

Hermitian categories, extension of scalars and systems of sesquilinear forms

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Abstract In this paper we define a notion of Witt group for sesquilinear forms in hermitian categories, which in turn provides a notion of Witt group for sesquilinear forms over rings with involution. We also study the extension of scalars for K -linear hermitian categories, where K is a field of characteristic $\neq 2$. We finally extend several results concerning sesquilinear forms to the setting of systems of such forms.

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Introduction

There exists a classical notion of Witt group for hermitian forms over rings with involution (see for instance [6] and [8]). However, there is no analogous notion for sesquilinear forms. In this paper we define, based on an equivalence of categories proven in [5], a notion of Witt group for sesquilinear forms in hermitian categories. In particular we obtain a notion of Witt group for sesquilinear forms over a ring with involution.

We then study the extension of scalars for K -linear hermitian categories, where K is a field of characteristic $\neq 2$, and we prove its injectivity in the case of an extension of odd degree and a hermitian category in which all idempotents split.

We also introduce the notion of system of sesquilinear forms over a ring with involution and we generalize several results proven in [5] to systems of sesquilinear forms.

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§1. Sesquilinear and hermitian forms over rings with involution

Let A be a ring. An *involution* on A is by definition an additive map $\sigma : A \rightarrow A$ such that $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a, b \in A$ and σ^2 is the identity. Let V be a right A -module of finite type. A *sesquilinear form* over (A, σ) is a biadditive map $s : V \times V \rightarrow A$ satisfying the condition $s(ax, by) = \sigma(a)s(x, y)b$ for all $x, y \in V$ and all $a, b \in A$. The *orthogonal sum* of two sesquilinear forms (V, s) and (V', s') is by definition the form $(V \oplus V', s \oplus s')$ defined by

$$(s \oplus s')(x \oplus x', y \oplus y') = s(x, y) + s'(x', y')$$

for all $x, y \in V$ and $x', y' \in V'$. Two sesquilinear forms (V, s) and (V', s') are called *isometric* if there exists an isomorphism of A -modules $f : V \xrightarrow{\sim} V'$ such that $s'(f(x), f(y)) = s(x, y)$ for all $x, y \in V$.

Let $V^* = \text{Hom}_A(V, A)$. Then V^* has a structure of right A -module given by $f(x) \cdot a = \sigma(a)f(x)$ for all $a \in A, f \in V^*$ and $x \in V$. We say that V is *reflexive* if the homomorphism of right A -modules $e_V : V \rightarrow V^{**}$ defined by $e_V(x)(f) = \sigma(f(x))$ for all $x \in V$ and $f \in V^*$ is bijective.

A sesquilinear form (V, s) over (A, σ) induces two homomorphisms of left A -modules $V \rightarrow V^*$, called its *left*, respectively *right adjoint*, namely $s_l : V \rightarrow V^*$ defined by $s_l(x)(y) = s(x, y)$ and $s_r : V \rightarrow V^*$ given by $s_r(x)(y) = \sigma(s(y, x))$ for all $x, y \in V$. We observe that $s_r = s_l^* e_V$.

Let $\epsilon = \pm 1$. A sesquilinear form (V, s) over (A, σ) is called ϵ -*hermitian* if V is a projective A -module and $\sigma(s(x, y)) = \epsilon s(y, x)$ for all $x, y \in V$, i.e. $s_l = \epsilon s_r$. A 1-hermitian form is also called a hermitian form. An ϵ -hermitian form (V, s) is called *unimodular* if s_l (or equivalently s_r) is bijective. There exists a classical notion of Witt group for unimodular ϵ -hermitian forms over (A, σ) (see e.g. [6]). Denote by $\text{Gr}^\epsilon(A, \sigma)$ the Grothendieck group of isometry classes of unimodular ϵ -hermitian forms over (A, σ) , with respect to the orthogonal sum. A unimodular ϵ -hermitian form over (A, σ) is called *hyperbolic* if it is isometric to a form $\mathbb{H}(V)$, where V is a finitely generated projective right A -module and

$$\mathbb{H}(V) : V \otimes V^* \rightarrow V^* \oplus V^{**}$$

$$x + y \mapsto y + \epsilon e_V(x), \quad \forall x \in V, y \in V^*.$$

The quotient of $\text{Gr}^\epsilon(A, \sigma)$ by the subgroup generated by the unimodular ϵ -hermitian forms is called the *Witt group of unimodular ϵ -hermitian forms* over (A, σ) and is denoted by $W^\epsilon(A, \sigma)$.

Let us denote by $\mathcal{S}(A, \sigma)$ ($\mathcal{H}^\epsilon(A, \sigma)$) the category of sesquilinear (respectively unimodular ϵ -hermitian) forms over (A, σ) . The morphisms of these categories are isometries. For simplicity let $\mathcal{H}(A, \sigma) = \mathcal{H}^1(A, \sigma)$.

§2. Hermitian categories

The aim of this section is to recall some basic notions about hermitian categories as presented in [8] (see also [6], [7]).

§2.1. Preliminaries

Let \mathcal{C} be an additive category. Let $*$: $\mathcal{C} \rightarrow \mathcal{C}$ be a duality functor, i.e. an additive contravariant functor with a natural isomorphism $(E_C)_{C \in \mathcal{C}} : \text{id} \rightarrow **$ such that $E_C^* E_C = \text{id}_{C^*}$ for all $C \in \mathcal{C}$. An additive category with a duality functor is called a *hermitian category*. A *sesquilinear form* in the category \mathcal{C} is a pair (C, s) , where C is an object of \mathcal{C} and $s : C \rightarrow C^*$. A sesquilinear form (C, s) is called *unimodular* if s is an isomorphism. Let $\epsilon = \pm 1$. An ϵ -*hermitian form* in the category \mathcal{C} is a sesquilinear form (C, s) such that $s = \epsilon s^* E_C$. A 1-hermitian form is also called a hermitian form. Orthogonal sums of forms are defined in the obvious way. Let (C, s) and (C', s') be two sesquilinear forms in \mathcal{C} . We say that these forms are *isometric* if there exists an isomorphism $f : C \xrightarrow{\sim} C'$ in the category \mathcal{C} such that $s = f^* s' f$.

