# PERIOD-INDEX AND $u$-INVARIANT QUESTIONS FOR FUNCTION FIELDS OVER COMPLETE DISCRETELY VALUED FIELDS

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Abstract. Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ the function field of a curve over $K$. Let $p$ be the characteristic of $\kappa$ and $\ell$ a prime not equal to $p$. If the Brauer $\ell$-dimensions of all finite extensions of $\kappa$ are bounded by $d$ and the Brauer $\ell$-dimensions of all extensions of $\kappa$ of transcendence degree at most 1 are bounded by $d+1$, then it is known that the Brauer $\ell$-dimension of $F$ is at most $d+2$ ([S1], [HHK1]). In this paper we give a bound for the Brauer $p$-dimension of $F$ in terms of the $p$-rank of $\kappa$. As an application, we show that if $\kappa$ is a perfect field of characteristic 2, then any quadratic form over $F$ in at least 9 variables is isotropic. If $\kappa$ is a finite field, this is a result of Heath-Brown/Leep ([HB], [Le]).

Let $K$ be a field. For a central simple algebra $A$ over $K$, the *period* of $A$ is the order of its class in the Brauer group of $K$ and the *index* of $A$ is the degree of the division algebra Brauer equivalent to $A$. The index of $A$ is denoted by $\mathrm{ind}(A)$ and period of $A$ by $\mathrm{per}(A)$. Let $K$ be a $p$-adic field and $F$ the function field of a curve over $K$. The question whether the index of a central simple algebra over $F$ divides the square of its period has remained open for a while. For indices which are coprime to $p$, an affirmative answer to this question is a theorem of Saltman ([S1]). To complete the answer, one needs to understand the relationship between the period and the index for algebras of period $p$ over $F$. One of the main results proved in this paper is the following

**Theorem 1.** *Let $K$ be a $p$-adic field and $F$ a function field of a curve over $K$. Then the index of any central simple algebra over $F$ divides the square of its period.*

Let $K$ be any field. For a prime $p$, we define the *Brauer $p$-dimension* of $K$, denoted by $\mathrm{Br}_p\mathrm{dim}(K)$, to be the smallest integer $d \geq 0$ such that for every finite extension $L$ of $K$ and for every central simple algebra $A$ over $L$ of period a power of $p$, $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^d$. The *Brauer dimension* of $K$, denoted by $\mathrm{Brdim}(K)$, is defined as the maximum of the Brauer $p$-dimension of $K$ as $p$ varies over all primes. Suppose the characteristic of $K$ is $p > 0$. If $[K : K^p] = p^n$, then $n$ is called the *$p$-rank* of $K$. A field of characteristic $p > 0$ is perfect if and only if its $p$-rank is 0. A theorem of Albert asserts that the Brauer $p$-dimension of a field $K$ of characteristic $p > 0$ is at most the $p$-rank of $K$ (cf. (1.2)).

In this paper, we discuss more generally the Brauer $p$-dimension of function fields of curves over a complete discretely valued field of characteristic 0 with residue field of characteristic $p > 0$.

We begin by bounding the Brauer dimension of complete discretely valued fields. Let $K$ be a complete discretely valued field and $\kappa$ its residue field. Suppose that $\mathrm{char}(K) = 0$ and $\mathrm{char}(\kappa) = p > 0$. Let $\ell$ be a prime. Suppose that $\mathrm{Br}_\ell\mathrm{dim}(\kappa) \leq d$. If $\ell \neq p$, then it is well known that $\mathrm{Br}_\ell\mathrm{dim}(K) \leq d+1$ (cf. [GS], Corollary 7.1.10). There seems to be no good connections between the Brauer $p$-dimension of $K$ and

the Brauer $p$-dimension of $\kappa$. For any $n \geq 0$, we give an example of a complete discretely valued field $K$ with $\mathrm{Br}_p\dim(K) \geq n$ and $\mathrm{Br}_p\dim(\kappa) = 0$. However there are bounds for the Brauer $p$-dimension of $K$ in terms of the $p$-rank of $\kappa$ and we prove the following

**Theorem 2.** *Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that $char(\kappa) = p > 0$ and the $p$-rank of $\kappa$ is $n$. Then $Br_p dim(K) \leq 1$ if $n = 0$ and $\frac{n}{2} \leq Br_p dim(K) \leq 2n$ if $n \geq 1$.*

Let $F$ be the function field of a curve over $K$. Let $\ell$ be a prime. Suppose that there exists $d$ such that $\mathrm{Br}_\ell\dim(\kappa) \leq d$ and $\mathrm{Br}_\ell\dim(\kappa(C)) \leq d+1$ for every curve $C$ over $\kappa$. It was proved in [HHK1] that if $\mathrm{char}(\kappa) \neq \ell$, then $\mathrm{Br}_\ell\dim(F) \leq d+2$. For $\ell = \mathrm{char}(\kappa)$, we prove the following

**Theorem 3.** *Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that $char(K) = 0$ and $char(\kappa) = p > 0$. Let $F$ be the function field of a curve over $K$. If the $p$-rank of $\kappa$ is $n$, then $Br_p dim(F) \leq 2n + 2$.*

We use the description of the Brauer group of a complete discretely valued field in the mixed characteristic case due to Kato ([K], [CT]) and the patching techniques of Harbater-Hartman-Krashen ([HHK1]) to prove our main results.

