A COPRODUCT STRUCTURE ON THE FORMAL AFFINE DEMAZURE ALGEBRA

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Abstract. In the present paper we generalize the coproduct structure on nil Hecke rings introduced and studied by Kostant-Kumar to the context of an arbitrary algebraic oriented cohomology theory and its associated formal group law. We then construct an algebraic model of the T-equivariant oriented cohomology of the variety of complete flags.

Contents

In [KK86], [KK90] and [Ku02] Kostant and Kumar introduced the language of (nil-)Hecke rings and described in a purely algebraic way equivariant cohomology and K-theory of flag varieties. In particular, for a given simply connected semisimple split linear algebraic group G with a maximal split torus T and Borel subgroup $B \supset T$, they established two isomorphisms (see [KK90, Thm. 4.4], [Ku02, Cor. 11.3.17])

$$S(\Lambda) \otimes_{S(\Lambda)^W} S(\Lambda) \xrightarrow{\simeq} H_T(G/B;\mathbb{Q}) \quad \text{and} \quad \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z}[\Lambda] \xrightarrow{\simeq} K_T(G/B),$$

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where $S(\Lambda)$ is the symmetric algebra of the group of characters Λ of T, $\mathbb{Z}[\Lambda]$ is the integral group ring, $S(\Lambda)^W$ and $\mathbb{Z}[\Lambda]^W$ denote the respective subrings of invariants under the action of the Weyl group W, and $H_T(G/B,\mathbb{Q})$, $K_T(G/B)$ are the T-equivariant cohomology and the K-theory of the variety of Borel subgroups G/B, respectively.

More precisely, they constructed algebraic models $\mathtt{\Lambda}$ and $\mathtt{\Psi}$ of the mentioned equivariant theories, dual respectively to the nil-Hecke and Hecke rings, together with explicit maps inducing the ring isomorphisms

$$(*)\qquad S(\Lambda)\otimes_{S(\Lambda)^W} S(\Lambda) \xrightarrow{\simeq} \mathtt{\Lambda} \quad\text{and}\quad \mathbb{Z}[\Lambda]\otimes_{\mathbb{Z}[\Lambda]^W}\mathbb{Z}[\Lambda] \xrightarrow{\simeq} \mathtt{\Psi},$$

where the product structure on $\mathtt{\Lambda}$ and $\mathtt{\Psi}$ is induced by the coproduct structure on the (nil-)Hecke ring. Indeed, they showed that all Hecke-type (Demazure, BGG) operators on $S(\Lambda)$ and $\mathbb{Z}[\Lambda]$ actually sit in a bigger but simpler ring in the so called twisted group algebra Q_W and, hence, they defined the respective coproduct to be the restriction of a natural one given on Q_W.

One goal of the present paper is to generalize their construction of the duals $\mathtt{\Lambda}$ and $\mathtt{\Psi}$ and of the coproduct for arbitrary one-dimensional commutative formal group law F (e.g. additive for $\mathtt{\Lambda}$, multiplicative for $\mathtt{\Psi}$, or universal) over a coefficient ring R. Namely, we define the coproduct on the formal affine Demazure algebra $\mathbf{D}_F$, already introduced in [HMSZ] as a generalization of the nil-Hecke ring, by restricting the natural coproduct given on a larger formal twisted group algebra.

As an application we provide a complete description of the algebra $\mathbf{D}_F$ in terms of generators and relations, hence, generalizing [HMSZ, Thm. 5.14]. Moreover, we show (see Theorem 11.4) that after inverting the torsion index $\mathfrak{t}$ of G or, more generally, under the assumption that the characteristic map is surjective, there is an isomorphism

$$R[\![\Lambda]\!]_F\otimes_{R[\![\Lambda]\!]_F^W} R[\![\Lambda]\!]_F \xrightarrow{\simeq} \mathbf{D}_F^*,$$

where $R[\![\Lambda]\!]_F$ is the formal group algebra introduced in [CPZ, §2] and $\mathbf{D}_F^*$ is the dual of $\mathbf{D}_F$. Note that specializing to the additive and multiplicative periodic formal group laws (see Example 1.1) we obtain the isomorphisms (*). Also note that the algebra $\mathbf{D}_F^*$ is defined under very few restrictions on the base ring R, e.g. $2\mathfrak{t}$ is not a zero divisor in R, hence allowing us to consider various formal group laws over finite fields as well.

According to Levine-Morel [LM07], to any algebraic oriented cohomology theory $\mathtt{h}(-)$ one can associate a formal group law F over the coefficient ring $R=\mathtt{h}(pt)$; moreover, there is a universal algebraic oriented cohomology theory $\Omega(-)$, called algebraic cobordism, corresponding to the universal formal group law U over the Lazard ring $\mathbb{L}=\Omega(pt)$. The associated T-equivariant theory, denoted by $\Omega_T(G/B)$, has been intensively studied during the last years by Deshpande, Heller, Kiritchenko, Krishna and Malagon-Lopez. In particular, using comparison results and the Atiyah-Hirzebruch spectral sequence for complex cobordism it was shown [KiKr, Thm. 5.1] that after inverting the torsion index the ring $\Omega_T(G/B)$ can be identified with $\mathbb{L}[\Lambda]_U\otimes_{\mathbb{L}[\Lambda]_U^W}\mathbb{L}[\Lambda]_U$. We also refer to [HHH] for similar results concerning T-equivariant topological complex oriented theories and to [GR12] for further discussion about the Borel model for $\Omega_T(G/B)$. In this context, our (Hecke-type) algebra $\mathbf{D}_F^*$ can be viewed as an algebraic model for the algebraic oriented T-equivariant cohomology $\mathtt{h}_T(G/B)$.

This paper is based on the techniques developed in [CPZ] and especially in [HMSZ] devoted to the unification of various existing geometric approaches to Hecke-type algebras. It is organized as follows:

In section 1 we recall basic properties of the formal group algebra $R[\![\Lambda]\!]_F$ following [CPZ, §2]. In section 2 we discuss the regularity of elements of $R[\![\Lambda]\!]_F$ corresponding to roots. In section 3 we study properties of the localization of $R[\![\Lambda]\!]_F$. In section 4 we recall the definition of formal Demazure operators. Section 5 is devoted to the twisted formal group algebra Q_W and the formal affine Demazure algebra $\mathbf{D}_F$. There, we provide an important formula (Lemma 5.4) expressing a product of Demazure elements in terms of the canonical basis of Q_W. We finish our preparation by relating the formal affine Demazure algebra to the subalgebra $\tilde{\mathbf{D}}_F$ of Q_W stabilizing $R[\![\Lambda]\!]_F$ (section 6) and we show that they coincide in most cases.

In section 7 we describe the formal affine Demazure algebra $\mathbf{D}_F$ in terms of generators and relations, hence, generalizing [HMSZ, Thm. 5.14]. We also establish an isomorphism (Theorem 7.9) between $\mathbf{D}_F$ and the algebra of Demazure operators on $R[\![\Lambda]\!]_F$. In key sections 8, 9 and 10 we introduce and study the coproduct structure on $\mathbf{D}_F$. For instance, we show (section 10) that the coproduct on $\mathbf{D}_F$ induces the coproduct on the cohomology ring considered in [CPZ, §7]. In section 11 we apply the obtained results to construct the dual $\mathbf{D}_F^*$ and to prove Theorem 11.4.

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1. Formal group algebras

We recall definition and basic properties of a formal group algebra, following [CPZ, §2].

Let F be a one dimensional formal group law over a commutative (unital) ring R (see [Ha78], [LM07, p. 4]). Given an integer $m \geq 0$ we use the notation

$$u +_F v = F(u,v), \quad m \cdot_F u := \underbrace{u +_F \cdots +_F u}_{m \text{ times}}, \quad \text{and} \quad (-m) \cdot_F u := -_F(m \cdot_F u),$$

where $-_F u$ denotes the formal inverse of u.

Let Λ be a lattice, i.e. a finitely generated free abelian group, and let $R[x_\Lambda]$ denote the polynomial ring over R with variables indexed by Λ. Let $\epsilon\colon R[x_\Lambda] \to R$ be the augmentation morphism which maps x_λ to 0 for each $\lambda \in \Lambda$. Let $R[\![x_\Lambda]\!]$ be the $\ker(\epsilon)$-adic completion of the polynomial ring $R[x_\Lambda]$. Let J_F be the closure of the ideal generated by x_0 and elements of the form $x_{\lambda_1+\lambda_2} - (x_{\lambda_1} +_F x_{\lambda_2})$ for all $\lambda_1, \lambda_2 \in \Lambda$.

Following [CPZ, Def. 2.4], we define the *formal group algebra* (or *formal group ring*) to be the quotient

$$R[\![\Lambda]\!]_F := R[\![x_\Lambda]\!]/J_F.$$

The class of x_λ in $R[\![\Lambda]\!]_F$ will be denoted identically. By definition, $R[\![\Lambda]\!]_F$ is a complete Hausdorff R-algebra with respect to the $\mathcal{I}_F$-adic topology, where $\mathcal{I}_F =$

$\ker \epsilon$ with $\epsilon\colon R[\![\Lambda]\!]_F \to R$ the induced augmentation map. We assume that $\mathcal{I}_F^{-i} = R[\![\Lambda]\!]_F$ if $i \geq 0$.

Recall from [CPZ, Cor. 2.13] that choosing a basis $\{\lambda_1, \ldots, \lambda_n\}$ of the lattice Λ gives a canonical isomorphism $R[\![\Lambda]\!]_F \simeq R[\![x_1, \ldots, x_n]\!]$ sending

$$x_{\sum m_i \lambda_i} \mapsto m_1 \cdot_F x_1 +_F \cdots +_F m_n \cdot_F x_n = m_1 x_1 + \cdots + m_n x_n + I^2,$$

where $I = (x_1, \ldots, x_n)$.

Example 1.1. (see [CPZ, Ex. 2.18 and Ex. 2.19])

For the *additive* formal group law over R given by $F_a(u, v) = u + v$ we have an R-algebra isomorphism $R[\![\Lambda]\!]_F \simeq S_R(\Lambda)^\wedge$, where $S_R^i(\Lambda)$ is the i-th symmetric power of Λ over R, the completion is at the kernel of the augmentation map $x_\lambda \mapsto 0$ and the isomorphism is induced by $x_\lambda \mapsto \lambda \in S_R^1(\Lambda)$.

Consider the group ring $R[\Lambda] := \{\sum_j a_j e^{\lambda_j} \mid a_j \in R,\ \lambda_j \in \Lambda\}$. Let $\mathrm{tr}\colon R[\Lambda] \to R$ be the trace map, i.e. a R-linear map sending every e^λ to 1. Let $R[\Lambda]^\wedge$ denote the completion of $R[\Lambda]$ at $\ker(\mathrm{tr})$.

For the *multiplicative periodic* formal group law over R given by $F_m(u, v) = u + v - \beta uv$, where β is an invertible element in R, we have an R-algebra isomorphism $R[\![\Lambda]\!]_F \simeq R[\Lambda]^\wedge$ induced by $x_\lambda \mapsto \beta^{-1}(1 - e^\lambda)$ and $e^\lambda \mapsto (1 - \beta x_\lambda) = (1 - \beta x_{-\lambda})^{-1}$.

Consider the *universal* formal group law $U(u, v) = u + v + a_{11}uv + \ldots$. Its coefficient ring is the Lazard ring $\mathbb{L}$. By universality there is a canonical ring homomorphism $\mathbb{L}[\![\Lambda]\!]_U \to S_{\mathbb{Z}}(\Lambda)^\wedge$ (resp. $\mathbb{L}[\![\Lambda]\!]_U \to \mathbb{Z}[\beta, \beta^{-1}][\Lambda]^\wedge$) given by sending all the coefficients a_{ij} of U to 0 (resp. all coefficients except a_{11} to 0 and $a_{11} \mapsto -\beta$).

We will need the following facts concerning formal group algebras. We refer to the appendix for the properties of regular elements.

Lemma 1.2. *Let $\Lambda \subseteq \Lambda'$ be two lattices. If $|\Lambda'/\Lambda|$ is regular in R, then the natural map $R[\![\Lambda]\!]_F \to R[\![\Lambda']\!]_F$ is an injection.*

Proof. Choose bases of Λ and Λ', and let A be the matrix expressing the vectors of the first in terms of the second. Then by Lemma 12.6, $\det(A) = \pm|\Lambda'/\Lambda|$. The lemma follows from Corollary 12.5 through isomorphisms $R[\![\Lambda]\!]_F \simeq R[\![x_1, \ldots, x_n]\!]$ and $R[\![\Lambda']\!]_F \simeq R[\![x_1, \ldots, x_n]\!]$ associated to the bases. □

Lemma 1.3. *Let $\Lambda = \Lambda_1 \oplus \Lambda_2$ be a direct sum of two lattices and let $\lambda \in \Lambda_1$. Then x_λ is regular in $R[\![\Lambda]\!]_F$ if and only if it is regular in $R[\![\Lambda_1]\!]_F$.*

Proof. By [CPZ, Thm. 2.1], there is a natural isomorphism of $R[\![\Lambda_1]\!]_F$-algebras $R[\![\Lambda_1 \oplus \Lambda_2]\!]_F \simeq (R[\![\Lambda_1]\!]_F)[\![\Lambda_2]\!]_F$. The result then follows by Corollary 12.2 applied to $f = x_\lambda \in R' = R[\![\Lambda_1]\!]_F$ and $R'[\![\Lambda_2]\!]_F \simeq R'[\![x_1, \ldots, x_l]\!]$ with l the rank of Λ_2. □

2. Root systems and regular elements

We recall several auxiliary facts concerning root datum following [SGA, Exp. XXI] and [Bo68]. We provide a criteria for regularity of a generator of the formal group algebra (see Lemma 2.2).

Following [SGA, Exp. XXI, §1.1] we define a *root datum* to be an embedding $\Sigma \hookrightarrow \Lambda^\vee$, $\alpha \mapsto \alpha^\vee$, of a non-empty finite subset Σ of a lattice Λ into its dual $\Lambda^\vee$ such that

- $\Sigma \cap 2\Sigma = \emptyset$, $\alpha^\vee(\alpha) = 2$ for all $\alpha \in \Sigma$, and

- $\beta - \alpha^\vee(\beta)\alpha \in \Sigma$ and $\beta^\vee - \beta^\vee(\alpha)\alpha^\vee \in \Sigma^\vee$ for all $\alpha, \beta \in \Sigma$, where $\Sigma^\vee$ denotes the image of Σ in $\Lambda^\vee$.

