Rationally trivial quadratic spaces are locally trivial:III

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Abstract

It is proved the following. Let R be a regular semi-local domain containing a field such that all the residue fields are infinite. Let K be the fraction field of R. If a quadratic space $(R^n, q : R^n \to R)$ over R is isotropic over K, then there is a unimodular vector $v \in R^n$ such that $q(v) = 0$. If $char(R) = 2$, then in the case of even n we assume that q is a non-singular space in the sense of [Kn] and in the case of odd $n > 2$ we assume that q is a semi-regular in the sense of [Kn].

1 Introduction

Let k be an infinite field,possibly $char(k) = 2$, and let X be a k-smooth irreducible affine scheme,let $x_1, x_2, \dots, x_s \in X$ be closed points. Let P be a free $k[X]$-module of rank $n > 0$. If n is odd, then let $(P, q : P \to k[X])$ be a semi-regular quadratic module over $k[X]$ in the sense of [Kn, Ch.IV, §3]. If n is even, then let $(P, q : P \to k[X])$ be a non-singular quadratic space in the sense of [Kn, Ch.I, (5.3.5))]. (In both cases it is equivalent of saying that the X-scheme $Q := \{q = 0\} \subset \mathbf{P}^{n-1}_X$ is smooth over X).

Let $p : Q \to X$ be the projection. For a nonzero element $g \in k[X]$ let $Q_g = p^{-1}(X_g)$. Let $U = \mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\dots,x_s\}})$. Set ${}_UQ = U \times_X Q$. For a k-scheme D equipped with k-morphisms $U \leftarrow D$ and $D \to X_g$ set ${}_DQ = {}_UQ \times_U D$ and $Q_{D,g} = D \times_{X_g} Q_g$.

1.0.1 Proposition. *If $n > 1$, then there exists a finite surjective étale k-morphism $U \leftarrow D$ of odd degree, a morphism $D \to X_g$ and an isomorphism of the D-schemes ${}_DQ \xleftarrow{\bar{\Phi}} Q_{D,g}$.*

Given this Proposition we may prove the following Theorem

1.0.2 Theorem (Main)**.** *Assume that $g \in k[X]$ is a non-zero element such that there is a section $s : X_g \to Q$ of the projection $Q_g \to X_g$. Then there is a section $s_U : U \to {}_UQ$ of the projection ${}_UQ \to U$.*

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Proof of Main Theorem. We will give a proof of the Theorem only in the local case and left to the reader the semi-local case. So, $s = 1$ and we will write x for x_1 and $\mathcal{O}_{X,x}$ for $\mathcal{O}_{X,\{x_1\}}$. If $g \in k[X] - m_x$, then there is nothing to prove. Now let $g \in m_x$ then by Proposition 1.0.1 there is a a finite surjective étale k-morphism $U \leftarrow D$ of odd degree, a morphism $D \to X_g$ and an isomorphism of the D-schemes ${}_DQ \xleftarrow{\bar{\Phi}} Q_{D,g}$.

The section s defines a section $s_D = (id, s) : D \to Q_{D,g}$ of the projection $Q_{D,g} \to D$. Further $\bar{\Phi} \circ s_D : D \to {}_DQ$ is a section of the projection ${}_DQ \to D$. Finally, if $p_1 : {}_DQ \to {}_UQ$ is the projection, then $p_1 \circ \bar{\Phi} \circ s_D : D \to {}_UQ$ is a U-morphism of U-schemes. Recall that $U \leftarrow D$ is a a finite surjective étale k-morphism of odd degree and U is local with an infinite residue field. Whence by a variant of Springer's theorem proven in [PR] there is a section $s_U : U \to {}_UQ$ of the projection ${}_UQ \to U$. (If char(k)=2 the proof a variant of Springer's theorem given in [PR] works well with a very mild modification). The Theorem is proven.