In the last section, we derive some consequences for the $u$-invariant of fields. The $u$-invariant of a field $L$ is the maximum dimension of anisotropic quadratic forms over $L$. Let $K$ be a complete discretely valued field with residue field $\kappa$. It is a theorem of Springer that $u(K) = 2u(\kappa)$. Let $F$ be a function field of a curve over $K$. Suppose that $\mathrm{char}(\kappa) \neq 2$. If $u(L) \leq d$ for every finite extension $L$ of $\kappa$ and $u(\kappa(C)) \leq 2d$ for every function field $\kappa(C)$ of a curve $C$ over $\kappa$, then in ([HHK1]), it is proved that $u(F) \leq 4d$. For a $p$-adic field $K$, this was proved in ([PS2]). Suppose $\kappa$ is a field of characteristic 2 with $[\kappa : \kappa^2] = n$. Then $u(\kappa) \leq 2n$ ([MMW], Corollary 1). Let $L$ be a finite extension of $\kappa$. Since $[L : L^2] = n$, we have $u(L) \leq 2n$. If $C$ is a curve over $\kappa$, then $[\kappa(C) : \kappa(C)^2] = 2n$ and hence $u(\kappa(C)) \leq 4n$. If $\mathrm{char}(K) = 2$, $[F : F^{*2}] = 4n$ and hence $u(F) \leq 8n$ ([MMW]. Corollary 1). Suppose that $\mathrm{char}(K) = 0$. If $\kappa$ is a finite field, then results of Heath-Brown ([HB) and Leep ([Le]) lead to $u(F) = 8$. However very little is known about $u(F)$ for general $\kappa$. We prove the following

**Theorem 4.** *Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that $char(K) = 0$ and $char(\kappa) = 2$. Let $F$ be a function field of a curve over $K$. If $\kappa$ is a perfect field, then $u(F) \leq 8$.*

This leads us to the following

**Conjecture.** *Suppose $K$ is a complete discretely valued field of characteristic 0 with residue field $\kappa$ of characteristic 2. If $F$ is a function field of a curve over $K$, then $u(F) \leq 8[\kappa : \kappa^2]$.*

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## 1. Module of differentials and Milnor $k$-groups

We begin by recalling two well-known results (1.1, 1.2) concerning the Brauer $\ell$-dimension of a field. Lemma 1.1 reduces the computation of the Brauer $\ell$-dimension to bounding indices of prime exponent algebras. Corollary 1.2 computes the Brauer $p$-dimension for fields of characteristic $p > 0$.

**Lemma 1.1.** *Let $k$ be any field and $\ell$ a prime. If for every central simple algebra $A$ of period $\ell$ over a finite extension $K$ of $k$, ind$(A)$ divides $\ell^d$, then $Br_\ell dim(k) \leq d$.*

*Proof.* Let $k'$ be a finite extension of $k$ and $A$ a central simple algebra over $k'$ of period $\ell^n$ for some $n$. We prove by induction on $n$ that ind$(A)$ divides $(\ell^n)^d$. The case $n = 1$ is the given hypothesis. Suppose that the lemma holds for $n - 1$. Let $A' = A^{\otimes \ell}$. Then per$(A') = \ell^{n-1}$. By the induction hypothesis ind$(A')$ divides $(\ell^{n-1})^d$. Thus there exists a finite extension $K$ of $k'$ of degree dividing $(\ell^{n-1})^d$ such that $A' \otimes_{k'} K$ is a matrix algebra. In particular per$(A \otimes_{k'} K) = \ell$ and by the hypothesis ind$(A \otimes_{k'} K)$ divides $\ell^d$. Thus there exists a finite extension $L$ of $K$ of degree dividing $\ell^d$ such that $A \otimes_{k'} L$ is a matrix algebra. Since $[L : k'] = [L : K][K : k']$ divides $\ell^d(\ell^{n-1})^d = (\ell^n)^d$, ind$(A)$ divides $(\ell^n)^d$. □

**Corollary 1.2.** *(Albert) Let $\kappa$ be a field of characteristic $p > 0$. Suppose that the $p$-rank of $\kappa$ is $n$. Then $Br_p dim(\kappa) \leq n$.*

*Proof.* Let $k'$ be a finite extension of $k$ and $A$ be a central simple algebra over $k'$ of period $p$. By (1.1), it is enough to show that ind$(A)$ divides $p^n$. By ([A], Chap. VII. Theorem 32), $A \otimes_{k'} k'^{1/p}$ is a matrix algebra and hence ind$(A)$ divides $[k'^{1/p} : k']$. Since $[k'^{1/p} : k'] = [k' : k'^p] = [k : k^p] = p^n$ ([B], A.V.135, Corollary 3), ind$(A)$ divides $p^n$. □

Let $\kappa$ be a field of characteristic $p > 0$. Let $\Omega^1_\kappa$ be the module of differentials on $\kappa$. Then the dimension of $\Omega^1_\kappa$ as a $\kappa$-vector space is equal to the $p$-rank of $\kappa$. Let $\Omega^2_\kappa$ be the second exterior power of $\Omega^1_\kappa$. Let $K_2(\kappa)$ be the Milnor $K$-group and $k_2(\kappa) = K_2(\kappa)/pK_2(\kappa)$. Then there is an injective homomorphism (cf., [CT], 3.0)

$$h^2_p : k_2(\kappa) \to \Omega^2_\kappa$$

given by

$$(a, b) \mapsto \frac{da}{a} \wedge \frac{db}{b}.$$

Suppose $\kappa = \kappa^p(a_1, \cdots, a_n)$. Then every element in $\Omega^2_\kappa$ is a linear combination of elements $da_i \wedge da_j$. In fact if $a_1, \cdots, a_n$ is a $p$-basis of $\kappa$, then $da_i \wedge da_j$, $1 \leq i < j \leq n$ is a basis of $\Omega^2_\kappa$ over $\kappa$.