The elements of Σ (resp. $\Sigma^\vee$) are called roots (resp. coroots).

The sublattice of Λ generated by Σ is called the *root lattice* and is denoted by Λ_r. The rank of $\Lambda_{\mathbb{Q}}$ is called the rank of the root datum. A root datum is called irreducible if it is not a direct sum of root data of smaller ranks. The sublattice of $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by all $\omega \in \Lambda_{\mathbb{Q}}$ such that $\alpha^\vee(\omega) \in \mathbb{Z}$ for all $\alpha \in \Sigma$ is called the *weight lattice* and is denoted by Λ_w. Observe that by definition $\Lambda_r \subseteq \Lambda_w$ and $\Lambda_r \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda_w \otimes_Z \mathbb{Q}$. A root datum is called semisimple if $\Lambda_{\mathbb{Q}} = \Lambda_r \otimes_{\mathbb{Z}} \mathbb{Q}$. From now on by a root datum we will always mean a semisimple one. Observe that in this case $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$.

The root lattice Λ_r admits a basis $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ such that each $\alpha \in \Sigma$ is a linear combination of α_i's with either all positive or all negative coefficients. So the set Σ splits into two disjoint subsets $\Sigma = \Sigma_+ \amalg \Sigma_-$, where Σ_+ (resp. Σ_-) is called the set of positive (resp. negative) roots. The roots α_i are called *simple roots*.

Given the set Π we define the set of *fundamental weights* $\{\omega_1, \ldots, \omega_n\} \subset \Lambda_w$ as $\alpha_i^\vee(\omega_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Fundamental weights form a basis of the weight lattice Λ_w. The matrix expressing simple roots in terms of fundamental weights is called the Cartan matrix of the root datum.

If $\Lambda = \Lambda_w$ (resp. $\Lambda = \Lambda_r$), then the root datum is called simply connected (resp. adjoint) and is denoted by $\mathcal{D}_n^{sc}$ (resp. $\mathcal{D}_n^{ad}$), where $\mathcal{D} = A, B, C, D, E, F, G$ is one of the Dynkin types and n is its rank. Observe that an irreducible root datum is uniquely determined by its Dynkin type and the lattice Λ.

Determinant of the Cartan matrix of an irreducible root datum coincides with $|\Lambda_w/\Lambda_r|$, and can be found in the tables at the end of [Bo68] under the name "indice de connexion". For future reference, we provide the list of determinants and the list of simply connected torsion primes, i.e. prime divisors of the torsion index of the associated simply connected root datum; these can be found in [D73, Prop. 8].

Type (s. connected)	A_l	B_l, $l \geq 3$	C_l	D_l, $l \geq 4$	G_2	F_4	E_6	E_7	E_8
Determinant	$l+1$	2	2	4	1	1	3	2	1
Torsion primes	$\emptyset$	2	$\emptyset$	2	2	2, 3	2, 3	2, 3	2, 3, 5

In several of the following statements, root datum C_n^{sc}, $n \geq 1$ need to be treated separately. Note that $A_1^{sc} = C_1^{sc}$ and $B_2^{sc} = C_2^{sc}$ are parts of these special cases.

Lemma 2.1. *Any root $\alpha \in \Sigma$ can be completed to a basis of Λ except if it is a long root in an irreducible component C_n^{sc}, $n \geq 1$, of the root datum, in which case, it can still be completed to a basis of Λ over $\mathbb{Z}[\frac{1}{2}]$.*

Proof. We may assume that the root datum is irreducible.

1. First, we can choose simple roots of Λ_r as $\Pi = \{\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n\}$ by [Bo68, Ch. IV, §1, Prop. 15], hence, proving the statement in the adjoint case ($\Lambda = \Lambda_r$).

2. If $\Lambda = \Lambda_w$, we express the root α in terms of the fundamental weights corresponding to Π as $\alpha = \sum_{i=1}^n k_i\omega_i$. Looking at the Cartan matrix we see that $k_i = -1$ for some i except in type G_2 or if $\alpha = \alpha_n$ is the long root in type C_n, $n \geq 1$. In other words, except in these two cases α can be completed to a basis of Λ_w by Lemma 12.7 as $(k_1, ..., k_n)$ is unimodular. Since the G_2 case is adjoint, it has already been considered. Finally, in the C_n case, since $k_i = 2$ for some i, the root α can be completed to a basis of Λ_w after inverting 2.

3. The remaining case $\Lambda_r \subsetneq \Lambda \subsetneq \Lambda_w$ can only happen in type D_n or A_n, both with $n \geq 3$, see the above table.

Pick a basis $\{\lambda_1, \ldots, \lambda_n\}$ of Λ and express $\alpha = \sum_{i=1}^n k_i \lambda_i$. Then, complete α to a basis $\{\alpha = e_1, e_2, \ldots, e_n\}$ of Λ_w, as in the previous step. Let $A = (a_{ij})$ be the matrix whose columns express the basis $\{\lambda_i\}$ in terms of the basis $\{e_i\}$. Applying A to the vector (k_i) gives $\alpha = e_1$. So $\sum_{j=1}^n a_{j1} k_j = 1$ which means that the row $(k_1, \ldots, k_n)$ is unimodular and, therefore, α can be completed to a basis of Λ by Lemma 12.7. □

Consider a formal group algebra $R[\![\Lambda]\!]_F$ associated to the lattice Λ of the root datum and a formal group law F over R.

Lemma 2.2. *For any root $\alpha \in \Sigma$ the element x_α is regular in $R[\![\Lambda]\!]_F$, except maybe if α is a long root in an irreducible component C_n^{sc}, $n \geq 1$, of the root datum. In that case, x_α is regular if and only if the formal series $2 \cdot_F x$ is regular in $R[\![x[\![$, which always holds if 2 is regular.*

Proof. If α is not a long root in C_n^{sc}, $n \geq 1$, then by Lemma 2.1, it can be completed to a basis of Λ, and $R[\![\Lambda]\!]_F \cong R[\![x_1, ..., x_n]\!]$ with $x_1 = x_\alpha$, so x_α is regular by Lemma 12.3. On the other hand, if α is a long root in C_n^{sc}, then it is part of a basis of roots, with associated fundamental weights part of a basis of Λ. If $n = 1$, then $\alpha = 2\omega$ where ω is the weight corresponding to α. If $n \geq 2$, then $\alpha = 2(\omega - \omega')$ where ω' is another weight. In both cases, we can therefore find an isomorphism $R[\![\Lambda]\!]_F \simeq R[\![x_1, \ldots, x_n]\!]$ such that x_α is mapped to $2 \cdot_F x_1$, which is regular if and only if it is already regular in $R[\![x_1]\!]$. If 2 is regular, then $2 \cdot_F x_1 = 2x_1 + x_1^2 y$ is regular by part (a) of Lemma 12.3. □

Remark 2.3. For some formal group laws x_α can indeed be a regular element even if 2 is a zero divisor and α is a long root in C_n^{sc}. Take the multiplicative formal group law $F(u, v) = u + v - uv$ over $R = \mathbb{Z}/2$ and the root datum C_1^{sc}. Then $x_\alpha = x_{2\omega} = 2x_\omega + x_\omega^2 = x_\omega^2$ is regular.

Lemma 2.4. *Let $\Lambda \subset \Lambda'$ be two lattices between Λ_r and Λ_w. If the determinant of the Cartan matrix is regular in R, then the natural map $R[\![\Lambda]\!]_F \to R[\![\Lambda']\!]_F$ is injective.*

Proof. Choose bases of Λ and Λ', and let A be the matrix expressing the first basis in terms of the second. Then the determinant of A divides the determinant of the Cartan matrix by composition of inclusions $\Lambda_r \subseteq \Lambda \subseteq \Lambda' \subseteq \Lambda_w$. The lemma then follows from Lemma 1.2. □

Remark 2.5. Here is a counter-example to the injectivity when the determinant of the Cartan matrix is not regular. For the type A_2, take $R = \mathbb{Z}/3$ and the additive formal group law. The map from $R[\![\Lambda_r]\!]_F \to R[\![\Lambda_w]\!]_F$ is not injective: if $\{\alpha_1, \alpha_2\}$ are simple roots, then $x_{\alpha_1 + 2\alpha_2} = x_{\alpha_1} + 2x_{\alpha_2} \neq 0$ is sent to $x_{3\omega_2} = 3x_{\omega_2} = 0$.

3. Localized formal group algebras

Consider a (semisimple) root datum $\Sigma \hookrightarrow \Lambda^\vee$. Various operators on the formal group algebra $R[\![\Lambda]\!]_F$, such as formal Demazure operators, are defined using formulas involving formal division by elements x_α, $\alpha \in \Sigma$, see for example [CPZ, Def. 3.5] in the case $\Lambda = \Lambda_w$. In the present section we study properties of the localization of $R[\![\Lambda]\!]_F$ at such elements. A key result here is Lemma 3.5 which will be used in the proof of Proposition 6.2.

Let $Q_{R,\Lambda}$ denote the localization of $R[\![\Lambda]\!]_F$ at the multiplicative subset generated by elements x_α for all $\alpha \in \Sigma$ (if the multiplicative subset contains zero, then $Q_{R,\Lambda}$ is trivial).

Remark 3.1. Observe that by [CPZ, Def. 3.12] $x_{-\alpha} = x_\alpha(-1 + x_{-\alpha}\kappa_\alpha)$ for some $\kappa_\alpha \in R[\![\Lambda]\!]_F$. Therefore, we can replace Σ by Σ_+ or Σ_-, since $-1 + x_{-\alpha}\kappa_\alpha$ is invertible in $R[\![\Lambda]\!]_F$.

Lemma 3.2. *If 2 is regular in R or the root datum doesn't have an irreducible component C_n^{sc}, $n \geq 1$, then the localization map $R[\![\Lambda]\!]_F \to Q_{R,\Lambda}$ is injective.*

Proof. By Lemma 2.2, the element x_α is regular for any root α, so we are localizing at a set consisting of regular elements. $\square$

Let r be a regular element of R. Then under the assumption of Lemma 3.2, there is a commutative diagram of inclusions

$$\begin{array}{ccccc} R[\![\Lambda]\!]_F & \subseteq & R[\![\Lambda]\!]_F[\tfrac{1}{r}] & \subseteq & R[\tfrac{1}{r}][\![\Lambda]\!]_F \\ \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\ Q_{R,\Lambda} & \subseteq & Q_{R,\Lambda}[\tfrac{1}{r}] & \subseteq & Q_{R[\frac{1}{r}],\Lambda} \end{array} \tag{3.1}$$

This can be seen by choosing a basis of Λ and, hence, identifying the respective formal group algebra with a ring of power series. Observe that if r is non-invertible, an element $\sum_{i\geq 0} \frac{x^i}{r^i}$ while being in $R[\frac{1}{r}][\![x]\!]$ is not in $R[\![x]\!][\frac{1}{r}]$, so the inclusions $R[\![\Lambda]\!]_F \subsetneq R[\![\Lambda]\!]_F[\frac{1}{r}] \subsetneq R[\frac{1}{r}][\![\Lambda]\!]_F$ are proper.

Lemma 3.3. *Let r be a regular element of R and let α be a root. Assume that 2 is invertible in R or α is not a long root of an irreducible component C_n^{sc}, $n \geq 1$. If $u \in R[\frac{1}{r}][\![\Lambda]\!]_F$ and $x_\alpha u \in R[\![\Lambda]\!]_F[\frac{1}{r}]$, then $u \in R[\![\Lambda]\!]_F[\frac{1}{r}]$.*

Proof. By Lemma 2.1 we complete α to a basis of Λ. Observe that if α is a long root of C_n^{sc}, then $x_\alpha = 2x_\omega + \mathcal{I}_F^2$. Therefore, the isomorphism $R[\![\Lambda]\!]_F \simeq R[\![x_1, \ldots, x_n]\!]$ determined by the choice of a basis can be modified so that it sends x_α to x_1.

The lemma then follows from the fact that if $u \in R[\frac{1}{r}][\![x_1, \ldots, x_n]\!]$ and $x_1 u \in R[\![x_1, \ldots, x_n]\!][\frac{1}{r}]$, then $u \in R[\![x_1, \ldots, x_n]\!][\frac{1}{r}]$. $\square$

Corollary 3.4. *Let r be a regular element of R. Assume that 2 is invertible in R or the root datum doesn't have an irreducible component C_n^{sc}, $n \geq 1$. Then we have $(R[\frac{1}{r}][\![\Lambda]\!]_F) \cap Q_{R,\Lambda}[\frac{1}{r}] = R[\![\Lambda]\!]_F[\frac{1}{r}]$ in $Q_{R[\frac{1}{r}],\Lambda}$, i.e. the right square of (3.1) is cartesian.*

Lemma 3.5. *Let r be a regular element of R. Assume that 2 is invertible in R or the root datum doesn't have an irreducible component C_n^{sc}, $n \geq 1$. Then we have $(R[\![\Lambda]\!]_F[\frac{1}{r}]) \cap Q_{R,\Lambda} = R[\![\Lambda]\!]_F$ in $Q_{R,\Lambda}[\frac{1}{r}]$, i.e. the left square of (3.1) is cartesian.*

Proof. Since the localization map is injective by Lemma 3.2, it is enough to show that if $u, v \in R[\![\Lambda]\!]_F$ satisfy $\frac{v}{\prod_\alpha x_\alpha} = \frac{u}{r^n}$, $\alpha \in \Sigma$, or equivalently, $r^n v = u \prod_\alpha x_\alpha$ in $R[\![\Lambda]\!]_F$, then r^n divides u. In $(R/r^n)[\![\Lambda]\!]_F = R[\![\Lambda]\!]_F/r^n$, we have $r^n v = 0$, and since by Lemma 2.2 the element x_α is regular in $(R/r^n)[\![\Lambda]\!]_F$ for any $\alpha \in \Sigma$, we also have $u = 0$. Thus u is divisible by r^n. $\square$

4. Formal Demazure operators

We now introduce formal Demazure operators following the approach mentioned in [CPZ, Rem. 3.6]. The formula (4.1) will be extensively used in section 9 in various computations involving coproducts.

Consider a root datum $\Sigma \hookrightarrow \Lambda^\vee$. For each $\alpha \in \Sigma$ we define a $\mathbb{Z}$-linear automorphism of Λ called a *reflection* by

$$s_\alpha \colon \lambda \mapsto \lambda - \alpha^\vee(\lambda)\alpha, \quad \lambda \in \Lambda.$$

The subgroup of linear automorphisms of Λ generated by reflections s_α, $\alpha \in \Sigma$, is called the *Weyl group* of the root datum and is denoted by W. Observe that W depends only on the Dynkin type.