Denote by $\mathcal{H}^\epsilon(\mathcal{C})$ the category of unimodular ϵ -hermitian forms in the category \mathcal{C} . The morphisms are isometries. For simplicity let $\mathcal{H}(\mathcal{C}) = \mathcal{H}^1(\mathcal{C})$.

The hyperbolic unimodular ϵ -hermitian forms in \mathcal{C} are the forms isometric to \mathbb{H}_Q , $Q \in \mathcal{C}$, given by

$$\mathbb{H}_Q = \epsilon E_Q \oplus \text{id}_{Q^*} : Q \oplus Q^* \rightarrow Q^{**} \oplus Q^* \simeq (Q \oplus Q^*)^*.$$

The quotient of the Grothendieck group of isometry classes of unimodular ϵ -hermitian forms in \mathcal{C} (with respect to the orthogonal sum) by the subgroup generated by the hyperbolic forms is called the Witt group of unimodular ϵ -hermitian forms in \mathcal{C} and is denoted by $W^\epsilon(\mathcal{C})$. For simplicity set $W(\mathcal{C}) = W^1(\mathcal{C})$.

We observe that if we take \mathcal{C} to be the category of reflexive right A -modules of finite type, then the notion of sesquilinear form coincides with the one defined in the preceding section. Analogously, if we take \mathcal{C} to be the category of projective right A -modules of finite type, then the notion of hermitian form coincides with the one defined in the preceding section.

Let $(\mathcal{M}, *)$ and $(\mathcal{M}', *)$ be two hermitian categories. A *duality preserving functor* from $(\mathcal{M}, *)$ to $(\mathcal{M}', *)$ is an additive functor $F : \mathcal{M} \rightarrow \mathcal{M}'$ together with a natural isomorphism $i = (i_M)_{M \in \mathcal{M}} : F^* \rightarrow *F$. This means that for any $M \in \mathcal{M}$ there exists a natural isomorphism $i_M : F(M^*) \xrightarrow{\sim} F(M)^*$ such that for all $N \in \mathcal{M}$ and $f \in \text{Hom}_{\mathcal{M}}(M, N)$ the following diagram commutes:

$$\begin{array}{ccc}
F(N^*) & \xrightarrow{F(f^*)} & F(M^*) \\
\downarrow i_N & & \downarrow i_M \\
F(N)^* & \xrightarrow{F(f)^*} & F(M)^*
\end{array}$$

For $\lambda = \pm 1$, a duality preserving functor F is called λ -hermitian if $i_{M^*}F(e_M) = \lambda i_M^*e_{F(M)}$ for all $M \in \mathcal{M}$. Let $\epsilon = \pm 1$. We recall from [6], pp. 80-81 that a λ -hermitian functor $F : \mathcal{M} \rightarrow \mathcal{M}'$ induces a functor

$$\text{Herm}^\epsilon(F) : \mathcal{H}^\epsilon(\mathcal{M}) \rightarrow \mathcal{H}^{\epsilon\lambda}(\mathcal{M}')$$

$$(M, h) \mapsto (F(M), i_M F(h)),$$

which preserves orthogonal sums and hyperbolicity. Therefore it induces a homomorphism between the corresponding Witt groups:

$$\overline{\text{Herm}}^\epsilon(F) : W^\epsilon(\mathcal{M}) \rightarrow W^{\epsilon\lambda}(\mathcal{M}').$$

If F is fully faithful, then $\text{Herm}^\epsilon(F)$ is also fully faithful. Moreover, if F is an equivalence of categories, then $\text{Herm}^\epsilon(F)$ is also an equivalence of categories and the induced group homomorphism $\overline{\text{Herm}}^\epsilon(F)$ is bijective.

§2.2. Transfer into the endomorphism ring

The aim of this subsection is to introduce the method of *transfer into the endomorphism ring*, which allows us to pass from the abstract setting of hermitian categories to that of a ring with involution, which is more concrete. This method will be extensively applied in section 4.

Let \mathcal{M} be a hermitian category and M be an object of \mathcal{M} , on which we suppose that there exists a unimodular ϵ_0 -hermitian form h_0 for a certain $\epsilon_0 = \pm 1$. Denote by E the endomorphism ring of M . According to [7], lemma 1.2, the form (M, h) induces on E an involution σ , defined by $\sigma(f) = h_0^{-1}f^*h_0$ for all $f \in E$.

We say that an *idempotent* $e \in E$ *splits* if there exist an object $M' \in \mathcal{M}$ and morphisms $i : M' \rightarrow M$, $j : M \rightarrow M'$ such that $ji = \text{id}_{M'}$ and $ij = e$.

Denote by $\mathcal{M}|_M$ the full subcategory of \mathcal{M} which has as objects all the objects of \mathcal{M} isomorphic to a direct summand of a finite direct sum of copies of M and by $\mathcal{P}(E)$ the category of finitely generated projective right E -modules. We consider the following functor:

$$\mathcal{F} = \text{Hom}(M, -) : \mathcal{M}|_M \rightarrow \mathcal{P}(E)$$

$$N \mapsto \text{Hom}(M, N), \quad \forall N \in \mathcal{M}|_M$$

$$f \mapsto \mathcal{F}(f), \quad \forall f \in \text{Hom}(N, N'), \quad \forall N, N' \in \mathcal{M}|_M,$$

where for all $g \in \text{Hom}(M, N)$, $\mathcal{F}(f)(g) = fg$. In [7], proposition 2.4 it has been proven that this functor is fully faithful and duality preserving with respect to the natural isomorphism $i = (i_N)_{N \in \mathcal{M}|_M} : \mathcal{F}^* \rightarrow {}^*\mathcal{F}$ given by $i_N(f) = \mathcal{F}(h_0^{-1}f^*e_N)$ for every $N \in \mathcal{M}|_M$ and $f \in \text{Hom}(M, N^*)$. In addition, if all the idempotents of $\mathcal{M}|_N$ split, then \mathcal{F} is an equivalence of categories. By computation we easily see that \mathcal{F} is ϵ_0 -hermitian.

§2.3. Ring extension and genus in hermitian categories

In this subsection we introduce the notions of extension of rings and genus in hermitian categories.