We now record a few facts about $\Omega^2_\kappa$ and $k_2(\kappa)$.

**Lemma 1.3.** *Let $a, b \in \kappa^*$ be $p$-dependent. Then $(a, b) = 0 \in k_2(\kappa)$.*

*Proof.* If $a$ is a $p^{\text{th}}$ power in $\kappa$, then $da = 0$. Suppose $a = \sum \lambda_i^p b^i$ for some $\lambda_i \in \kappa$. Then $da = (\sum \lambda_i^p i b^{i-1})db$ and $da \wedge db = 0$. In particular $\frac{da}{a} \wedge \frac{db}{b} = 0 \in \Omega^2_\kappa$. Since $h^2_p((a, b)) = \frac{da}{a} \wedge \frac{db}{b}$ and $h^2_p$ is injective, we have $(a, b) = 0 \in k_2(\kappa)$. □

**Lemma 1.4.** *Suppose that $\kappa = \kappa^p(a_1, \cdots a_n)$. Then the natural homomorphism $k_2(\kappa) \to k_2(\kappa(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}}))$ is zero.*

*Proof.* Let $(a, b) \in k_2(\kappa)$. Let $\kappa' = \kappa(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}})$. If $a$ is a $p^{\text{th}}$ power in $\kappa'$, then the image of $(a, b) \in k_2(\kappa')$ is zero. Suppose that $a$ is not a $p^{\text{th}}$ power in $\kappa'$. Since $\kappa'^p = \kappa^p(a_1, \cdots, a_{n-1})$, $\kappa = \kappa'^p(a_n)$ and hence $[\kappa : \kappa'^p] \leq p$. Since $a \notin \kappa'^p$, we have $\kappa = \kappa'^p(a) = \kappa^p(a_1, \cdots, a_{n-1}, a)$. In particular $a$ and $b$ are $p$-dependent over $\kappa'$ and hence, by (1.3), $(a, b) = 0 \in k_2(\kappa')$. Since every element in $k_2(\kappa)$ is a sum of elements of the form $(a, b)$, the image of $k_2(\kappa)$ in $k_2(\kappa')$ is zero. □

**Lemma 1.5.** *Let $a_1, \cdots, a_n \in \kappa^*$ be $p$-independent over $\kappa$ and $\kappa' = \kappa(\sqrt[p^{r_1}]{a_1}, \cdots, \sqrt[p^{r_n}]{a_n})$. Then every element in the kernel of the natural homomorphism $\Omega^2_\kappa \to \Omega^2_{\kappa'}$ is of the form $da_1 \wedge f_1 + \cdots + da_n \wedge f_n$ for some $f_i \in \Omega^1_\kappa$.*

*Proof.* Let $B \subset \kappa^*$ be such that $B \cap \{a_1, \cdots a_n\} = \emptyset$ and $B \cup \{a_1, \cdots, a_n\}$ is a $p$-basis of $\kappa$. Then $B \cup \{\sqrt[p^{r_1}]{a_1}, \cdots, \sqrt[p^{r_n}]{a_n}\}$ is a $p$-basis of $\kappa'$. Let $\alpha \in \Omega^2_\kappa$ be in the kernel of $\Omega^2_\kappa \to \Omega^2_{\kappa'}$. Then $\alpha = \sum \lambda_{ij} da_i \wedge da_j$ with $1 \le i < j \le m$ and $a_{n+1}, \cdots, a_m \in B$, $\lambda_i \in \kappa$. Since the image of $da_i \wedge da_j$ in $\Omega^2_{\kappa'}$ is zero for $1 \le i \le n$ and the image of $\alpha$ in $\Omega^2_{\kappa'}$ is zero, the image of $\sum \lambda_{ij} da_i \wedge da_j$, $n+1 \le i < j \le m$, in $\Omega^2_{\kappa'}$ is zero. Since $B$ is $p$-independent over $\kappa'$ and $a_{n+1}, \cdots, a_m \in B$, $da_i \wedge da_j$, $n+1 \le i < j \le m$ are linearly independent over $\kappa'$ and hence $\lambda_{ij} = 0$ for $n+1 \le i < j \le m$. Thus $\alpha = \sum \lambda_{ij} da_i \wedge da_j$ with $1 \le i < j \le n$. □

**Lemma 1.6.** *Let $a_1, \cdots, a_{2n} \in \kappa^*$ be $p$-independent in $\kappa$. Let $\kappa'$ be an extension of $\kappa$ of degree $d$ and $\lambda_1, \cdots, \lambda_n \in \kappa^*$. If $d < p^n$, then the image of $\lambda_1 da_1 \wedge da_2 + \cdots + \lambda_n da_{2n-1} \wedge da_{2n}$ in $\Omega^2_{\kappa'}$ is non-zero.*