Fix a formal group law F over R. There is an induced action of the Weyl group W on $R[\![\Lambda]\!]_F$ via R-algebra automorphisms defined by $w(x_\lambda) = x_{w(\lambda)}$ for all $w \in W$ and $\lambda \in \Lambda$.

Lemma 4.1. *If x_α is regular in $R[\![\Lambda]\!]_F$, then the difference $u - s_\alpha(u)$ is uniquely divisible by x_α for each $u \in R[\![\Lambda]\!]_F$ and $\alpha \in \Sigma$.*

Proof. Divisibility follows from the proof of [CPZ, Cor. 3.4]. Uniqueness follows from the regularity of x_α. □

Definition 4.2. For each $\alpha \in \Sigma$ such that x_α is regular in $R[\![\Lambda]\!]_F$ we define an R-linear operator Δ_α on $R[\![\Lambda]\!]_F$, called a *formal Demazure operator*,

$$\Delta_\alpha(u) := \tfrac{u - s_\alpha(u)}{x_\alpha}, \quad u \in R[\![\Lambda]\!]_F.$$

Remark 4.3. Observe that in [CPZ, Def. 3.5] we defined the Demazure operator on $R[\![\Lambda]\!]_F$ for $\Lambda = \Lambda_w$ by specializing the Demazure operator defined over the Lazard ring. If x_α is regular, then this definition coincides with Definition 4.2.

Definition 4.4. We say that $R[\![\Lambda]\!]_F$ is Σ-*regular* if for each $\alpha \in \Sigma$, the element x_α is regular in $R[\![\Lambda]\!]_F$.

Unless otherwise stated, we shall always assume that the formal group algebra $R[\![\Lambda]\!]_F$ is Σ-regular. Observe that this immediately implies that $R[\![\Lambda]\!]_F$ injects into the localization $Q_{R,\Lambda}$ (cf. Lemma 3.2).

Remark 4.5. Note that Lemma 2.2 implies that, $R[\![\Lambda]\!]_F$ is Σ-regular if 2 is regular in R or if the root datum doesn't contain an irreducible component C_n^{sc}.

We fix a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$. Let $s_i = s_{\alpha_i}$, $i \in [n] = \{1, \ldots, n\}$ denote the corresponding (simple) reflections. One of the basic facts concerning the Weyl group W is that it is generated by simple reflections. Given a sequence $I = (i_1, ..., i_l)$ of length $|I| = l$ with $i_j \in [n]$, we set $w(I) = s_{i_1} \ldots s_{i_l}$ to be the product of the respective simple reflections. A sequence I is called a *reduced sequence* of $w \in W$ if I is of minimal length among all sequences J such that $w = w(J)$. The length of I_w is called the length of w and is denoted by $\ell(w)$. Given a sequence I we define

$$\Delta_I = \Delta_{i_1} \circ \ldots \circ \Delta_{i_l}, \quad \text{where } \Delta_i := \Delta_{\alpha_i}.$$

Remark 4.6. If F is the additive or multiplicative formal group law, then Δ_{I_w} doesn't depend on a choice of the reduced sequence of w. In this case, it coincides with the classical Demazure operator Δ_w of [D73, §3 and §9]. For other formal group laws Δ_{I_w} depends on a choice of I_w (see [CPZ, Thm. 3.9]).

Definition 4.7. We define R-linear operators $B_i^{(j)}\colon R[\![\Lambda]\!]_F \to R[\![\Lambda]\!]_F$, where $j \in \{-1,0,1\}$ and $i \in [n]$, as

$$B_i^{(-1)} := \Delta_i, \quad B_i^{(0)} := s_i, \quad \text{and} \quad B_i^{(1)} := \text{multiplication by } (-x_i) := -x_{\alpha_i}.$$

Observe that we have $B_i^{(j)}(\mathcal{I}_F^m) \subset \mathcal{I}_F^{m+j}$, where $\mathcal{I}_F$ is the augmentation ideal.

Let $I = (i_1, \ldots, i_l)$ be a sequence of length l, and let E be a subset of $[l]$. We denote by $I_{|E}$ the subsequence of I consisting of all i_j's with $j \in E$.

Lemma 4.8. *Let $I = (i_1, \ldots, i_l)$. Then for any $u, v \in R[\![\Lambda]\!]_F$ we have*

$$\Delta_I(uv) = \sum_{E_1,E_2 \subset [l]} p^I_{E_1,E_2} \Delta_{I_{|E_1}}(u) \Delta_{I_{|E_2}}(v) \tag{4.1}$$

where $p^I_{E_1,E_2} = B_1 \circ \ldots \circ B_l(1) \in \mathcal{I}_F^{|E_1|+|E_2|-l}$ with the operator $B_j\colon R[\![\Lambda]\!]_F \to R[\![\Lambda]\!]_F$ defined as

$$B_j = \begin{cases} B_{i_j}^{(1)} \circ B_{i_j}^{(0)}, & \text{if } j \in E_1 \cap E_2, \\ B_{i_j}^{(-1)}, & \text{if } j \notin E_1 \cup E_2, \\ B_{i_j}^{(0)}, & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on the length l of I. For $l = 1$ and $I = (i)$ the formula holds since

$$\begin{aligned} \Delta_i(uv) &= \Delta_i(u)v + u\Delta_i(v) - x_i\Delta_i(u)\Delta_i(v) \\ &= -\Delta_i(1)uv + s_i(1)u\Delta_i(v) + s_i(1)\Delta_i(u)v - x_i\Delta_i(u)\Delta_i(v) \end{aligned}$$

by [CPZ, Prop. 3.8.(4)] and because $\Delta_i(1) = 0$ and $s_i(1) = 1$.

Setting $I' = (i_2, \ldots, i_l)$, we obtain by induction

$$\Delta_I(uv) = \Delta_{i_1}(\Delta_{I'}(uv)) = \Delta_{i_1}\Big(\sum_{E_1,E_2 \subset [l-1]} p^{I'}_{E_1,E_2} \Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\Big).$$

For fixed $E_1, E_2 \subset [l-1]$, by [CPZ, Prop. 3.8.(4)] we have

$$\begin{aligned} \Delta_{i_1}\big(p^{I'}_{E_1,E_2}\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\big) =& \Delta_{i_1}(p^{I'}_{E_1,E_2})\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v) \\ &+ s_{i_1}(p^{I'}_{E_1,E_2})\Delta_{i_1}\big(\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\big), \end{aligned}$$

and

$$\begin{aligned} \Delta_{i_1}\big(\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\big) =& (\Delta_{i_1} \circ \Delta_{I'_{|E_1}})(u)\Delta_{I'_{|E_2}}(v) + \Delta_{I'_{|E_1}}(u)(\Delta_{i_1} \circ \Delta_{I'_{|E_2}})(v) \\ &- x_{i_1}(\Delta_{i_1} \circ \Delta_{I'_{|E_1}})(u)(\Delta_{i_1} \circ \Delta_{I'_{E_2}})(v). \end{aligned}$$

Combining these two identities, we obtain

$$\begin{aligned} \Delta_{i_1}\big(p^{I'}_{E_1,E_2}\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\big) &= B_{i_1}^{(-1)}(p^{I'}_{E_1,E_2}) \cdot \Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v) \\ &+ B_{i_1}^{(0)}(p^{I'}_{E_1,E_2}) \cdot [(\Delta_{i_1} \circ \Delta_{I'_{|E_1}})(u)\Delta_{I'_{|E_2}}(v) + \Delta_{I'_{|E_1}}(u)(\Delta_{i_1} \circ \Delta_{I'_{|E_2}})(v)] \\ &+ (B_{i_1}^{(1)} \circ B_{i_1}^{(0)})(p^{I'}_{E_1,E_2}) \cdot (\Delta_{i_1} \circ \Delta_{I'_{|E_1}})(u)(\Delta_{i_1} \circ \Delta_{I'_{|E_2}})(v) \end{aligned}$$

and using the notation $E+1=\{e+1\mid e\in E\}$ for a set of integers E we get

$$\begin{aligned}&= B_{i_1}^{(-1)}(p^{I'}_{E_1,E_2})\cdot\Delta_{I_{|E_1+1}}(u)\Delta_{I_{|E_2+1}}(v)\\ &\quad + B_{i_1}^{(0)}(p^{I'}_{E_1,E_2})\cdot[\Delta_{I_{|\{1\}\cup(E_1+1)}})(u)\Delta_{I_{E_2+1}}(v)+\Delta_{I_{|E_1+1}}(u)\Delta_{I_{|\{1\}\cup(E_2+1)}}(v)]\\ &\qquad\qquad + (B_{i_1}^{(1)}\circ B_{i_1}^{(0)})(p^{I'}_{E_1,E_2})\cdot(\Delta_{I_{|\{1\}\cup(E_1+1)}})(u)\Delta_{I_{|\{1\}\cup(E_2+1)}}(v).\end{aligned}$$

Since a subset of $[l]$ is of the form $E+1$ or $\{1\}\cup(E+1)$ with $E\subseteq[l-1]$, the result follows by induction. □

Remark 4.9. For any simple root α we have $s_\alpha(1)=1$ and $\Delta_\alpha(1)=0$. Therefore, if there is an operator of type $B^{(-1)}$ applied to 1 before applying all the operators of type $B^{(1)}\circ B^{(0)}$, then $p^I_{E_1,E_2}=0$. In particular, if $E_1\cap E_2=\emptyset$ and $E_1\cup E_2\neq[l]$, then $p^I_{E_1,E_2}=0$.

5. Twisted formal group algebra and Demazure elements

We recall the notions of a twisted formal group algebra and a formal affine Demazure algebra introduced in [HMSZ, §5]. The purpose of this section is to generalize [KK86, Prop. 4.3] and [KK90, Prop. 2.6] to the context of arbitrary formal group laws (see Lemma 5.4), hence, providing a general formula expressing a product of Demazure elements in terms of the canonical basis of the twisted formal group algebra.

As in section 3, given a root datum $\Sigma\hookrightarrow\Lambda^\vee$ consider the localization $Q=Q_{R,\Lambda}$ of the formal group algebra $S=R[\![\Lambda]\!]_F$. Since the Weyl group W preserves the set Σ, its action on S extends to Q. Following to [KK86, §4.1] and [HMSZ, Def. 5.1] we define the *twisted formal group algebra* to be the R-module $Q_W:=Q\otimes_R R[W]$ with the multiplication given by

$$(q\delta_w)(q'\delta_{w'})=qw(q')\delta_{ww'}\ \text{ for all } w,w'\in W \text{ and } q,q'\in Q$$

(extended by linearity), where δ_w denotes the element in $R[W]$ corresponding to w (so we have $\delta_w\delta_{w'}=\delta_{ww'}$ for $w,w'\in W$) and δ_1 denotes 1. Observe that Q_W is a free left Q-module (via the left multiplication) with basis $\{\delta_w\}_{w\in W}$, but Q_W is not a Q-algebra as $\delta_1 Q=Q\delta_1$ is not central in Q_W.

Remark 5.1. The twisted formal group algebra can be defined for any W-action on Q, where W is a unital monoid and $w(q_1\cdot q_2)=w(q_1)\cdot w(q_2)$ for any $q_1,q_2\in Q$ and $w\in W$.

Remark 5.2. In [KK86, §4.1] (resp. [KK90, §2.1]) Q denoted the field of fractions of a symmetric algebra (resp. of an integral group ring) of the weight lattice Λ_w and Q_W was defined using the right Q-module notation, i.e. $Q_W=\mathbb{Z}[W]\otimes_{\mathbb{Z}}Q$. Using our terminology, the case of [KK86] (resp. of [KK90]) corresponds to the additive formal group law (resp. multiplicative periodic formal group law) and the simply connected root datum.

In [HMSZ], Q denoted the localization of $R[\![\Lambda_w]\!]_F$ at all elements x_λ, where $\lambda\in\Lambda_w$.

Definition 5.3. Following [KK86, I_{24}] and [HMSZ, Def. 5.2] for each root $\alpha\in\Sigma$ we define the *Demazure element*

$$X_\alpha:=x_\alpha^{-1}(1-\delta_{s_\alpha})=x_\alpha^{-1}-\delta_{s_\alpha}x_{-\alpha}^{-1}\in Q_W.$$

Given a set of simple roots $\{\alpha_1, \dots, \alpha_n\}$ for any sequence $I = (i_1, ..., i_l)$ from $[n]$ we denote

$$\delta_I = \delta_{s_{i_1} \dots s_{i_l}} \text{ and } X_I := X_{i_1} \cdot \dots \cdot X_{i_l}, \text{ where } X_i = X_{\alpha_i}.$$

There is an *anti-involution* $(\text{-})^t$ on the R-algebra Q_W defined by

$$q\delta_w \mapsto (q\delta_w)^t := w^{-1}(q)\delta_{w^{-1}}, \quad w \in W,\ q \in Q.$$

Observe that $(qx)^t = x^t q$ for $x \in Q_W$ and $q \in Q$, so it is neither right Q-linear nor left Q-linear.

We obtain the following generalization of [KK86, Prop. 4.3]:

Lemma 5.4. *Given a reduced sequence I_v of $v \in W$ of length l let*

$$X_{I_v} = \sum_{w \in W} a_{v,w}\delta_w = \sum_{w \in W} \delta_w a'_{v,w}$$

for some $a_{v,w}, a'_{v,w} \in Q$. Then

(a) $a_{v,w} = 0$ unless $w \le v$ with respect to the Bruhat order on W,
(b) $a_{v,v} = (-1)^l \prod_{\alpha \in v(\Sigma_-) \cap \Sigma_+} x_\alpha^{-1} = a'_{v,v^{-1}}$,
(c) $a'_{v,w} = 0$ unless $w \ge v^{-1}$.

Proof. We proceed by induction on the length l of v.

The lemma obviously holds for $I_v = \emptyset$ the empty sequence, since $X_\emptyset = 1$.

Let $I_v = (i_1, \dots i_l)$ be a reduced sequence of v and let $\beta = \alpha_{i_1}$. Then $(i_2, \dots, i_n)$ is a reduced sequence of $v' = s_\beta v$ and we have

(1) $w \le v'$ implies $w \le v$ and $s_\beta w \le v$;
(2) $\{\beta\} \cup s_\beta\big(v'(\Sigma_-) \cap \Sigma_+\big) = v(\Sigma_-) \cap \Sigma_+$;
(3) $w^{-1} \le v$ if and only $w \ge v^{-1}$.