Let \mathcal{C} be an additive category and R be a commutative ring with unity. By extension of scalars from \mathbb{Z} to R we obtain a new category, denoted by $\mathcal{C} \otimes_{\mathbb{Z}} R$ and called the *extension of the category \mathcal{C} to the ring R* . Its objects are the same as those of \mathcal{C} and for two such objects M and N set

$$\text{Hom}_{\mathcal{C} \otimes_{\mathbb{Z}} R}(M, N) = \text{Hom}_{\mathcal{C}}(M, N) \otimes_{\mathbb{Z}} R.$$

It is straightforward to check that in this way we obtain an additive category. If, in addition, there is a duality functor $*$ in the category \mathcal{C} , then $\mathcal{C} \otimes_{\mathbb{Z}} R$ becomes a hermitian category by setting $(f \otimes a)^* = f^* \otimes a$ for all $M, N \in \mathcal{C}$, $f \in \text{Hom}_{\mathcal{C}}(M, N)$ and $a \in R$.

For each prime spot p of \mathbb{Q} we denote by \mathcal{C}_p the category $\mathcal{C} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Let \mathcal{C} be an additive category. Two sesquilinear forms $s : M \rightarrow M^*$ and $t : N \rightarrow N^*$ in the category \mathcal{C} are said to be in the same *genus* if for all primes p of \mathbb{Q} there exist $F_p \in \text{Hom}_{\mathcal{C}}(M, N)_p$ and $G_p \in \text{Hom}_{\mathcal{C}}(N, M)_p$ such that $F_p G_p$ and $G_p F_p$ are equal to the identity and in addition $F_p^*(t \otimes 1)F_p = s \otimes 1$. This is equivalent to saying that the extended sesquilinear forms $(M, s \otimes 1)$ and $(N, t \otimes 1)$ become isometric in the extended category \mathcal{C}_p for every p .

§2.4. K -linear hermitian categories and scalar extension

The aim of this section is to give an introduction to the theory of K -linear hermitian categories. Their consideration is motivated by the idea of defining a notion of scalar extension in hermitian categories.

2.4.1. Definition Let K be a field. A *K -linear category* is an additive category \mathcal{M} such that for every $M, N \in \mathcal{M}$, the set $\text{Hom}_{\mathcal{M}}(M, N)$ has a structure of finite-dimensional K -vector space such that the composition of morphisms is K -bilinear. An additive functor $F : \mathcal{M} \rightarrow \mathcal{N}$ between two K -linear categories is called *K -linear* if for any $M, N \in \mathcal{M}$, the induced map

$$F_{M,N} : \text{Hom}_{\mathcal{M}}(M, N) \rightarrow \text{Hom}_{\mathcal{N}}(F(M), F(N))$$

is K -linear.

A K -linear hermitian category is a K -linear category together with a K -linear duality functor.

Let $(\mathcal{M}, *)$ be a K -linear hermitian category and consider a finite field extension L of K . We define the *extension of the category \mathcal{M} to L* as being the category \mathcal{M}_L with the same objects as \mathcal{M} and with the morphisms given by

$$\mathrm{Hom}_{\mathcal{M}_L}(M, N) = \mathrm{Hom}_{\mathcal{M}}(M, N) \otimes_K L$$

for all $M, N \in \mathcal{M}$. It is clear that the category \mathcal{M}_L is L -linear.

If in \mathcal{M} there is a K -linear duality functor $*$ then we can define a duality functor in \mathcal{M}_L in the following way: on objects it coincides with the duality functor of \mathcal{M} and for morphisms set $(f \otimes a)^* = f^* \otimes a$ for all $f \in \mathrm{Hom}_{\mathcal{M}}(M, N)$, $M, N \in \mathcal{M}$ and $a \in L$ and then extend by additivity.

The *scalar extension functor* from \mathcal{M} to \mathcal{M}_L is defined by

$$\begin{aligned} \mathcal{R}_{L/K} : \mathcal{M} &\rightarrow \mathcal{M}_L \\ M &\rightarrow M, \quad \forall M \in \mathcal{M} \\ f &\mapsto f \otimes 1, \quad \forall f \in \mathrm{Hom}(M, N). \end{aligned}$$

It is straightforward to check that the functor $\mathcal{R}_{L/K}$ is 1-hermitian. As we have seen in §2.1, it induces for every $\epsilon = \pm 1$ a group homomorphism

$$\overline{\mathrm{Herm}}^\epsilon(\mathcal{R}_{L/K}) : W^\epsilon(\mathcal{M}) \rightarrow W^\epsilon(\mathcal{M}_L),$$

called *restriction map*.

§3. Sesquilinear forms over rings with involution and hermitian categories

In this section we prove that the category of sesquilinear forms over a ring with involution is equivalent with the category of unimodular hermitian forms in a suitably constructed hermitian category. We also define a notion of Witt group for sesquilinear forms in hermitian categories. In particular we obtain a notion of Witt group for sesquilinear forms over rings with involution, which generalizes the analogous notion for unimodular hermitian forms.

§3.1. An equivalence of categories

Let \mathcal{M} be an additive category. On the model of [5], §3, we construct the *category of double arrows of \mathcal{M}* , denoted by $\mathcal{M}^{(2)}$. Its objects are of the form (M, N, f, g) , where $M, N \in \mathcal{M}$ and $f, g \in \mathrm{Hom}(M, N)$. A morphism from (M, N, f, g) to

(M', N', f', g') is a pair (ϕ, ψ) , where $\phi \in \text{Hom}(M, M')$ and $\psi \in \text{Hom}(N, N')$ satisfying the conditions $\psi f = f' \phi$ and $\psi g = g' \phi$. Then $\mathcal{M}^{(2)}$ is obviously an additive category. If in addition there is a duality functor $*$ in \mathcal{M} , then $\mathcal{M}^{(2)}$ becomes a hermitian category by defining the dual of (M, N, f, g) as being (N^*, M^*, g^*, f^*) and setting $E_{(M,N,f,g)} = (e_M, e_N)$.