*Proof.* Since $\Omega^2_\kappa \to \Omega^2_{\kappa_1}$ is injective for any separable extension $\kappa_1$ of $\kappa$, by replacing $\kappa$ by the separable closure of $\kappa$ in $\kappa'$, we assume that $\kappa'$ is purely inseparable over $\kappa$. Then $\kappa' = \kappa(\sqrt[p^{r_1}]{b_1}, \cdots, \sqrt[p^{r_n}]{b_m})$ for some $b_i \in \kappa^*$ with $\{b_1, \cdots, b_m\}$ $p$-independent over $\kappa$. Since the kernels of the homomorphisms $\Omega^2_\kappa \to \Omega^2_{\kappa'}$ and $\Omega^2_\kappa \to \Omega^2_{\kappa(\sqrt[p]{b_1}, \cdots, \sqrt[p]{b_m})}$ are equal by (1.5), we assume that $\kappa' = \kappa(\sqrt[p]{b_1}, \cdots, \sqrt[p]{b_m})$. Since $[\kappa' : \kappa] = p^m < p^n$, we have $m \le n-1$. Without loss of generality we assume that $m = n-1$.

Suppose that $\{a_1, \cdots, a_{2n}\}$ is a $p$-basis of $\kappa$. Let $r$ be the maximum such that $\{b_1, \cdots, b_{n-1}, a_{i_1}, a_{i_2} \cdots, a_{i_r}\}$ is $p$-independent with no two $a_{i_s}$ in $\{a_{2j-1}, a_{2j}\}$. By reindexing, we assume that $\{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}\}$ is $p$-independent. Then, for each $i \ge 2r+3$, $\{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}, a_i\}$ is $p$-dependent. Since $\{a_1, \cdots, a_{2n}\}$ is $p$-basis of $\kappa$, there exists $1 \le t_1 < t_2 < \cdots < t_q \le r+1$ such that $\{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}, a_{2t_1}, \cdots, a_{2t_q}\}$ is a $p$-basis of $\kappa$. After reshuffling the indices, we assume that $t_1 = 1, \cdots, t_q = q$ and $B = \{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}, a_2, a_4, \cdots a_{2q},\}$ is a $p$-basis of $\kappa$ with $q \le r+1$. Then $\{\sqrt[p]{b_1}, \cdots, \sqrt[p]{b_{n-1}}, a_1, a_3, \cdots, a_{2r+1}, a_2, a_4, \cdots, a_{2q}\}$ is a $p$-basis of $\kappa'$

Since $B$ is a $p$-basis of $\kappa$, every element of $\Omega^2_\kappa$ can be written as a linear combination of $dx \wedge dy$, $x, y \in B$. We now compute the coefficient of $da_1 \wedge da_2$ in the expansion of $da_{2i+1} \wedge da_{2i+2}$ as a linear combination of $dx \wedge dy$, $x, y \in B$. Let $1 \le i \le r$. Since $a_{2i+1} \in B$, the coefficient of $da_1 \wedge da_2$ in the expansion of $da_{2i+1} \wedge da_{2i+2}$ is zero. Let $i > r$. Since $a_{2i+1}$ and $a_{2i+2}$ are $p$-dependent over $\{b_1, \cdots, b_{n-1}, a_1, a_3, \cdots, a_{2r+1}\}$, in the expansion of $da_{2i+1}$ and $da_{2i+2}$ there is no $da_2$ term. Hence, there is no $da_1 \wedge da_2$ term in the expansion of $da_{2i+1} \wedge da_{2i+2}$.

Thus, the coefficient of $da_1 \wedge da_2$ in the expansion of $\alpha = \lambda_1 da_1 \wedge da_2 + \cdots + \lambda_n da_{2n-1} \wedge da_{2n}$ as a linear combination of $dx \wedge dy$ with $x, y \in B$ is $\lambda_1$. Since $\{\sqrt[p]{b_1}, \cdots, \sqrt[p]{b_{n-1}}, a_1, a_3, \cdots, a_{2r+1}, a_2, a_4, \cdots, a_{2q}\}$ is a $p$-basis of $\kappa'$, the image of $\alpha$ in $\Omega^2_{\kappa'}$ is non-zero.

Let $\{a_1, \cdots, a_{2n}\}$ be any $p$-independent subset of $\kappa$. Let $B' \subset \kappa$ be such that $B' \cup \{a_1, \cdots, a_{2n}\}$ is a $p$-basis of $\kappa$ and $B' \cap \{a_1, \cdots, a_{2n}\} = \emptyset$. Let $\kappa_1$ be the extension of $\kappa$ obtained by adjoining $\sqrt[p^d]{b}$ for all $b \in B'$ and $d \ge 1$. Then $\{a_1, \cdots, a_{2n}\}$ is a $p$-basis of $\kappa_1$. Then $\kappa_1\kappa'$ is an extension of $\kappa_1$ of degree $< p^n$. Hence the image of $\lambda_1 dc_1 \wedge dc_2 + \cdots + \lambda_n dc_{2n-1} \wedge dc_{2n}$ in $\Omega^2_{\kappa'\kappa_1}$ is non-zero. In particular, the image of $\lambda_1 dc_1 \wedge dc_2 + \cdots + \lambda_n dc_{2n-1} \wedge dc_{2n}$ in $\Omega^2_{\kappa'}$ is non-zero. □

## 2. Brauer $p$-dimension of a complete discretely valued field

In the section we give a bound for the Brauer $p$-dimension of a complete discrete valued field of characteristic zero with residue of characteristic $p > 0$, in terms of the $p$-rank of the residue field.