Indeed, the properties (1) and (3) are consequences of the fact that elements smaller than v are the elements w obtained by taking a subsequence of a reduced sequence of v, by [De77, Th. 1.1, III, (ii)]. Property (2) follows from [Bo68, Ch. VI, §1, No 6, Cor. 2].

We then compute

$$X_{I_v} = x_\beta^{-1}(1 - \delta_\beta) \sum_{w \le v'} a_{v',w}\delta_w = \sum_{w \le v'} x_\beta^{-1} a_{v',w}\delta_w - \sum_{w \le v'} x_\beta^{-1} s_\beta(a_{v',w})\delta_{s_\beta w}$$
$$\overset{(1)}{=} -x_\beta^{-1} s_\beta(a_{v',v'})\delta_v + \sum_{w < v} a_{w,v}\delta_w.$$

So part (a) holds and $a_{v,v} = -x_\beta^{-1} s_\beta(a_{v',v'})$. Hence, the expression of $a_{v,v}$ in part (b) follows from property (2). Property (3) implies (c) by applying the anti-involution sending δ_w to $\delta_{w^{-1}}$ and, thus, identifying $a'_{v,w} = a_{v,w^{-1}}$. $\square$

Remark 5.5. Observe that the coefficient $a_{v,v}$ doesn't depend on the choice of a reduced sequence I_v of v.

Corollary 5.6. *(cf.* [KK86, Cor. 4.5]*) For each $w \in W$, let I_w be a reduced sequence of w. The elements $(X_{I_w})_{w \in W}$ form a basis of Q_W as a left (resp. right) Q-module. The element δ_v decomposes as $\sum_{w \le v} b_{w,v} X_{I_w}$ with $b_{w,v}$ in Q. Furthermore,*

$$b_{w,w} = \prod_{\alpha \in v(\Sigma_-) \cap \Sigma_+} (-1)^l x_\alpha.$$

Proof. The matrix $(a_{v,w})_{(v,w)\in W^2}$, resp. $(a'_{v,w})_{(v,w)\in W^2}$, is triangular with invertible coefficients on the diagonal (resp. the anti-diagonal) by Lemma 5.4. It expresses elements X_{I_w} in terms of the basis $(\delta_w)_{w\in W}$ of the left (resp. right) Q-module Q_W.

The decomposition of δ_w follows from the fact that the inverse of a triangular matrix is also triangular with inverse coefficients on the diagonal. □

We are now ready to define a key object of the present paper.

Definition 5.7. We define the *formal affine Demazure algebra* $\mathbf{D}_F$ to be the R-subalgebra of Q_W generated by elements of the formal group algebra S and by Demazure elements X_α for all $\alpha \in \Sigma$.

The following lemma shows that it is the same object as the one considered in [HMSZ, Definition 5.3].

Lemma 5.8. *The Demazure algebra $\mathbf{D}_F$ coincides with the R-subalgebra of Q_W generated by elements of S and Demazure elements X_i, for $i \in [l]$.*

Proof. Since the Weyl group is generated by reflections and $\delta_{s_i} = 1 - x_{\alpha_i}X_i$, we have $R[W] \subseteq \mathbf{D}_F$. Since any $\alpha \in \Sigma$ can be written as $\alpha = w(\alpha_i)$ for some simple root α_i, the lemma then follows from the formula $\delta_w X_{\alpha_i}\delta_{w^{-1}} = X_{w(\alpha_i)}$. □

6. Invariant presentation of the formal affine Demazure algebra

In the present section using the action of Demazure elements on $R[\![\Lambda]\!]_F$ via Demazure operators we consider the algebra $\tilde{\mathbf{D}}_F$ of elements of Q_W fixing S. It is closely related to the formal affine Demazure algebra $\mathbf{D}_F$. We show that the algebra $\tilde{\mathbf{D}}_F$ is a free $R[\![\Lambda]\!]_F$-module with a basis given by products of Demazure elements (see Corollary 6.3), hence, generalizing [KK86, Thm.4.6.(a)].

The localization $Q = Q_{R,\Lambda}$ of $S = R[\![\Lambda]\!]_F$ has a structure of left Q_W-module defined by (cf. [KK86, I_{33}])

$$(\delta_w q)q' = w(qq') \text{ for } w \in W \text{ and } q, q' \in Q.$$

Let $\tilde{\mathbf{D}}_F$ denote the R-subalgebra of Q_W preserving S when acting on the left, i.e.

$$\tilde{\mathbf{D}}_F := \{x \in Q_W \mid x \cdot S \subseteq S\}.$$

By definition we have $S \subset \tilde{\mathbf{D}}_F$ and $X_\alpha \in \tilde{\mathbf{D}}_F$, since X_α acts on S by the Demazure operator Δ_α. Therefore, in this case $\mathbf{D}_F \subseteq \tilde{\mathbf{D}}_F$.

Let $\mathfrak{t}$ be the *torsion index* of the root datum as defined in [D73, §5]. Its prime factors are the torsion primes listed in the table in §2 together with the prime divisors of $|\Lambda_w/\Lambda|$, by [D73, §5, Prop. 6].

Let I_0 be a reduced sequence of the longest element w_0 and let $N = \ell(w_0)$. Recall that by [CPZ, 5.2] there exists an element $u_0 \in \mathcal{I}_F^N \subset S$ such that $\epsilon\Delta_{I_0}(u_0) = \mathfrak{t}$. Moreover, u_0 satisfies the following property: if $|I| \le N$, then

$$\epsilon\Delta_I(u_0) = \begin{cases} \mathfrak{t}, & \text{if } I \text{ is a reduced sequence of } w_0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us recall a key result for future reference.

Lemma 6.1. *For each $w \in W$, let I_w be a reduced sequence of w. Then if the torsion index $\mathfrak{t}$ is invertible in R, the matrix $(\Delta_{I_v}\Delta_{I_w}(u_0))_{(v,w)\in W\times W}$ with coefficients in S is invertible. Thus, if $\mathfrak{t}$ is just regular, the kernel of this matrix is trivial.*

Proof. It readily follows from the above property of u_0, see [CPZ, Prop. 6.6] and its proof. □

Proposition 6.2. *Assume that $\mathfrak{t}$ is regular in R. Also assume that $2 \in R^\times$ or that either all or none of the irreducible components of the root datum are C_n^{sc}, $n \geq 1$.*

Then the R-algebra $\tilde{\mathbf{D}}_F$ is the left S-submodule of Q_W generated by $\{X_{I_w}\}_{w\in W}$.

Proof. Observe that the hypotheses of the proposition imply that S is Σ-regular, so $\tilde{\mathbf{D}}_F$ is well-defined.

By Corollary 5.6 we can write $y \in \tilde{\mathbf{D}}_F$ as $y = \sum_v c_v X_{I_v}$, where $c_v \in Q$. It is enough to prove that $c_v \in S$ for each v. Apply y successively to the elements $\Delta_{I_w}(u_0)$, for all $w \in W$. By definition of $\mathbf{D}_F$ all resulting elements $\sum_v c_v \Delta_{I_v}\Delta_{I_w}(u_0)$ are in S. By Lemma 6.1, the matrix $(\Delta_{I_v}\Delta_{I_w}(u_0))_{(v,w)\in W\times W}$ with coefficients in S becomes invertible after inverting $\mathfrak{t}$. This implies that $c_v \in S[\frac{1}{\mathfrak{t}}]$ for each $v \in W$.

If all irreducible components of the root system are C_n^{sc}, $n \geq 1$, we are finished since $\mathfrak{t} = 1$. In the remaining two cases (2 is invertible or there are no components C_n^{sc}) we have $S[\frac{1}{\mathfrak{t}}] \cap Q = S$ by Lemma 3.5. □

Corollary 6.3. *Under the hypotheses of Proposition 6.2 the elements $\{X_{I_w}\}_{w\in W}$ form a basis of the left S-module $\tilde{\mathbf{D}}_F$.*

Proof. The elements X_{I_w} are independent over S, since they are independent over Q and the map $S \to Q$ is injective by Lemma 3.2. □

Example 6.4. Here is an example where the torsion index $\mathfrak{t} = 2$ is regular in $R = \mathbb{Z}$ while the remaining assumptions of Proposition 6.2 are not satisfied and its conclusion does not hold.

Consider the additive formal group law over $\mathbb{Z}$ and a direct sum of root data C_1^{ad} and C_1^{sc} with the respective lattices Λ_r and Λ'_w. Let α be the simple root of Λ_r, and let ω' be the fundamental weight of Λ'_w. Consider the isomorphism $\mathbb{Z}[\![\Lambda \oplus \Lambda']\!]_F \simeq \mathbb{Z}[\![x,y]\!]$ defined by $x_\alpha \mapsto x$ and $x_{\omega'} \mapsto y$. Observe that $Q \simeq \mathbb{Z}[\![x,y]\!][\frac{1}{x}, \frac{1}{2y}]$, so it contains $\frac{1}{2}$.

Consider the simple reflection s_α. By definition $s_\alpha(y) = y$ and $s_\alpha(x) = -x$. Therefore, Δ_α acts on Q linearly over $\mathbb{Z}[\![y]\!]$ and $\Delta_\alpha(x^i) = (x^i + (-x)^i)/x$ for each $i \geq 1$. So if $z = \sum_{i\geq 0} a_i x^i$ with $a_i \in \mathbb{Z}[\![y]\!]$, then $\Delta_\alpha(z) = \sum_{i\geq 1} a_i(1 + (-1)^i)x^{i-1}$ is divisible by 2. Hence, $\frac{1}{2}X_\alpha$ is in $\tilde{\mathbf{D}}_F$, while it is not in the left S-submodule, since $\frac{1}{2} \notin \mathbb{Z}[\![x,y]\!]$. Observe that the proposition holds if $\mathbb{Z}$ is replaced by $\mathbb{Z}[\frac{1}{2}]$.

7. Presentation in terms of generators and relations

In the present section we describe the formal affine Demazure algebra $\mathbf{D}_F$ in terms of generators and relations. The main result (Theorem 7.5) generalizes [HMSZ, Thm. 5.14]. We also establish an isomorphism (Theorem 7.9) between the formal affine Demazure algebra and the algebra of Demazure operators on $R[\![\Lambda]\!]_F$ of [CPZ, §4].

Lemma 7.1. *Assume that $\mathfrak{t}$ is regular in R. Let I and I' be reduced sequences of the same element $w \in W$. Then in the formal affine Demazure algebra $\mathbf{D}_F$ we have*

$$X_I - X_{I'} = \sum_{v<w} c_v X_{I_v} \quad \textit{for some } c_v \in S.$$

Proof. Since the definition of the twisted formal group algebra is functorial in the lattice, we may assume that each irreducible component of our root datum is adjoint. In particular, it is not C_n^{sc} and, hence, satisfies all the hypotheses of 6.2. Then, by Proposition 6.2, the difference $X_I - X_{I'} \in \tilde{\mathbf{D}}_F$ can be written as a linear combination $\sum_{w\in W} c_v X_{I_v}$, $c_v \in S$. Since S injects into Q by Lemma 3.2, it suffices to check that $c_v = 0$ unless $v \le w$ (in the Bruhat order) when this linear combination is the one obtained in Q_W by Corollary 5.6. We therefore compute:

$$X_I - X_{I'} \overset{\text{Lem. 5.4 (a)}}{=} \sum_{u\le w} a_u\delta_u - \sum_{u\le w} a'_u\delta_u \overset{\text{Lem. 5.4 (b)}}{=} \sum_{u<w}(a_u - a'_u)\delta_u$$
$$\overset{\text{Cor. 5.6}}{=} \sum_{u<w}(a_u - a'_u)\sum_{v\le u} b_v X_{I_v} = \sum_{v<w} b_v\Big(\sum_{v\le u<w}(a_u - a'_u)\Big)X_{I_v}. \qquad \square$$

Let us examine commutation relations between Demazure elements of simple roots of root datum of rank 1 or 2.

Example 7.2. In the rank 1 case we have [HMSZ, (5.1)]

$$X_\alpha^2 = \kappa_\alpha X_\alpha = X_\alpha\kappa_\alpha \tag{7.1}$$

where $\kappa_\alpha = \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}$ actually lives in S [CPZ, Def. 3.12].

Example 7.3. For the rank 2 case let α_1 and α_2 be simple roots and let W be the Weyl group generated by $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$. Let $m = 2, 3, 4$ or 6 be the order of $s_1 s_2$, i.e. the length of the longest element

$$w_0^{\alpha_1,\alpha_2} = \underbrace{s_1 s_2 s_1\cdots}_{m\text{ times}} = \underbrace{s_2 s_1 s_2\cdots}_{m\text{ times}}.$$

Any element $w \neq w_0^{\alpha_1,\alpha_2}$ in W has a unique reduced sequence, of the form either $(1,2,1\cdots)$ or $(2,1,2\cdots)$. Let I_w be that sequence. In this case, Lemma 7.1 says that

$$\underbrace{X_1X_2X_1\cdots}_{m\text{ times}} - \underbrace{X_2X_1X_2\cdots}_{m\text{ times}} = \sum_{w<w_0^{\alpha_1,\alpha_2}} \eta_w^{\alpha_1,\alpha_2} X_{I_w}, \quad \text{where } \eta_w^{\alpha_1,\alpha_2}\in S. \tag{7.2}$$

Remark 7.4. In [HMSZ, Proposition 5.8] several explicit formulas were given for the coefficients $\eta_w^{\alpha_i,\alpha_j}$ $(i \neq j)$ appearing in similar decompositions with coefficients on the right. Moreover, case by case it was shown that $\eta_w^{\alpha_1,\alpha_2} \in S$ for root data of types A_2, B_2 and D_2 (the G_2 case was left open). The formula (7.2) provides a uniform proof of this fact for all types including G_2.

The following theorem describes the twisted formal group algebra Q_W and the formal affine Demazure algebra $\mathbf{D}_F$ in terms of generators and relations. It generalizes [HMSZ, Theorem 5.14].