We define a functor $F : \mathcal{S}(\mathcal{M}) \rightarrow \mathcal{H}(\mathcal{M}^{(2)})$ as follows. Let (M, s) be a sesquilinear form in \mathcal{M} . Then (M, M^*, s, s^*e_M) is an object of $\mathcal{M}^{(2)}$ and it is easy to check that (e_M, id_{M^*}) defines a unimodular hermitian form on it. Set $F(M, s) = ((M, M^*, s, s^*e_M), (e_M, \text{id}_{M^*}))$. For an isometry $\phi : (M, s) \rightarrow (M', s')$ of sesquilinear forms in \mathcal{M} set $F(\phi) = (\phi, \phi^{*-1})$.

We also define a functor G in the opposite direction: for every object (M, N, f, g) of $\mathcal{M}^{(2)}$ and any unimodular hermitian form (ξ_1, ξ_2) on it set $G((M, N, f, g), (\xi_1, \xi_2)) = (M, \xi_2 f)$. For a morphism (λ_1, λ_2) between two unimodular hermitian forms in $\mathcal{M}^{(2)}$ define $G(\lambda_1, \lambda_2) = \lambda_1$.

3.1.1. Theorem *The functors F and G realize an equivalence between the categories $\mathcal{S}(\mathcal{M})$ and $\mathcal{H}(\mathcal{M}^{(2)})$.*

Proof. The proof is completely analogous to the one of [5], theorem 4.1.

§3.2. Hyperbolic sesquilinear forms

Using the functor G we can define a notion of hyperbolicity for sesquilinear forms in \mathcal{M} , which will in turn lead to a notion of Witt group.

3.2.1. Definition A sesquilinear form in \mathcal{M} is called *hyperbolic* if it is isometric to a form $G(P, s)$, where $(P, s) \in \mathcal{H}(\mathcal{M}^{(2)})$ is hyperbolic.

Let (A, σ) be a ring with involution. Taking \mathcal{M} to be the category of reflexive right A -modules of finite type we obtain a notion of hyperbolicity for sesquilinear forms over (A, σ) .

We have the following explicit characterization of hyperbolic sesquilinear forms over (A, σ) :

3.2.2 Proposition *A sesquilinear form (V, s) over (A, σ) is hyperbolic if and only if there exist two reflexive right A -modules of finite type M and N and two A -linear homomorphisms $f, g : M \rightarrow N$ such that (V, s) is isometric to the form $(M \oplus N^*, \tilde{H}_{(M,N,f,g)})$, given by*

$$\tilde{H}_{(M,N,f,g)} : (M \oplus N^*) \times (M \oplus N^*) \rightarrow A$$

$$(x_1 + y_1, x_2 + y_2) \mapsto y_1(g(x_2)) + \sigma(y_2(f(x_1))), \quad \forall x_1, x_2 \in M, \quad \forall y_1, y_2 \in N^*.$$

Proof. For all $Q = (M, N, f, g) \in \mathcal{M}^{(2)}$ the hyperbolic unimodular hermitian form \mathbb{H}_Q in $\mathcal{M}^{(2)}$ is defined by

$$\mathbb{H}_Q = E_Q \oplus \text{id}_{Q^*} : Q \oplus Q^* \rightarrow Q^{**} \oplus Q^* \simeq (Q \oplus Q^*)^*,$$

hence $G(\mathbb{H}_Q)$ is given by

$$G(\mathbb{H}_Q) : M \oplus N^* \rightarrow M^* \oplus N^{**}$$

$$x + y \mapsto g^*(y) + e_N(f(x)), \quad \forall x \in M, \quad \forall y \in N^*.$$

It is easy to identify this form with \tilde{H}_Q . \square

Particularizing the last proposition, we obtain:

3.2.3. Proposition

- a) *The hyperbolic hermitian forms over (A, σ) are given, up to isometry, by the forms $(M \oplus N^*, \tilde{H}_{(M,N,f,f)})$, where M and N are two reflexive right A -modules of finite type and $f : M \rightarrow N$ is an A -linear homomorphism.*
- b) *The hyperbolic unimodular sesquilinear forms over (A, σ) are given, up to isometry, by $(M \oplus N^*, \tilde{H}_{(M,N,f,g)})$, where M and N are two reflexive right A -modules of finite type and $f, g : M \rightarrow N$ are two A -linear isomorphisms.*
- c) *The hyperbolic unimodular hermitian forms over (A, σ) are given, up to isometry, by $(M \oplus M^*, \mathbb{H}_M)$, where M is a reflexive right A -module of finite type and*

$$\mathbb{H}_M : (M \oplus M^*) \times (M \oplus M^*) \rightarrow A$$

$$(x_1 + y_1, x_2 + y_2) \mapsto y_1(x_2) + \sigma(y_2(x_1)), \quad \forall x_1, y_1 \in M, \quad \forall y_1, y_2 \in M^*.$$

Proof. It is straightforward to check that the form $(M \oplus N^*, \tilde{H}_{(M,N,f,g)})$ is hermitian if and only if $f = g$ and that it is unimodular if and only if f and g are isomorphisms. The point c) follows from a) and b) and from the fact that the forms $(M \oplus N^*, \tilde{H}_{(M,N,f,f)})$ and $(M \oplus M^*, \mathbb{H}_M)$ are isometric via the isomorphism of A -modules $\text{id}_M \oplus f^* : M \oplus N^* \xrightarrow{\sim} M \oplus M^*$. \square

3.2.4. Definition Let $\text{Gr}_S(\mathcal{M})$ be the Grothendieck group of isometry classes of sesquilinear forms in \mathcal{M} , with respect to the orthogonal sum. It is easy to check that the isometry classes of hyperbolic sesquilinear forms form a subgroup of $\text{Gr}_S(\mathcal{M})$, which we denote by $\mathbb{H}_S(\mathcal{M})$. The *Witt group of sesquilinear forms* in the category \mathcal{M} is defined to be the quotient

$$W_S(\mathcal{M}) = \text{Gr}_S(\mathcal{M}) / \mathbb{H}_S(\mathcal{M}).$$

Taking \mathcal{M} to be the category of reflexive right A -modules of finite type we obtain a notion of Witt group for sesquilinear forms over (A, σ) . We observe that in the case of unimodular hermitian forms our definition coincides with the well-known one.

