{\mathrm{gen}})$, $\mathrm{Im}(\theta) \subset \mathrm{Ker}(i) = \mathrm{Im}(\delta)$. By Lemma 2.3, j is an isomorphism, hence ν is surjective.

Note that U_{gen} is a localization of U, hence $\mathrm{Pic}(U_{\mathrm{gen}}) = 0$ as $\mathrm{Pic}(U) = 0$. It follows that ν is injective. □

An algebraic group G over a field F is called *universally special* if $H^1(K, G) = \{1\}$ for every field extension K/F, i.e., all G-torsors over any field extension of F are trivial.

Corollary 2.5. *If the group G is universally special, then* $\mathrm{Pic}(G) = 0$.

3. Invariants with values in $\mathbb{Q}/\mathbb{Z}(j)$

In this section we find a description for the group of cohomological invariants with values in $\mathbb{Q}/\mathbb{Z}(j)$ by identifying the image of the embedding θ in (2.1).

3a. **Balanced elements.** Let G be an algebraic group over a field F. We assume that G is connected if F is finite. Let $E \longrightarrow X$ be a G-torsor such that $E(F) \neq \emptyset$. We write p_1 and p_2 for the two projections $E^2/G := (E \times E)/G \longrightarrow X$.

Lemma 3.1. *Let K/F be a field extension and $x_1, x_2 \in X(K)$. Then the G-torsors $E(x_1)$ and $E(x_2)$ over K are isomorphic if and only if there is a point $y \in (E^2/G)(K)$ such that $p_1(y) = x_1$ and $p_2(y) = x_2$.*

Proof. "$\Rightarrow$": By construction, we have G-equivariant morphisms $f_i : E(x_i) \longrightarrow E$ for $i = 1, 2$. Choose an isomorphism $h : E(x_1) \xrightarrow{\sim} E(x_2)$ of G-torsors over K. The morphism $(f_1, f_2 h) : E(x_1) \longrightarrow E^2$ yields the required point $\mathrm{Spec}\, K = E(x_1)/G \longrightarrow E^2/G$.

"$\Leftarrow$": The pull-back of $E \longrightarrow X$ with respect to any projection $E^2/G \longrightarrow X$ coincides with the G-torsor $E^2 \longrightarrow E^2/G$, hence

$$E(x_1) = x_1^*(E) = y^* p_1^*(E) \simeq y^*(E^2) \simeq y^* p_2^*(E) = x_2^*(E) = E(x_2). \qquad \square$$

Let H be a (contravariant) functor from the category of integral schemes over F and dominant morphisms to the category of abelian groups. We have the two maps $p_i^* : H(X) \longrightarrow H(E^2/G)$, $i = 1, 2$. An element $h \in H(X)$ is called *balanced* if $p_1^*(h) = p_2^*(h)$. We write $H(X)_{\mathrm{bal}}$ for the subgroup of balanced elements in $H(X)$. In other words, $H(X)_{\mathrm{bal}} = h_0\big(H(E^\bullet/G)\big)$ in the notation of Appendix A-IV.

We can view H as a (covariant) functor $\mathsf{Fields}_F \longrightarrow \mathsf{Sets}$ taking a field K to $H(K) := H\big(\operatorname{Spec} K\big)$.

Lemma 3.2. *Let $h \in H(X)_{\mathrm{bal}}$ be a balanced element, K/F a field extension and I a G-torsor over* $\operatorname{Spec}(K)$*. Let $x \in X(K)$ be a point such that $E(x) \simeq I$. Then the element $x^*(h)$ in $H(K)$ does not depend on the choice of x.*

Proof. Let $x_1, x_2 \in X(K)$ be two points such that $E(x_1) \simeq E(x_2)$. By Lemma 3.1, there is a point $y \in (E^2/G)(K)$ such that $p_1(y) = x_1$ and $p_2(y) = x_2$. Therefore

$$x_1^*(h) = y^*\big(p_1^*(h)\big) = y^*\big(p_2^*(h)\big) = x_2^*(h). \qquad \square$$

It follows from Lemma 3.2 that if the torsor $E \longrightarrow X$ is classifying with $E(F) \neq \emptyset$, then every element $h \in H(X)_{\mathrm{bal}}$ determines an H-invariant i_h of G as follows. Let I be a G-torsor over a field extension K/F. We claim that there is a point $x \in X(K)$ such that $E(x) \simeq I$. If K is infinite, this follows from the definition of the classifying G-torsor. If K is finite then all G-torsors over K are trivial by [25], as G is connected. Since $E(K) \neq \emptyset$, we can take for x the image in $X(K)$ of any point in $E(K)$. Defining $i_h(E) = x^*(h) \in H(K)$, we have a group homomorphism

$$H(X)_{\mathrm{bal}} \longrightarrow \operatorname{Inv}(G, H), \quad h \mapsto i_h.$$

3b. **Cohomology with values in $\mathbb{Q}/\mathbb{Z}(j)$.** For every integer $j \geq 0$, the coefficients $\mathbb{Q}/\mathbb{Z}(j)$ are defined as the direct sum over all prime integers p of the objects $\mathbb{Q}_p/\mathbb{Z}_p(j)$ in the derived category of sheaves of abelian groups on the big étale site of $\operatorname{Spec} F$, where

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \operatorname*{colim}_n \mu_{p^n}^{\otimes j}$$

if $p \neq \operatorname{char} F$, with μ_{p^n} the sheaf of $(p^n)^{\text{th}}$ roots of unity, and

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \operatorname*{colim}_n W_n\Omega^j_{log}[-j]$$

if $p = \operatorname{char} F > 0$, with $W_n\Omega^j_{log}$ the sheaf of *logarithmic de Rham-Witt differentials* (see [23]).

We write $H^m\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the étale cohomology of a scheme X with values in $\mathbb{Q}/\mathbb{Z}(j)$. Then

$$H^m\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)\{p\} = \operatorname*{colim}_n H^m\big(X, \mu_{p^n}^{\otimes j}\big)$$

if $p \neq \operatorname{char} F$ and

$$H^m\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)\{p\} = \operatorname*{colim}_n H^{m-j}\big(X, W_n\Omega^j_{log}\big)$$

if $p = \operatorname{char} F > 0$. In the latter case, the group $W_n\Omega^j_{log}(F)$ is canonically isomorphic to $K^M_j(F)/p^nK^M_j(F)$, where $K^M_j(F)$ is Milnor's K-group of F (see [1, Cor. 2.8]), hence by [21] and [16, Part II, Appendix A], $H^s\big(F, W_n\Omega^j_{log}\big)$ is isomorphic to

$$H^s\big(F, K^M_j(F_{\mathrm{sep}})/p^nK^M_j(F_{\mathrm{sep}})\big) = \begin{cases} K^M_j(F)/p^nK^M_j(F), & \text{if } s = 0; \\ H^2\big(F, K^M_j(F_{\mathrm{sep}})\big)_{p^n}, & \text{if } s = 1; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that in the case $p = \operatorname{char} F > 0$, we have

$$H^m\big(F, \mathbb{Q}/\mathbb{Z}(j)\big)\{p\} = \begin{cases} K^M_j(F) \otimes (\mathbb{Q}_p/\mathbb{Z}_p), & \text{if } m = j; \\ H^2\big(F, K^M_j(F_{\mathrm{sep}})\big)\{p\}, & \text{if } m = j+1; \\ 0, & \text{otherwise.} \end{cases}$$

The *motivic complexes* $\mathbb{Z}(j)$, for $j = 0, 1, 2$, of étale sheaves on a smooth scheme X were defined in [26] and [27] by S. Lichtenbaum. We write $H^*\big(X, \mathbb{Z}(j)\big)$ for the étale cohomology groups of X with values in $\mathbb{Z}(j)$.

The complex $\mathbb{Z}(0)$ is equal to the constant sheaf $\mathbb{Z}$ and $\mathbb{Z}(1) = \mathbb{G}_{m,X}[-1]$, thus $H^n\big(X, \mathbb{Z}(1)\big) = H^{n-1}(X, \mathbb{G}_{m,X})$. In particular, $H^3\big(X, \mathbb{Z}(1)\big) = \operatorname{Br}(X)$, the cohomological Brauer group of X. The complex $\mathbb{Z}(2)$ is concentrated in degrees 1 and 2 and there is a product map $\mathbb{Z}(1) \otimes^L \mathbb{Z}(1) \longrightarrow \mathbb{Z}(2)$.

The exact triangle in the derived category of étale sheaves

$$\mathbb{Z}(j) \longrightarrow \mathbb{Q} \otimes \mathbb{Z}(j) \longrightarrow \mathbb{Q}/\mathbb{Z}(j) \longrightarrow \mathbb{Z}(j)[1]$$

yields the connecting homomorphism

$$H^i\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^{i+1}\big(X, \mathbb{Z}(j)\big),$$

which is an isomorphism if $X = \operatorname{Spec}(F)$ for a field F and $i > j$ [22, Lemme 1.1].

Write $\mathcal{H}^n\big(\mathbb{Q}/\mathbb{Z}(j)\big)$ for the Zariski sheaf on a smooth scheme X associated to the presheaf $U \mapsto H^n\big(U, \mathbb{Q}/\mathbb{Z}(j)\big)$ of étale cohomology groups.

Let G be an algebraic group over F. We assume that G is connected if F is a finite field and write $\operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the group of *degree n invariants* of G for the functor $K \mapsto H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$. Note that Property 2.1 holds for this functor by [16, Part 2, Prop. A.9].

Choose a classifying G-torsor $E \longrightarrow X$ with E a G-rational variety such that $E(F) \neq \emptyset$. Applying the construction given in §3a to the functor $U \mapsto H^0_{\mathrm{Zar}}\big(U, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$, we get a homomorphism

$$\varphi : H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}} \longrightarrow \operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big).$$

Theorem 3.3. *Let G be an algebraic group over a field F. We assume that G is connected if F is a finite field. Let $E \longrightarrow X$ be a classifying G-torsor with E a G-rational variety such that $E(F) \neq \emptyset$. Then the homomorphism φ is an isomorphism.*

Proof. Let $E_{\mathrm{gen}} \longrightarrow F(X)$ be the generic fiber of the classifying G-torsor $E \longrightarrow X$. Note that since the pull-back of $E \longrightarrow X$ with respect to any of the two projections $E^2/G \longrightarrow X$ coincides with the G-torsor $E^2 \longrightarrow E^2/G$, the pull-backs of the generic G-torsor $E_{\mathrm{gen}} \longrightarrow \operatorname{Spec} F(X)$ with respect to the two morphisms $\operatorname{Spec} F(E^2/G) \longrightarrow \operatorname{Spec} F(X)$ induced by the projections are isomorphic. It follows that for every invariant $i \in \operatorname{Inv}\bigl(G, H^*(\mathbb{Q}/\mathbb{Z}(j))\bigr)$ we have

$$p_1^*\bigl(i(E_{\mathrm{gen}})\bigr) = i\bigl(p_1^*(E_{\mathrm{gen}})\bigr) = i\bigl(p_2^*(E_{\mathrm{gen}})\bigr) = p_2^*\bigl(i(E_{\mathrm{gen}})\bigr)$$

in $H^*\bigl(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j)\bigr)$, i.e., $i(E_{\mathrm{gen}}) \in H^*\bigl(F(X), \mathbb{Q}/\mathbb{Z}(j)\bigr)_{\mathrm{bal}}$. By Proposition A.9, $\partial_x(h) = 0$ for every point $x \in X$ of codimension 1, hence

$$\theta(i) = i(E_{\mathrm{gen}}) \in H^0_{\mathrm{Zar}}\bigl(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\bigr)_{\mathrm{bal}}$$

by Proposition A.10. By Theorem 2.2, θ is injective and by construction, the composition $\theta \circ \varphi$ is the identity. It follows that φ is an isomorphism. $\square$

Write $\overline{H}^0_{\mathrm{Zar}}\bigl(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\bigr)$ for the factor group of $H^0_{\mathrm{Zar}}\bigl(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\bigr)$ by the natural image of $H^n\bigl(F, \mathbb{Q}/\mathbb{Z}(j)\bigr)$.

Corollary 3.4. *The isomorphism φ yields an isomorphism*

$$\overline{H}^0_{\mathrm{Zar}}\bigl(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\bigr)_{\mathrm{bal}} \xrightarrow{\sim} \operatorname{Inv}^n\bigl(G, \mathbb{Q}/\mathbb{Z}(j)\bigr)_{\mathrm{norm}}.$$

4. Degree 3 invariants of algebraic tori

In this section we prove the main theorem that describes degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$.

4a. **Algebraic tori.** Let F be a field and $\Gamma = \operatorname{Gal}(F_{\mathrm{sep}}/F)$ the absolute Galois group of F. An *algebraic torus* of dimension n over F is an algebraic group T such that T_{sep} is isomorphic to the product of n copies of the multiplicative group $\mathbb{G}_m$ (see [11] or [35]). For an algebraic torus T over a field F, we write $\widehat{T}_{\mathrm{sep}}$ for the Γ-module of characters $\operatorname{Hom}(T_{\mathrm{sep}}, \mathbb{G}_m)$. The group $\widehat{T}_{\mathrm{sep}}$ is a Γ-*lattice*, i.e., a free abelian group of finite rank with a continuous Γ-action. The contravariant functor $T \mapsto \widehat{T}_{\mathrm{sep}}$ is an anti-equivalence between the category of algebraic tori and the category of Γ-lattices: the torus T and the group $T(F)$ can be reconstructed from the lattice $\widehat{T}_{\mathrm{sep}}$ by the formulas

$$T = \operatorname{Spec}\bigl(F_{\mathrm{sep}}[\widehat{T}_{\mathrm{sep}}]\bigr)^{\Gamma},$$

$$T(F) = \operatorname{Hom}_{\Gamma}(\widehat{T}_{\mathrm{sep}}, F_{\mathrm{sep}}^{\times}) = \bigl(\widehat{T}^{\circ}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times}\bigr)^{\Gamma},$$

where $\widehat{T}^{\circ}_{\mathrm{sep}} = \operatorname{Hom}(\widehat{T}_{\mathrm{sep}}, \mathbb{Z})$.

We write $\widehat{T}$ for the character group $\operatorname{Hom}(T, \mathbb{G}_m) = (\widehat{T}_{\mathrm{sep}})^{\Gamma}$ and T° for the *dual torus* having character lattice $\widehat{T}^{\circ}_{\mathrm{sep}}$.

A torus T is called *quasi-split* if T is isomorphic to the group of invertible elements of an étale F-algebra, or equivalently, the Γ-lattice $\widehat{T}_{\mathrm{sep}}$ is permutation, i.e., $\widehat{T}_{\mathrm{sep}}$ has a Γ-invariant $\mathbb{Z}$-basis. An *invertible* torus is a direct factor of a quasi-split torus.