Theorem 7.5. *Let $\Sigma \hookrightarrow \Lambda^\vee$ be a root datum and let F be a formal group law over R. Assume that the formal group algebra $S = R[\![\Lambda]\!]_F$ is Σ-regular and the torsion index $\mathfrak{t}$ is regular in R. Let $Q = Q_{R,\Lambda}$ denote the localization of S. Given a set of simple roots $\{\alpha_1,\ldots,\alpha_n\}$ with associated simple reflections $\{s_1,\ldots,s_n\}$, let $m_{i,j}$ denote the order of the product s_is_j in the Weyl group.*

Then elements $q \in Q$ (resp. $q \in S$) and the Demazure elements $X_i = X_{\alpha_i}$ satisfy the following relations ($\forall i, j \in [n]$)

$$X_i q = \Delta_i(q) + s_i(q) X_i, \tag{7.3}$$

$$X_i^2 = \kappa_{\alpha_i} X_i, \tag{7.4}$$

$$\underbrace{X_i X_j X_i \cdots}_{m_{i,j}\ times} - \underbrace{X_j X_i X_j \cdots}_{m_{i,j}\ times} = \sum_{w < w_0^{\alpha_i,\alpha_j}} \eta_w^{\alpha_i,\alpha_j} X_{I_w}, \ \eta_w^{\alpha_i,\alpha_j} \in S. \tag{7.5}$$

Here $w_0^{\alpha_1,\alpha_2}$ is defined in Example 7.3. These relations together with the ring law in S and the fact that the X_i are R-linear form a complete set of relations in the twisted formal group algebra Q_W (resp. the formal affine Demazure algebra $\mathbf{D}_F$).

Proof. The third relation follows from (7.2). The first two relations follow from [HMSZ, Theorem 5.14]. The presentation of the twisted formal group algebra Q_W follows from Theorem 5.14, loc. cit. The presentation of the formal affine Demazure algebra $\mathbf{D}_F$ follows similarly, because of the fact that $\eta_w^{\alpha_i,\alpha_j} \in S$. □

Corollary 7.6. *If $\mathfrak{t}$ is regular in R and S is Σ-regular, then for any sequence I, we can write $X_I = \sum_{v \in W} a_{I,I_v} X_{I_v}$ for some $a_{I,I_v} \in S$ such that:*

1. *If I is a reduced decomposition of $w \in W$, then $a_{I,I_v} = 0$ unless $v \leq w$, and $a_{I,I_w} = 1$.*
2. *If I is not reduced, then $a_{I,I_v} = 0$ for all v such that $\ell(v) \geq |I|$.*

and this decomposition is unique.

Proof. Uniqueness holds because the $(X_{I_w})_{w \in W}$ are linearly independent since they are linearly independent over Q, and the map $S \to Q$ is injective. (1) holds by Theorem 7.1 with $I' = I_w$. If I is not reduced, then we proceed by induction on its length, which is at least 2. We write I as $(i) \cup I'$. If I' is not reduced we are done by induction and applying identity (7.3) to move coefficients to the left of X_i. If I' is a reduced sequence of $w' \in W$, then

$$X_i X_{I'} \overset{(1)}{=} X_i \sum_{v \leq w'} a_v X_{I_v} \overset{(7.3)}{=} \sum_{v < w'} X_i(a_v) X_{I_v} + X_i X_{I_{w'}} + \sum_{v < w'} a_v X_i X_{I_v}.$$

By induction, the third term is irrelevant. Since $(i) \cup I_{w'}$ is not reduced, there is a reduced sequence I'' of w' with first term i by the exchange property [Bo68, Ch. IV, §1, no 5, Prop. 4]. Using part (1), and identity (7.4), $X_i X_{I_{w'}}$ also decomposes as a linear combination of X_{I_v} with v of length at most $\ell(I_{w'}) = |I'|$ and we are done. □

As an immediate consequence we obtain the following generalization of [HMSZ, Lem. 5.13] and [KK86, Thm. 4.6.(a)].

Proposition 7.7. *If $\mathfrak{t}$ is regular in R and S is Σ-regular, then the R-algebra $\mathbf{D}_F$ is free as a left S-submodule of Q_W, with basis $(X_{I_w})_{w \in W}$.*

Proof. We've already seen that the (X_{I_w}) are linearly independent. Let P denote the left S-submodule of Q_W generated by $\{X_{I_w}\}_{w \in W}$. By definition we have $P \subseteq \mathbf{D}_F$. It suffices to prove that P is a R-subalgebra, which reduces to showing that $X_i q X_{I_w} \in P$ for any $i \in [n]$, $w \in W$ and $q \in S$. Using (7.3), it reduces further to showing that $X_i X_{I_w} \in P$, which holds by Corollary 7.6. □

Remark 7.8. Under the hypotheses of 6.2, Proposition 6.2 and Proposition 7.7 imply that $\mathbf{D}_F = \tilde{\mathbf{D}}_F$.

We now explain the relationship between the algebras $\mathbf{D}_F$, $\tilde{\mathbf{D}}_F$, Q_W and R-linear endomorphisms of Q and S. Following [CPZ, Definition 4.5] let $\mathcal{D}(\Lambda)_F$ denote the subalgebra of $\mathrm{End}_R(S)$ generated by Demazure operators Δ_α for all $\alpha \in \Sigma$ and left multiplications by elements in S.

Let $\phi\colon Q_W \to \mathrm{End}_R(Q)$ be the R-algebra homomorphism induced by the left action of Q_W on Q. By definition the formal group algebra S acts on the left on both R-algebras, ϕ is S-linear, and $\phi(X_\alpha) = \Delta_\alpha$ for each $\alpha \in \Sigma$.

We then have the following commutative diagram, where $\mathrm{Stab}(S)$ is the R-subalgebra of $\mathrm{End}_R(Q)$ mapping S into itself, res_S is the restriction of an endomorphism of Q to S and $\phi_{\mathbf{D}_F}$ is the restriction of ϕ to $\mathbf{D}_F$

$$\begin{tikzcd}
\mathbf{D}_F \arrow[dd, "\phi_{\mathbf{D}_F}"'] & \subseteq & \tilde{\mathbf{D}}_F \arrow[d] & \subseteq & Q_W \arrow[d, "\phi"] \\
 & & \mathrm{Stab}(S) \arrow[d] & \subseteq & \mathrm{End}_R(Q) \arrow[d, "\mathrm{res}_S"] \\
\mathcal{D}(\Lambda)_F & \subseteq & \mathrm{End}_R(S) & \subseteq & \mathrm{Hom}_R(S,Q)
\end{tikzcd} \tag{7.6}$$

Observe that the image of $\phi_{\mathbf{D}_F}$ is $\mathcal{D}(\Lambda)_F$ since R-algebra generators of $\mathbf{D}_F$ map to R-algebra generators of $\mathcal{D}(\Lambda)_F$.

Theorem 7.9. *In Diagram (7.6), if $\mathfrak{t}$ is regular in R and S is Σ-regular, then the map $\mathrm{res}_S \circ \phi$ is injective and, therefore, $\phi\colon Q_W \to \mathrm{End}_R(Q)$ is injective and the map $\phi_{\mathbf{D}_F}\colon \mathbf{D}_F \to \mathcal{D}(\Lambda)_F$ is an isomorphism.*

Proof. Since Q_W is a free Q-module, it injects into the respective twisted formal group algebra over $R[1/\mathfrak{t}]$, so we can assume that the torsion index is invertible in R. Let y be an element acting by zero on S. We can express it as $y = \sum_{v\in W} a_v X_{I_v}$ by Corollary 5.6. Applying y successively to all $\Delta_{I_w}(u_0)$, $w \in W$, we conclude that $a_v = 0$ for each $v \in W$ by Lemma 6.1. This shows that $\mathrm{res}_S \circ \phi$ is injective, and the other two facts follow by diagram chase. □

Remark 7.10. Note that by [CPZ, Theorem 4.11], once reduced sequences I_w have been chosen for all $w \in W$, the family $\{\Delta_{I_w}\}_{w\in W}$ forms a basis of $\mathcal{D}(\Lambda)_F$ as a S-module, so the S-module $P \subseteq \mathbf{D}_F$ from the proof of Proposition 7.7 already surjects to $\mathcal{D}(\Lambda)_F$ via $\phi_{\mathbf{D}_F}$, even if $\mathfrak{t}$ is not regular in R.

8. Coproduct on the twisted formal group algebra

In the present section we introduce and study various (co)products on the twisted formal group algebra Q_W. We follow the ideas of [KK86, (4.14)] and [KK90, (2.14)]. In this section we don't assume that S is Σ-regular.

Let W be a group and let R be a commutative ring. The group ring $R[W]$ is a Hopf R-algebra endowed with a cocommutative coproduct $\triangle_{R[W]}\colon R[W] \to R[W] \otimes R[W]$ defined by $\delta_w \mapsto \delta_w \otimes \delta_w$. Its counit $\varepsilon_{R[W]}$ sends any δ_w to 1 (unless otherwise specified, all modules and tensor products are over R).

Assume that W acts on Q by means of R-linear automorphisms. Let $Q_W = Q \otimes R[W]$ be the twisted formal group algebra considered in §5. Let $\upsilon_Q\colon Q \to Q_W$ and $\upsilon_{R[W]}\colon R[W] \to Q_W$ denote the natural inclusions of R-algebras defined by $q \mapsto q \otimes 1$ and $\delta_w \mapsto 1 \otimes \delta_w$ respectively.

Definition 8.1. We define a coproduct $\triangle_{Q_W}\colon Q_W \to Q_W \otimes Q_W$ to be the composition of R-linear morphisms

$$Q \otimes R[W] \xrightarrow{\mathrm{id}_Q \otimes \triangle_{R[W]}} Q \otimes R[W] \otimes R[W] \xrightarrow{\mathrm{id}_{Q \otimes R[W]} \otimes \upsilon_{R[W]}} (Q \otimes R[W]) \otimes (Q \otimes R[W]).$$

We endow the R-module $Q_W \otimes Q_W$ with the usual tensor product algebra structure, i.e. the product is given by $(z_1 \otimes z_1') \cdot (z_2 \otimes z_2') \mapsto (z_1 z_2 \otimes z_1' z_2')$.

Lemma 8.2. *The coproduct $\triangle_{Q_W}$ is associative and is a morphism of R-algebras, i.e. $\triangle_{Q_W}(z \cdot z') = \triangle_{Q_W}(z) \cdot \triangle_{Q_W}(z')$.*

Proof. By linearity, it suffices to check these properties on elements of the form $q\delta_w$ with $q \in Q$ and $w \in W$. By definition we have $\triangle_{Q_W}(q\delta_w) = q\delta_w \otimes \delta_w$ Therefore, we obtain

$$\triangle_{Q_W}(q\delta_w \cdot q'\delta_{w'}) = \triangle_{Q_W}(qw(q')\delta_{ww'}) = qw(q')\delta_{ww'} \otimes \delta_{ww'} =$$
$$q\delta_w \cdot q'\delta_{w'} \otimes \delta_w \cdot \delta_{w'} = \triangle_{Q_W}(q\delta_w) \cdot \triangle_{Q_W}(q'\delta_{w'}). \qquad \square$$

Remark 8.3. The coproduct $\triangle_{Q_W}$ has no counit in general, and it is not cocommutative, because coefficients in Q are arbitrarily put on the left.

Lemma 8.4. *The map $\upsilon_{R[W]}$ is a map of R-coalgebras (without counits), i.e. $\triangle_{Q_W} \circ \upsilon_{R[W]} = (\upsilon_{R[W]} \otimes \upsilon_{R[W]}) \circ \triangle_{R[W]}$.*

Proof. It is straightforward on R-basis elements δ_w of $R[W]$. $\square$

Lemma 8.5. *The map $\mathrm{id}_Q \otimes \triangle_{R[W]}\colon Q_W \to Q_W \otimes R[W]$ is a morphism of R-algebras (with units), and so is the coproduct $\triangle_{Q_W}$.*

Proof. The first claim can be checked directly on elements $q\delta_w$, and the second one follows since the second map in the composition defining $\triangle_{Q_W}$ is a morphism of R-algebras. $\square$

Remark 8.6. Observe that the product in Q_W is, therefore, a morphism of R-coalgebras (without units), since it is the same condition. Similarly, the unit of Q_W preserves coproducts.

We now define a coproduct on Q_W viewed as a *left* Q-module. Beware that Q is not central in Q_W, so Q_W is not a Q-algebra in the usual sense. The convention is still that unlabeled tensor products are over R.

Note that the tensor product $Q_W \otimes_Q Q_W$ of left Q-modules has a natural structure of left Q-module via action $q \cdot (z \otimes z') = qz \otimes z' = z \otimes qz'$.

Let $\pi : Q_W \otimes Q_W \to Q_W \otimes_Q Q_W$ be the projection map. It is a morphism of left Q-modules for both actions of Q on the source, on the left or right factor. The Q-submodule $\ker(\pi)$ is generated by elements of the form $qz \otimes z' - z \otimes qz' = (q \otimes 1 - 1 \otimes q) \cdot (z \otimes z')$. Let $\iota = \mathrm{id}_{Q_W} \otimes \upsilon_{R[W]}$. The morphism of left Q-modules

$$\psi = \pi \circ \iota\colon Q_W \otimes R[W] \to Q_W \otimes_Q Q_W$$

is an isomorphism.

Definition 8.7. We define the product $\odot$ on $Q_W \otimes_Q Q_W$ by transporting the product of the tensor product algebra $Q_W \otimes R[W]$ through the isomorphism ψ, i.e.

$$z \odot z' := \psi(\psi^{-1}(z) \cdot \psi^{-1}(z')) \qquad \text{for } z, z' \text{ in } Q_W \otimes_Q Q_W.$$

Remark 8.8. Note that the morphism ψ is a morphism of R-algebras, and so is ι, but π does *not* respect products in general. We therefore have an inclusion ι of R-algebras, that is split by $\psi^{-1} \circ \pi$, a map of Q-modules, but not a map of R-algebras. Still, π restricted to $\mathrm{im}(\iota)$ is a morphism of R-algebras, because it is the inverse of the morphism of R-algebras ι.

Definition 8.9. We define a coproduct $\triangle\colon Q_W \to Q_W \otimes_Q Q_W$ on Q_W viewed as a Q-module to be the morphism of left Q-modules given by the composition $\pi \circ \triangle_{Q_W} = \psi \circ (\mathrm{id}_Q \otimes \triangle_{R[W]})$.

We therefore have $\triangle(q\delta_w) = q\delta_w \otimes \delta_w = \delta_w \otimes q\delta_w$.

Proposition 8.10. *The coproduct $\triangle$ is Q-linear, associative and cocommutative, with counit $\varepsilon : Q_W \to Q$ defined by $x \mapsto x(1)$ (action of Q_W on Q defined in section 6). Therefore, Q_W is a cocommutative coalgebra in the category of left Q-modules*

Proof. Since all maps involved are morphisms of left Q-modules, it suffices to check it on Q-basis elements δ_w of Q_W. All three properties are then straightforward. $\square$

Remark 8.11. The counit ε satisfies that for any $q \in Q$ and any $x \in Q_W$, we have $\varepsilon(xq) = x(q)$.