§3.3. Finiteness results concerning the genus of sesquilinear forms

In this subsection we prove a finiteness result concerning the genus of sesquilinear forms, based on the methods of [5], §9. For a ring A we denote by $T(A)$ the \mathbb{Z} -torsion subgroup of A . If R is a ring containing \mathbb{Z} , then we say that A is R -finite if $A_R = A \otimes_{\mathbb{Z}} R$ is a finitely generated R -module and $T(A)$ is finite.

3.3.1. Theorem *Let A be a ring and σ be an involution on A . Let (V, s) be a sesquilinear form over (A, σ) and assume that $\text{End}_A(V)$ is \mathbb{Q} -finite. Then the genus of (V, s) contains only a finite number of isometry classes of sesquilinear forms.*

Proof. The functor F defined in §3.1 induces a bijection between the genus of (V, s) and the genus of $F(V, s)$. Denote by E the endomorphism ring of (V, V^*, s_l, s_r) in \mathcal{M} . Since $\text{End}_A(V)$ is \mathbb{Q} -finite and E is a subring of $\text{End}_A(V) \times \text{End}_A(V^*) \simeq \text{End}_A(V) \times \text{End}_A(V)$, E is \mathbb{Q} -finite too. From [3], theorem 3.4 it follows that the genus of $F(V, s)$ contains only a finite number of isometry classes of unimodular hermitian forms, which implies that the genus of (V, s) contains only a finite number of isometry classes of sesquilinear forms.

§4. The restriction map for K -linear hermitian categories in odd degree extensions

It is well-known that if K is a field of characteristic $\neq 2$ and L/K is an extension of odd degree, then the restriction map $r_{L/K} : W(K) \rightarrow W(L)$ is injective. The aim of this section is to prove an analogous result for K -linear hermitian categories.

Let K be a field of characteristic $\neq 2$ and \mathcal{M} be a K -linear hermitian category. Let L be a finite extension of K , \mathcal{M}_L be the extension of the category \mathcal{M} to L and $\mathcal{R}_{L/K} : \mathcal{M} \rightarrow \mathcal{M}_L$ be the scalar extension functor, as defined in §2.5.

4.1. Theorem *Suppose that all the idempotents of \mathcal{M} split and that the extension L/K is of odd degree. Then for all $\epsilon = \pm 1$ the map*

$$\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}) : W^\epsilon(\mathcal{M}) \rightarrow W^\epsilon(\mathcal{M}_L)$$

is injective.

This result will follow as an immediate corollary from the following one.

4.2. Theorem *Let M be an object of \mathcal{M} that admits a unimodular hermitian or skew-hermitian form. Suppose that all the idempotents of $\mathcal{M}|_M$ split and that the extension L/K is of odd degree. Then for all $\epsilon = \pm 1$, the map*

$$\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}^M) : W^\epsilon(\mathcal{M}|_M) \rightarrow W^\epsilon(\mathcal{M}_L|_M)$$

is injective.

Proof of theorem 4.2 Denote by h_0 a unimodular ϵ_0 -hermitian form on M . Since the category \mathcal{M} is K -linear, the endomorphism ring E of M in \mathcal{M} has a structure of finite-dimensional K -algebra. According to [7], lemma 1.2, h_0 induces on E a K -linear involution, denoted by σ . Clearly the endomorphism ring E' of M in \mathcal{M}_L equals $E \otimes_K L$ and is a finite-dimensional L -algebra. We also observe that (M, h_{0L}) is a unimodular ϵ_0 -hermitian form in \mathcal{M}_L and that it induces by [7], lemma 1.2 the involution $\sigma \otimes \text{id}_L$ on E' . We have an obvious 1-hermitian functor of extension of scalars

$$\mathcal{R}_{L/K}^M : \mathcal{M}|_M \rightarrow \mathcal{M}_L|_M.$$

We recall from §2.2 that there are two fully faithful ϵ_0 -hermitian functors

$$\mathcal{F} : \mathcal{M}|_M \rightarrow \mathcal{P}(E),$$

$$\mathcal{F}' : \mathcal{M}_L|_M \rightarrow \mathcal{P}(E \otimes_K L).$$

We denote by $R_{L/K}$ the obvious functor of scalar extension

$$R_{L/K} : \mathcal{P}(E) \rightarrow \mathcal{P}(E \otimes_K L).$$

It is straightforward to prove that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_L|_M & \xrightarrow{\mathcal{F}'} & \mathcal{P}(E \otimes_K L) \\ \mathcal{R}_{L/K}^M \uparrow & & \uparrow R_{L/K} \\ \mathcal{M}|_M & \xrightarrow{\mathcal{F}} & \mathcal{P}(E) \end{array}$$

For $\epsilon = \pm 1$ the above diagram induces a commutative diagram on the level of unimodular hermitian forms:

$$\begin{array}{ccc} \mathcal{H}^\epsilon(\mathcal{M}_L|_M) & \xrightarrow{\text{Herm}^\epsilon(\mathcal{F}')} & \mathcal{H}^{\epsilon\epsilon_0}(E \otimes_K L, \sigma \otimes \text{id}_L) \\ \text{Herm}^\epsilon(\mathcal{R}_{L/K}^M) \uparrow & & \uparrow \text{Herm}^{\epsilon\epsilon_0}(R_{L/K}) \\ \mathcal{H}^\epsilon(\mathcal{M}|_M) & \xrightarrow{\text{Herm}^\epsilon(\mathcal{F})} & \mathcal{H}^{\epsilon\epsilon_0}(E, \sigma) \end{array}$$

and one on the level of Witt groups:

$$\begin{array}{ccc}
W^\epsilon(\mathcal{M}_L|_M) & \xrightarrow{\overline{\text{Herm}}^\epsilon(\mathcal{F}')} & W^{\epsilon\epsilon_0}(E \otimes_K L, \sigma \otimes \text{id}_L) \\
\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}^M) \uparrow & & \uparrow \overline{\text{Herm}}^{\epsilon\epsilon_0}(R_{L/K}) \\
W^\epsilon(\mathcal{M}|_M) & \xrightarrow{\overline{\text{Herm}}^\epsilon(\mathcal{F})} & W^{\epsilon\epsilon_0}(E, \sigma)
\end{array}$$