A torus T is called *flasque* (respectively, *coflasque*) if $H^1(L, \widehat{T}^\circ_{\mathrm{sep}}) = 0$ (respectively, $H^1(L, \widehat{T}_{\mathrm{sep}}) = 0$) for every finite field extension L/K. A *flasque resolution* of a torus S is an exact sequence of tori $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ with T flasque and P quasi-split. By [11, §4], or [35, §4.7], the torus T in the flasque resolution is invertible if and only if S is a direct factor of a rational torus.

4b. **Products.** Let T be a torus over F and let $\widehat{T}(i)$ denote the complex $\widehat{T}_{\mathrm{sep}} \otimes \mathbb{Z}(i)$ of étale sheaves over F for $i = 0, 1, 2$. Thus, $\widehat{T}(0) = \widehat{T}_{\mathrm{sep}}$ and $\widehat{T}(1) = \big(\widehat{T}_{\mathrm{sep}} \otimes F^\times_{\mathrm{sep}}\big)[-1] = T^\circ(F_{\mathrm{sep}})[-1]$.

Let S and T be algebraic tori over F and let i and j be nonnegative integers with $i + j \le 2$. For any smooth variety X over F, we have the product map

$$(\widehat{S}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma \otimes H^p\big(X, \widehat{S}^\circ(i)\big) \otimes H^q\big(X, \widehat{T}^\circ(j)\big) \longrightarrow H^{p+q}\big(X, \mathbb{Z}(i+j)\big) \tag{4.1}$$

taking $a \otimes b \otimes c$ to $a \cup b \cup c$, via the canonical pairings between $\widehat{S}_{\mathrm{sep}}$ and $\widehat{S}^\circ_{\mathrm{sep}}$, $\widehat{T}_{\mathrm{sep}}$ and $\widehat{T}^\circ_{\mathrm{sep}}$, and the product map $\mathbb{Z}(i) \otimes^L \mathbb{Z}(j) \longrightarrow \mathbb{Z}(i+j)$.

Recall that there is an isomorphism $H^n\big(F, \mathbb{Z}(k)\big) \simeq H^{n-1}\big(F, \mathbb{Q}/\mathbb{Z}(k)\big)$ for $n > k$. In particular, we have the cup-product map

$$(\widehat{S}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma \otimes H^p(F, S) \otimes H^q(F, T) \longrightarrow H^3\big(F, \mathbb{Q}/\mathbb{Z}(2)\big) \tag{4.2}$$

if $p + q = 2$.

If $S = T^\circ$ is the dual torus, then $(\widehat{S}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma = \mathrm{End}_\Gamma(\widehat{T}_{\mathrm{sep}})$ contains the canonical element 1_T. We then have the product map

$$H^p\big(X, \widehat{T}(i)\big) \otimes H^q\big(X, \widehat{T}^\circ(j)\big) \longrightarrow H^{p+q}\big(X, \mathbb{Z}(i+j)\big) \tag{4.3}$$

and in particular, the product maps

$$H^1(F, \widehat{T}_{\mathrm{sep}}) \otimes H^1(F, T) \longrightarrow H^2\big(F, \mathbb{Q}/\mathbb{Z}(1)\big) = \mathrm{Br}(F), \tag{4.4}$$

$$H^1(F, T^\circ) \otimes H^1(F, T) \longrightarrow H^3\big(F, \mathbb{Q}/\mathbb{Z}(2)\big), \tag{4.5}$$

$$H^2(F, T^\circ) \otimes H^0(F, T) \longrightarrow H^3\big(F, \mathbb{Q}/\mathbb{Z}(2)\big), \tag{4.6}$$

taking $a \otimes b$ to $1_T \cup a \cup b$.

The Picard group of T is canonically isomorphic to $H^1(F, \widehat{T}_{\mathrm{sep}})$ [35, §4.3]. The map ν in Theorem 2.4 is given by the cup-product (4.4) with the class of the T-torsor $U_{\mathrm{gen}} \longrightarrow \mathrm{Spec}(K)$ in $H^1(K, T)$ (see [13, Prop. 2.9] or [3, Lemma 2.6]). Thus, we have proved the following corollary.

Corollary 4.1. *Let T be a torus over F. Then the map*

$$H^1(F, \widehat{T}_{\mathrm{sep}}) \longrightarrow \mathrm{Inv}(T, \mathrm{Br})_{\mathrm{norm}},$$

taking an element a to the invariant $b \mapsto a_K \cup b$ for $b \in H^1(K, T)$ and K/F is an isomorphism.

As T is a commutative group, the set $H^1(K,T)$ is an abelian group. An invariant $i \in \mathrm{Inv}(T,H)$ for a functor H is called *linear* if $i_K : H^1(K,T) \longrightarrow H(K)$ is a group homomorphism for every K/F. Clearly, every linear invariant is normalized. By Corollary 4.1, every normalized Br-invariant of a torus is linear. In the next section we will see that a normalized degree 3 invariant of a torus need not be linear.

4c. **Main theorem.** Let T be a torus over F and choose a standard classifying T-torsor $U \longrightarrow U/T$ such that the codimension of $V \setminus U$ in V is at least 3. Such a torsor exits by [14, Lemma 9].

By [32, Prop. 6.10], there is an exact sequence

$$F_{\mathrm{sep}}[U]^\times/F_{\mathrm{sep}}^\times \longrightarrow \widehat{T}_{\mathrm{sep}} \longrightarrow \mathrm{Pic}\big((U/T)_{\mathrm{sep}}\big) \longrightarrow \mathrm{Pic}(U_{\mathrm{sep}}).$$

The codimension assumption implies that the side terms are trivial, hence the map $\widehat{T}_{\mathrm{sep}} \longrightarrow \mathrm{Pic}\big((U/T)_{\mathrm{sep}}\big)$ is an isomorphism. It follows that the classifying T-torsor $U \longrightarrow U/T$ is universal in the sense of [7] (see Proposition B.1).

Write $K_*(F)$ for the (Quillen) K-groups of F and $\mathcal{K}_*$ for the Zariski sheaf associated to the presheaf $U \mapsto K_*(U)$. Then the groups $H^n_{\mathrm{Zar}}\big(U/G, \mathcal{K}_2\big)$ are independent of the choice of the classifying torsor (cf., [14]). So we write $H^n_{\mathrm{Zar}}\big(BT, \mathcal{K}_2\big)$ for this group (see Appendix A-IV). As T_{sep} is a split torus, by the Künneth formula (see Example A.5),

$$H^n_{\mathrm{Zar}}\big(BT_{\mathrm{sep}}, \mathcal{K}_2\big) = \begin{cases} K_2(F_{\mathrm{sep}}), & \text{if } n=0; \\ \mathrm{Pic}\big((U/G)_{\mathrm{sep}}\big) \otimes F_{\mathrm{sep}}^\times = \widehat{T}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^\times = T^\circ(F_{\mathrm{sep}}), & \text{if } n=1; \\ \mathrm{CH}^2\big((U/G)_{\mathrm{sep}}\big) = S^2(\widehat{T}_{\mathrm{sep}}), & \text{if } n=2. \end{cases}$$

Applying the calculation of the $\mathcal{K}$-cohomology groups to the standard classifying T-torsor $U^i \longrightarrow U^i/G$ for every $i > 0$ instead of $U \longrightarrow U/T$, by Proposition B.3, we have the exact sequence

(4.7)
$$0 \longrightarrow H^1(F,T^\circ) \xrightarrow{\alpha} \overline{H}^4\big(U^i/G, \mathbb{Z}(2)\big) \longrightarrow \overline{H}^4\big((U^i/G)_{\mathrm{sep}}, \mathbb{Z}(2)\big)^\Gamma \longrightarrow H^2(F,T^\circ),$$

where $\overline{H}^4\big(U^i/G, \mathbb{Z}(2)\big)$ is the factor group of $H^4\big(U^i/G, \mathbb{Z}(2)\big)$ by $H^4\big(F, \mathbb{Z}(2)\big)$, the map α is given by $\alpha(a) = q^*(a) \cup [U^i]$ with $q : U^i/G \longrightarrow \operatorname{Spec} F$ the structure morphism, $[U^i]$ the class of the T-torsor $U^i \longrightarrow U^i/G$ in $H^1(U^i/G, T)$, and the cup-product is taken for the pairing (B.6).

Taking the sequences (4.7) for all i, we get the exact sequence of cosimplicial groups

$$0 \longrightarrow H^1(F,T^\circ) \xrightarrow{\alpha} \overline{H}^4\big(U^\bullet/T, \mathbb{Z}(2)\big) \longrightarrow \overline{H}^4\big((U^\bullet/T)_{\mathrm{sep}}, \mathbb{Z}(2)\big)^\Gamma \longrightarrow H^2\big(F,T^\circ\big).$$

The first and the last cosimplicial groups in the sequence are constant, hence by Lemma A.2, the sequence

$$(4.8) \quad 0 \longrightarrow H^1(F,T^\circ) \xrightarrow{\alpha} \overline{H}^4\big(U/T, \mathbb{Z}(2)\big)_{\mathrm{bal}} \longrightarrow \overline{H}^4\big((U/T)_{\mathrm{sep}}, \mathbb{Z}(2)\big)^\Gamma_{\mathrm{bal}} \longrightarrow H^2(F,T^\circ)$$

is exact.

The following theorem was proved in [23, Th. 1.1]:

Theorem 4.2. *Let X be a smooth variety over F. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(X) \longrightarrow H^4\big(X, \mathbb{Z}(2)\big) \longrightarrow H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))\big) \longrightarrow 0.$$

By Theorem 4.2, there is an exact sequence of cosimplicial groups

$$0 \longrightarrow \mathrm{CH}^2(U^\bullet/T) \longrightarrow \overline{H}^4\big(U^\bullet/T, \mathbb{Z}(2)\big) \longrightarrow \overline{H}^0_{\mathrm{Zar}}\big(U^\bullet/T, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))\big) \longrightarrow 0.$$

As the functor CH^2 is homotopy invariant, by Lemma A.4, the first group in the sequence is constant. In view of Lemma A.2, and following the notation for the $\mathcal{K}$-cohomology, the sequence

$$0 \longrightarrow \mathrm{CH}^2(BT) \longrightarrow \overline{H}^4\big(U/T, \mathbb{Z}(2)\big)_{\mathrm{bal}} \longrightarrow \overline{H}^0_{\mathrm{Zar}}\big(U/T, \mathcal{H}(\mathbb{Q}/\mathbb{Z}(2))\big)_{\mathrm{bal}} \longrightarrow 0 \tag{4.9}$$

is exact. By Corollary 3.4, the last group in the sequence is canonically isomorphic to $\mathrm{Inv}\big(T, H^3(\mathbb{Q}/\mathbb{Z}(2))\big)_{\mathrm{norm}}$.

As the torus T_{sep} is split, all the invariants of T_{sep} are trivial hence the sequence (4.9) over F_{sep} yields an isomorphism

$$\overline{H}^4\big((U/T)_{\mathrm{sep}}, \mathbb{Z}(2)\big)_{\mathrm{bal}} \simeq \mathrm{CH}^2(BT_{\mathrm{sep}}) \simeq S^2(\widehat{T}_{\mathrm{sep}}). \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we get the following diagram with an exact row and column:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^1(F, T^0) \arrow[d, "\alpha"'] \arrow[dr] & & \\
0 \arrow[r] & \mathrm{CH}^2(BT) \arrow[r] \arrow[dr] & \overline{H}^4\big(U/T, \mathbb{Z}(2)\big)_{\mathrm{bal}} \arrow[r] \arrow[d] & \mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \arrow[r] & 0. \\
 & & S^2(\widehat{T}_{\mathrm{sep}})^\Gamma \arrow[d] & & \\
 & & H^2(F, T^0) & &
\end{tikzcd}$$

Write $\mathrm{Dec} = \mathrm{Dec}(\widehat{T}_{\mathrm{sep}})$ for the subgroup of *decomposable elements* in $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ (see Appendix A-II).

Lemma 4.3. *The image of the homomorphism* $\mathrm{CH}^2(BT) \longrightarrow \mathrm{CH}^2(BT_{\mathrm{sep}})^\Gamma = S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ *in the diagram coincides with* Dec.

Proof. Consider the Grothendieck ring $K_0(BT)$ of the category of T-equivariant vector bundles over $\mathrm{Spec}(F)$, or equivalently, of the category of finite dimensional representations of T. If T is split, every representation of T is a direct sum of one-dimensional representations. Therefore, there is an isomorphism between the group ring $\mathbb{Z}[\widehat{T}]$ of all formal finite sums $\sum_{x\in\widehat{T}} a_x e^x$ and $K_0(BT)$, taking e^x with $x \in \widehat{T}$ to the class of the 1-dimensional representation given by x. In general, for every torus T, we have $K_0(BT_{\mathrm{sep}}) = \mathbb{Z}[\widehat{T}_{\mathrm{sep}}]$ and $K_0(BT) = \mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma = K_0(BT_{\mathrm{sep}})^\Gamma$ (see [28, §9]). The group $\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma$ is generated by the sums $\sum_{i=1}^n e^{\gamma_i x}$, where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are representatives of the left cosets of an arbitrary open subgroup Γ' in Γ and $x \in (\widehat{T}_{\mathrm{sep}})^{\Gamma'}$.

The equivariant Chern classes were defined in [14, §2.4]. The first Chern class $c_1 : K_0(BT_{\mathrm{sep}}) \longrightarrow \mathrm{CH}^1(BT_{\mathrm{sep}}) = \widehat{T}_{\mathrm{sep}}$ takes e^x to x. In the diagram

$$\begin{tikzcd}
\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma \arrow[r, equal] \arrow[d] & K_0(BT) \arrow[r, "c_2"] \arrow[d] & \mathrm{CH}^2(BT) \arrow[d] & \\
\mathbb{Z}[\widehat{T}_{\mathrm{sep}}] \arrow[r, equal] & K_0(BT_{\mathrm{sep}}) \arrow[r, "c_2"] & \mathrm{CH}^2(BT_{\mathrm{sep}}) \arrow[r, equal] & S^2(\widehat{T}_{\mathrm{sep}})
\end{tikzcd}$$

the second Chern class maps c_2 are surjective by [15, Lemma C.3]. It follows from the formula $c_2(a+b) = c_2(a) + c_1(a)c_1(b) + c_2(b)$ that the composition

$$\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma = K_0(BT) \longrightarrow K_0(BT_{\mathrm{sep}}) \xrightarrow{c_2} \mathrm{CH}^2(BT_{\mathrm{sep}}) = S^2(\widehat{T}_{\mathrm{sep}}) \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})/(\widehat{T})^2$$

is a homomorphism and its image is generated by the elements (see Appendix A-II)

$$c_2\Big(\sum_{i=1}^n e^{\gamma_i x}\Big) = \sum_{i<j} (\gamma_i x)(\gamma_j x) = \mathrm{Qtr}(x). \qquad \square$$

By the restriction-corestriction argument, the kernel of the homomorphism $\mathrm{CH}^2(BT) \longrightarrow \mathrm{CH}^2(BT_{\mathrm{sep}})^\Gamma = S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ coincides with the torsion subgroup $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ in $\mathrm{CH}^2(BT)$.