□

The Main Theorem has the following corollaries

1.0.3 Corollary (Main1). *Let $\mathcal{O}_{X,\{x_1,x_2,\dots,x_s\}}$ be the semi-local ring as above and let $k(X)$ be the rational function field on X. Let P be a free $\mathcal{O}_{X,\{x_1,x_2,\dots,x_s\}}$-module of rank $n > 1$ and $q : P \to \mathcal{O}_{X,\{x_1,x_2,\dots,x_s\}}$ be a form over $\mathcal{O}_{X,\{x_1,x_2,\dots,x_s\}}$ as above, that is the $\mathcal{O}_{X,\{x_1,x_2,\dots,x_s\}}$-scheme $Q := \{q = 0\} \subset \mathbf{P}^{n-1}_{\mathcal{O}_{X,x}}$ is smooth over $\mathcal{O}_{X,x}$. If the equation $q = 0$ has a non-trivial solution over $k(X)$, then it has a unimodular solution over $\mathcal{O}_{X,\{x_1,x_2,\dots,x_s\}}$.*

1.0.4 Corollary (Main2). *Let R be a semi-local regular domain containing a field and R is such that all the residue fields are infinite. Let K be the fraction field of R. Let P be a free R-module of rank $n > 1$ and $q : P \to R$ be a quadratic form over R such that the R-scheme $Q := \{q = 0\} \subset \mathbf{P}^{n-1}_R$ is smooth over R. If the equation $q = 0$ has a non-trivial solution over K, then it has a unimodular solution over R.*

1.0.5 Corollary (Main3). *Let R be a semi-local regular domain containing a field and R is such that all the residue fields are infinite. Let K be the fraction field of R. Let P be a free R-module of even rank $n > 0$ and $q : P \to R$ be a quadratic form over R such that the R-scheme $Q := \{q = 0\} \subset \mathbf{P}^{n-1}_R$ is smooth over R. Let $u \in R^\times$ be a unit. If u is represented by q over K, then u is represented by q already over R.*

If $1/2 \in R$, then the same holds for a quadratic space of an arbitrary rank.

Proof of Proposition 1.0.1. The following Lemma is a corollary from Lemma 2.2.1 and Proposition 3.1.7. from [Kn]

1.0.6 Lemma. *For $n > 1$ there exists an affine open subset X^0 containing x and a Galois étale cover $\tilde{X}^0 \xrightarrow{\pi} X^0$ such that the $k[X^0]$-module $P \otimes_{k[X]} k[X^0]$ coincides with $k[X^0]^n$ and $\pi^*(q)$ is proportional to the quadratic space $\perp_{i=1}^m T_iT_{i+m}$ in the case $n = 2m$ and is proportional to the semi-regular quadratic module $\perp_{i=1}^m T_iT_{i+m} \perp T_n^2$ in the case $n = 2m + 1$.*

By this Lemma we may and will assume that $P = k[X]^n$ and that we are given with a Galois étale cover $\pi : \tilde{X} \xrightarrow{\pi} X$ such that the quadratic space $\pi^*(q)$ is proportional to a split quadratic space. Let Γ be the Galois group of $\tilde{X}$ over X. Let $\tilde{U} = \pi^{-1}(U) \subset \tilde{X}$.

Let $\overline{U \times X} := (\tilde{U} \times \tilde{X})/\Delta(\Gamma)$. Clearly, $U \times X = (\tilde{U} \times \tilde{X})/(\Gamma \times \Gamma)$. Let $\rho : \overline{U \times X} \to U \times X$ be the obvious map.

Let $p_2 : U \times X \to X$ be projection to X and $p_1 : U \times X \to U$ be the projection to U. The quadratic spaces $p_1^*(q)$ and $p_2^*(q)$ over $U \times X$ are not proportional in general. However the following Proposition holds (see Appendix, Lemma 2.0.8)

1.0.7 Proposition. *The quadratic spaces $\rho^*(p_2^*(q))$ and $\rho^*(p_1^*(q))$ are proportional.*

Further by [PSV, Prop. 3.3, Prop. 3.4] and [PaSV] we may find an open X' in X containing x and an open affine $S \subset \mathbf{P}^{d-1}$ (d=dim(X)) and a smooth morphism $f' : X' \to S$ making X' into a smooth relative curve over S with the geometrically irreducible fibres. Moreover we may find f' such that $f'|_{X' \cap Z} : Z' = X' \cap Z \to S$ is finite, where Z is the vanishing locus of $g \in k[X]$. Moreover f' can be written as $pr_S \circ \Pi' = f'$, where $\Pi' : X' \to \mathbf{A}^1 \times S$ is a finite surjective morphism. Set $\tilde{X}' = \pi^{-1}(X')$. Replacing notation we write X for X', $\tilde{X}$ for $\tilde{X}'$, Z for Z', $f : X \to S$ for $f' : X' \to S$, $\Pi : X \to \mathbf{A}^1 \times S$ for $\Pi' : X' \to \mathbf{A}^1 \times S$.