Let $R$ be a complete discrete valuation ring with field of fractions $K$ and residue field $\kappa$. Let $\nu$ be the discrete valuation on $K$ given by $R$ and $\pi$ a parameter in $R$. Suppose that $\mathrm{char}(K) = 0$ and $\mathrm{char}(\kappa) = p > 0$ and that $K$ contains a primitive $p^{th}$ root of unity. Fixing a primitive $p^{\mathrm{th}}$ root of unity $\zeta \in K^*$, for $a, b \in K^*$, let $(a, b) \in {}_p\mathrm{Br}(K)$ be the class of the cyclic $K$-algebra defined by $x^p = a, y^p = b$ and $xy = \zeta yx$. Let $N = \nu(p)p/(p-1)$. Let $\mathrm{br}(K)_0 = {}_p\mathrm{Br}(K)$. For $i \geq 1$, let $U_i = \{u \in R^* \mid u \equiv 1 \bmod (\pi)^i\}$ and $\mathrm{br}(K)_i$ be the subgroup of ${}_p\mathrm{Br}(K)$ generated by cyclic algebras $(u, a)$ with $u \in U_i$ and $a \in K^*$. Since $R$ is complete, for $n > N$, every element in $U_n$ is a $p^{\mathrm{th}}$ power and $\mathrm{br}(K)_n = 0$.

Let $\Omega^1_\kappa$ be the module of differentiales on $\kappa$. For any $c \in \kappa$, let $\tilde{c} \in R$ be a lift of $c$. For $i \geq 1$, the maps

$$\Omega^1_\kappa \to \mathrm{br}(K)_i/\,\mathrm{br}(K)_{i+1}$$

given by $x\frac{dy}{y} \mapsto (1 + \tilde{x}\pi^i, \tilde{y})$ and

$$\kappa \to \mathrm{br}(K)_i/\,\mathrm{br}(K)_{i+1}$$

given by $z \mapsto (\pi, 1 + \tilde{z}\pi^i)$ yield a surjective homomorphism

$$\rho_i : \Omega^1_\kappa \oplus \kappa \to \mathrm{br}(K)_i/\,\mathrm{br}(K)_{i+1}$$

([K], Thm. 2, cf. [CT], 4.3.1).

Let $K_2(\kappa)$ be the Milnor $K$-group and $k_2(\kappa) = K_2(\kappa)/pK_2(\kappa)$. The maps

$$k_2(\kappa) \to \mathrm{br}(K)_0/\,\mathrm{br}(K)_1$$

given by $(x, y) \mapsto (\tilde{x}, \tilde{y})$ and

$$\kappa^*/\kappa^{*p} \to \mathrm{br}(K)_0/\,\mathrm{br}(K)_1$$

given by $(z) \mapsto (\pi, \tilde{z})$ yield an isomorphism

$$\rho_0 : k_2(\kappa) \oplus \kappa^*/\kappa^{*p} \to \mathrm{br}(K)_0/\,\mathrm{br}(K)_1$$

([K], Thm.2, cf. [CT], 4.3.1).

**Lemma 2.1.** *Let $R$, $K$ and $\kappa$ be as above. Suppose that $\kappa = \kappa^p(a_1, \cdots, a_n)$ for some $a_1, \cdots, a_n \in \kappa$. Let $u_1, \cdots, u_n \in R$ be lifts of $a_1, \cdots, a_n$. Let $\alpha \in {}_p\mathrm{Br}(K)$. Then, there exists $u \in R^*$ such that $(\alpha - (\pi, u)) \otimes K(\sqrt[p]{u_1}, \cdots, \sqrt[p]{u_{n-1}}) \in \mathrm{br}(K(\sqrt[p]{u_1}, \cdots, \sqrt[p]{u_{n-1}}))_1$.*

*Proof.* Since $\rho_0$ is surjective, there exists $x_i, y_i, a \in \kappa^*$ such that $\rho_0(\sum_i (x_i, y_i) - (a)) = \alpha \in \mathrm{br}(K)_0/\,\mathrm{br}(K)_1$. In particular, if $u$ is a lift of $a$ in $R$,

$$\rho_0(\sum_i (x_i, y_i)) = \alpha - (\pi, u) \in \mathrm{br}(K)_0/\,\mathrm{br}(K)_1.$$

Let $K' = K(\sqrt[p]{u_1}, \cdots, \sqrt[p]{u_n - 1})$. Then $K'$ is also a complete discretely valued field with residue field $\kappa' = \kappa(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}})$. By the functoriality of the map $\rho_0$, we have $\rho_0(\sum(x_i, y_i)) = \alpha - (\pi, u) \in \mathrm{br}(K')_0/\,\mathrm{br}(K')_1$. By (1.4), the image of $\sum_i (x_i, y_i)$ is zero in $k_2(\kappa')$. Hence $\alpha - (\pi, u) = \rho_0(\sum(x_i, y_i)) = 0 \in \mathrm{br}(K')_0/\,\mathrm{br}(K')_1$. In particular, $\alpha - (\pi, u) \in \mathrm{br}(K')_1$. $\square$