Lemma 8.12. *The coproduct $\triangle$ is a morphism of R-algebras.*

Proof. We have $\triangle = \psi \circ (\mathrm{id}_Q \otimes \triangle_{R[W]})$ and both maps are maps of R-algebras. $\square$

Lemma 8.13. *The kernel $\ker(\pi)$ is a right ideal, and is also stable by multiplication on the left by elements in the image of $\triangle_{Q_W}$. In particular $\ker(\pi) \cap \mathrm{im}(\triangle_{Q_W})$ is a double-sided ideal of the R-subalgebra $\mathrm{im}(\triangle_{Q_W})$.*

Proof. It is a right ideal, since it is additively generated by elements of the form $(q \otimes 1 - 1 \otimes q) \cdot (z \otimes z')$, with $q \in Q$ and $z, z' \in Q_W$. To prove the second part of the claim, it suffices to prove that any element of the form $\triangle_{Q_W}(x) \cdot (q \otimes 1 - 1 \otimes q)$ with $q \in Q$ and $x \in Q_W$ is a sum of elements of the form $(q' \otimes 1 - 1 \otimes q) \cdot \triangle_{Q_W}(x)$ for some $q' \in Q$ and $x' \in Q_W$. It is enough to check it when $x = q_w\delta_w$, in which case one has $q' = w(q)$. $\square$

Remark 8.14. Note that this left stability uses the cocommutativity of $\triangle_{R[W]}$ in an essential way.

Corollary 8.15. *The R-submodule $\mathrm{im}(\triangle_{Q_W}) + \ker(\pi) \subseteq Q_W \otimes Q_W$ is an R-subalgebra and the morphism π restricted to $\mathrm{im}(\triangle_{Q_W}) + \ker(\pi)$ is a morphism of R-algebras.*

Proposition 8.16. *For elements $y \in Q_W \otimes R[W]$ and $z \in Q_W \otimes Q_W$, we have*

$$\pi(z) \odot \psi(y) = \pi(z \cdot \iota(y)) \tag{8.1}$$

and if $\iota(y) \cdot \ker(\pi) \subseteq \ker(\pi)$, we also have

$$\psi(y) \odot \pi(z) = \pi(\iota(y) \cdot z). \tag{8.2}$$

In particular, for any $x \in Q_W$, we have

$$\pi(z) \odot \triangle(x) = \pi\big(z \cdot \triangle_{Q_W}(x)\big) \quad \textit{and} \quad \triangle(x) \odot \pi(z) = \pi\big(\triangle_{Q_W}(x) \cdot z\big). \tag{8.3}$$

Proof. We have

$$\pi(z)\odot\psi(y)=\psi(\psi^{-1}\pi(z)\cdot y)=\pi\iota(\psi^{-1}\pi(z)\cdot y)$$
$$=\pi(\iota\psi^{-1}\pi(z)\cdot\iota(y))=\pi((z+\ker(\pi))\cdot\iota(y))=\pi(z\cdot\iota(y)+\ker(\pi))=\pi(z\cdot\iota(y)).$$

Here we have used that $\ker(\pi)$ is a right ideal. Similarly, reversing the product we obtain (8.2). For identity (8.3), just set $y=(\mathrm{id}_Q\otimes\triangle_{R[W]})(x)$ and use Lemma 8.13 to satisfy the extra assumption. □

9. Coproduct on the formal affine Demazure algebra

In the present section we define the coproduct on the formal affine Demazure algebra $\mathbf{D}_F$ by restricting the coproduct $\triangle$ on Q_W (see Theorem 9.2) and compute it for Demazure elements (see Proposition 9.5).

All results of the present section assume that $\mathfrak{t}$ is regular in R and the formal group algebra S is Σ-regular (Definition 4.4).

Lemma 9.1. *The induced map $\mathbf{D}_F\otimes_S\mathbf{D}_F\to Q_W\otimes_Q Q_W$ of left S-modules is injective.*

Proof. The S-module $\mathbf{D}_F\otimes_S\mathbf{D}_F$ is free with basis $(X_{I_v}\otimes X_{I_w})_{(v,w)\in W^2}$, by Proposition 7.7. On the other hand this basis is also a Q-basis of $Q_W\otimes_Q Q_W$ by Corollary 5.6. Since S is Σ-regular, by Lemma 3.2, S injects in Q and we are done. □

We can therefore identify $\mathbf{D}_F\otimes_S\mathbf{D}_F$ with the S-submodule image $\pi(\mathbf{D}_F\otimes\mathbf{D}_F)$ in $Q_W\otimes_Q Q_W$. Beware, however, that through this identification, the product $\odot$ does *not* correspond to the usual product of $\mathbf{D}_F\otimes\mathbf{D}_F$, and $\mathbf{D}_F\otimes_S\mathbf{D}_F$ is *not* stable by the product $\odot$.

By direct computation we obtain in $Q_W\otimes Q_W$ that $\triangle_{Q_W}(X_\alpha)=$

$$\begin{aligned}&X_\alpha\otimes1+1\otimes X_\alpha-x_\alpha X_\alpha\otimes X_\alpha+(\frac{1}{x_\alpha}\otimes1-1\otimes\frac{1}{x_\alpha})(\delta_{s_\alpha}\otimes1-\delta_{s_\alpha}\otimes\delta_{s_\alpha})\\ &=X_\alpha\otimes1+1\otimes X_\alpha-X_\alpha\otimes x_\alpha X_\alpha+(\frac{1}{x_\alpha}\otimes1-1\otimes\frac{1}{x_\alpha})(1\otimes1-1\otimes\delta_{s_\alpha}).\end{aligned}\tag{9.1}$$

It implies that in $Q_W\otimes_Q Q_W$ we have

$$\triangle(X_\alpha)=X_\alpha\otimes1+1\otimes X_\alpha-x_\alpha X_\alpha\otimes X_\alpha.\tag{9.2}$$

Theorem 9.2. *The coproduct $\triangle\colon Q_W\to Q_W\otimes_Q Q_W$ restricts to an S-linear coproduct $\triangle\colon\mathbf{D}_F\to\mathbf{D}_F\otimes_S\mathbf{D}_F$ with counit $\varepsilon\colon\mathbf{D}_F\to S$ obtained by restricting the counit $\varepsilon\colon Q_W\to Q$. One has $\varepsilon(X_{I_w})=\delta_{w,1}$ (the Kronecker symbol).*

Proof. Since $\triangle$ is Q-linear and is a morphism of R-algebras, it suffices to show that $\triangle(X_\alpha)\odot\pi(z)\in\pi(\mathbf{D}_F\otimes\mathbf{D}_F)$ for any root α and any $z\in\mathbf{D}_F\otimes\mathbf{D}_F$ (in particular for $z=1\otimes1$). We thus compute

$$\begin{aligned}\triangle(X_\alpha)\odot\pi(z)&\overset{(8.3)}{=}\pi(\triangle_{Q_W}(X_\alpha)\cdot z)\\&=\pi\big((1\otimes X_\alpha+X_\alpha\otimes1-x_\alpha X_\alpha\otimes X_\alpha+\ker(\pi))\cdot z\big)\\&\overset{\text{Lem. }8.13}{=}\pi\big((1\otimes X_\alpha+\Delta_\alpha\otimes1-x_\alpha X_\alpha\otimes X_\alpha)\cdot z+\ker(\pi)\big)\\&=\pi\big((1\otimes X_\alpha+\Delta_\alpha\otimes1-x_\alpha X_\alpha\otimes X_\alpha)\cdot z\big)\end{aligned}$$

and the last term is in $\pi(\mathbf{D}_F\otimes\mathbf{D}_F)$, since the product is term by term in $Q_W\otimes Q_W$.

The counit ε on Q_W obviously maps $\mathbf{D}_F$ to S by remark 8.11 since $\mathbf{D}_F \subseteq \tilde{\mathbf{D}}_F$. One has $\varepsilon(X_{I_w}) = \Delta_{I_w}(1) = \delta_{w,1}$. $\square$

Given a sequence I in $[n]$ we want to provide a formula for the coproduct of the element X_I. By [HMSZ, Lemma 5.5] we have the following relation

$$X_\alpha \cdot q = \Delta_\alpha(q) + s_\alpha(q) \cdot X_\alpha, \qquad \text{for all } q \in Q \text{ and } \alpha \in \Sigma. \tag{9.3}$$

and, more generally, we obtain

Lemma 9.3. *For any sequence I and for any $q \in Q$, we have*

$$X_I \cdot q = \sum_{E \subseteq [l]} \phi_{I,E}(q) \cdot X_{I_{|E}} \tag{9.4}$$

where $\phi_{I,E} = B_1 \circ \cdots \circ B_l$ with

$$B_j = \begin{cases} B_{i_j}^{(0)}, & \text{if } j \in E, \\ B_{i_j}^{(-1)} & \text{otherwise.} \end{cases}$$

Proof. Induction using (9.3). $\square$

Lemma 9.4. *For any sequence $I = (i_1, \ldots, i_l)$ in $[n]$ we have in $Q_W \otimes Q_W$*

$$\prod_{k=1}^{l} (X_{i_k} \otimes 1 + 1 \otimes X_{i_k} - x_{i_k} X_{i_k} \otimes X_{i_k}) = \sum_{E_1, E_2 \subset [l]} (p^I_{E_1,E_2} X_{I_{|E_1}}) \otimes X_{I_{|E_2}} \tag{9.5}$$

where the element $p^I_{E_1,E_2} \in S$ is defined in Lemma 4.8.

Proof. We prove the formula by induction on the length l of I. For $l = 1$, it holds by identity (9.2). Let $I' = (i_2, \ldots, i_l)$. For fixed $E_1, E_2 \subseteq [l-1]$, and for $p \in S$ we have

$$\begin{aligned}
&(1 \otimes X_{i_1} + X_{i_1} \otimes 1 - x_{i_1} X_{i_1} \otimes X_{i_1}) \cdot \big((pX_{I'_{|E_1}}) \otimes X_{I'_{|E_2}}\big) \\
&= (pX_{I'_{|E_1}}) \otimes (X_{i_1} X_{I'_{|E_2}}) + (X_{i_1} p X_{I'_{|E_1}}) \otimes X_{I'_{|E_2}} - (x_{i_1} X_{i_1} p X_{I'_{|E_1}}) \otimes (X_{i_1} X_{I'_{|E_2}}) \\
&\overset{(9.3)}{=} (pX_{I'_{|E_1}}) \otimes (X_{i_1} X_{I'_{|E_2}}) + (\Delta_{i_1}(p) X_{I'_{|E_1}}) \otimes X_{I'_{|E_2}} + (s_{i_1}(p) X_{i_1} X_{I'_{|E_1}}) \otimes X_{I'_{|E_2}} \\
&\quad - (x_{i_1} \Delta_{i_1}(p) X_{I'_{|E_1}}) \otimes (X_{i_1} X_{I'_{|E_2}}) - (x_{i_1} s_{i_1}(p) X_{i_1} X_{I'_{|E_1}}) \otimes (X_{i_1} X_{I'_{|E_2}}) \\
&= (\Delta_{i_1}(p) X_{I'_{|E_1}}) \otimes X_{I'_{|E_2}} + (s_{i_1}(p) X_{i_1} X_{I'_{|E_1}}) \otimes X_{I'_{|E_2}} \\
&\quad + (s_{i_1}(p) X_{i_1} X_{I'_{|E_1}}) \otimes X_{I'_{|E_2}} - (x_{i_1} s_{i_1}(p) X_{i_1} X_{I'_{|E_1}}) \otimes (X_{i_1} X_{I'_{|E_2}})
\end{aligned}$$

and the result follows by induction as in the end of the proof of Lemma 4.8. $\square$

Proposition 9.5 (cf. Proposition 4.15 [KK86])**.** *For any $I = (i_1, \ldots, i_l)$ we have*

$$\triangle(X_I) = \sum_{E_1, E_2 \subset [l]} p^I_{E_1,E_2} X_{I_{|E_1}} \otimes X_{I_{|E_2}}. \tag{9.6}$$

Proof. We have

$$\begin{aligned}
&\triangle(X_I) = \triangle(X_{i_1}) \odot \cdots \odot \triangle(X_{i_l}) = \pi \triangle_{Q_W}(X_{i_1}) \odot \cdots \odot \pi \triangle_{Q_W}(X_{i_l}) \\
&\overset{(9.2)}{=} \pi(1 \otimes X_{i_1} + X_{i_1} \otimes 1 - x_{i_1} X_{i_1} \otimes X_{i_1}) \odot \cdots \odot \pi(1 \otimes X_{i_l} + X_{i_l} \otimes 1 - x_{i_l} X_{i_l} \otimes X_{i_l}) \\
&\overset{\text{Cor. } 8.15}{=} \pi\big((1 \otimes X_{i_1} + X_{i_1} \otimes 1 - x_{i_1} X_{i_1} \otimes X_{i_1}) \cdots (1 \otimes X_{i_l} + X_{i_l} \otimes 1 - x_{i_l} X_{i_l} \otimes X_{i_l})\big) \\
&\overset{(9.5)}{=} \sum_{E_1, E_2 \subset [l]} p^I_{E_1,E_2} X_{I_{|E_1}} \otimes X_{I_{|E_2}}. \quad \square
\end{aligned}$$

Remark 9.6. Note that the only subtlety in the proof of the previous formula is to check that products of elements $\triangle(X_i)$ can indeed be computed term-wise in the tensor product, which is the content of Corollary 8.15.

10. Coproduct on the algebra of Demazure operators

In the present section we lift the coproduct on $\epsilon\mathcal{D}(\Lambda)_F$ defined in [CPZ] to a natural coproduct on $\mathcal{D}(\Lambda)_F$, and show it is isomorphic to the coproduct $\triangle$ on $\mathbf{D}_F$ through the isomorphism $\phi_{\mathbf{D}_F}$ of Theorem 7.9.

Unless otherwise specified, all tensor products are still over R, all Hom and End groups are homomorphisms of R-modules. When M and N are topological R-modules, let $\mathrm{Hom}^0(N,M)$ (resp. $\mathrm{End}^0(M)$) denote the R-submodule of $\mathrm{Hom}(N,M)$ (resp. of $\mathrm{End}(M)$) of continuous homomorphisms. When M is a topological module over S, and R acts through S, $\mathrm{Hom}(N,M)$ has an obvious structure of left S-module, by $s\cdot f=(x\mapsto sf(x))$. Then $\mathrm{Hom}^0(N,M)$ is an S-submodule of $\mathrm{Hom}(N,M)$, and in particular $\mathrm{End}^0(S)$ is an S-submodule of $\mathrm{End}(S)$. By construction, $\mathcal{D}(\Lambda)_F$ is an S-submodule of $\mathrm{End}^0(S)$. Let $\Upsilon_1:\mathcal{D}(\Lambda)_F\to\mathrm{End}^0(S)$ denote the inclusion.