Let (N, g) and $(N', g') \in \mathcal{H}^\epsilon(\mathcal{M}|_M)$ be such that

$$\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}^M)(N, g) = \overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}^M)(N', g')$$

in $W^\epsilon((\mathcal{M}_L)|_M)$. It follows that

$$\overline{\text{Herm}}^\epsilon(\mathcal{F}')(\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}^M)(N, g)) = \overline{\text{Herm}}^\epsilon(\mathcal{F}')(\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}^M)(N', g'))$$

in $W^{\epsilon\epsilon_0}(E \otimes_K L, \sigma \otimes \text{id}_L)$. As the last diagram above commutes, we obtain

$$\overline{\text{Herm}}^{\epsilon\epsilon_0}(R_{L/K})(\overline{\text{Herm}}^\epsilon(\mathcal{F})(N, g)) = \overline{\text{Herm}}^{\epsilon\epsilon_0}(R_{L/K})(\overline{\text{Herm}}^\epsilon(\mathcal{F})(N', g')).$$

By [4], proposition 1.2, the map $\overline{\text{Herm}}^{\epsilon\epsilon_0}(R_{L/K})$ is injective, so we deduce that $\overline{\text{Herm}}^\epsilon(\mathcal{F})(N, g) = \overline{\text{Herm}}^\epsilon(\mathcal{F})(N', g')$ in $W^{\epsilon\epsilon_0}(E, \sigma)$. Since in the category $\mathcal{M}|_M$ all idempotents split, the functor \mathcal{F} is an equivalence of categories and hence the group homomorphism $\overline{\text{Herm}}^\epsilon(\mathcal{F}) : W^\epsilon(\mathcal{M}|_M) \rightarrow W^{\epsilon\epsilon_0}(E, \sigma)$ is bijective (see §2.1). It follows that $(N, g) = (N', g')$ in $W^\epsilon(\mathcal{M}|_M)$. We conclude that the map $\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}^M)$ is injective.

Proof of theorem 4.1 Let $(N, g) \in \mathcal{H}^\epsilon(\mathcal{M})$ be such that $\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K})(N, g)$ is hyperbolic. This means that $(N, g \otimes 1)$ is a hyperbolic unimodular ϵ -hermitian form in the category \mathcal{M}_L , so in the category $\mathcal{M}_L|_N$ too. Hence $\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}^N)(N, g) = 0$ and from the injectivity of the map $\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K}^N)$ we deduce that (N, g) is hyperbolic in $\mathcal{M}|_N$, so in \mathcal{M} too. In conclusion the map $\overline{\text{Herm}}^\epsilon(\mathcal{R}_{L/K})$ is injective.

§5. Systems of sesquilinear forms

In this section we generalize results proven in [5] to systems of sesquilinear forms. We first construct an equivalence of categories as in [5], §4.

§5.1. Preliminaries

Let A be a ring with an involution σ and I be a set. A *system of sesquilinear forms* over (A, σ) is $(V, (s_i)_{i \in I})$, where V is a reflexive right A -module of finite type and for all $i \in I$, (V, s_i) is a sesquilinear form over (A, σ) . A morphism between two systems of sesquilinear forms $(V, (s_i)_{i \in I})$ and $(V', (s'_i)_{i \in I})$ consists of an isomorphism of A -modules $f : V \xrightarrow{\sim} V'$ such that for every $i \in I$ and $x, y \in V$ we have $s'_i(f(x), f(y)) = s_i(x, y)$. Let us denote by $\mathcal{S}^{(I)}(A, \sigma)$ the category of systems of sesquilinear forms over (A, σ) .

Denote by J the disjoint union of two copies of I . We define the *category of J -arrows between reflexive A -modules* as being the category $\mathcal{M}^{(J)}$ constructed in the following way: its objects are of the form $(V, W, (f_i, g_i)_{i \in I})$, where V and W are two reflexive A -modules of finite type and for all $i \in I$, $f_i, g_i : V \rightarrow W$ are homomorphisms of A -modules. A morphism from $(V, W, (f_i, g_i)_{i \in I})$ to $(V', W', (f'_i, g'_i)_{i \in I})$ is a pair (ϕ, ψ) , where $\phi : V \xrightarrow{\sim} V'$ and $\psi : W \xrightarrow{\sim} W'$ are isomorphisms of A -modules such that for all $i \in I$ we have $\psi f_i = f'_i \phi$ and $\psi g_i = g'_i \phi$. By defining direct sums in the obvious way we see that $\mathcal{M}^{(J)}$ is an additive category. Let $(W^*, V^*, (g_i^*, f_i^*)_{i \in I})$ be the dual of $(V, W, (f_i, g_i)_{i \in I})$ and set $E_{(V, W, (f_i, g_i)_{i \in I})} = (e_V, e_W)$. This defines a duality on the category $\mathcal{M}^{(J)}$. We observe that if the set I has one element, then $\mathcal{M}^{(J)}$ coincides with the category constructed in [5], §3.

We define a functor $\Psi : \mathcal{S}^{(I)}(A, \sigma) \rightarrow \mathcal{M}^{(J)}$ in the following way: Let $(V, (s_i)_{i \in I})$ be an object of $\mathcal{S}^{(I)}(A, \sigma)$ and for all $i \in I$ let $s_{il} : V \rightarrow V^*$ and $s_{ir} : V \rightarrow V^*$ be the left, respectively the right adjoint of (V, s_i) (cf. §1). Then $(V, V^*, (s_{il}, s_{ir})_{i \in I})$ is an object of $\mathcal{M}^{(J)}$ and it is easy to check that (e_V, id_{V^*}) defines a unimodular hermitian form on it. Let $\Psi(V, (s_i)_{i \in I}) = ((V, V^*, (s_{il}, s_{ir})_{i \in I}), (e_V, \text{id}_{V^*}))$. For a morphism $\varphi : (V, (s_i)_{i \in I}) \rightarrow (V', (s'_i)_{i \in I})$ of systems of sesquilinear forms over (A, σ) set $\Psi(\varphi) = (\varphi, \varphi^{*-1})$.