**Proposition 2.2.** *Let $R$, $K$, $\kappa$ and $\pi$ be as above. Suppose that $\kappa = \kappa^p(a_1, \cdots, a_n)$ for some $a_1, \cdots, a_n \in \kappa$. Let $u_1, \cdots, u_n \in R$ be lifts of $a_1, \cdots, a_n$. Let $\alpha \in \mathrm{br}(K)_1$. Then there exist $\lambda, \lambda_1, \cdots, \lambda_n \in R^*$ such that*

$$\alpha = (\lambda_1, u_1) + \cdots + (\lambda_n, u_n) + (\pi, \lambda).$$

*Proof.* Let $\alpha \in \mathrm{br}(K)_1$. First we show, by induction on $i$, that for each $i \geq 0$, there exist $x_{i1}, \cdots, x_{in}, x_i \in R^*$ such that $\alpha - (x_{i1}, u_1) - \cdots (x_{in}, u_n) - (\pi, x_i) \in \mathrm{br}(K)_{i+1}$. Since $\alpha \in \mathrm{br}(K)_1$, we take $x_{0j} = x_0 = 1$, $1 \leq j \leq n$. Suppose that $i \geq 1$ and there exist $x_{i1}, \cdots, x_{in}, x_i \in R^*$ such that $\alpha - (x_{i1}, u_1) - \cdots (x_{in}, u_n) - (\pi, x_i) \in \mathrm{br}(K)_{i+1}$. Since the homomorphism $\rho_{i+1} : \Omega^1_\kappa \oplus \kappa \to \mathrm{br}(K)_{i+1}/\mathrm{br}(K)_{i+2}$ is surjective, there exist $w \in \Omega^1_\kappa$ and $a \in \kappa$ such that

$$\rho_{i+1}(w, a) = \alpha - (x_{i1}, u_1) - \cdots (x_{in}, u_n) - (\pi, x_i) \in \mathrm{br}(K)_{i+1}/\mathrm{br}(K)_{i+2}.$$

Since $\kappa = \kappa^p(a_1, \cdots, a_n)$, $\Omega_\kappa$ is generated by $\frac{da_i}{a_i}$, $1 \leq i \leq n$ and hence $w = \sum_i b_i \frac{da_i}{a_i}$ for some $b_i \in \kappa$. Thus

$$\rho_{i+1}(w, a) = (1 + \tilde{b}_1 \pi^{i+1}, u_1) + \cdots (1 + \tilde{b}_n \pi^{i+1}, u_n) + (\pi, 1 + \tilde{a}\pi^{i+1}).$$

In particular, $\alpha - (x_{i1}, u_1) - \cdots - (x_{in}, u_n) - (\pi, x_i) - (1 + \tilde{b}_1 \pi^{i+1}, u_1) - \cdots - (1 + \tilde{b}_n \pi^{i+1}, u_n) - (\pi, 1 + \tilde{a}\pi^{i+1}) \in \mathrm{br}(K)_{i+2}$. Let $x_{(i+1)j} = x_{ij}(1 + \tilde{b}_j \pi^{i+1})$ for $1 \leq j \leq n$ and $x_{i+1} = x_i(1 + \tilde{a}\pi^{i+1})$. Since $(x, yz) = (x, y) + (x, z) \in {}_p\mathrm{Br}(K)$, we have $\alpha - (x_{(i+1)1}, u_1) - \cdots - (x_{(i+1)n}, u_n) - (\pi, x_{i+1}) \in \mathrm{br}(K)_{i+2}$.

Since $\mathrm{br}(K)_i = 0$ for $i > N$, we have $\alpha = (x_{(N+1)1}, u_1) + \cdots + (x_{(N+1)n}, u_n) + (\pi, x_{N+1})$. □

**Corollary 2.3.** *Let $K$ and $\kappa$ be as above. Suppose that the $p$-rank of $\kappa$ is $n$. Let $D$ be a central simple algebra over $K$ of period $p$. If $D$ represents an element in $\mathrm{br}(K)_1$, then ind$(D)$ divides $p^{n+1}$.*

*Proof.* Let $\alpha \in \mathrm{br}(K)_1$ be the class of $D$. By (2.2), there exist $\lambda, \lambda_1, \cdots, \lambda_n \in R^*$ such that $\alpha = (\lambda_1, u_1) + \cdots + (\lambda_n, u_n) + (\pi, \lambda)$. Hence $\alpha \otimes K(\sqrt[p]{u_1}, \cdots \sqrt[p]{u_n}, \sqrt[p]{\pi}) = 0$ and the index of $D$ divides $p^{n+1}$. □

**Theorem 2.4.** *Let $K$ be a complete discretely valued field with residue field $\kappa$. Let $R$ be the valuation ring of $K$ and $\pi \in R$ be a parameter. Suppose that char$(K) = 0$, char$(\kappa) = p$ and the $p$-rank of $\kappa$ is $n$. Let $a_1, \cdots, a_n \in \kappa$ be such that $\kappa = \kappa^p(a_1, \cdots, a_n)$ and $u_1, \cdots, u_n \in R$ be lifts of $a_1, \cdots, a_n$. Let $D$ be a central simple algebra over $K$ of period $p$. If $n = 0$, then $D \otimes K(\sqrt[p]{\pi})$ is a matrix algebra and if $n \geq 1$, then $D \otimes K(\sqrt[p^2]{u_1}, \cdots \sqrt[p^2]{u_{n-1}}, \sqrt[p]{u_n}, \sqrt[p]{\pi})$ is a matrix algebra.*

*Proof.* Let $\zeta$ be a primitive $p^{\text{th}}$ root of unity and $K' = K(\zeta)$. Since $[K' : K]$ is coprime to $p$, $\mathrm{ind}(D) = \mathrm{ind}(D \otimes K')$. Since $K'$ is finite extension of a complete discretely valued field $K$, $K'$ is also a complete discrete valued field with residue field $\kappa'$ a finite extension of $\kappa$. In particular, $p$-rank$(\kappa') = p$-rank$(\kappa)$. Thus, by replacing, $K$ by $K'$, we assume that $K$ contains a primtiive $p^{\text{th}}$ root of unity. Let $\alpha \in {}_p\mathrm{Br}(K)$ be the class of $D$.