The following theorem describes degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$:

Theorem 4.4. *Let T be an algebraic torus a field F. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(BT)_{\mathrm{tors}} \longrightarrow H^1(F, T^0) \xrightarrow{\alpha}$$
$$\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \mathrm{Dec} \longrightarrow H^2(F, T^0).$$

The homomorphism α is given by $\alpha(a)(b) = a_K \cup b$ for every $a \in H^1(F, T^0)$ and $b \in H^1(K, T)$ and every field extension K/F, where the cup-product is defined in (4.5).

Proof. The exactness of the sequence follows from the diagram before Lemma 4.3. It remains to describe the map α. Take an $a \in H^1(F, T^0)$ and consider the invariant i defined by $i(b) = a_K \cup b$, where the cup-product is given by

(4.5). We need to prove that $i = \alpha(a)$. Choose a standard classifying T-torsor $U \longrightarrow U/T$ and set $K = F(U/T)$. Let U_{gen} be the generic fiber of the classifying torsor. By Theorem 2.2, it suffices to show that $i(U_{\mathrm{gen}}) = \alpha(a)(U_{\mathrm{gen}})$ over K. This follows from the description of the map α in the exact sequence (4.7). □

4d. **Torsion in** $\mathrm{CH}^2(BT)$**.** We investigate the group $\mathrm{CH}^2(BT)_{\mathrm{tors}}$, the first term of the exact sequence in Theorem 4.4.

Let S be an algebraic torus over F. Using the Gersten resolution, [30, Prop. 5.8] we identify the group $H^0(S_{\mathrm{sep}}, \mathcal{K}_2)$ with a subgroup in $K_2\big(F_{\mathrm{sep}}(S)\big)$. Set $\overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2) := H^0(S_{\mathrm{sep}}, \mathcal{K}_2)/K_2(F_{\mathrm{sep}})$. By [16, Part 2, §5.7], we have an exact sequence

$$0 \longrightarrow \widehat{S}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times} \longrightarrow \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2) \xrightarrow{\lambda} \Lambda^2 \widehat{S}_{\mathrm{sep}} \longrightarrow 0 \tag{4.11}$$

of Γ-modules, where $\lambda\big(\{e^x, e^y\}\big) = x \wedge y$ for $x, y \in \widehat{S}_{\mathrm{sep}}$.

Consider the Γ-homomorphism

$$\begin{aligned} \gamma : \Lambda^2 \widehat{S}_{\mathrm{sep}} &\longrightarrow \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2) \\ x \wedge y &\longmapsto \{e^x, e^y\} - \{e^y, e^x\}. \end{aligned}$$

We have $\lambda \circ \gamma = 2 \cdot \mathrm{Id}$, hence the connecting homomorphism

$$\partial : H^i(F, \Lambda^2 \widehat{S}_{\mathrm{sep}}) \longrightarrow H^{i+1}(F, \widehat{S}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times}) \tag{4.12}$$

satisfies $2\partial = 0$.

Lemma 4.5. *If S is an invertible torus, then sequence of Γ-modules (4.11) is split.*

Proof. We can assume that S is quasi-split. Let $\{x_1, x_2, \dots, x_m\}$ be a permutation basis for $\widehat{S}_{\mathrm{sep}}$. Then the elements $x_i \wedge x_j$ for $i < j$ form a $\mathbb{Z}$-basis for $\Lambda^2 \widehat{S}_{\mathrm{sep}}$. The map $\Lambda^2 \widehat{S}_{\mathrm{sep}} \longrightarrow \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2)$, taking $x_i \wedge x_j$ to $\{e^{x_i}, e^{x_j}\}$ is a splitting for γ. □

Let

$$1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$$

be a coflasque resolution of T, i.e., P is a quasi-split torus and Q is a coflasque torus (see [11]). The torus P is an open set in the affine space of a separable F-algebra on which T acts linearly. Hence $P \longrightarrow Q$ is a standard classifying T-torsor. By Theorem 2.2, the natural map

$$\theta : \mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big) \longrightarrow H^3\big(F(Q), \mathbb{Q}/\mathbb{Z}(2)\big)$$

is injective.

Consider the spectral sequence (B.11) for the variety $X = Q$. We have $H^1(Q_{\mathrm{sep}}, \mathcal{K}_2) = 0$ by [16, Part 2, Cor. 5.6]. In view of Proposition B.4, we have an injective homomorphism

$$\beta : H^2\big(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)\big) \longrightarrow \overline{H}^4\big(Q, \mathbb{Z}(2)\big) \tag{4.13}$$

such that the composition of β with the homomorphism

$$H^2(F,Q^\circ) \longrightarrow H^2\big(F,\overline{H}^0(Q_{\mathrm{sep}},\mathcal{K}_2)\big)$$

is given by the cup-product with the class of the identity in $H^0(Q,Q)$.

Lemma 4.6. *The group* $\mathrm{CH}^2(Q)$ *is trivial.*

Proof. By [28, Th. 9.1], the Grothendieck group $K_0(Q)$ is generated by the classes of the sheaves $i_*(P)$, where P is an invertible sheaf on X_L, L/F a finite separable field extension and $i : X_L \longrightarrow X$ is the natural morphism. By definition of a coflasque torus,

$$\mathrm{Pic}(Q_L) = H^1(L,\widehat{Q}_{\mathrm{sep}}) = 0.$$

It follows that every invertible sheaf on Q_L is trivial, hence $K_0(Q) = \mathbb{Z}\cdot 1$. Since the group $\mathrm{CH}^2(Q)$ is generated by the second Chern classes of vector bundles on Q [15, Lemma C.3], $\mathrm{CH}^2(Q) = 0$. □

It follows from Proposition A.10, Theorem 4.2, and Lemma 4.6 that the homomorphism

$$\kappa : \overline{H}^4\big(Q,\mathbb{Z}(2)\big) \longrightarrow \overline{H}^4\big(F(Q),\mathbb{Z}(2)\big) = \overline{H}^3\big(F(Q),\mathbb{Q}/\mathbb{Z}(2)\big) \tag{4.14}$$

is injective.

Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q^\circ(F_{\mathrm{sep}}) & \longrightarrow & \overline{H}^0(Q_{\mathrm{sep}},\mathcal{K}_2) & \longrightarrow & \Lambda^2\widehat{Q}_{\mathrm{sep}} & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & Q^\circ(F_{\mathrm{sep}}) & \longrightarrow & P^\circ(F_{\mathrm{sep}}) & \longrightarrow & T^\circ(F_{\mathrm{sep}} & \longrightarrow & 0
\end{array}$$

where s is the composition of the natural map $\overline{H}^0(Q_{\mathrm{sep}},\mathcal{K}_2) \longrightarrow \overline{H}^0(P_{\mathrm{sep}},\mathcal{K}_2)$ and a splitting of $P^\circ(F_{\mathrm{sep}}) \longrightarrow \overline{H}^0(P_{\mathrm{sep}},\mathcal{K}_2)$ (see Lemma 4.5).

We have the following diagram

$$\begin{array}{ccccccc}
 & & & & H^1(F,T^\circ) & \xrightarrow{\alpha} & \mathrm{Inv}^3\big(T,\mathbb{Q}/\mathbb{Z}(2)\big)\\
 & & & \nearrow{\scriptstyle t^*} & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \theta}\\
H^1\big(F,\overline{H}^0(Q_{\mathrm{sep}},\mathcal{K}_2)\big) & \longrightarrow & H^1(F,\Lambda^2\widehat{Q}_{\mathrm{sep}}) & \xrightarrow{\partial} & H^2(F,Q^\circ) & \xrightarrow{\sigma} & \overline{H}^3\big(F(Q),\mathbb{Q}/\mathbb{Z}(2)\big)
\end{array}$$

where σ is the composition of the maps in (4.13) and (4.14):

$$H^2(F,Q^0) \longrightarrow H^2\big(F,\overline{A}^0(Q_{\mathrm{sep}},K_2)\big) \xrightarrow{\beta} \overline{H}^4\big(Q,\mathbb{Z}(2)\big) \xrightarrow{\kappa} \overline{H}^4\big(F(Q),\mathbb{Z}(2)\big) = \overline{H}^3\big(F(Q),\mathbb{Q}/\mathbb{Z}(2)\big).$$

Note that the connecting map ∂_1 is injective as $H^1(F,P^\circ) = 0$ since P° is a quasi-split torus. As $2\partial = 0$ in (4.12), we have $2t^* = 0$.

The commutativity of the triangle follows from the definition of t^*. We claim that the square in the diagram is anti-commutative. Note that $\partial_2(\xi) = [P_{\mathrm{gen}}]$, where $\partial_2 : H^0(F,Q) \longrightarrow H^1(F,T)$ is the connecting homomorphism, P_{gen} is the generic fiber of the morphism $P \longrightarrow Q$, and $\xi \in H^0(K,Q)$ is the generic point of Q with $K = F(Q)$. It follows from the description of the maps α and β in (4.7) and (4.13), respectively, and Lemma A.1 that

$$\sigma\big(\partial_1(a)\big) = \partial_1(a)_K \cup \xi = (-a_K) \cup \partial_2(\xi) = (-a_K) \cup [P_{\mathrm{gen}}] = -\theta\big(\alpha(a)\big).$$

for every $a \in H^1(F,T^\circ)$.

The maps β and κ are injective, hence the bottom sequence in the diagram is exact. Thus, we have an exact sequence

$$H^1\big(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)\big) \longrightarrow H^1\big(F, \Lambda^2 \widehat{Q}_{\mathrm{sep}}\big) \longrightarrow \mathrm{Ker}(\alpha) \longrightarrow 0$$

and $2 \cdot \mathrm{Ker}(\alpha) = 2 \cdot \mathrm{Im}(t^*) = 0$. Furthermore, $\mathrm{Ker}(\alpha) \simeq \mathrm{CH}^2(BT)_{\mathrm{tors}}$ and the group $H^1(F, \Lambda^2 \widehat{Q}_{\mathrm{sep}})$ is finite.

We have proved:

Theorem 4.7. *Let $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ be a coflasque resolution of a torus T. Then there is an exact sequence*

$$H^1\big(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)\big) \longrightarrow H^1\big(F, \Lambda^2 \widehat{Q}_{\mathrm{sep}}\big) \longrightarrow \mathrm{CH}^2(BT)_{\mathrm{tors}} \longrightarrow 0.$$

Moreover, $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ is a finite group satisfying $2 \cdot \mathrm{CH}^2(BT)_{\mathrm{tors}} = 0$.

Corollary 4.8. *If T° is a direct factor of a rational torus, or if T is split over a cyclic field extension, then $\mathrm{CH}^2(BT)_{\mathrm{tors}} = 0$, i.e., the map α in Theorem 4.4 is injective.*

Proof. The exact sequence $1 \longrightarrow Q^\circ \longrightarrow P^\circ \longrightarrow T^\circ \longrightarrow 1$ is a flasque resolution of T°. If T° is a direct factor of a rational torus, or if T is split over a cyclic field extension, the torus Q° ,and hence Q, is invertible (see §4a and [35, §4, Th. 3]). By Lemma 4.5, the sequence (4.11) for the torus Q is split, hence the first map in Theorem 4.7 is surjective. ☐

4e. **Special tori.** Let T be an algebraic torus over a field F. The *tautological invariant* of the torus $T^\circ \times T$ is the normalized invariant taking a pair $(a,b) \in H^1(K,T^\circ) \times H^1(K,T)$ to the cup-product $a \cup b \in H^3\big(K, \mathbb{Q}/\mathbb{Z}(2)\big)$ defined in (4.5).

The following theorem shows that if a torus T has only trivial degree 3 normalized invariants with values in $\mathbb{Q}/\mathbb{Z}(2)$ universally, i.e., over all field extensions of F, then T has no non-constant invariants at all by the simple reason: every T-torsor over a field is trivial. Note that it follows from Theorem 2.4 that T has no degree 2 normalized invariants with values in $\mathbb{Q}/\mathbb{Z}(1)$ universally if and only if T is coflasque.

Theorem 4.9. *Let T be an algebraic torus over a field F. Then the following are equivalent:*

1. $\mathrm{Inv}^3\big(T_K, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} = 0$ *for every field extension K of F.*
2. *The tautological invariant of the torus $T^\circ \times T$ is trivial.*
3. *The torus T is invertible.*

(4) *The torus T is universally special.*

Proof. (1) $\Rightarrow$ (2): Let K/F be a field extension and $a \in H^1(K, T^\circ)$. By assumption, the degree 3 normalized invariant $i = \alpha(a)$ with values in $\mathbb{Q}/\mathbb{Z}(2)$, defined by $i(b) = a \cup b$ for every $b \in H^1(K, T)$, is trivial. In other words, the tautological invariant of the torus $T^\circ \times T$ is trivial.

(2) $\Rightarrow$ (3): The image of the tautological invariant in the group $S^2(\widehat{T}^\circ_{\mathrm{sep}} \oplus \widehat{T}_{\mathrm{sep}})^\Gamma / \operatorname{Dec}$ is represented by the identity $1_{\widehat{T}}$ in the direct factor $(\widehat{T}^\circ_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma = \operatorname{End}_\Gamma(\widehat{T}_{\mathrm{sep}})$ of $S^2(\widehat{T}^\circ_{\mathrm{sep}} \oplus \widehat{T}_{\mathrm{sep}})^\Gamma$ (see Appendix A-II). The projection of Dec on the direct summand $(\widehat{T}^\circ_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma$ is generated by the traces $\operatorname{Tr}(a \otimes b)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in (\widehat{T}^\circ_{\mathrm{sep}})^{\Gamma'}$ and $b \in (\widehat{T}_{\mathrm{sep}})^{\Gamma'}$. Hence $1_{\widehat{T}} = \operatorname{Tr}(a_i \otimes b_i)$ for some open subgroups $\Gamma_i \subset \Gamma$, $a_i \in (\widehat{T}^\circ)^{\Gamma_i}$ and $b_i \in (\widehat{T})^{\Gamma_i}$. If $P_i = \mathbb{Z}[\Gamma/\Gamma_i]$, then a_i can be viewed as a Γ-homomorphism $\widehat{T} \longrightarrow P_i$ and b_i can be viewed as a Γ-homomorphism $P_i \longrightarrow \widehat{T}$ such that the composition

$$\widehat{T} \xrightarrow{(b_i)} P \xrightarrow{(a_i)} \widehat{T},$$

where $P = \coprod P_i$, is the identity. It follows that $\widehat{T}$ is a direct summand of a permutation Γ-module P and hence T is invertible.