5.1.1. Theorem *The functor Ψ is an equivalence of categories between $\mathcal{S}^{(I)}(A, \sigma)$ and $\mathcal{H}(\mathcal{M}^{(J)})$.*

Proof. The proof is analogous to the one of [5], theorem 4.1. The functor $\Phi : \mathcal{H}(\mathcal{M}^{(J)}) \rightarrow \mathcal{S}(A, \sigma)$ which realizes, together with Ψ , the desired equivalence of categories is defined in the following way: Let $((V, W, (f_i, g_i)_{i \in I}), (\varphi, \psi))$ be an object of $\mathcal{H}(\mathcal{M}^{(J)})$. For every $i \in I$ define a sesquilinear form $s_i : V \times V \rightarrow A$ by $s_i(x, y) = (\psi g_i)(x)(y)$ for all $x, y \in V$ and set $\Phi((V, W, (f_i, g_i)_{i \in I}), (\varphi, \psi)) = (V, (s_i)_{i \in I})$. For a morphism $\lambda = (\lambda_1, \lambda_2)$ between two objects of $\mathcal{H}(\mathcal{M}^{(J)})$ set $\Phi(\lambda) = \lambda_1$.

If $(V, (s_i)_{i \in I})$ is a system of sesquilinear forms over (A, σ) , then we denote by $q(V, (s_i)_{i \in I})$ the corresponding object $(V, V^*, (s_{il}, s_{ir})_{i \in I})$ of the category $\mathcal{M}^{(J)}$. We next describe, following [5], §5, the set of isometry classes of systems of sesquilinear forms over (A, σ) corresponding by theorem 5.1.1 to a given object of the category $\mathcal{M}^{(J)}$.

Let us fix a system of sesquilinear forms $(V_0, (s_{0i})_{i \in I})$ over (A, σ) and consider the unimodular hermitian form $\eta_{V_0} = (e_{V_0}, \text{id}_{V_0^*})$ on $Q_0 = q(V_0, (s_{0i})_{i \in I})$. Let E be the endomorphism ring of the object Q_0 in $\mathcal{M}^{(J)}$. The form η_{V_0} induces an involution \sim on E , defined by $\tilde{f} = \eta_{V_0}^{-1} f^* \eta_{V_0}$ for all $f \in E$, where f^* denotes the dual of f in $\mathcal{M}^{(J)}$. As in [5], §5 we denote by $H(\sim, E^\times)$ the set of equivalence classes for the equivalence relation defined on $\{f \in E \mid \tilde{f} = f\}$ by: $f \equiv f'$ if there exists a $g \in E^\times$ such that $\tilde{g} f g = f'$. Analogously to [5], theorem 5.1 we obtain:

5.1.2. Theorem *The set of isometry classes of systems of sesquilinear forms $(V, (s_i)_{i \in I})$ over (A, σ) such that $q(V, (s_i)_{i \in I}) \simeq Q_0$ is in bijection with $H(\sim, E^\times)$.*

Proof. The proof is analogous to the one of [5], theorem 5.1. For every system of sesquilinear forms $(V, (s_i)_{i \in I})$ over (A, σ) such that $q(V, (s_i)_{i \in I}) \simeq Q_0$ the unimodular hermitian form (e_V, id_{V^*}) on $q(V, (s_i)_{i \in I})$ induces a unimodular hermitian form η_V on Q_0 . The desired bijection is given by:

$$\begin{aligned} \{[(V, (s_i)_{i \in I})] \mid (V, (s_i)_{i \in I}) \in \mathcal{S}^{(I)}(A, \sigma), q(V, (s_i)_{i \in I}) \simeq Q_0\} &\rightarrow H(\sim, E^\times) \\ [(V, (s_i)_{i \in I})] &\mapsto \eta_{V_0}^{-1} \eta_V. \end{aligned}$$

§5.2. Witt's cancellation theorem

Let K be a field of characteristic $\neq 2$, A be a finite-dimensional K -algebra and σ be an involution on A . Analogously to [5], theorem 6.1, a cancellation theorem holds for systems of sesquilinear forms over (A, σ) :

5.2.1. Theorem *Let $(V, (s_i)_{i \in I})$, $(V', (s'_i)_{i \in I})$ and $(V'', (s''_i)_{i \in I})$ be systems of sesquilinear forms over (A, σ) such that*

$$(V', (s'_i)_{i \in I}) \oplus (V, (s_i)_{i \in I}) \simeq (V'', (s''_i)_{i \in I}) \oplus (V, (s_i)_{i \in I}).$$

Then we have $(V', (s'_i)_{i \in I}) \simeq (V'', (s''_i)_{i \in I})$.

Due to the equivalence between the categories $\mathcal{S}^{(I)}(A, \sigma)$ and $\mathcal{H}(\mathcal{M}^{(J)})$ given by theorem 5.1.1, it is enough to prove that Witt's cancellation theorem holds in the category $\mathcal{H}(\mathcal{M}^{(J)})$. This can be proven as in [5], proposition 6.2.

§5.3. Springer's theorem

In this subsection we prove an analogue of Springer's theorem for systems of sesquilinear forms defined over finite-dimensional algebras with involution.

Let K be a field of characteristic $\neq 2$, A be a finite-dimensional K -algebra and σ be a K -linear involution on A . We also consider a finite extension L of K , the finite-dimensional L -algebra $A_L = A \otimes_K L$ and the L -linear involution $\sigma_L = \sigma \otimes \text{id}_L$ on A_L . If $(V, (s_i)_{i \in I})$ is a system of sesquilinear forms over (A, σ) , then we denote by $(V, (s_i)_{i \in I})_L = (V_L, ((s_i)_L)_{i \in I})$ the system of sesquilinear forms over (A_L, σ_L) obtained by extension of scalars.

5.3.1. Theorem *Suppose that L/K is an extension of odd degree. Let $(V, (s_i)_{i \in I})$ and $(V', (s'_i)_{i \in I})$ be two systems of sesquilinear forms over (A, σ) . If $(V, (s_i)_{i \in I})_L$ and $(V', (s'_i)_{i \in I})_L$ are isometric over (A_L, σ_L) , then $(V, (s_i)_{i \in I})$ and $(V', (s'_i)_{i \in I})$ are isometric over (A, σ) .*

Proof. The proof is analogous to the one of [5], theorem 7.1 and uses the equivalence of categories given by theorem 5.1.1.