Suppose $n = 0$. Then $\kappa = \kappa^p$ and $k_2(\kappa) = 0$. Since $\rho_0 : k_2(\kappa) \oplus \kappa^*/\kappa^{*p} \to \mathrm{br}(K)_0/\mathrm{br}(K)_1$ is an isomorphism, ${}_p\mathrm{Br}(K) = \mathrm{br}(K)_1$. Thus, by (2.2), $\alpha = (\pi, u)$. In particular $D \otimes K(\sqrt[p]{\pi})$ is a matrix algebra.

Suppose that $n \geq 1$. Since $p$-rank$(\kappa) = n$, there exist $a_1, \cdots, a_n \in \kappa^*$ such that $\kappa = \kappa^p(a_1, \cdots, a_n)$. Let $u_1, \cdots, u_n \in R$ be lifts of $a_1, \cdots, a_n$ and $\pi \in R$ a parameter. Let $K_1 = K(\sqrt[p]{u_1}, \cdots, \sqrt[p]{u_{n-1}})$. Then $K_1$ is also a complete discrete valued field with

residue field $\kappa_1 = \kappa(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}})$. Let $R_1$ be the valuation ring of $K_1$. Then $\pi$ is a parameter in $R_1$. By (2.1), there exists $u \in R^*$ such that $(\alpha - (\pi, u)) \otimes K_1 \in \mathrm{br}(K_1)_1$. Since $\kappa_1^p = \kappa^p(a_1, \cdots, a_{n-1})$, we have $\kappa_1 = \kappa_1^p(\sqrt[p]{a_1}, \cdots, \sqrt[p]{a_{n-1}}, a_n)$. Since $\alpha - (\pi, u) \in \mathrm{br}(K_1)_1$, by (2.2), there exist $\lambda_1, \cdots, \lambda_n, \lambda \in R_1$ such that

$$\alpha - (\pi, u) = (\lambda_1, \sqrt[p]{u_1}) + \cdots + (\lambda_{n-1}, \sqrt[p]{u_{n-1}}) + (\lambda_n, u_n) + (\pi, \lambda).$$

Hence

$$\alpha = (\lambda_1, \sqrt[p]{u_1}) + \cdots + (\lambda_{n-1}, \sqrt[p]{u_{n-1}}) + (\lambda_n, u_n) + (\pi, u\lambda).$$

In particular $D \otimes K(\sqrt[p^2]{u_1}, \cdots \sqrt[p^2]{u_{n-1}}, \sqrt[p]{u_n}, \sqrt[p]{\pi})$ is a matrix algebra. □

**Corollary 2.5.** *Let $K$, $\kappa$ and $n$ be as in (2.4). Then $Br_p dim(K)$ is 1 if $n = 0$ and $Br_p dim(K) \leq 2n$ if $n \geq 1$.*

*Proof.* Let $K'$ be a finite extension of $K$. Let $D$ be a central simple algebra over $K'$ of period $p$. Since $K'$ is also a complete discretely valued field with the $p$-rank of the residue field equal to $n$, corollary follows by (2.4) and (1.6). □

**Lemma 2.6.** *Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that $char(K) = 0$, $char(\kappa) = p > 0$ and $[\kappa : \kappa^p] \geq 2n$. Then $Br_p dim(K) \geq n$.*

*Proof.* Let $a_1, \cdots, a_{2n} \in \kappa^*$ be $p$-independent. Let $u_1, \cdots, u_{2n} \in K^*$ be the lifts of $a_1, \cdots, a_{2n}$. Let $D = (u_1, u_2) + \cdots + (u_{2n-1}, u_{2n})$. We claim that $\mathrm{ind}(D) = p^n$. This would show that $\mathrm{Br}_p\mathrm{dim}(K) \geq n$.

Let $K_1$ be an extension of $K$ of degree at most $p^{n-1}$. Since $K$ is a complete discretely valued field, $K_1$ is also a complete discretely valued field with residue field $\kappa_1$ and $[\kappa_1 : \kappa] \leq [L : K] \leq p^{n-1}$. Then, by ( 1.6), the image of $da_1 \wedge da_2 + \cdots + da_{2n-1} \wedge da_{2n}$ in $\Omega^2_{\kappa_1}$ is non-zero. Since $h^2_p((a_1, a_2) + \cdots + (a_{2n-1}, a_{2n})) = da_1 \wedge da_2 + \cdots + da_{2n-1} \wedge da_{2n}$ is nonzero in $\Omega^2_{\kappa_1}$, $(a_1, a_2) + \cdots + (a_{2n-1}, a_{2n})$ is non-zero in $k_2(\kappa_1)$. Since $\rho_0((a_1, a_2) + \cdots + (a_{2n-1}, a_{2n}))$ is the class of $D \otimes_K K_1$ in $\mathrm{br}(K_1)_0/\mathrm{br}(K)_1$ and $\rho_0$ is injective, $D \otimes_K K_1$ is non-trivial in ${}_p\mathrm{Br}(K_1)$. Hence $\mathrm{ind}(D) \geq p^n$. Since $D$ is a product of $n$ cyclic algebras, $\mathrm{ind}(D) = p^n$. □