Let $\overline{U \times_S X} := (\tilde{U} \times_S \tilde{X})/\Delta(\Gamma)$. Clearly, $U \times_S X = (\tilde{U} \times_S \tilde{X})/(\Gamma \times \Gamma)$. Let

$$\rho_S : \overline{U \times_S X} \to U \times_S X$$

be the obvious map.

Let $p_X : U \times_S X \to X$ be projection to X and $p_U : U \times_S X \to U$ be the projection to U. By Proposition 1.0.7 the quadratic spaces $\rho_S^*(p_X^*(q))$ and $\rho_S^*(p_U^*(q))$ are ...

Now the pull-back of Π be means of the morphism $U \hookrightarrow X \to S$ defines a finite surjective morphism $\Theta : U \times_S X \to \mathbf{A}^1 \times U$. So, $\Theta \circ \rho_S : \overline{U \times_S X} \to \mathbf{A}^1 \times U$ is a finite surjective morphism of U-schemes. The U-scheme $\overline{U \times_S X}$ is smooth over U since $U \times_S X$ is smooth over U and ρ_S is étale. The subscheme $\Delta(\tilde{U})/\Delta(\Gamma) \subset \overline{U \times_S X}$ projects isomorphically onto U. So, we are given with a section $\tilde{\Delta}$ of the morphism

$$\overline{U \times_S X} \xrightarrow{\rho_S} U \times_S X \xrightarrow{p_U} U.$$

The recollection from the latter paragraph shows that we are under the hypotheses of Lemma 3.0.9 from Appendix B for the relative U-curve $\mathcal{X} := \overline{U \times_S X}$ and its closed subset $\mathcal{Z} := \rho_S^{-1}(U \times_S Z)$. (If to be more accurate, then one should take the connected component $\mathcal{X}^c$ of $\mathcal{X}$ containing $\tilde{\Delta}(U)$ and the closed subset $\mathcal{Z} \cap \mathcal{X}^c$ of $\mathcal{X}^c$).

By Lemma 3.0.9 there exists an open subscheme $\mathcal{X}^0 \hookrightarrow \mathcal{X}$ and a finite surjective morphism $\alpha : \mathcal{X}^0 \to \mathbf{A}^1 \times U$ such that α is étale over $0 \times U$ and $1 \times U$ and $\alpha^{-1}(0 \times U) = \tilde{\Delta}(U) \coprod D_0$. Moreover if we define D_1 as $\alpha^{-1}(1 \times U)$, then $D_1 \cap Z = \emptyset$ and $D_0 \cap Z = \emptyset$. One has $[D_1 : U] = [D_0 : U] + 1$. Thus either $[D_1 : U]$ is odd or $[D_0 : U]$ is odd.

Assume $[D_1 : U]$ is odd. Then the morphism $1 \times U \xleftarrow{\alpha|_{D_1}} D_1$, the morphism $D_1 \xrightarrow{p_X \circ \rho_S} X - Z$ and the isomorphism $\bar{\Phi} := \Phi|_{D_1}$ satisfy the conclusion of the Proposition 1.0.1 (here Φ is from the Proposition 1.0.7). The Proposition is proven.

□

2 Appendix A: Equating Lemma

Let k be a field, X be a k-smooth affine scheme, G be a reductive k-group, $\mathcal{G}/X$ be a principle G-bundle over X. Let $\pi : \tilde{X} \to X$ be a finite étale Galois cover of X with a Galois group Γ and let $s : \tilde{X} \to \mathcal{G}$ be an X-scheme morphism (*in other words* $\mathcal{G}$ *splits over* $\tilde{X}$). Let $\overline{X \times X} := (\tilde{X} \times \tilde{X})/\Delta(\Gamma)$. Clearly, $X \times X = (\tilde{X} \times \tilde{X})/(\Gamma \times \Gamma)$. Let $\pi : \overline{X \times X} \to X \times X$ be the obvious map. Observe that the map $\tilde{X} \times \tilde{X} \to \overline{X \times X}$ is an étale Galois cover with the Galois group Γ.