(3) $\Rightarrow$ (4): Obvious as every invertible torus is universally special.

(4) $\Rightarrow$ (1): Obvious. $\square$

4f. **Linear and quadratic invariants.** Let T be a torus over F. By Theorem 4.4, we have a natural homomorphism to the group of linear invariants:

$$\alpha : H^1(F, T^\circ) \longrightarrow \operatorname{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{lin}}.$$

Let S and T be algebraic tori over F. For every field extension K/F, the cup-product (4.2) yields a homomorphism

$$\varepsilon : (\widehat{T}^{\otimes 2}_{\mathrm{sep}})^\Gamma \longrightarrow \operatorname{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)$$

defined by $\varepsilon(a)(b) = a_K \cup b \cup b$ for $a \in (\widehat{T}^{\otimes 2}_{\mathrm{sep}})^\Gamma$ and $b \in H^1(K, T)$.

We say that an invariant $i \in \operatorname{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)$ is *quadratic* if the function $h(a, b) := i(a + b) - i(a) - i(b)$ is bilinear and $h(a, a) = 2i(a)$ for all a and b. For example, the tautological invariant of the torus $T^\circ \times T$ in §4e is quadratic. We write $\operatorname{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{quad}}$ for the subgroup of all quadratic invariants in $\operatorname{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)$. The image of ε is contained in $\operatorname{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{quad}}$.

Write D for the subgroup of $(\widehat{T}^{\otimes 2}_{\mathrm{sep}})^\Gamma$ generated by $\operatorname{Dec}(\widehat{T}_{\mathrm{sep}}, \widehat{T}_{\mathrm{sep}})$ (the latter defined in Appendix A-II) and the image of $1 - \tau$, where τ is the exchange automorphism of $(\widehat{T}^{\otimes 2}_{\mathrm{sep}})^\Gamma$.

Lemma 4.10. *The composition of ε with the map $\operatorname{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big) \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \operatorname{Dec}$ in Theorem 4.4 is induced by the natural homomorphism $\widehat{T}^{\otimes 2}_{\mathrm{sep}} \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})$. Moreover, the map ε vanishes on D.*

Proof. Let $U \longrightarrow U/T =: X$ be a standard classifying T-torsor as in §4c. Consider the commutative diagram

$$\begin{tikzcd}
(\widehat{T}_{\mathrm{sep}}^{\otimes 2})^{\Gamma} \otimes H^1(X,T)^{\otimes 2} \arrow[r, "prod"] \arrow[d] & \overline{H}^4\big(X, \mathbb{Z}(2)\big)_{\mathrm{bal}} \arrow[r] \arrow[d] & \mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \\
\widehat{T}_{\mathrm{sep}}^{\otimes 2} \otimes H^1(X_{\mathrm{sep}},T)^{\otimes 2} \arrow[r, "prod"] \arrow[d, "\eta"', "\wr"] & \overline{H}^4\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)_{\mathrm{bal}} \arrow[d, "\wr"] & \\
\widehat{T}_{\mathrm{sep}}^{\otimes 2} \otimes (\widehat{T}_{\mathrm{sep}}^{\circ})^{\otimes 2} \otimes \widehat{T}_{\mathrm{sep}}^{\otimes 2} \arrow[r, "\kappa"] & S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec} &
\end{tikzcd}$$

where the product maps are given by (4.1), η identifies $H^1(X_{\mathrm{sep}}, T) = \widehat{T}_{\mathrm{sep}}^{\circ} \otimes \mathrm{Pic}(X_{\mathrm{sep}})$ with $\widehat{T}_{\mathrm{sep}}^{\circ} \otimes \widehat{T}_{\mathrm{sep}}$ and κ is given by the pairing between the first and second factors. Write $[U]$ for the class of the classifying torsor in $H^1(X,T)$. The image of $[U]$ in $H^1(X_{\mathrm{sep}}, T_{\mathrm{sep}}) = \widehat{T}_{\mathrm{sep}}^{\circ} \otimes \widehat{T}_{\mathrm{sep}} = \mathrm{End}(\widehat{T}_{\mathrm{sep}})$ is the identity $1_{\widehat{T}_{\mathrm{sep}}}$. Hence for every $a \in (\widehat{T}_{\mathrm{sep}}^{\otimes 2})^{\Gamma}$, the image of $a \otimes [U] \otimes [U]$ under the diagonal map in the diagram coincides with the canonical image of a in $S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec}$.

Clearly, $\varepsilon\tau = \varepsilon$, hence ε is trivial on the image of $1-\tau$. Let $\Gamma' \subset \Gamma$ be an open subgroup and $a, b \in (\widehat{T}_{\mathrm{sep}})^{\Gamma'} = \mathrm{CH}^1(X_{\mathrm{sep}})^{\Gamma'} = \mathrm{CH}^1(X_L)$, where $L = (F_{\mathrm{sep}})^{\Gamma'}$. Then $\mathrm{Tr}(a \otimes b)$ is the image of the trace map $\mathrm{CH}^2(X_L) \longrightarrow \mathrm{CH}^2(X)$ and therefore, $\varepsilon\big(\mathrm{Tr}(a \otimes b)\big)$ belongs to the image of an element from $\mathrm{CH}^2(X) = \mathrm{CH}^2(BT)$ in the sequence (4.9) and hence is trivial in $\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)$. □

Both compositions of the natural map $(\widehat{T}_{\mathrm{sep}}^{\otimes 2})^{\Gamma}/D \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec}$ with the map $S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec} \longrightarrow (\widehat{T}_{\mathrm{sep}}^{\otimes 2})^{\Gamma}/D$ induced by $ab \mapsto a \otimes b + b \otimes a$ are multiplication by 2. Hence the kernel and the cokernel of $(\widehat{T}_{\mathrm{sep}}^{\otimes 2})^{\Gamma}/D \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec}$ have exponent 2. It follows that 2 times the homomorphism $S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec} \longrightarrow H^2(F, T^0)$ from Theorem 4.4 is trivial.

Theorem 4.4 also yields:

Theorem 4.11. *Let p be an odd prime and let T be an algebraic torus over F. Then*

$$\mathrm{Inv}^3\big(T, \mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{norm}} = \mathrm{Inv}^3\big(T, \mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{lin}} \oplus \mathrm{Inv}^3\big(T, \mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{quad}}$$

and there are natural isomorphisms $\mathrm{Inv}^3\big(T, \mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{lin}} \simeq H^1(F, T^{\circ})\{p\}$ *and* $\mathrm{Inv}^3\big(T, \mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{quad}} \simeq \big(S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\,\mathrm{Dec}\big)\{p\}$.

Example 4.12. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of n elements with the natural action of the symmetric group S_n. A continuous surjective group homomorphism $\Gamma \longrightarrow S_n$ yields a separable field extension L/F of degree n. Consider the torus $T = R_{L/F}(\mathbb{G}_{m,L})/\,\mathbb{G}_m$, where $R_{L/F}$ is the Weil restriction (see [35, Ch. 1, §3.12]). Note that the generic maximal torus of the group

$\mathbf{PGL}_n$ is of this form (see §5b). The character lattice $\widehat{T}_{\mathrm{sep}}$ is the kernel of the augmentation homomorphism $\mathbb{Z}[X] \longrightarrow \mathbb{Z}$.

The dual torus T° is the norm one torus $R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. For every field extension K/F, we have:

$$H^1(K,T) = \mathrm{Br}(KL/K), \qquad H^1(K,T^\circ) = K^\times/N(KL)^\times,$$

where $KL := K \otimes L$, N is the norm map for the extension KL/K and $\mathrm{Br}(KL/K) = \mathrm{Ker}\big(\mathrm{Br}(K) \longrightarrow \mathrm{Br}(KL)\big)$. The pairing

$$K^\times/N(KL)^\times \otimes \mathrm{Br}(KL/K) \longrightarrow H^3\big(F, \mathbb{Q}/\mathbb{Z}(2)\big)$$

defines linear degree 3 invariants of both T and T°.

We claim that $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma/\,\mathrm{Dec} = 0$ and $S^2(\widehat{T}^\circ_{\mathrm{sep}})^\Gamma/\,\mathrm{Dec} = 0$, i.e., T and T° have no nontrivial quadratic degree 3 invariants. We have $\widehat{T}^\circ_{\mathrm{sep}} = \mathbb{Z}[X]/\mathbb{Z}N_X$, where $N_X = \sum x_i$. The group $S^2(\widehat{T}^\circ_{\mathrm{sep}})^\Gamma$ is generated by $S := \sum_{i<j} x_i \cdot x_j$. As $S \in \mathrm{Dec}$, we have $S^2(\widehat{T}^\circ_{\mathrm{sep}})^\Gamma/\,\mathrm{Dec} = 0$.

Let $D = \sum x_i^2$ and $E := \mathrm{Qtr}(x_1 - x_2) = 2S - (n-1)D$, where the quadratic map Qtr is defined in Appendix A-II. The group $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ is generated by E if n is even and by $E/2$ if n is odd. A computation shows that $nE/2 = \mathrm{Qtr}(nx_1 - N_X)$. It follows that the generator of $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ belongs to Dec, hence $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma/\,\mathrm{Dec}$ is trivial.

Note that as the torus T is rational, it follows from Theorem 4.4 and Corollary 4.8 that $\mathrm{Inv}^3\big(T^\circ, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \simeq \mathrm{Br}(L/F)$.

5. Unramified invariants

Let K/F be a field extension and v a discrete valuation of K over F with valuation ring O_v. We say that an element $a \in H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ is *unramified with respect to* v if a belongs to the image of the map $H^n\big(O_v, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ (see [10]). We write $H^n_{\mathrm{nr}}\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the subgroup of the elements in $H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ that are unramified with respect to all discrete valuations of K over F. We have a natural homomorphism

$$H^n\big(F, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(K, \mathbb{Q}/\mathbb{Z}(j)\big). \tag{5.1}$$

A dominant morphism of varieties $Y \longrightarrow X$ yields a homomorphism

$$H^n_{\mathrm{nr}}\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(F(Y), \mathbb{Q}/\mathbb{Z}(j)\big).$$

Proposition 5.1. *Let K/F be a purely transcendental field extension. Then the homomorphism (5.1) is an isomorphism.*

Proof. The statement is well known for the p-components if $p \neq \mathrm{char}\, F$ (see, for example, [10, Prop. 1.2]). It suffices to consider the case $K = F(t)$ and prove the surjectivity of (5.1). The coniveau spectral sequence for the projective line $\mathbb{P}^1$ (see Appendix (A.1)) yields an exact sequence

$$H^n\big(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \coprod_{x \in \mathbb{P}^1} H^{n+1}_x\big(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)\big)$$

and, therefore, a surjective homomorphism $H^n\big(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$. By the projective bundle theorem (classical if $p \neq \mathrm{char}(F)$ and [17, Th. 2.1.11] if $p = \mathrm{char}(F) > 0$), we have

$$H^n\big(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)\big) = H^n\big(F, \mathbb{Q}/\mathbb{Z}(j)\big) \oplus H^{n-2}\big(F, \mathbb{Q}/\mathbb{Z}(j-1)\big)t,$$

where t is a generator of $H^2\big(\mathbb{P}^1, \mathbb{Z}(1)\big) = \mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$. As t vanishes over the generic point of $\mathbb{P}^1$, the result follows. $\square$

Let G be an algebraic group over F. Choose a standard classifying G-torsor $U \longrightarrow U/G$. An invariant $i \in \mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ is called *unramified* if the image of i under $\theta : \mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big)$ is unramified. This is independent of the choice of standard classifying torsor. Indeed, if $U' \longrightarrow U'/G$ is another standard classifying torsor G-torsor, then $(U \times V')/G \longrightarrow U/G$ and $(V \times U')/G \longrightarrow U'/G$ are vector bundles. Hence the field $F\big((U \times U')/G\big)$ is a purely transcendental extension of $F(U/G)$ and $F(U'/G)$ and by Proposition 5.1,

$$H^n_{\mathrm{nr}}\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^n_{\mathrm{nr}}\big(F((U{\times}U')/G), \mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^n_{\mathrm{nr}}\big(F(U'/G), \mathbb{Q}/\mathbb{Z}(j)\big).$$

We write $H^n_{\mathrm{nr}}\big(F(BG), \mathbb{Q}/\mathbb{Z}(j)\big)$ for this common value and $\mathrm{Inv}^n_{\mathrm{nr}}\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the subgroup of unramified invariants. Similarly, we write $\mathrm{Br}_{\mathrm{nr}}\big(F(BG)\big)$ for the *unramified Brauer group* $H^2_{\mathrm{nr}}\big(F(BG), \mathbb{Q}/\mathbb{Z}(1)\big)$.

Proposition 5.2. *The map* $\mathrm{Inv}^n_{\mathrm{nr}}\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(F(BG), \mathbb{Q}/\mathbb{Z}(j)\big)$ *induced by* θ *is an isomorphism.*

Proof. By Theorem 3.3, it suffices to show that $H^n_{\mathrm{nr}}\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big) \subset H^n\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{bal}}$. We follow Totaro's approach (see [16, p. 99]). Consider the open subscheme W of $(U^2/G) \times \mathbb{A}^1$ of all triples (u, u', t) such that $(2-t)u + (t-1)u' \in U$. We have the projection $q : W \longrightarrow U^2/G$, the morphisms $f : W \longrightarrow U/G$ defined by $f(u, u', t) = (2-t)u + (t-1)u'$, and $h_i : U^2/G \longrightarrow W$ defined by $h_i(u, u') = (u, u', i)$ for $i = 1$ and 2. The composition $f \circ h_i$ is the projection $p_i : U^2/G \longrightarrow U/G$ and $q \circ h_i$ is the identity of U^2/G.