Let $S\otimes S$ (resp. $S\otimes S\otimes S$) be endowed with the topology defined by the $\mathcal{I}_F^i\otimes\mathcal{I}_F^j$, $i,j\in\mathbb{N}$ (by the $\mathcal{I}_F^i\otimes\mathcal{I}_F^j\otimes\mathcal{I}_F^k$, $i,j,k\in\mathbb{N}$) as a basis of neighborhoods of zero. Let

$$\begin{array}{rccc}\Upsilon_2: & \mathcal{D}(\Lambda)_F\otimes_S\mathcal{D}(\Lambda)_F & \to & \mathrm{Hom}^0(S\otimes S,S)\\ & D_1\otimes D_2 & \mapsto & ((r\otimes s)\mapsto D_1(r)D_2(s))\end{array}$$

and similarly

$$\begin{array}{rccc}\Upsilon_3: & \mathcal{D}(\Lambda)_F\otimes_S\mathcal{D}(\Lambda)_F\otimes_S\mathcal{D}(\Lambda)_F & \to & \mathrm{Hom}^0(S\otimes S\otimes S,S)\\ & D_1\otimes D_2\otimes D_3 & \mapsto & ((r\otimes s\otimes t)\mapsto D_1(r)D_2(s)D_3(t)).\end{array}$$

Lemma 10.1. *If the torsion index $\mathfrak{t}$ is regular in R, the maps Υ_1,Υ_2 and Υ_3 are injective.*

Proof. By definition, Υ_1 is an inclusion. For Υ_2, by [CPZ, Theorem 4.11], once reduced sequences I_w have been chosen for all $w\in W$, the family $\{\Delta_{I_w}\}_{w\in W}$ forms a basis of $\mathcal{D}(\Lambda)_F$ as an S-module. Let us assume that $\sum_{v,w}a_{v,w}\Delta_{I_v}\otimes\Delta_{I_w}$ is mapped to the zero morphism. Thus, for any $r,s\in S$, we have $\sum_w\sum_v a_{v,w}\Delta_{I_v}(r)\Delta_{I_w}(s)=0$. Replacing successively s by $\Delta_{I_{w'}}(u_0)$ for all $w'\in W$, we obtain that for any w', $\sum_w\zeta_w(r)\Delta_{I_w}(\Delta_{I_{w'}}(u_0))=0$ with $\zeta_w(r)=\sum_v a_{v,w}\Delta_{I_v}(r)$. So, given w, for any $r\in S$, we have $\zeta_w(r)=0$ by Lemma 6.1. Replacing r successively by $\Delta_{I_{v'}}(u_0)$ for all $v'\in W$, we similarly obtain $a_{v,w}=0$.

The proof for Υ_3 is the same, but with one extra step for the last factor. $\square$

Another way of phrasing the previous lemma is that an element of $\mathcal{D}(\Lambda)_F\otimes_S\mathcal{D}(\Lambda)_F$ is characterized by how it acts on $S\otimes S$ (with values in S), and similarly with three factors.

Lemma 10.2. *If $\mathfrak{t}$ is regular in R, then the preimage by Υ_1 of $\mathrm{Hom}^0(S,\mathcal{I}_F)$ is $\mathcal{I}_F\mathcal{D}(\Lambda)_F$.*

Proof. Let $d\in\mathcal{D}(\Lambda)_F$ such that $d(s)\in\mathcal{I}_F$ for any $s\in S$. Decompose $d=\sum_w a_w\Delta_{I_w}$ by [CPZ, Theorem 4.11]. After inverting the torsion index in R, by Lemma 6.1, we obtain that for any $w\in W$, the element a_w is in $\mathcal{I}_{F,R[1/\mathfrak{t}]}$, the augmentation ideal of $R[\frac{1}{\mathfrak{t}}][\![\Lambda]\!]_F$. Since $\mathcal{I}_{F,R[1/\mathfrak{t}]}\cap S=\mathcal{I}_F$ inside $R[\frac{1}{\mathfrak{t}}][\![\Lambda]\!]_F$, we are done. $\square$

Let $\chi : \mathrm{End}^0(S) \to \mathrm{Hom}^0(S \otimes S, S)$ be the morphism of left S-modules sending $f \in \mathrm{End}^0(S)$ to $((u \otimes v) \mapsto f(uv))$.

Theorem 10.3. *Assume that $\mathfrak{t}$ is regular. Along Υ_2, the morphism χ factors (uniquely) through a morphism of left S-modules $\triangle_{\mathcal{D}(\Lambda)_F} : \mathcal{D}(\Lambda)_F \to \mathcal{D}(\Lambda)_F \otimes_S \mathcal{D}(\Lambda)_F$, which is thus the unique such morphism satisfying the formula*

$$\Upsilon_2(\triangle_{\mathcal{D}(\Lambda)_F}(f))(u \otimes v) = f(uv). \tag{10.1}$$

It is an associative and commutative S-linear coproduct, with counit $\varepsilon_{\mathcal{D}(\Lambda)_F}$ mapping f to $f(1)$ (evaluation at 1).

Proof. Since $\mathcal{D}(\Lambda)_F \otimes_S \mathcal{D}(\Lambda)_F$ injects by Υ_2 in $\mathrm{Hom}^0(S \otimes S, S)$, uniqueness is clear. Now χ restricts by [CPZ, Lemma 7.1 (1)]. Associativity follows from the fact that both compositions involved $\mathcal{D}(\Lambda)_F \to \mathcal{D}(\Lambda)_F \otimes_S \mathcal{D}(\Lambda)_F \otimes_S \mathcal{D}(\Lambda)_F$ composed with Υ_3 send $D \in \mathcal{D}(\Lambda)_F$ to $((u, v, w) \mapsto D(uvw))$, so they are equal by injectivity of Υ_3. Commutativity follows similarly with Υ_2 instead of Υ_3, and the fact that $\varepsilon_{\mathcal{D}(\Lambda)_F}$ is a (left) counit holds because $(\varepsilon_{\mathcal{D}(\Lambda)_F} \otimes \mathrm{id}_S) \circ \triangle_{\mathcal{D}(\Lambda)_F}(f)$ sends f to $(v \mapsto f(1 \cdot v) = f(v))$. It is also a right counit by the same argument on the right side. $\square$

Theorem 10.4. *When the torsion index $\mathfrak{t}$ is regular in R and S is Σ-regular, the map $\phi_{\mathbf{D}_F} : \mathbf{D}_F \to \mathcal{D}(\Lambda)_F$ of Theorem 7.9 is an isomorphism of S-coalgebras with counits.*

Proof. Since $\mathfrak{t}$ is regular, the $(X_{I_w})_{w \in W}$ are an S-basis of $\mathbf{D}_F$. The $(\Delta_{I_w})_{w \in W}$ are an S-basis of $\mathcal{D}(\Lambda)_F$ in any case. The identification of $\triangle$ and $\triangle_{\mathcal{D}(\Lambda)_F}$ through $\phi_{\mathbf{D}_F}$ then follows from formulas (9.6) and (4.1), via the uniqueness of Theorem 10.3. $\square$

The coproduct that appears in [CPZ] is then constructed as follows. The augmentation map $\epsilon : R[\![\Lambda]\!]_F \to R$ induces a map of R-modules $\epsilon_* \colon \mathcal{D}(\Lambda)_F \to \mathrm{Hom}_R(S, R)$ given by $f \mapsto \epsilon \circ f$. Let $\epsilon\mathcal{D}(\Lambda)_F$ be the image of $\mathcal{D}(\Lambda)_F$ by this map. As in the proof of [CPZ, Thm. 7.3] we define an R-linear coproduct

$$\triangle^\epsilon \colon \epsilon\mathcal{D}(\Lambda)_F \to \epsilon\mathcal{D}(\Lambda)_F \otimes_R \epsilon\mathcal{D}(\Lambda)_F$$

by

$$\triangle^\epsilon(f)(u \otimes v) := f(uv), \quad f \in \epsilon\mathcal{D}(\Lambda)_F,\ u, v \in R[\![\Lambda]\!]_F.$$

Here the property (which follows from [CPZ, Prop. 3.8])

$$\epsilon\Delta_i(u \cdot v) = \epsilon\Delta_i(u) \cdot \epsilon(v) + \epsilon(u) \cdot \epsilon\Delta_i(v), \quad u, v \in R[\![\Lambda]\!]_F$$

guarantees that the coproduct $\triangle^\epsilon$ is well defined (cf. [CPZ, Lem. 7.1]) and

$$\triangle^\epsilon(\epsilon\Delta_i) = \epsilon\Delta_i \otimes \epsilon + \epsilon \otimes \epsilon\Delta_i.$$

Through $\epsilon : S \to R$, we can restrict $\epsilon\mathcal{D}(\Lambda)_F$ to an S-module. Then $\epsilon\mathcal{D}(\Lambda)_F \otimes_R \epsilon\mathcal{D}(\Lambda)_F = \epsilon\mathcal{D}(\Lambda)_F \otimes_S \epsilon\mathcal{D}(\Lambda)_F$, and $\triangle^\epsilon$ is an S-linear coproduct. The morphism $\epsilon_* : \mathcal{D}(\Lambda)_F \to \epsilon\mathcal{D}(\Lambda)_F$ is clearly a morphism of S-coalgebras. In other words, the diagram

$$\begin{tikzcd}
\mathbf{D}_F \arrow[r, "\phi_{\mathbf{D}_F}"] \arrow[d, "\triangle"] & \mathcal{D}(\Lambda)_F \arrow[r, "\epsilon_*"] \arrow[d, "\triangle_{\mathcal{D}(\Lambda)_F}"] & \epsilon\mathcal{D}(\Lambda)_F \arrow[d, "\triangle^\epsilon"] \\
\mathbf{D}_F \otimes_S \mathbf{D}_F \arrow[r, "\phi_{\mathbf{D}_F}^{\otimes 2}"] & \mathcal{D}(\Lambda)_F \otimes_S \mathcal{D}(\Lambda)_F \arrow[r, "\epsilon_*^{\otimes 2}"] & \epsilon\mathcal{D}(\Lambda)_F \otimes_R \epsilon\mathcal{D}(\Lambda)_F
\end{tikzcd} \tag{10.2}$$

commutes.

Remark 10.5. The significance of the coproduct $\triangle^\epsilon$ follows from the following fact:

Following [CPZ, §6] consider the dual $\epsilon\mathcal{D}(\Lambda)_F^* = \mathrm{Hom}_R(\epsilon\mathcal{D}(\Lambda)_F, R)$ (in loc. cit. $\epsilon\mathcal{D}(\Lambda)_F^* = \mathcal{H}(M)_F$). Then by [CPZ, Thm. 7.3 and Thm. 13.12] the coproduct $\triangle^\epsilon$ on $\epsilon\mathcal{D}(\Lambda)_F$ induces (by duality) an R-algebra structure on $\epsilon\mathcal{D}(\Lambda)_F^*$. Moreover, under the hypotheses of [CPZ, Thm. 13.3] there is a natural ring isomorphism $\epsilon\mathcal{D}(\Lambda)_F^* \simeq \mathtt{h}(G/B)$, where $\mathtt{h}(G/B)$ is the algebraic oriented cohomology of the variety of complete flags G/B.

11. The dual of a formal affine Demazure algebra

In the present section we consider the dual $(\mathcal{D}(\Lambda)_F)^\star$ of the algebra of Demazure operators, or equivalently the dual of the formal affine Demazure algebra $\mathbf{D}_F^\star$ and prove Theorem 11.4.

We will need the following classical facts, that the reader can easily verify anyway.

Let S be a commutative unital ring and let D be a free left S-module of finite type. Assume that D is an S-coalgebra, with linear coproduct $\triangle\colon D \to D \otimes_S D$ and counit $\varepsilon : D \to S$. Then the dual $D^\star = Hom_S(D, S)$ has a natural structure of S-algebra with unit, given by dualizing the coproduct and using the natural isomorphism $D^\star \otimes_S D^\star \simeq (D \otimes_S D)^\star$, and dualizing the counit, i.e. ε, seen as an element of $D^\star$, is the unit. It is commutative if and only if D is cocommutative. Let $p^\theta_{\theta_1,\theta_2} \in S$ be the coefficients of the coproduct on a basis $\{x_\theta\}_{\theta\in\Theta}$ of D (Θ is therefore finite), i.e. for every $\theta \in \Theta$,

$$\triangle(x_\theta) = \sum_{\theta_1,\theta_2\in\Theta} p^\theta_{\theta_1,\theta_2} x_{\theta_1} \otimes x_{\theta_2}.$$

On the dual basis $(x^\star_\theta)_{\theta\in\Theta}$ of $(x_\theta)_{\theta\in\Theta}$, the product of $D^\star$ is given by

$$x^\star_{\theta_1} x^\star_{\theta_2} = \sum_{\theta\in\Theta} p^\theta_{\theta_1,\theta_2} x^\star_\theta.$$

For the rest of this section, we assume that $2\mathfrak{t}$ is regular in R. It implies that S is Σ-regular by Lemma 2.2.

We apply the above construction to $D = \mathcal{D}(\Lambda)_F$, with its coalgebra structure given by Theorem 10.3. This yields a commutative algebra with unit $\mathcal{D}(\Lambda)_F^\star$. Given choices of a reduced decompositions I_w for each element $w \in W$, we have a basis $(\Delta_{I_w})_{w\in W}$ of $\mathcal{D}(\Lambda)_F$.

Remark 11.1. Instead of $\mathcal{D}(\Lambda)_F$, we can use $\mathbf{D}_F$, with its coalgebra structure given by Theorem 9.2 and its basis $(X_{I_w})_{w\in W}$. Since we have $\mathbf{D}_F \simeq \mathcal{D}(\Lambda)_F$ as S-coalgebras by Theorem 10.4, we dually have $(\mathcal{D}(\Lambda)_F)^\star \simeq \mathbf{D}_F^\star$ as algebras, and it won't make any difference. In particular, Theorem 11.4 also holds for $\mathbf{D}_F$.

The R-dual $(\epsilon\mathcal{D}(\Lambda)_F)^* = \mathrm{Hom}_R(\epsilon\mathcal{D}(\Lambda)_F, R)$ is an R-algebra with unit by dualizing the coproduct $\triangle^\epsilon$ of section 10. By [CPZ, Prop. 5.4], the $\epsilon\Delta_{I_w}$, $w \in W$, form a basis of $\epsilon\mathcal{D}(\Lambda)_F$. Using $\epsilon : S \to R$, we can restrict $(\epsilon\mathcal{D}(\Lambda)_F)^*$ to an S-algebra with unit.