Let $q_i : \tilde{X} \times \tilde{X} \to \tilde{X}$ be projection to the i-th factor and let $p_i : X \times X \to X$ be projection to the i-th factor. The principal G bundles $\mathcal{G}_1 := p_1^*(\mathcal{G})$ and $\mathcal{G}_2 := p_2^*(\mathcal{G})$ over $X \times X$ are not isomorphic in general. However the following Proposition holds

2.0.8 Lemma. *The principal G-bundles $\pi^*(\mathcal{G}_1)$ and $\pi^*(\mathcal{G}_2)$ are isomorphic and moreover there is such an isomorphism $\Phi : \pi^*(\mathcal{G}_1) \to \pi^*(\mathcal{G}_2)$ that the restriction of Φ to the subscheme $X = \Delta(\tilde{X})/(\Gamma) \subset \overline{X \times X}$ is the identity isomorphism.*

Proof. The morphism $s : \tilde{X} \to \mathcal{G}$ gives rise to a 1-cocycle $a : \Gamma \to G(\tilde{X})$ defined as follows: given $\gamma \in \Gamma$ consider the composition $s \circ \gamma$ and set $a_\gamma \in G(\tilde{X})$ to be a unique element with $a_\gamma \cdot s = s \circ \gamma$ in $G(\tilde{X})$.

It's straight forward to check that the 1-cocycle corresponding to the principal G bundle $\pi^*(\mathcal{G}_1)$ and the morphism $\tilde{X} \times \tilde{X} \xrightarrow{q_1} \tilde{X} \xrightarrow{s} \mathcal{G}$ coincides with the one

$$\Gamma \xrightarrow{a} G(\tilde{X}) \xrightarrow{q_1^*} G(\tilde{X} \times \tilde{X}).$$

Similarly the 1-cocycle corresponding to the principal G bundle $\pi^*(\mathcal{G}_2)$ and the morphism $\tilde{X} \times \tilde{X} \xrightarrow{q_2} \tilde{X} \xrightarrow{s} \mathcal{G}$ coincides with the one

$$\Gamma \xrightarrow{a} G(\tilde{X}) \xrightarrow{q_2^*} G(\tilde{X} \times \tilde{X}).$$

Let $b \in G(\tilde{X} \times \tilde{X})$ be an element defined by the equality $b \cdot (s \circ q_2) = s \circ q_1$. To prove that the principal G bundles $\pi^*(\mathcal{G}_1)$ and $\pi^*(\mathcal{G}_2)$ are isomorphic it suffices to check that for every $\gamma \in \Gamma$ the following relation holds in $G(\tilde{X} \times \tilde{X})$

$$^{\gamma}b \cdot q_2^*(a)(\gamma) \cdot b^{-1} = q_1^*(a)(\gamma), \tag{1}$$

where $q_i^*(a)(\gamma) := q_i^* \circ a$ for $i = 1, 2$.

To prove the relation (1) it suffices to check the following one in $\mathcal{G}(\tilde{X} \times \tilde{X})$

$$(^{\gamma}b \cdot q_2^*(a)(\gamma) \cdot b^{-1}) \cdot (s \circ q_1) = q_1^*(a)(\gamma) \cdot (s \circ q_1). \tag{2}$$

One has the following chain of relations

$$(^{\gamma}b \cdot q_2^*(a)(\gamma) \cdot b^{-1}) \cdot (s \circ q_1) = (\,^{\gamma}b \cdot q_2^*(a)(\gamma)) \cdot (s \circ q_2) = \,^{\gamma}b \cdot \,^{\gamma}(s \circ q_2) =$$

$$= \,^{\gamma}(s \circ q_1) = s \circ q_1 \circ (\gamma \times \gamma)$$

The first one follows from the definition of the element b, the second one follows from the commutativity of the diagram

$$\begin{array}{ccc}
G \times \mathcal{G} & \xrightarrow{\nu} & \mathcal{G} \\
{\scriptstyle (a(\gamma),s)}\uparrow & & \uparrow{\scriptstyle s} \\
\tilde{X} & \xrightarrow{\gamma} & \tilde{X} \\
{\scriptstyle q_2}\uparrow & & \uparrow{\scriptstyle q_2} \\
\tilde{X} \times \tilde{X} & \xrightarrow{\gamma\times\gamma} & \tilde{X} \times \tilde{X},
\end{array}$$

the third one follows from the commutativity of the diagram

$$\begin{array}{ccc}
G \times \mathcal{G} & \xrightarrow{\nu} & \mathcal{G} \\
{\scriptstyle (b,s\circ q_2)}\uparrow & & \uparrow{\scriptstyle s} \\
\tilde{X} \times \tilde{X} & \xrightarrow{q_1} & \tilde{X} \\
{\scriptstyle \gamma\times\gamma}\uparrow & & \\
\tilde{X} \times \tilde{X}. & &
\end{array}$$