Let w_i be the generic point of the pre-image of i with respect to the projection $W \longrightarrow \mathbb{A}^1$ and write O_i for the local ring of W at w_i. The morphisms q, f, and h_i yield F-algebra homomorphisms $F(U^2/G) \longrightarrow O_i$, $F(U/G) \longrightarrow O_i$ and $O_i \longrightarrow F(U/G)$. Note that by Proposition A.11, we have $H^n_{\mathrm{nr}}\big(F(W), \mathbb{Q}/\mathbb{Z}(j)\big) \subset$

$H^n\big(O_i, \mathbb{Q}/\mathbb{Z}(j)\big)$. In the commutative diagram

$$\begin{tikzcd}
H^n_{\mathrm{nr}}\big(F(U/G),\mathbb{Q}/\mathbb{Z}(j)\big) \arrow[r, "f^*"] \arrow[d, hook] & H^n_{\mathrm{nr}}\big(F(W),\mathbb{Q}/\mathbb{Z}(j)\big) \arrow[d, hook] & H^n_{\mathrm{nr}}\big(F(U^2/G),\mathbb{Q}/\mathbb{Z}(j)\big) \arrow[l, "q^*"', "\sim"] \arrow[d, hook] \\
H^n\big(F(U/G),\mathbb{Q}/\mathbb{Z}(j)\big) \arrow[r, "f^*"] \arrow[dr, "p_i^*"'] & H^n\big(O_i,\mathbb{Q}/\mathbb{Z}(j)\big) \arrow[d, "h_i^*"] & H^n\big(F(U^2/G),\mathbb{Q}/\mathbb{Z}(j)\big) \arrow[l, "q^*"'] \arrow[dl, equal] \\
 & H^n\big(F(U^2/G),\mathbb{Q}/\mathbb{Z}(j)\big) &
\end{tikzcd}$$

the top right map q^* is an isomorphism by Proposition 5.1 since the field extension $F(W)/F(U^2/G)$ is purely transcendental. It follows that the restriction of p_i^* on $H^n_{\mathrm{nr}}\big(F(U/G),\mathbb{Q}/\mathbb{Z}(j)\big)$ coincides with $(q^*)^{-1}\circ f^*$ and hence is independent of i. □

5a. **Unramified invariants of tori.**

Proposition 5.3. *If T is a flasque torus, then every invariant in* $\mathrm{Inv}^n\big(G,\mathbb{Q}/\mathbb{Z}(j)\big)$ *is unramified.*

Proof. Consider an exact sequence of tori $1\longrightarrow T\longrightarrow P\longrightarrow Q\longrightarrow 1$ with P quasi-split. Choose a smooth projective compactification X of Q (see [6]). As T is flasque, by [11, Prop. 9], there is a T-torsor $E\longrightarrow X$ extending the T-torsor $P\longrightarrow Q$. The torsor E is classifying and T-rational. Choose an invariant in $\mathrm{Inv}^n\big(G,\mathbb{Q}/\mathbb{Z}(j)\big)$ and consider its image a in $H^n\big(F(X),\mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{bal}}$ (see Theorem 3.3). We show that a is unramified with respect to every discrete valuation v on $F(X)$ over F (cf., [5, Prop. 2.1.8]). By Proposition A.9, a is unramified with respect to the discrete valuation associated to every point $x\in X$ of codimension 1, i.e., $\partial_x(a)=0$.

As X is projective, the valuation ring O_v of the valuation v dominates a point $x\in X$. It follows from Proposition A.11 that a belongs to the image of $H^n\big(O_{X,x},\mathbb{Q}/\mathbb{Z}(j)\big)\longrightarrow H^n\big(F(X),\mathbb{Q}/\mathbb{Z}(j)\big)$. As the local ring $O_{X,x}$ is a subring of O_v, a belongs to the image of $H^n\big(O_v,\mathbb{Q}/\mathbb{Z}(j)\big)\longrightarrow H^n\big(F(X),\mathbb{Q}/\mathbb{Z}(j)\big)$ and hence a is unramified with respect to v. □

Let T be a torus over F. By [12, Lemma 0.6], there is an exact sequence of tori $1\longrightarrow T\longrightarrow T'\longrightarrow P\longrightarrow 1$ with T' flasque and P quasi-split. The following theorem computes the unramified invariants of T in terms of the invariants of T'.

Theorem 5.4. *There is a natural isomorphism*

$$\mathrm{Inv}^n_{\mathrm{nr}}\big(T,\mathbb{Q}/\mathbb{Z}(j)\big)\simeq \mathrm{Inv}^n\big(T',\mathbb{Q}/\mathbb{Z}(j)\big).$$

Proof. Choose an exact sequence $1\longrightarrow T'\longrightarrow P'\longrightarrow S\longrightarrow 1$ with P' a quasi-split torus. Let S' is the cokernel of the composition $T\longrightarrow T'\longrightarrow P'$. We have an exact sequence $1\longrightarrow P\longrightarrow S'\longrightarrow S\longrightarrow 1$. As P is quasi-split, the latter exact sequence splits at the generic point of S and therefore, $F(S')$

is a purely transcendental field extension of $F(S)$. It follows from Propositions 5.1, 5.2, and 5.3 that

$$\operatorname{Inv}^n_{\mathrm{nr}}\big(T,\mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^n_{\mathrm{nr}}\big(F(S'),\mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^n_{\mathrm{nr}}\big(F(S),\mathbb{Q}/\mathbb{Z}(j)\big) \simeq \operatorname{Inv}^n_{\mathrm{nr}}\big(T',\mathbb{Q}/\mathbb{Z}(j)\big) = \operatorname{Inv}^n\big(T',\mathbb{Q}/\mathbb{Z}(j)\big). \quad \square$$

The following corollary is essentially equivalent to [12, Prop. 9.5] in the case when F is of zero characteristic.

Corollary 5.5. *The isomorphism* $\operatorname{Inv}(T,\mathrm{Br}) \xrightarrow{\sim} \operatorname{Pic}(T) = H^1(F,\widehat{T})$ *identifies* $\operatorname{Inv}_{\mathrm{nr}}(T,\mathrm{Br})$ *with the subgroup* $H^1(F,\widehat{T}')$ *of* $H^1(F,\widehat{T})$ *of all elements that are trivial when restricted to all cyclic subgroups of the decomposition group of* T.

Proof. The description of $H^1(F,\widehat{T}')$ as a subgroup of $H^1(F,\widehat{T})$ is given in [12, Prop. 9.5], and this part of the proof is characteristic free. □

In view of Propositions 5.1 and 5.2 we can calculate the group of unramified cohomology for the function field of an arbitrary torus in terms of the invariants of a flasque torus:

Theorem 5.6. *Let S be a torus over F and let $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ be a flasque resolution of S, i.e., T is flasque and P is quasi-split. Then there is a natural isomorphism*

$$H^n_{\mathrm{nr}}\big(F(S),\mathbb{Q}/\mathbb{Z}(j)\big) \simeq \operatorname{Inv}^n\big(T,\mathbb{Q}/\mathbb{Z}(j)\big).$$

Corollary 5.7. *A torus S has no nonconstant unramified degree 3 cohomology with values in $\mathbb{Q}/\mathbb{Z}(2)$ universally, i.e., $H^3_{\mathrm{nr}}\big(K(S),\mathbb{Q}/\mathbb{Z}(2)\big) = H^3\big(K,\mathbb{Q}/\mathbb{Z}(2)\big)$ for any field extension K/F, if and only if S is a direct factor of a rational torus.*

Proof. If S is a direct factor of a rational torus, then S has no nonconstant unramified cohomology by Proposition 5.1.

Conversely, let $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ be a flasque resolution of S. By Theorem 5.6, $\operatorname{Inv}^3\big(T_K,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} = 0$ for every K/F. It follows from Theorem 4.9 that T is invertible and hence S is a factor of a rational torus (see §4a). □

5b. **The Brauer invariant for semisimple groups.** The following theorem was proved by Bogomolov [2, Lemma 5.7] in characteristic zero:

Theorem 5.8. *Let G be a (connected) semisimple group over a field F. Then* $\operatorname{Inv}_{\mathrm{nr}}(G,\mathrm{Br}) = \operatorname{Inv}(G,\mathrm{Br})_{\mathrm{const}} = \mathrm{Br}(F)$ *and* $\mathrm{Br}_{\mathrm{nr}}\big(F(BG)\big) = \mathrm{Br}(F)$.

Proof. Let $G' \longrightarrow G$ be a simply connected cover of G and C the kernel of $G' \longrightarrow G$. By Theorem 2.4, we have

$$\operatorname{Inv}(G,\mathrm{Br})_{\mathrm{norm}} = \operatorname{Pic}(G) = \widehat{C}(F).$$

As the map $\widehat{C}(F) \longrightarrow \widehat{C}(F_{\mathrm{sep}})$ is injective, we can replace F by F_{sep} and assume that the group G is split.

Consider the variety $\mathcal{T}$ of maximal tori in G and the closed subscheme $\mathcal{X} \subset G \times \mathcal{T}$ of all pairs (g, T) with $g \in T$. The generic fiber of the projection $\mathcal{X} \longrightarrow \mathcal{T}$ is the *generic torus* T_{gen} of G. Then T_{gen} is a maximal torus of G_K, where $K := F(\mathcal{T})$. We have an exact sequence

$$1 \longrightarrow C_K \longrightarrow T'_{\mathrm{gen}} \longrightarrow T_{\mathrm{gen}} \longrightarrow 1,$$

where T'_{gen} is the generic torus of G'.

The restriction $\mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} \longrightarrow \mathrm{Inv}(T_{\mathrm{gen}}, \mathrm{Br})_{\mathrm{norm}}$ can be identified with the connecting homomorphism $\widehat{C} \longrightarrow H^1(K, \widehat{T}_{\mathrm{gen}})$. Note that the decomposition group of T_{gen} coincides with the Weyl group W of G by [34, Th. 1], hence $H^1(K, \widehat{T}_{\mathrm{gen}}) \simeq H^1(W, \widehat{T}_{\mathrm{gen}})$.

Let w be a Coxeter element in W.[1] It is the product of reflections with respect to all simple roots (in some order). By [20, Lemma, p. 76], 1 is not an eigenvalue of w on the space of weights $\widehat{T'}_{\mathrm{gen}} \otimes \mathbb{R}$. Let W_0 be the cyclic subgroup in W generated by w. It follows that the first term in the exact sequence

$$(\widehat{T'}_{\mathrm{gen}})^{W_0} \longrightarrow \widehat{C} \longrightarrow H^1(W_0, \widehat{T}_{\mathrm{gen}})$$

is trivial, i.e., the second map is injective. Hence every nonzero character χ in $\widehat{C}$ restricts to a nonzero element in $H^1(W_0, \widehat{T}_{\mathrm{gen}})$. It follows that the image of χ in $H^1(W, \widehat{T}_{\mathrm{gen}})$ is ramified by Theorem 5.5, hence so is χ. $\square$

Appendix A. Generalities

A-I. **Proof of Theorem 2.2.** Suppose that $i(E_{\mathrm{gen}}) = 0$ for an H-invariant i of G. Let K/F be a field extension and $I \longrightarrow \operatorname{Spec} K$ a G-torsor. We need to show that $i(I) = 0$ in $H(K)$.

Suppose first that K is infinite. Find a point $x \in X(K)$ such that I is isomorphic to the pull-back of the classifying torsor with respect to x. Let x' be a rational point of X_K above x and write O for the local ring $O_{X_K, x'}$. The K-algebra O is a regular local ring with residue field K. Therefore, the completion $\widehat{O}$ is isomorphic to $K[[t_1, t_2, \ldots, t_n]]$ over K. Let L be the quotient field of $\widehat{O}$, a field extension of $K(X)$. We have the following diagram of morphisms with a commutative square and three triangles:

$$\begin{tikzcd}
 & \operatorname{Spec} K \arrow[d, shift left] \arrow[dr, "x"] & \\
\operatorname{Spec} L \arrow[ur] \arrow[r] \arrow[d] & \operatorname{Spec} \widehat{O} \arrow[u, shift left] \arrow[r] \arrow[d] & X \\
\operatorname{Spec} K(X) \arrow[r] & \operatorname{Spec} O \arrow[ur] &
\end{tikzcd}$$

The pull-back of the classifying torsor $E \longrightarrow X$ via $\operatorname{Spec} K(X) \longrightarrow X$ is $(E_{\mathrm{gen}})_{K(X)}$. The G-torsor I is the pull-back of $E \longrightarrow X$ with respect to x. Let

[1] We owe the idea to use the Coxeter element and the reference below to S. Garibaldi.

$\widehat{E}$ be the pull-back of $E \longrightarrow X$ via $\operatorname{Spec}\widehat{O} \longrightarrow X$. Therefore, I is the pull-back of $\widehat{E}$. By a theorem of Grothendieck [19, Prop. 8.1], $\widehat{E}$ is the pull-back of I with respect to $\operatorname{Spec}\widehat{O} \longrightarrow \operatorname{Spec}(K)$. It follows that $I_L \simeq (E_{\mathrm{gen}})_L$ as torsors over L. Hence the images of $i(I)$ and $i(E_{\mathrm{gen}})$ in $H(L)$ are equal and therefore, $i(I)_L = 0$. By Property 2.1, we have $i(I) = 0$.

If K is finite, we replace F by $F(\!(t)\!)$ and K by $K(\!(t)\!)$. By the first part of the proof, $i(I)$ belongs to the kernel of $H(K) \longrightarrow H\big(K(\!(t)\!)\big)$ and hence is trivial by Property 2.1 again.

A-II. **Decomposable elements.** Let Γ be a profinite group and A a Γ-lattice. Write A^Γ for the subgroup of Γ-invariant elements in A. Let $\Gamma' \subset \Gamma$ be an open subgroup and choose representatives $\gamma_1, \gamma_2, \dots, \gamma_n$ for the left cosets of Γ' in Γ. We have the *trace map* $\operatorname{Tr} : A^{\Gamma'} \longrightarrow A^\Gamma$ defined by $\operatorname{Tr}(a) = \sum_{i=1}^n \gamma_i a$.

Let $S^2(A)$ be the symmetric square of A. Consider the *quadratic trace map* $\operatorname{Qtr} : A^{\Gamma'} \longrightarrow S^2(A)^\Gamma$ defined by $\operatorname{Qtr}(a) = \sum_{i<j} (\gamma_i a)(\gamma_j a)$. Write $\operatorname{Dec}(A)$ for the subgroup of *decomposable elements* in $S^2(A)^\Gamma$ generated by the square $(A^\Gamma)^2$ of A^Γ and the elements $\operatorname{Qtr}(a)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in A^{\Gamma'}$.

Let B be another Γ-lattice. We write $\operatorname{Dec}(A, B)$ for the subgroup of $(A\otimes B)^\Gamma$ generated by elements of the form $\operatorname{Tr}(a \otimes b)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in A^{\Gamma'}$, $b \in B^{\Gamma'}$.