§5.4. Weak Hasse principle

The well-known weak Hasse principle states that if two quadratic forms defined over a global field k of characteristic $\neq 2$ become isometric over all the completions of k , then they are already isometric over k . The aim of this section is to generalize this result to the case of systems of sesquilinear forms defined over a skew field with involution. Throughout this section we suppose that I is finite.

For results concerning the weak Hasse principle for systems of hermitian or quadratic forms over fields see [1], respectively [2]. A weak Hasse principle for sesquilinear forms defined over a skew field with involution has been proven in [5].

Let K be a field of characteristic $\neq 2$, D be a finite-dimensional skew field with center K and σ be an involution on D . Denote by k the fixed field of σ in K . Then either $k = K$ (when σ is said to be of the *first kind*) or K is a quadratic extension of k and the restriction of σ to K is the non-trivial automorphism of K over k (in which case σ is said to be of the *second kind* or a *unitary involution*).

Suppose that k is a global field. For every prime spot p of k , let k_p be the completion of k at p , $K_p = K \otimes_k k_p$ and $D_p = D \otimes_k k_p$. Then D_p is an algebra with center K_p and consider on it the involution $\sigma_p = \sigma \otimes \text{id}_{k_p}$. Then from any sesquilinear form (V, s) over (D, σ) we obtain by extension of scalars a sesquilinear form (V_p, s_p) over (D_p, σ_p) , where $V_p = V \otimes_k k_p$ is a free right D_p -module of rank $\dim_D(V)$. Hence we obtain a notion of extension of scalars for systems of sesquilinear forms over (D, σ) by setting $(V, (s_i)_{i \in I})_p = (V_p, ((s_i)_p)_{i \in I})$.

We say that the *weak Hasse principle* holds for systems of sesquilinear forms over (D, σ) if any two systems of sesquilinear forms $(V, (s_i)_{i \in I})$ and $(V', (s'_i)_{i \in I})$ over (D, σ) that become isometric over all the completions of k (i.e. $(V_p, ((s_i)_p)_{i \in I}) \simeq (V'_p, ((s'_i)_p)_{i \in I})$ over (D_p, σ_p) for every prime spot p of k) are already isometric over (D, σ) .

For the problem of determining when the weak Hasse principle holds we will use the ideas developed in [5]. We denote by $\mathcal{M}^{(J)}$ ($(\mathcal{M}_p)^{(J)}$) the category of J -arrows between reflexive D - (resp. D_p -) modules. Let us fix a system of sesquilinear forms s_0 over (D, σ) . Consider a system of sesquilinear forms s over (D, σ) that becomes isometric to s_0 over all the completions of k . Denote by $q(s)$ the object of $\mathcal{M}^{(J)}$ corresponding to s by the equivalence of categories given in theorem 5.1.1. For every prime spot p of k we have $(s_0)_p \simeq s_p$, hence $q((s_0)_p) \simeq q(s_p)$. Since q commutes with base change, we obtain $q(s_0)_p \simeq q(s)_p$ and since I is finite, it follows that $q(s_0) \simeq q(s)$.

Denote by E the endomorphism ring of $q(s_0)$ in \mathcal{M} . The unimodular hermitian form $\eta_{s_0} = (e_{V_0}, \text{id}_{V_0^*})$ defined on $q(s_0)$ induces an involution \sim on E . For every p let $E_p = E \otimes_k k_p$, on which we consider the involution $\sim_p = \sim \otimes \text{id}_{k_p}$. By theorem 5.1.2 it is clear that if the localisation map

$$\Phi : H(\sim, E^\times) \rightarrow \prod_p H(\sim_p, E_p^\times)$$

is injective, then the weak Hasse principle holds for systems of sesquilinear forms over (D, σ) . As in [5], §8, we prove the following:

5.4.1. Theorem *If σ is a unitary involution, then the map Φ is injective and hence the weak Hasse principle holds for systems of sesquilinear forms over (D, σ) .*

§5.5. Finiteness results

In this section we generalize theorem 3.3.1 and the results proven in [5], §9 to systems of sesquilinear forms. Following §2.3 we say that two systems of sesquilinear forms over (A, σ) are in the same genus if they become isometric over all the extensions of A to \mathbb{Z}_p , where p is a prime.

Fix a system of sesquilinear forms $(V, (s_i)_{i \in I})$ over (A, σ) and denote by $q(V, (s_i)_{i \in I})$ the corresponding object $(V, V^*, (s_{il}, s_{ir})_{i \in I})$ of $\mathcal{M}^{(J)}$ and by E its endomorphism ring in $\mathcal{M}^{(J)}$. It is straightforward to prove the following results using the methods of [5], §9 and the equivalence of categories $F : \mathcal{S}^{(I)}(A, \sigma) \rightarrow \mathcal{H}(\mathcal{M}^{(J)})$ given by theorem 5.1.1:

5.5.1. Theorem *If there exists a non-zero integer m such that $\text{End}_A(V)$ is $\mathbb{Z}[1/m]$ -finite, then there exist only finitely many isometry classes of systems of sesquilinear forms on V .*

5.5.2. Theorem *Let N be an object of $\mathcal{M}^{(J)}$ and assume that there exists a non-zero integer m such that $\text{End}_{\mathcal{M}^{(J)}}(N)$ is $\mathbb{Z}[1/m]$ -finite. Then there exist only finitely many isometry classes of systems of sesquilinear forms $(V, (s_i)_{i \in I})$ over (A, σ) such that $q(V, (s_i)_{i \in I}) \simeq N$.*

5.5.3. Theorem *If there exists a non-zero integer m such that $\text{End}_A(V)$ is $\mathbb{Z}[1/m]$ -finite, then $(V, (s_i)_{i \in I})$ contains only finitely many isometry classes of orthogonal summands.*

5.5.4. Theorem *If $\text{End}_A(V)$ is \mathbb{Q} -finite, then the genus of $(V, (s_i)_{i \in I})$ contains only a finite number of isometry classes of systems of sesquilinear forms.*

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