Thus $({}^{\gamma}b \cdot q_2^*(a)(\gamma) \cdot b^{-1}) \cdot (s \circ q_1) = s \circ q_1 \circ (\gamma \times \gamma)$. The right hand side of the relation (2) is equal to $s \circ q_1 \circ (\gamma \times \gamma)$ as well, as follows from the commutativity of the diagram

$$\begin{array}{ccc}
G \times \mathcal{G} & \xrightarrow{\nu} & \mathcal{G} \\
{\scriptstyle (a(\gamma),s)}\uparrow & & \uparrow{\scriptstyle s} \\
\tilde{X} & \xrightarrow{\gamma} & \tilde{X} \\
{\scriptstyle q_1}\uparrow & & \uparrow{\scriptstyle q_1} \\
\tilde{X} \times \tilde{X} & \xrightarrow{\gamma\times\gamma} & \tilde{X} \times \tilde{X}.
\end{array}$$

So, the relation (2) holds. Whence the relation (1) holds. Whence the principal G bundles $\pi^*(\mathcal{G}_1)$ and $\pi^*(\mathcal{G}_2)$ are isomorphic.

The composite $\tilde{X} \xrightarrow{\Delta} \tilde{X} \times \tilde{X} \xrightarrow{q_2} \tilde{X} \xrightarrow{s} \mathcal{G}$ equals s and equals the composite $s \circ q_1 \circ \Delta$. Whence $\Delta^*(b) = 1 \in G(\tilde{X})$. This shows that the restriction to $X = \Delta(X)/\Delta(\Gamma)$ of the isomorphism $\pi^*(\mathcal{G}_1)$ and $\pi^*(\mathcal{G}_2)$ corresponding to the element b is the identity isomorphism. The Lemma is proved.

□

3 Appendix B: a variant of geometric lemma

Let k be an infinite field, Y be a k-smooth algebraic variety, $y \in Y$ be a point, $\mathcal{O} = \mathcal{O}_{Y,y}$ be the local ring, $U = \mathrm{Spec}(\mathcal{O})$. Let $\mathcal{X}/U$ be a U-smooth relative curve with geometrically connected fibres equipped with a finite surjective morphism $\pi : \mathcal{X} \to \mathbf{A}^1 \times U$ and equipped with a section $\Delta : U \to \mathcal{X}$ of the projection $p : \mathcal{X} \to U$. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed subset finite over U. The following Lemma is a variant of Lemma 5.1 from [OP].

3.0.9 Lemma. *There exists an open subscheme $\mathcal{X}^0 \hookrightarrow \mathcal{X}$ and a finite surjective morphism $\alpha : \mathcal{X}^0 \to \mathbf{A}^1 \times U$ such that α is étale over $0 \times U$ and $1 \times U$ and $\alpha^{-1}(0 \times U) = \Delta(U) \coprod D_0$. Moreover if we define D_1 as $\alpha^{-1}(1 \times U)$, then $D_1 \cap \mathcal{Z} = \emptyset$ and $D_0 \cap \mathcal{Z} = \emptyset$.*

Proof. Let $\bar{\mathcal{X}}$ be the normalization of the scheme $\mathbf{P}^1 \times U$ in the function field $k(\mathcal{X})$ of $\mathcal{X}$. Let $\bar{\pi} : \bar{\mathcal{X}} \to \mathbf{P}^1 \times U$ be the morphism. Let $\mathcal{X}_\infty = \pi^{-1}(\infty \times U)$ be the set theoretic preimage of $\infty \times U$. Let $\bar{p} : \bar{\mathcal{X}} \to U$ be the structure map. Let $u \in U$ be the closed point and $\bar{X}_u = \bar{\mathcal{X}} \times_U u$.

Let , $L' = \bar{\pi}^*(\mathcal{O}_{\mathbf{P}^1 \times U}(1))$, $L'' = \mathcal{O}_{\bar{X}}(\Delta(U))$. Let $D_\infty = (\pi^*)(\infty \times U)$ be the pull-back of the Cartier divisor $\infty \times U \subset \mathbf{P}^1 \times U$. Choose and fix a closed embedding $i : \bar{\mathcal{X}} \hookrightarrow \mathbf{P}^n \times U$ of U-schemes. Set $L = i^*(\mathcal{O}_{\mathbf{P}^n \times U}(1))$.