There is a natural isomorphism

$$S^2(A \oplus B) \simeq S^2(A) \oplus (A \otimes B) \oplus S^2(B).$$

Moreover, the equality

$$\operatorname{Qtr}(a + b) = \operatorname{Qtr}(a) + \big(\operatorname{Tr}(a)\operatorname{Tr}(b) - \operatorname{Tr}(ab)\big) + \operatorname{Qtr}(b)$$

yields the decomposition

$$\operatorname{Dec}(A \oplus B) \simeq \operatorname{Dec}(A) \oplus \operatorname{Dec}(A, B) \oplus \operatorname{Dec}(B).$$

A-III. **Cup-products.** Let $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ be an exact sequence of tori. We consider the connecting maps

$$\partial_1 : H^p\big(F, \widehat{T}(i)\big) \longrightarrow H^{p+1}\big(F, \widehat{Q}(i)\big)$$

for the exact sequence $0 \longrightarrow \widehat{Q}_{\mathrm{sep}} \longrightarrow \widehat{P}_{\mathrm{sep}} \longrightarrow \widehat{T}_{\mathrm{sep}} \longrightarrow 0$ of character Γ-lattices and

$$\partial_2 : H^q\big(F, \widehat{Q}^\circ(j)\big) \longrightarrow H^{q+1}\big(F, \widehat{T}^\circ(j)\big)$$

for the dual sequence of lattices (see notation in §4b).

Lemma A.1. *Let $a \in H^p\big(F, \widehat{T}(i)\big)$ and $b \in H^q\big(F, \widehat{Q}^\circ(j)\big)$ with $i+j \le 2$. Then $\partial_1(a) \cup b = (-1)^{p+1} a \cup \partial_2(b)$ in $H^{p+q+1}\big(F, \mathbb{Z}(i+j)\big)$, where the cup-product is defined in (4.3).*

Proof. By [4, Ch. V, Prop. 4.1], the elements $\partial_1(1_T)$ and $\partial_2(1_Q)$ in

$$H^1\big(F, \widehat{T}^{\circ}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}}\big) = \mathrm{Ext}^1_{\Gamma}\big(\widehat{T}_{\mathrm{sep}}, \widehat{Q}_{\mathrm{sep}}\big)$$

differ by a sign. Write τ for the isomorphism induced by permutation of the factors. By the standard properties of the cup-product, we have

$$\begin{aligned}
\partial_1(a) \cup b &= 1_T \cup \partial_1(a) \cup b \\
&= \partial_1(1_T) \cup a \cup b \\
&= (-1)^{pq}\tau\big(\partial_1(1_T) \cup b \cup a\big) \\
&= (-1)^{pq+1}\tau\big(\partial_2(1_Q) \cup b \cup a\big) \\
&= (-1)^{pq+1}\tau\big(1_Q \cup \partial_2(b) \cup a\big) \\
&= (-1)^{p+1} 1_Q \cup a \cup \partial_2(b) \\
&= (-1)^{p+1} a \cup \partial_2(b).
\end{aligned}$$

□

A-IV. **Cosimplicial abelian groups.** Let $A^\bullet$ be a cosimplicial abelian group

$$A^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} A^1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; A^2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots$$

and write $h_*(A^\bullet)$ for the homology groups of the associated complex of abelian groups. In particular,

$$h_0(A^\bullet) = \mathrm{Ker}\,\big[(d^0 - d^1) : A^0 \longrightarrow A^1\big].$$

We say that the cosimplicial abelian group $A^\bullet$ is *constant* if for every i, all the coface maps $d_j : A^{i-1} \longrightarrow A^i$, $j = 0, 1, \dots i$, are isomorphisms. In this case all the d_j are equal as $d_j = s_j^{-1} = d_{j+1}$, where the s_j are the codegeneracy maps. For a constant cosimplicial abelian group $A^\bullet$, we have $h_0(A^\bullet) = A^0$ and $h_i(A^\bullet) = 0$ for all $i > 0$. We will need the following straightforward statement.

Lemma A.2. *Let* $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow D^\bullet$ *be an exact sequence of cosimplicial abelian groups with* $A^\bullet$ *a constant cosimplicial group. Then the sequence of groups* $0 \longrightarrow A^0 \longrightarrow h_0(B^\bullet) \longrightarrow h_0(C^\bullet) \longrightarrow h_0(D^\bullet)$ *is exact.*

Let H be a contravariant functor from the category of schemes over F to the category of abelian groups. We say that H is *homotopy invariant* if for every vector bundle $E \longrightarrow X$ over F, the induced map $H(X) \longrightarrow H(E)$ is an isomorphism.

For an integer $d > 0$ consider the following property of the functor H:

Property A.3. *For every closed subscheme Z of a scheme X with* $\mathrm{codim}_X(Z) \geq d$*, the natural homomorphism $H(X) \longrightarrow H(X \setminus Z)$ is an isomorphism.*

Let G be an algebraic group over a field F and choose a standard classifying G-torsor $U \longrightarrow U/G$. Let U^i denote the product of i copies of U. We have the G-torsors $U^i \longrightarrow U^i/G$.

Consider the cosimplicial abelian group $A^\bullet = H(U^\bullet/G)$ with $A^i = H(U^{i+1}/G)$ and coface maps $A^{i-1} \longrightarrow A^i$ induced by the projections $U^{i+1}/G \longrightarrow U^i/G$.

Lemma A.4. *Let H be a homotopy invariant functor satisfying Property A.3 for some d. Let $U \longrightarrow U/G$ be a standard classifying G-torsor and U' an open subset of a G-representation V'.*
1. *If $\operatorname{codim}_{V'}(V' \setminus U') \geq d$, then the natural homomorphism $H(U/G) \longrightarrow H\big((U \times U')/G\big)$ is an isomorphism.*
2. *If $\operatorname{codim}_V(V \setminus U) \geq d$, then the cosimplicial group $H(U^\bullet/G)$ is constant.*

Proof. 1. The scheme $(U \times U')/G$ is an open subset of the vector bundle $(U \times V')/G$ over U/G with complement of codimension at least d. The map in question is the composition $H(U/G) \longrightarrow H\big((U \times V')/G\big) \longrightarrow H\big((U \times U')/G\big)$ and both maps in the composition are isomorphisms since H is homotopy invariant and satisfies Property A.3.

2. By the first part of the lemma applied to the G-torsor $U^i \longrightarrow U^i/G$ and $U' = U$, the map $H(U^i/G) \longrightarrow H(U^{i+1}/G)$ induced by a projection $U^{i+1}/G \longrightarrow U^i/G$ is an isomorphism. □

By Lemma A.4, if H is a homotopy invariant functor satisfying Property A.3 for some d, then the group $H(U/G)$ does not depend on the choice of the representation V and the open set $U \subset V$ provided $\operatorname{codim}_V(V \setminus U) \geq d$. Following [33], we denote this group by $H(BG)$.

Example A.5. The split torus $T = (\mathbb{G}_m)^n$ over F acts freely on the product U of n copies of $\mathbb{A}^{r+1} \setminus \{0\}$ with $U/T \simeq (\mathbb{P}^r)^n$, i.e., BT is "approximated" by the varieties $(\mathbb{P}^r)^n$ if "$r \gg 0$." We then have $\mathrm{CH}^*(BT) = S^*(\widehat{T})$, where S^* represents the symmetric algebra and $\widehat{T}$ is the character group of T (see [14, p. 607]). In particular, $\mathrm{Pic}(BT) = \mathrm{CH}^1(BT) = \widehat{T}$. More generally, by the Künneth formula [15, Prop. 3.7],

$$H^*_{\mathrm{Zar}}(BT, \mathcal{K}_*) \simeq \mathrm{CH}^*(BT) \otimes K_*(F) \simeq S^*(\widehat{T}) \otimes K_*(F),$$

where $K_n(F)$ is the Quillen K-group of F and $\mathcal{K}_n$ is the Zariski sheaf associated to the presheaf $U \mapsto K_n(U)$.

A-V. **Étale cohomology.** Let A be a sheaf of abelian groups on the big étale site over F. For a scheme X and a closed subscheme $Z \subset X$ we write $H^*_Z(X, A)$ for the étale cohomology group of X with support in Z and values in A [29, Ch. III, §1]. Write $X^{(i)}$ for the set of points in X of codimension i. For a point $x \in X^{(1)}$ set

$$H^*_x(X, A) = \operatorname*{colim}_{x \in U} H^*_{\overline{\{x\}} \cap U}(U, A),$$

where the colimit is taken over all open subsets $U \subset X$ containing x. If X is a variety, write

$$\partial_x : H^*\big(F(X), A\big) \longrightarrow H^{*+1}_x(X, A)$$

for the residue homomorphisms arising from the *coniveau spectral sequence* [9, 1.2]

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H^{p+q}_x(X, A) \Rightarrow H^{p+q}(X, A). \tag{A.1}$$

Let $f : Y \longrightarrow X$ be a dominant morphism of varieties over F, $y \in Y^{(1)}$, and $x = f(y)$. If $x \in X^{(1)}$, there is a natural homomorphism $f_y^* : H_x^*(X, A) \longrightarrow H_y^*(Y, A)$. The following lemma is straightforward.

Lemma A.6. *Let $f : Y \longrightarrow X$ be a dominant morphism of varieties over F, $y \in Y^{(1)}$ and $x = f(y)$.*

(1) *If x is the generic point of X, then the composition*

$$H^*\big(F(X), A\big) \xrightarrow{f^*} H^*\big(F(Y), A\big) \xrightarrow{\partial_y} H_y^{*+1}(Y, A)$$

is trivial.

(2) *If $x \in X^{(1)}$, the diagram*

$$\begin{array}{ccc} H^*\big(F(X), A\big) & \xrightarrow{\partial_x} & H_x^{*+1}(X, A) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f_y^*} \\ H^*\big(F(Y), A\big) & \xrightarrow{\partial_y} & H_y^{*+1}(Y, A). \end{array}$$

is commutative.

Lemma A.7. *Let X be a geometrically irreducible variety, $Z \subset X$ a closed subvariety of codimension 1, and x the generic point of Z. Let P be a variety over F such that $P(K)$ is dense in P for every field extension K/F with K infinite, and let y be the generic point of $Z \times P$ in $Y := X \times P$. Then the homomorphism $f_y^* : H_x^*(X, A) \longrightarrow H_y^*(Y, A)$ induced by the projection $f : Y \longrightarrow X$ is injective.*

Proof. Assume first that the field F is infinite. An element $\alpha \in H_x^*(X, A)$ is represented by an element $h \in H_{Z \cap U}^*(U, A)$ for a nonempty open set $U \subset X$ containing x. If α belongs to the kernel of $f_y^* : H_x^*(X, A) \longrightarrow H_y^*(Y, A)$, then there is an open subset $W \subset U \times P$ containing y such that h belongs to the kernel of the composition

$$g : H_{Z \cap U}^*(U, A) \longrightarrow H_{(Z \cap U) \times P}^*(U \times P, A) \longrightarrow H_{(Z \times P) \cap W}^*(W, A).$$

As F is infinite, by the assumption on P, there is a rational point $t \in P$ in the image of the dominant composition $(Z \times P) \cap W \hookrightarrow Z \times P \longrightarrow P$. We have $(U \times t) \cap W = U' \times t$ for an open subset $U' \subset U$ such that $x \in U'$. Composing g with the homomorphism $H_{(Z \times P) \cap W}^*(W, A) \longrightarrow H_{Z \cap U'}^*(U', A)$ induced by the morphism $(U', Z \cap U') \longrightarrow (W, (Z \times P) \cap W)$, $u \mapsto (u, t)$, we see that h belongs to the kernel of the restriction homomorphism $H_{Z \cap U}^*(U, A) \longrightarrow H_{Z \cap U'}^*(U', A)$, hence the image of α in $H_x^*(X, A)$ is trivial.

Suppose now that F is a finite field. Choose a prime integer p and an infinite algebraic pro-p-extension L/F. By the first part of the proof, the statement holds for the variety X_L over L. By the restriction-corestriction argument, $\mathrm{Ker}(f_y^*)$ is a p-primary torsion group. Since this holds for every prime p, we have $\mathrm{Ker}(f_y^*) = 0$. □

Corollary A.8. *Let $E \longrightarrow X$ be a G-torsor over a geometrically irreducible variety X with E a G-rational variety and consider the first projection $p : E^2/G \longrightarrow X$. Let $x \in X$ and $y \in E^2/G$ be points of codimension 1 such that $p(y) = x$. Then the homomorphism $p_y^* : H_x^*(X, A) \longrightarrow H_y^*(E^2/G, A)$ is injective.*

Proof. Choose a linear G-space V and a nonempty G-variety U that is G-isomorphic to open subschemes of E and V. We can replace the variety E^2/G by $(E \times U)/G$, an open subscheme in the vector bundle $(E \times V)/G$ over X. Shrinking X around x we may assume that the vector bundle is trivial, i.e., $(E \times U)/G$ is isomorphic to an open subscheme in $X \times V$. The statement then follows from Lemma A.7. □

Proposition A.9. *In the conditions of Corollary A.8, let $h \in H^*\big(F(X), A\big)_{\mathrm{bal}}$. Then $\partial_x(h) = 0$ for every point $x \in X$ of codimension 1.*

Proof. Let $y \in E^2/G$ be the point of codimension 1 such that $p_1(y) = x$. As $p_2(y)$ is the generic point of X, by Lemma A.6(1), $\partial_y(h') = 0$, where $h' = p_1^*(h) = p_2^*(h)$ in $H^*\big(F(E^2/G), A\big)$. It follows from Lemma A.6(2) that $\partial_x(h)$ is in the kernel of $(p_1)_y^* : H_x^*(X, A) \longrightarrow H_y^*(E^2/G, A)$ and hence is trivial by Corollary A.8. □

The sheaf $\mathcal{H}^*\big(\mathbb{Q}/\mathbb{Z}(j)\big)$ defined in §3 has a flasque resolution related to the Cousin complex by [9, §2] (for the p-components with $p \neq \operatorname{char} F$) and [18, Th. 1.4] (for the p-component with $p = \operatorname{char} F > 0$):

$$0 \longrightarrow \mathcal{H}^n\big(\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \coprod_{x \in X^{(0)}} i_{x*} H_x^n\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \coprod_{x \in X^{(1)}} i_{x*} H_x^{n+1}\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \cdots,$$

where $i_x : \operatorname{Spec} F(x) \longrightarrow X$ are the canonical morphisms. In particular, we have:

Proposition A.10. *Let X be a smooth variety over F. The sequence*

$$0 \longrightarrow H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^*(\mathbb{Q}/\mathbb{Z}(j))\big) \longrightarrow H^*\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\partial} \coprod_{x \in X^{(1)}} H_x^{*+1}\big(X, \mathbb{Q}/\mathbb{Z}(j)\big),$$

where $\partial = \coprod \partial_x$, is exact.