Lemma 11.2. *The map $\xi : (\mathcal{D}(\Lambda)_F)^\star \to (\epsilon\mathcal{D}(\Lambda)_F)^*$ sending f to the map $\epsilon d \mapsto \epsilon f(d)$ is well defined, and it is a surjective map of S-algebras with units.*

Proof. If $\epsilon d_1 = \epsilon d_2$, as elements of $\mathrm{Hom}^0(S, R)$, or in other words $(d_1 - d_2) \in \mathrm{Hom}^0(S, \mathcal{I}_F)$, then Lemma 10.2 shows that $d_1 - d_2 \in \mathcal{I}_F \mathcal{D}(\Lambda)_F$, and thus $\epsilon f(d_1) = \epsilon f(d_2)$ in R for any $f \in \mathrm{Hom}_S(\mathcal{D}(\Lambda)_F, S)$. This proves that ξ is well-defined.

It is clearly a morphism of S-modules. Checking that it respects the products is a simple exercise from their definitions as duals to the coproducts, using the commutativity of Diagram (10.2) and the fact that ϵ is a morphism of rings.

Finally, ξ is surjective, because it sends the (dual) basis $\Delta^\star_{I_w}$ of $(\mathcal{D}(\Lambda)_F)^\star$ to the (dual) basis $\epsilon\Delta^*_{I_w}$ of $(\epsilon\mathcal{D}(\Lambda)_F)^*$. □

For any $s \in S$, let $\mathrm{ev}_s \in \mathbf{D}^\star_F$ be the map $d \mapsto d(s)$.

Lemma 11.3. *The product on $\mathcal{D}(\Lambda)_F$ satisfies $\mathrm{ev}_{s_1} \cdot \mathrm{ev}_{s_2} = \mathrm{ev}_{s_1 s_2}$.*

Proof. By Theorem 10.3, the coproduct $\triangle_{\mathcal{D}(\Lambda)_F}$ on $\mathcal{D}(\Lambda)_F$ satisfies formula (10.1). Thus

$$\begin{aligned}(\mathrm{ev}_{s_1} \cdot \mathrm{ev}_{s_2})(d) &= (\mathrm{ev}_{s_1} \otimes \mathrm{ev}_{s_2})(\triangle_{\mathcal{D}(\Lambda)_F}(d)) = \Upsilon_2(\triangle_{\mathcal{D}(\Lambda)_F}(d))(s_1 \otimes s_2)\\ &= d(s_1 s_2) = \mathrm{ev}_{s_1 s_2}(d).\end{aligned}$$

On $\mathbf{D}^\star_F$, the property then follows using Theorem 10.4. □

Let $c_S : S \to \mathcal{D}(\Lambda)^\star_F$ be the R-linear map sending $s \in S$ to ev_s. By the previous lemma, it is actually an R-algebra map. Since the subring $S^W \subseteq S \subseteq \mathcal{D}(\Lambda)_F$ of elements fixed by the Weyl group W is central[1] in $\mathcal{D}(\Lambda)_F$, the map c_S is in fact S^W-linear.

The map c_S would correspond in cohomology to an equivariant characteristic map. In [CPZ, Def. 6.1], there is a (non-equivariant) characteristic map $c_R : S \to (\epsilon\mathcal{D}(\Lambda)_F)^*$, sending $s \in S$ to $\mathrm{ev}_s : \epsilon d \mapsto \epsilon d(s)$. They are related by $c_R = \xi \circ c_S$. Note that c_R is a map of R-algebras, either by [CPZ, Theorem 7.3] or by Lemma 11.3 and the fact that ξ is a morphism of S-algebras and S acts on $\epsilon\mathcal{D}(\Lambda)^*_F$ via R through ϵ.

The tensor product $S \otimes_{S^W} S$ is an S-algebra via the first component. We now consider the S-algebra map

$$\rho_S : S \otimes_{S^W} S \to \mathcal{D}(\Lambda)^\star_F, \qquad s_1 \otimes s_2 \mapsto s_1 c_S(s_2)$$

and the R-algebra map

$$\rho_R : R \otimes_{S^W} S \to \epsilon\mathcal{D}(\Lambda)^*_F, \qquad r \otimes s \mapsto r c_R(s_2).$$

where R is considered an S^W-module via the augmentation map ϵ. These two maps are related by the commutative diagram

$$\begin{tikzcd} S \otimes_{S^W} S \arrow[r, "\rho_S"] \arrow[d, "\epsilon\otimes\mathrm{id}_S"'] & \mathcal{D}(\Lambda)^\star_F \arrow[d, "\xi"] \\ R \otimes_{S^W} S \arrow[r, "\rho_R"] & (\epsilon\mathcal{D}(\Lambda)_F)^* \end{tikzcd}$$

The following theorem generalizes [KK90, Thm.4.4] and [Ku02, Cor. 11.3.17] to the context of an arbitrary formal group law. It is also related to [KiKr, Theorem 4.7], see Remark 11.6.

Theorem 11.4. *If $2\mathfrak{t}$ is regular in R, the following conditions are equivalent.*

[1]$\mathcal{D}(\Lambda)_F$ is by definition generated by elements that all commute with S^W, see [CPZ, Prop. 3.14]. This is where we first use that 2 is regular.

(a) *The map $\rho_S : S \otimes_{S^W} S \to \mathcal{D}(\Lambda)^\star_F$ is an isomorphism (of S-algebras).*
(b) *The map $\rho_R : R \otimes_{S^W} S \to (\epsilon\mathcal{D}(\Lambda)_F)^*$ is an isomorphism (of R-algebras).*
(c) *The characteristic map c_R (of R-algebras) is surjective.*

In particular all are true if the torsion index $\mathfrak{t}$ is invertible, by [CPZ, Theorem 6.4].

Proof. By the above diagram, if ρ_S is surjective, then so is ρ_R. Furthermore, by definition of ρ_R, we have $c_R = \rho_R \circ (1 \otimes \mathrm{id}_S)$. Since the map $1 \otimes \mathrm{id}_S$ is surjective (because S^W surjects to R via ϵ), we have that c_R is surjective if and only if ρ_R is surjective. We now need the following lemma.

Lemma 11.5. *Let N be the length of w_0, the longest element in W. If c_R is surjective, then there is an element $u_0' \in S$ such that for any sequence I, we have*

$$\epsilon\Delta_I(u_0') = \begin{cases} 0, & \text{if } I \text{ is of length } < N \text{ or is of length } N \text{ but non reduced,} \\ 1, & \text{if } I \text{ is of length } N \text{ and is reduced.} \end{cases}$$

Proof. Choose a reduced decomposition I_w for every $w \in W$. By definition of c_R, if it is surjective, there is an element u_0' such that $\epsilon X_{I_w}(u_0') = 1$ if $w = w_0$ and zero if not. Then, decomposing X_I as in Corollary 7.6, we have $X_I(u_0') = \sum_v \epsilon(a_{I,I_v})\epsilon X_{I_v}(u_0') = \epsilon(a_{I,I_{w_0}})$ and the conditions of Corollary 7.6 imply the result. □

Returning to the proof of Theorem 11.4, let us conversely show that if c_R is surjective, then both ρ_R and ρ_S are isomorphisms. The lemma implies that the matrix $(\epsilon X_{I_v} X_{I_w}(u_0'))_{(v,w)\in W\times W}$ (with coefficients in R) is invertible, since it is triangular with 1 on the diagonal. Thus, the matrix $(X_{I_v} X_{I_w}(u_0'))_{(v,w)\in W\times W}$ (with coefficients in S) is invertible, since the kernel of $\epsilon : S \to R$ is in the radical of S. As in [CPZ, Theorem 6.7], this implies that the $\Delta_{I_w}(u_0')$ form a basis of S as an S^W-module. Thus, the map ρ_S (resp. ρ_R) is between two free S-modules (resp. R-modules) of the same rank $|W|$, so it is an isomorphism if it is surjective. It suffices to show the surjectivity of ρ_S, since it implies that of ρ_R. The image $\rho_S(1 \otimes X_{I_v}(u_0'))$ of a basis element of $S \otimes_{S^W} S$ decomposes as $\sum_w X_{I_w} X_{I_v}(u_0') X^\star_{I_w}$ on the (dual) basis of $(\mathcal{D}(\Lambda)_F)^\star$, so by invertibility of the above matrix, we are done. □

Remark 11.6. When $F = U$ is the universal formal group law over the Lazard ring, by [KiKr, Thm. 5.1] the algebra $S \otimes_{S^W} S$ can be identified (up to a completion) with the equivariant algebraic cobordism $\Omega_T(G/B)$ after inverting $\mathfrak{t}$. Therefore, one may think of the dual algebra $(\mathcal{D}(\Lambda)_F)^\star$ or $\mathbf{D}^\star_U$ as an algebraic model of $\Omega_T(G/B)$. In [KiKr, Theorem 4.6], a theorem similar to Theorem 11.4 is shown, using topological inputs and inverting the torsion index. Here, apart from the fact that the treatment is completely algebraic, the main improvement is that for other formal group laws, it might not be necessary to invert the torsion index for the equivalent conditions of Theorem 11.4 to hold. For example, in the case of the periodic multiplicative formal group law $x+y-xy$ of K-theory, the characteristic map is already surjective over $\mathbb{Z}$ for simply connected root data [Pi72].

Remark 11.7. Note that in Lemma 11.5, the element u_0' has no reason to be in $\mathcal{I}_F^N$. For example, for the periodic multiplicative formal group law $x + y - xy$ over $\mathbb{Z}$, for simply connected root data with non-trivial torsion index (ex: G_2^{sc}), such an element u_0' cannot be found in $\mathcal{I}_F^N$.

Remark 11.8. In this section the algebra $(\mathcal{D}(\Lambda)_F)^\star$ can be replaced everywhere by $\mathbf{D}^\star_F$.

12. Appendix

Recall that an element of a commutative, associative unital ring R is called regular if it is not a zero divisor or, in other words, that multiplication by this element is an injective ring endomorphism.

Lemma 12.1. *Let f be a zero divisor in the polynomial ring $R[x_1, \dots, x_n]$. Then there exists an element $r \neq 0$ in R such that $rf = 0$.*

Proof. See [MC42], Theorem 3 and the comment following it. $\square$

Let $I = (x_1, \dots, x_n)$ denote the ideal in the ring $R[\![x_1, \dots, x_n]\!]$ of formal power series generated by $x_1, \dots, x_n$.

Corollary 12.2. *If $f \in R[\![x_1, \dots, x_n]\!]$ is a zero divisor, then there exists an element $r \neq 0$ in R such that all coefficients of the lowest degree part of f are annihilated by r.*

Proof. Take a nonzero $g \in R[\![x_1, \dots, x_n]\!]$ such that $fg = 0$. Let $f = f_i + I^{i+1}$ and $g = g_j + I^{j+1}$ where f_i and g_j are the lowest degree parts (homogeneous and nonzero). Then $fg = f_i g_j + I^{i+j+1}$ and, hence, $f_i g_j = 0$. The result then follows by Lemma 12.1. $\square$

Lemma 12.3. *Let $f = a_1x_1 + \cdots + a_nx_n + I^2$, $a_i \in R$, be an element of $R[\![x_1, \dots, x_n]\!]$.*

(a) If a_i is regular in R for some i, then f is regular in $R[\![x_1, \dots, x_n]\!]$.

(b) If the vector of coefficients $(a_1, \dots, a_n)$ can be completed to a basis of R^n, then f is regular in $R[\![x_1, \dots, x_n]\!]$.

Proof. Assume that f is a zero divisor in $R[\![x_1, \dots, x_n]\!]$. Then by Corollary 12.2 there is $r \neq 0$ in R that annihilates each a_i. This contradicts the assumptions of (a) and (b). $\square$

Observe that there exist examples of vectors consisting of zero divisors that can be completed to a basis, so that (a) does not imply (b).

Let $A = (a_{ij})$ be an $n \times n$-matrix with coefficients in R. Consider a map $\Psi_A \colon R[\![x_1, \dots, x_n]\!] \to R[\![x_1, \dots, x_n]\!]$ defined by sending each x_i to an element of the form $\sum a_{ij}x_j + I^2$.

Lemma 12.4. *The matrix A is invertible if and only if the map Ψ_A is invertible.*

Proof. See [Bo58, Ch. IV, §4, No 7, Prop. 10]. $\square$

Corollary 12.5. *If the determinant of the matrix A is regular, then the map Ψ is injective.*

Proof. Let T be the total ring of fractions of R. There is a canonical injection $R \hookrightarrow T$. The matrix A viewed as a matrix with coefficients in T has an invertible determinant. Therefore, by Lemma 12.4 the respective map $\Psi_A \colon T[\![x_1, \dots, x_n]\!] \to T[\![x_1, \dots, x_n]\!]$ is invertible. The injectivity of Ψ_A over R then follows by the commutative diagram

$$\begin{tikzcd} R[\![x_1, \dots, x_n]\!] \arrow[r, "\Psi_A"] \arrow[d, hook] & R[\![x_1, \dots, x_n]\!] \arrow[d, hook] \\ T[\![x_1, \dots, x_n]\!] \arrow[r, "\Psi_A", "\simeq"'] & T[\![x_1, \dots, x_n]\!] \end{tikzcd}.$$

$\square$

By a lattice we call a finitely generated free Abelian group, i.e. a free $\mathbb{Z}$-module of finite rank.

Lemma 12.6. *Given an inclusion of lattices $\Lambda \subseteq \Lambda'$ of the same rank n, an $n \times n$ matrix A expressing a basis of Λ in terms of a basis of Λ' satisfies* $\det(A) = \pm|\Lambda'/\Lambda|$, *the order of the group Λ'/Λ up to a sign.*

Proof. It follows from [Bo58, Ch. VII, §4, No 6, Cor. 1]. □

An n-tuple of integers $(k_1, \ldots, k_n)$ is called a unimodular row if there exist integers $(a_1, \ldots, a_n)$ such that $\sum_{i=1}^{n} k_i a_i = 1$.

Lemma 12.7. *Any unimodular row $(k_1, \ldots, k_n)$ can be completed to a basis of $\mathbb{Z}^n$.*

Proof. The kernel of the map $\mathbb{Z}^n \to \mathbb{Z}$ sending $(a_1, \ldots, a_n)$ to $\sum_i k_i a_i$ is a direct factor of $\mathbb{Z}^n$, and it is torsion free so it is free. □

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