The sheaf L is very ample. Thus the sheaf $L'' \otimes L$ is very amply as well. So, there exists a closed embedding $i'' : \bar{\mathcal{X}} \hookrightarrow \mathbf{P}^{n''} \times U$ of U-schemes such that $L'' \otimes L = (i'')^*(\mathcal{O}_{\mathbf{P}^{n''} \times U}(1))$.
Using Bertini theorem choose a hyperplane $H'' \subset \mathbf{P}^{n''} \times U$ such that
(a'') $H'' \cap \Delta(U) = \emptyset$, $H'' \cap \mathcal{Z} = \emptyset$, $H'' \cap D_\infty = \emptyset$.
Define a Cartier divisor D'' on $\bar{\mathcal{X}}$ as the the closed subscheme $H'' \cap \bar{\mathcal{X}}$ of $\bar{\mathcal{X}}$.
Regard $D''_1 := D'' \coprod D_\infty$ as a Cartier divisor on $\bar{\mathcal{X}}$. Clearly, one has $\mathcal{O}_{\bar{\mathcal{X}}}(D''_1) = L'' \otimes L \otimes L'$.

The sheaf L is very ample. Thus the sheaf $L' \otimes L$ is very ample as well. So, there exists a closed embedding $i' : \bar{\mathcal{X}} \hookrightarrow \mathbf{P}^{n'} \times U$ of U-schemes such that $L' \otimes L = (i')^*(\mathcal{O}_{\mathbf{P}^{n'} \times U}(1))$.
Using Bertini theorem choose a hyperplane $H' \subset \mathbf{P}^{n'} \times U$ such that
(a') $H' \cap \Delta(U) = \emptyset$, $H' \cap \mathcal{Z} = \emptyset$, $H' \cap D''_1 = \emptyset$;
(b') the scheme theoretic intersection $H' \cap \bar{X}_u$ is a $k(u)$-smooth scheme.
Define a Cartier divisor D' on $\bar{\mathcal{X}}$ as the closed subscheme $D' = H' \cap \bar{\mathcal{X}}$ of $\bar{\mathcal{X}}$.
Regard $D'_1 := D' \coprod \Delta(U)$ as a Cartier divisor on $\bar{\mathcal{X}}$. Clearly, one has $\mathcal{O}_{\bar{\mathcal{X}}}(D'_1) = L' \otimes L \otimes L''$.

Observe that D' is an essentially k-smooth scheme finite and étale over U. Let s' and s'' be global sections of $L' \otimes L \otimes L''$ such that the vanishing locus of s' is the Cartier divisor D'_1 and the vanishing locus of s'' is the Cartier divisor D''_1. Clearly $D'_1 \cap D''_1 = \emptyset$. Thus $f = [s' : s''] : \bar{\mathcal{X}} \to \mathbf{P}^1$ is a regular morphism of U-schemes. Set

$$\bar{\alpha} = (f, \bar{p}) : \bar{\mathcal{X}} \to \mathbf{P}^1 \times U.$$

Clearly, $\bar{\alpha}$ is a finite surjective morphism. Set $\mathcal{X}^0 = \bar{\alpha}^{-1}(\mathbf{A}^1 \times U)$ and

$$\alpha = \bar{\alpha}|_{\mathcal{X}^0} : \mathcal{X}^0 \to \mathbf{A}^1 \times U.$$

Clearly, α is a finite surjective morphism and $\mathcal{X}^0$ is an open subscheme of $\mathcal{X}$. Since α is a finite surjective morphism and $\mathcal{X}^0$, $\mathbf{A}^1 \times U$ are regular schemes the morphism α is flat by a theorem of Grothendieck. Since D'_1 is finite étale over U the morphism α is étale over $0 \times U$. So, we may choose a point $1 \in \mathbf{P}^1$ such that the α is étale over $1 \times U$ and $(\alpha)^{-1}(1 \times U) \cap \mathcal{Z} = \emptyset$. If we set $D_0 = D'_1$, then $\alpha^{-1}(0 \times U) = \Delta(U) \coprod D_0$ and $D_0 \cap \mathcal{Z} = \emptyset$. The Lemma is proven.

□

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