Proposition A.11. *Let X be a smooth variety over F and $x \in X$. The sequence*

$$0 \longrightarrow H^*\big(O_{X,x}, \mathbb{Q}/\mathbb{Z}(j))\big) \longrightarrow H^*\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\partial} \coprod_{\substack{x' \in X^{(1)} \\ x' \in \overline{\{x\}}}} H_{x'}^{*+1}\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$$

is exact.

Appendix B. Spectral sequences

B-I. **Hochschild-Serre spectral sequence.** Let

$$\mathcal{A} \xrightarrow{W} \mathcal{B} \xrightarrow{V} \mathcal{C}$$

be additive left exact functors between abelian categories with enough injective objects. If W takes injective objects to V-acyclic ones, there is a spectral sequence

$$E_2^{p,q} = R^p V\big(R^q W(A)\big) \Rightarrow R^{p+q}(VW)(A)$$

for every complex A in $\mathcal{A}$ bounded from below.

We have exact triangles in the derived category of $\mathcal{B}$:

$$\tau_{\leq n} RW(A) \longrightarrow RW(A) \longrightarrow \tau_{\geq n+1} RW(A) \longrightarrow \tau_{\leq n} RW(A)[1], \tag{B.1}$$

$$\tau_{\leq n-1} RW(A) \longrightarrow \tau_{\leq n} RW(A) \longrightarrow R^n W(A)[-n] \longrightarrow \tau_{\leq n-1} RW(A)[1]. \tag{B.2}$$

The filtration on $R^n(VW)(A)$ is defined by

$$F^j R^n(VW)(A) = \operatorname{Im}\big(R^n V(\tau_{\leq(n-j)} RW(A)) \longrightarrow R^n V(RW(A)) = R^n(VW)(A)\big).$$

As $\tau_{\geq n+1} RW(A)$ is acyclic in degrees $\leq n$, the morphism

$$R^n V\big(\tau_{\leq n} RW(A)\big) \longrightarrow R^n V(RW(A)) = R^n(VW)(A)$$

is an isomorphism, in particular, $F^0 R^n(VW)(A) = R^n(VW)(A)$.

The *edge* homomorphism is defined as the composition

$$R^n(VW)(A) \xrightarrow{\sim} R^n V\big(\tau_{\leq n} RW(A)\big) \longrightarrow R^n V\big(R^n W(A)[-n]\big) = V\big(R^n W(A)\big).$$

Moreover, the kernel $F^1 R^n(VW)(A)$ of the edge homomorphism is isomorphic to $R^n V\big(\tau_{\leq n-1} RW(A)\big)$. We define the morphism d_n as the composition

$$d_n : F^1 R^n(VW)(A) \longrightarrow R^n V\big(R^{n-1} W(A)[-n+1]\big) = R^1 V\big(R^{n-1} W(A)\big) = E_2^{1,n-1}.$$

B-II. **First spectral sequence.** Let X be a smooth variety over a field F. We have the functors

$$\mathrm{Sheaves}_{\acute{e}t}(X) \xrightarrow{q_*} \mathrm{Sheaves}_{\acute{e}t}(F) \xrightarrow{V} Ab,$$

where q_* is the push-forward map for the structure morphism $q : X \longrightarrow \operatorname{Spec}(F)$ and $V(M) = H^0(F, M)$.

Consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p\big(F, H^q(X_{\mathrm{sep}}, \mathbb{Z}(2)\big) \Rightarrow H^{p+q}\big(X, \mathbb{Z}(2)\big). \tag{B.3}$$

Set $\Delta(i) := Rq_*\big(\mathbb{Z}(i)\big)$ for $i = 1$ or 2. Then $\Delta(i)$ is the complex of étale sheaves on F concentrated in degrees ≥ 1. The j^{th} term $F^j H^n(X, \mathbb{Z}(i))$ of the filtration on $H^n\big(X, \mathbb{Z}(i)\big)$ coincides with the image of the canonical homomorphism

$$H^n\big(F, \tau_{\leq(n-j)} \Delta(i)\big) \longrightarrow H^n\big(F, \Delta(i)\big) = H^n\big(X, \mathbb{Z}(i)\big).$$

Let M be a Γ-lattice viewed as an étale sheaf over F. Note that there are canonical isomorphisms

$$H^*\big(F, M^\circ \otimes \Delta(i)\big) = \operatorname{Ext}^*_F\big(M, \Delta(i)\big) = \operatorname{Ext}^*_X\big(q^* M, \mathbb{Z}(i)\big), \tag{B.4}$$

where $M^\circ := \mathrm{Hom}(M,\mathbb{Z})$ is the dual lattice.

Consider also the following product map

$$\mathbb{Z}(1)\otimes^L \Delta(1) \longrightarrow Rq_*\big(q^*\mathbb{Z}(1)\otimes^L \mathbb{Z}(1)\big) \longrightarrow Rq_*\big(\mathbb{Z}(1)\otimes^L \mathbb{Z}(1)\big) \longrightarrow Rq_*\big(\mathbb{Z}(2)\big).$$

The complex $\mathbb{Z}(1)\otimes^L \tau_{\leq 2}\Delta(1)$ is trivial in degrees > 3, hence we have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}(1)\otimes^L \tau_{\leq 2}\Delta(1) \arrow[r, "prod"] \arrow[d] & \tau_{\leq 3}Rq_*\big(\mathbb{Z}(2)\big) \arrow[d] \arrow[r, phantom, "="] & \tau_{\leq 3}\Delta(2) \\
\mathbb{Z}(1)\otimes^L \Delta(1) \arrow[r, "prod"] & Rq_*\big(\mathbb{Z}(2)\big) \arrow[r, phantom, "="] & \Delta(2).
\end{tikzcd}$$

There are canonical morphisms from (B.2):

$$h_2 : \tau_{\leq 2}\Delta(1)[2] \longrightarrow H^2\big(X_{\mathrm{sep}},\mathbb{Z}(1)\big),$$

$$h_3 : \tau_{\leq 3}\Delta(1)[3] \longrightarrow H^3\big(X_{\mathrm{sep}},\mathbb{Z}(2)\big).$$

Consider an element

$$\delta \in H^1(F, M\otimes F_{\mathrm{sep}}^\times) = \mathrm{Ext}^1_F(M^\circ,\mathbb{G}_{m,F}) = \mathrm{Ext}^2_F\big(M^\circ,\mathbb{Z}(1)\big),$$

and view δ as a morphism

$$\delta : M^\circ \longrightarrow \mathbb{Z}(1)[2]$$

in $D^+\big(\mathrm{Sheaves}_{\acute{e}t}(F)\big)$.

The following diagram

$$\begin{tikzcd}
M^\circ\otimes\Delta(1)[2] \arrow[r, "\delta\otimes 1"] & \mathbb{Z}(1)\otimes^L\Delta(1)[4] \arrow[r, "prod"] & \Delta(2)[4] \\
M^\circ\otimes\tau_{\leq 2}\Delta(1)[2] \arrow[r, "\delta\otimes 1"] \arrow[u, "(1\otimes i_2)[2]"] \arrow[d, "1\otimes h_2"] & \mathbb{Z}(1)\otimes^L\tau_{\leq 2}\Delta(1)[4] \arrow[r, "prod"] \arrow[u, "(1\otimes i_2)[4]"] \arrow[d, "1\otimes h_2"] & \tau_{\leq 3}\Delta(2)[4] \arrow[u, "(i_3)[4]"] \arrow[d, "h_3"] \\
M^\circ\otimes H^2\big(X_{\mathrm{sep}},\mathbb{Z}(1)\big) \arrow[r, "\delta\otimes 1"] & \mathbb{Z}(1)\otimes^L H^2\big(X_{\mathrm{sep}},\mathbb{Z}(1)\big)[2] \arrow[r, "prod"] & H^3\big(X_{\mathrm{sep}},\mathbb{Z}(2)\big)[1],
\end{tikzcd}$$

where $i_2 : \tau_{\leq 2}\Delta(1) \longrightarrow \Delta(1)$ and $i_3 : \tau_{\leq 3}\Delta(2) \longrightarrow \Delta(2)$ are natural morphisms, is commutative.

By (B.4), we have

$$H^0\big(F, M^\circ\otimes\Delta(1)[2]\big) = \mathrm{Ext}^2_F\big(M,\Delta(1)\big) = \mathrm{Ext}^2_X\big(q^*M,\mathbb{Z}(1)\big).$$

Furthermore, the diagram above yields a commutative square

$$\begin{tikzcd}
\mathrm{Ext}^2_X\big(q^*M,\mathbb{Z}(1)\big) \arrow[r, "q^*(\delta)\,\cup\,-"] \arrow[d, "d_2"] & F^1H^4\big(X,\mathbb{Z}(2)\big) \arrow[d, "d_4"] \\
\mathrm{Hom}_\Gamma\big(M, H^2(X_{\mathrm{sep}},\mathbb{Z}(1)\big) \arrow[r, "j"] & H^1\big(F, H^3(X_{\mathrm{sep}},\mathbb{Z}(2))\big),
\end{tikzcd}$$

where d_2 is the edge map coming from the spectral sequence

$$\operatorname{Ext}_F^p\big(M, H^q(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \Rightarrow \operatorname{Ext}_X^{p+q}\big(q^*M, \mathbb{Z}(1)\big) \tag{B.5}$$

and j coincides with the composition

$$\operatorname{Hom}_\Gamma\big(M, H^2(X_{\mathrm{sep}}, \mathbb{Z}(1)\big) = H^0\big(F, M^\circ \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \xrightarrow{\delta \cup -}$$
$$H^1\big(F, F_{\mathrm{sep}}^\times \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \xrightarrow{\rho} H^1\big(F, H^3\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)\big),$$

with ρ given by the product map.

Now suppose the group $H^2\big(X_{\mathrm{sep}}, \mathbb{Z}(1)\big)$, which is canonically isomorphic to $\operatorname{Pic}(X_{\mathrm{sep}})$, is a lattice. Let $M = \operatorname{Pic}(X_{\mathrm{sep}})$ and consider the torus T over F with $\widehat{T}_{\mathrm{sep}} = M$. It follows that

$$\delta \in H^1(F, T^\circ) = H^1(F, \widehat{T}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^\times) = H^2\big(F, \widehat{T}_{\mathrm{sep}} \otimes \mathbb{Z}(1)\big),$$

where T° is the dual torus. Note that $\delta \cup 1_M = \delta$, where

$$1_M \in H^0\big(F, M^\circ \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) = \operatorname{End}_\Gamma(M)$$

is the identity.

The top map in the last diagram is given by the pairing

$$H^1(X, T^0) \otimes H^1(X, T) \longrightarrow F^1H^4\big(X, \mathbb{Z}(2)\big), \tag{B.6}$$

$$a \otimes b \mapsto a \cup b \tag{B.7}$$

defined as the cup-product in (4.3),

$$H^2\big(X, \widehat{T}(1)\big) \otimes H^2\big(X, \widehat{T}^\circ(1)\big) \longrightarrow F^1H^4\big(X, \mathbb{Z}(2)\big),$$

if we identify $\operatorname{Ext}_X^2(q^*M, \mathbb{Z}(1))$ with $H^2\big(X, \widehat{T}^\circ(1)\big) = H^1(X, T)$.

In this case, the homomorphism

$$\rho : H^1(F, T^\circ) \longrightarrow H^1\big(F, H^3\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)\big) \tag{B.8}$$

is given by the product homomorphism

$$T^\circ(F_{\mathrm{sep}}) = F_{\mathrm{sep}}^\times \otimes \widehat{T}_{\mathrm{sep}} = F_{\mathrm{sep}}^\times \otimes \operatorname{Pic}(X_{\mathrm{sep}}) \longrightarrow H^3(X_{\mathrm{sep}}, \mathbb{Z}(2)).$$

A T-torsor $E \longrightarrow X$ is called *universal* if the class of E in $H^1(X, T) = \operatorname{Ext}_X^2\big(q^*M, \mathbb{Z}(1)\big)$ satisfies $d_2\big([E]\big) = 1_M$ (see [7]).

Commutativity of the previous diagram gives:

Proposition B.1. *Let X be a smooth variety over F such that* $\operatorname{Pic}(X_{\mathrm{sep}})$ *is a lattice. Let T be the torus over F satisfying $\widehat{T}_{\mathrm{sep}} = \operatorname{Pic}(X_{\mathrm{sep}})$ and let E be a universal T-torsor over X with the class $[E] \in H^1(X, T)$. Then for every $\delta \in H^1(F, T^\circ)$, we have*

$$d_4\big(q^*(\delta) \cup [E]\big) = \rho(\delta),$$

where $d_4 : F^1H^4\big(X, \mathbb{Z}(2)\big) \longrightarrow H^1\big(F, H^3(X_{\mathrm{sep}}, \mathbb{Z}(2))\big)$ is the map induced by the Hochschild-Serre spectral sequence (B.3) and the cup-product is taken for the pairing (B.6).

B-III. **Second spectral sequence.** We assume that $H^3\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big) = 0$, hence in particular $E_2^{0,3} = 0$ in the spectral sequence (B.3) and so $E_\infty^{2,2} \subset E_2^{2,2}$. Therefore, we have a canonical map

$$e_4 : F^2H^4\big(X, \mathbb{Z}(2)\big) \longrightarrow E_\infty^{2,2} \hookrightarrow E_2^{2,2} = H^2\big(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2)\big).$$

Let N be a Γ-lattice. Consider an element

$$\gamma \in H^2(F, N \otimes F_{\mathrm{sep}}^\times) = \mathrm{Ext}_F^2(N^\circ, \mathbb{G}_{m,F}) = \mathrm{Ext}_F^3\big(N^\circ, \mathbb{Z}(1)\big),$$

and view γ as a morphism

$$\gamma : N^\circ \longrightarrow \mathbb{Z}(1)[3]$$

in $D^+\big(\mathrm{Sheaves}_{ét}(F)\big)$.

As above, the commutative diagram

$$\begin{tikzcd}
N^\circ \otimes \Delta(1)[1] \arrow[r, "\gamma\otimes 1"] & \mathbb{Z}(1) \otimes^L \Delta(1)[4] \arrow[r, "prod"] & \Delta(2)[4] \\
N^\circ \otimes \tau_{\le 1}\Delta(1)[1] \arrow[r, "\gamma\otimes 1"] \arrow[u, "(1\otimes i_1)[1]"] \arrow[d, "1\otimes h_1"] & \mathbb{Z}(1) \otimes^L \tau_{\le 1}\Delta(1)[4] \arrow[r, "prod"] \arrow[u, "(1\otimes i_1)[4]"] \arrow[d, "1\otimes h_1"] & \tau_{\le 2}\Delta(2)[4] \arrow[u, "(i_2)[4]"] \arrow[d, "h_2"] \\
N^\circ \otimes H^1\big(X_{\mathrm{sep}}, \mathbb{Z}(1)\big) \arrow[r, "\gamma\otimes 1"] & \mathbb{Z}(1) \otimes^L H^1\big(X_{\mathrm{sep}}, \mathbb{Z}(1)\big)[3] \arrow[r, "prod"] & H^2\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)[2],
\end{tikzcd}$$

where i_1, i_2, h_1 and h_2 are defined in a similar fashion as in §B-II, yields a commutative square

$$\begin{tikzcd}
\mathrm{Ext}_X^1(q^*N, \mathbb{Z}(1)) \arrow[r, "q^*(\gamma)\,\cup\,-"] \arrow[d, "d_1"] & F^2H^4\big(X, \mathbb{Z}(2)\big) \arrow[d, "e_4"] \\
\mathrm{Hom}_\Gamma\big(N, H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)\big) \arrow[r, "k"] & H^2\big(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2))\big),
\end{tikzcd}$$

where d_1 is the edge map coming from the spectral sequence

$$\mathrm{Ext}_F^p\big(N, H^q(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \Rightarrow \mathrm{Ext}_X^{p+q}\big(q^*N, \mathbb{Z}(1)\big)$$

and k coincides with the composition

$$\mathrm{Hom}_\Gamma\big(N, H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)\big) = H^0\big(F, N^\circ \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \xrightarrow{\gamma\,\cup\,-}$$
$$H^2\big(F, F_{\mathrm{sep}}^\times \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \longrightarrow H^2\big(F, H^2\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)\big)$$

with the last homomorphism given by the product map.

Suppose N is a Γ-lattice in $F_{\mathrm{sep}}[X]^\times$ such that the composition $N \hookrightarrow F_{\mathrm{sep}}[X]^\times \longrightarrow F_{\mathrm{sep}}[X]^\times / F_{\mathrm{sep}}^\times$ is an isomorphism. Consider the torus Q with $\widehat{Q}_{\mathrm{sep}} = N$, so that $\gamma \in H^2(F, Q^\circ)$.

Note that $\gamma \cup i_N = \gamma$, where

$$i_N \in H^0\big(F, N^\circ \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) = \mathrm{Hom}_\Gamma(N, F_{\mathrm{sep}}[X]^\times)$$

is the embedding.

The top map in the previous diagram is given by the pairing

$$\begin{aligned} H^2(X, Q^0) \otimes H^0(X, Q) &\longrightarrow F^2H^4\big(X, \mathbb{Z}(2)\big) \\ a \otimes b &\longmapsto a \cup b, \end{aligned} \tag{B.9}$$

defined as the cup-product in (4.3),

$$H^3\big(X, \widehat{Q}(1)\big) \otimes H^1\big(X, \widehat{Q}^\circ(1)\big)\big) \longrightarrow H^4\big(X, \mathbb{Z}(2)\big),$$

if we identify $\mathrm{Ext}^1_X\big(q^*N, \mathbb{Z}(1)\big)$ with $H^1\big(X, \widehat{Q}^\circ(1)\big) = H^0(X, Q)$.

The inclusion of $\widehat{Q}_{\mathrm{sep}}$ into $F_{\mathrm{sep}}[X]^\times$ yields a morphism $\varepsilon : X \longrightarrow Q$ that can be viewed as an element of $H^0(X, Q)$. Consider the map

$$\mu : H^2(F, Q^\circ) \longrightarrow H^2\big(F, H^2\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)\big) \tag{B.10}$$

given by composition with the product homomorphism

$$Q^\circ(F_{\mathrm{sep}}) = F^\times_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}} \longrightarrow F^\times_{\mathrm{sep}} \otimes H^1\big(X_{\mathrm{sep}}, \mathbb{Z}(1)\big) \longrightarrow H^2\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big).$$

We have proved:

Proposition B.2. *Let X be a smooth variety over F such that $H^3\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big) = 0$. Let N be a Γ-lattice in $F_{\mathrm{sep}}[X]^\times$ such that the composition $N \hookrightarrow F_{\mathrm{sep}}[X]^\times \longrightarrow F_{\mathrm{sep}}[X]^\times/F^\times_{\mathrm{sep}}$ is an isomorphism. Let Q be the torus over F satisfying $\widehat{Q}_{\mathrm{sep}} = N$. Then for every $\gamma \in H^2(F, Q^\circ)$, we have*

$$e_4\big(q^*(\gamma) \cup \varepsilon\big) = \mu(\gamma),$$

where $e_4 : F^2H^4\big(X, \mathbb{Z}(2)\big) \longrightarrow H^2\big(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2))\big)$ is the map induced by the Hochschild-Serre spectral sequence (B.3) and the cup-product is taken for the pairing (B.9).

B-IV. **Relative étale cohomology.** Let X be a smooth variety over F. Following [23, §3], we define the relative étale cohomology groups as follows. Recall that $\Delta(i) = Rq_*\big(\mathbb{Z}(i)\big)$ for $i = 1$ and 2, where $q : X \longrightarrow \mathrm{Spec}(F)$ is the structure morphism, and let $\Delta'(i)$ be the cone of the natural morphism $\mathbb{Z}(i) \longrightarrow \Delta(i)$ in $D_+\big(\mathrm{Sheaves}_{\acute{e}t}(F)\big)$. Define $H^*\big(X/F, \mathbb{Z}(2)\big) := H^*\big(F, \Delta'(2)\big)$. There is an infinite exact sequence

$$\ldots \longrightarrow H^i\big(F, \mathbb{Z}(2)\big) \longrightarrow H^i\big(X, \mathbb{Z}(2)\big) \longrightarrow H^i\big(X/F, \mathbb{Z}(2)\big) \longrightarrow H^{i+1}\big(F, \mathbb{Z}(2)\big) \longrightarrow \cdots$$

If X has a rational point, we have

$$H^i\big(X/F, \mathbb{Z}(2)\big) = \overline{H}^i\big(X, \mathbb{Z}(2)\big) := H^i\big(X, \mathbb{Z}(2)\big)/H^i\big(F, \mathbb{Z}(2)\big).$$

There is a Hochschild-Serre type spectral sequence [23, §3]

$$E_2^{p,q} = H^p\big(F, H^q(X_{\mathrm{sep}}/F_{\mathrm{sep}}, \mathbb{Z}(2))\big) \Rightarrow H^{p+q}\big(X/F, \mathbb{Z}(2)\big), \tag{B.11}$$

and we have by [23, Lemma 3.1] that

$$H^q\big(X_{\mathrm{sep}}/F_{\mathrm{sep}}, \mathbb{Z}(2)\big) = \begin{cases} 0, & \text{if } q \le 0; \\ \text{uniquely divisible group}, & \text{if } q = 1; \\ \overline{H}^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2), & \text{if } q = 2; \\ H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2), & \text{if } q = 3. \end{cases}$$

It follows that $E_2^{p,q} = 0$ if $q \le 1$ and $p > 0$. Comparing the spectral sequences (B.3) and (B.11), by Proposition B.1 we have:

Proposition B.3. *Let X be a smooth variety over F such that $X(F) \neq \emptyset$. If $H^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2) = K_2(F_{\mathrm{sep}})$, then the spectral sequence (B.11) yields an exact sequence*

$$0 \longrightarrow H^1\big(F, H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)\big) \xrightarrow{\alpha} \overline{H}^4\big(X, \mathbb{Z}(2)\big) \longrightarrow \overline{H}^4\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)^{\Gamma} \longrightarrow H^2\big(F, H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)\big).$$

If, moreover, the group $\mathrm{Pic}(X_{\mathrm{sep}})$ is a lattice and T is the torus over F such that $\widehat{T}_{\mathrm{sep}} = \mathrm{Pic}(X_{\mathrm{sep}})$, then $\alpha\big(\rho(\delta)\big) = q^*(\delta) \cup [E]$ for every $\delta \in H^1(F, T^\circ)$, where ρ is defined in (B.8) and E is a universal T-torsor over X.

Comparing the spectral sequences (B.3) and (B.11), by Proposition B.2 we have:

Proposition B.4. *Let X be a smooth variety over F such that $X(F) \neq \emptyset$. If $H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2) = 0$, then the spectral sequence (B.11) yields an exact sequence*

$$0 \longrightarrow H^2\big(F, \overline{H}^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)\big) \xrightarrow{\beta} \overline{H}^4\big(X, \mathbb{Z}(2)\big) \longrightarrow \overline{H}^4\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)^{\Gamma} \longrightarrow H^3\big(F, \overline{H}^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)\big).$$

If N is a Γ-lattice in $F_{\mathrm{sep}}[X]^\times$ such that the composition $N \hookrightarrow F_{\mathrm{sep}}[X]^\times \longrightarrow F_{\mathrm{sep}}[X]^\times / F_{\mathrm{sep}}^\times$ is an isomorphism and Q is the torus over F satisfying $\widehat{Q}_{\mathrm{sep}} = N$, then $\beta\big(\mu(\gamma)\big) = q^*(\gamma) \cup \varepsilon$ for every $\gamma \in H^2(F, Q^\circ)$, where μ is defined in (B.10) and $\varepsilon \in H^0(X, Q)$ is given by the inclusion of $\widehat{Q}_{\mathrm{sep}}$ into $F_{\mathrm{sep}}[X]^\times$.

References

[1] S. Bloch and K. Kato, *p-adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. (1986), no. 63, 107–152.

[2] F. A. Bogomolov, *The Brauer group of quotient spaces of linear representations*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 3, 485–516, 688.

[3] M. Borovoi and C. Demarche, *Manin obstruction to strong approximation for homogeneous spaces*, arXiv:0912.0408v2 (2010), 52 pages.

[4] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, With an appendix by David A. Buchsbaum, Reprint of the 1956 original.

[5] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Amer. Math. Soc., Providence, RI, 1995, pp. 1–64.

[6] J.-L. Colliot-Thélène, D. Harari, and A. N. Skorobogatov, *Compactification équivariante d'un tore (d'après Brylinski et Künnemann)*, Expo. Math. **23** (2005), no. 2, 161–170.

[7] J.-L. Colliot-Thélène and J.-J. Sansuc, *La descente sur les variétés rationnelles. II*, Duke Math. J. **54** (1987), no. 2, 375–492.

[8] J.-L. Colliot-Thélène, *Résolutions flasques des groupes réductifs connexes*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 5, 331–334.

[9] J.-L. Colliot-Thélène, R. Hoobler, and B. Kahn, *The Bloch-Ogus-Gabber theorem*, Algebraic K-theory (Toronto, ON, 1996), Amer. Math. Soc., Providence, RI, 1997, pp. 31–94.

[10] J.-L. Colliot-Thélène and M. Ojanguren, *Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford*, Invent. Math. **97** (1989), no. 1, 141–158.

[11] J.-L. Colliot-Thélène and J.-J.Sansuc, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 175–229.

[12] J.-L. Colliot-Thélène and J.-J.Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra **106** (1987), no. 1, 148–205.

[13] J.-L. Colliot-Thélène and F. Xu, *Brauer-Manin obstruction for integral points of homogeneous spaces and representation by integral quadratic forms*, Compos. Math. **145** (2009), no. 2, 309–363, With an appendix by Dasheng Wei and Xu.

[14] D. Edidin and W. Graham, *Equivariant intersection theory*, Invent. Math. **131** (1998), no. 3, 595–634.

[15] H. Esnault, B. Kahn, M. Levine, and E. Viehweg, *The Arason invariant and mod* 2 *algebraic cycles*, J. Amer. Math. Soc. **11** (1998), no. 1, 73–118.

[16] R. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological invariants in Galois cohomology*, American Mathematical Society, Providence, RI, 2003.

[17] M. Gros, *Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique*, Mém. Soc. Math. France (N.S.) (1985), no. 21, 87.

[18] M. Gros and N. Suwa, *La conjecture de Gersten pour les faisceaux de Hodge-Witt logarithmique*, Duke Math. J. **57** (1988), no. 2, 615–628.

[19] A. Grothendieck, *Schémas en groupes. III: Structure des schémas en groupes réductifs*, Springer-Verlag, Berlin, 1962/1964, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153.

[20] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[21] O. Izhboldin, *On p-torsion in K_*^M for fields of characteristic p*, Algebraic K-theory, Amer. Math. Soc., Providence, RI, 1991, pp. 129–144.

[22] B. Kahn, *Descente galoisienne et K_2 des corps de nombres*, K-Theory **7** (1993), no. 1, 55–100.

[23] B. Kahn, *Applications of weight-two motivic cohomology*, Doc. Math. **1** (1996), No. 17, 395–416.

[24] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits.

[25] S. Lang, *Algebraic groups over finite fields*, Amer. J. Math. **78** (1956), 555–563.

[26] S. Lichtenbaum, *The construction of weight-two arithmetic cohomology*, Invent. Math. **88** (1987), no. 1, 183–215.

[27] S. Lichtenbaum, *New results on weight-two motivic cohomology*, The Grothendieck Festschrift, Vol. III, Birkhäuser Boston, Boston, MA, 1990, pp. 35–55.

[28] A. S. Merkurjev and I. A. Panin, *K-theory of algebraic tori and toric varieties*, K-Theory **12** (1997), no. 2, 101–143.

[29] J. S. Milne, *Étale cohomology*, Princeton University Press, Princeton, N.J., 1980.

[30] D. Quillen, *Higher algebraic K-theory. I*, (1973), 85–147. Lecture Notes in Math., Vol. 341.

[31] M. Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), No. 16, 319–393 (electronic).

[32] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. Reine Angew. Math. **327** (1981), 12–80.

[33] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.

[34] V. E. Voskresenskiĭ, *Maximal tori without affect in semisimple algebraic groups*, Mat. Zametki **44** (1988), no. 3, 309–318, 410.

[35] V. E. Voskresenskiĭ, *Algebraic groups and their birational invariants*, Translations of Mathematical Monographs, vol. 179, American Mathematical Society, Providence, RI, 1998, Translated from the Russian manuscript by Boris Kunyavski [Boris È. Kunyavskiĭ].

Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA

E-mail address: blinstein@math.ucla.edu

Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA

E-mail address: merkurev@math.ucla.edu