Combining (2.4) and (2.6), we have the following

**Theorem 2.7.** *Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that $char(K) = 0$, $char(\kappa) = p > 0$ and the $p$-rank of $\kappa$ is $n$. If $n = 0$, then $Br_p dim(K) \leq 1$ and if $n \geq 1$, then $\frac{n}{2} \leq Br_p dim(K) \leq 2n$.*

**Example 2.8.** Let $k$ be a purely transcendental extension of the finite field $\mathbf{F}_p$ of transcendence degree 2n and $\kappa$ the separable closure of $k$. Let $K$ be a complete discretely valued field of characteristic 0 with residue field $\kappa$. Then the Brauer $p$-dimension of $\kappa$ is 0 ([A], Ch.IV, §7, Theorem 18) and $\mathrm{Br}_p\mathrm{dim}(K) \geq n$ (2.6). Note that the $p$-rank of $\kappa$ is $2n$. Thus in the mixed characteristic case, the bound on the Brauer dimension of the residue field should be replaced by the bound on the $p$-rank of the residue field in order to get a good bound on the Brauer dimension of a complete discretely valued field.

## 3. The main theorem

Let $R$ be an integral domain and $K$ its field of fractions. Let $A$ be a central simple algebra over $K$. We say that $A$ is *unramified on $R$* if there exists an Azumaya algebra $\mathscr{A}$ over $R$ such that $\mathscr{A} \otimes_R K$ is Brauer equivalent to $A$. If $P$ is a prime ideal of $R$ and $A$ is unramified on $R_P$, then we say that $A$ is *unramified at $P$*. If $\nu$ is a discrete

valuation of $K$ with $R$ as the valuation ring at $\nu$ and $A$ is unramified on $R$, then we also say that $A$ is *unramified at* $\nu$.

Let $\mathscr{X}$ be a regular integral scheme with function field $K$ and $A$ a central simple algebra over $K$. Let $x \in \mathscr{X}$ be a point. If $A$ is unramified on the local ring $\mathscr{O}_{\mathscr{X},x}$ at $x$, then we say that $A$ is *unramified at* $x$. If $A$ is not unramified at $x$, then we say that $A$ is *ramified at* $x$. The *ramification divisor* of $A$ on $\mathscr{X}$ is the divisor $\sum x$, where sum is taken over the codimension one points $x$ of $\mathscr{X}$ with $A$ ramified at $x$. The *support* of the ramification divisor of $A$ is simply the union of codimension one points of $\mathscr{X}$ where $A$ is ramified.

Let $T$ be a complete discrete valuation ring with field of fractions $K$ and $t \in T$ a parameter. Let $\mathscr{X}$ be an excellent regular, integral, proper scheme over $\mathrm{Spec}(T)$ of dimension 2 with function field $F$ and reduced special fibre $Y$. For a closed point $P$ of $\mathscr{X}$, let $\mathscr{O}_{\mathscr{X},P}$ denote the local ring at $P$, $\hat{\mathscr{O}}_{\mathscr{X},P}$ the completion of the regular local ring $\mathscr{O}_{\mathscr{X},P}$ at its maximal ideal and $F_P$ the field of fractions of $\hat{\mathscr{O}}_{\mathscr{X},P}$. For an open subset $U$ of an irreducible component of $Y$, let $R_U$ be the ring consisting of elements in $F$ which are regular on $U$. Then $T \subset R_U$. Let $\hat{R}_U$ be the $(t)$-adic completion of $R_U$ and $F_U$ the field of fractions of $\hat{R}_U$ (cf. [HHK1]). In this section we give a bound for the Brauer $p$-dimension of $F$ in terms of the $p$-rank of the residue field of $T$.

We begin with the following results (3.1, 3.2, 3.3 and 3.4) which are well-known and we include them for the sake of completeness.

**Lemma 3.1.** *Let $B$ be a regular local ring with field of fractions $K$, residue field $\kappa$ and maximal ideal $m$. Let $n$ be a natural number and $u \in B$ a unit such that $[\kappa(\sqrt[n]{\overline{u}}) : \kappa] = n$. Then $B[\sqrt[n]{u}]$ is a regular local ring with residue field $\kappa(\sqrt[n]{\overline{u}})$.*

*Proof.* Since $[\kappa(\sqrt[n]{\overline{u}}) : \kappa] = n$, $B[\sqrt[n]{u}]/mB[\sqrt[n]{u}] \simeq \kappa(\sqrt[n]{\overline{u}})$ is a field. Thus $m$ generates the maximal ideal of $B[\sqrt[n]{u}]$. Since the dimension of $B[\sqrt[n]{u}]$ is equal to the dimension of $B$, $B[\sqrt[n]{u}]$ is a regular local ring. □

**Lemma 3.2.** *Let $B$ be a regular local ring with field of fractions $K$, residue field $\kappa$ and maximal ideal $m$. Let $\pi \in m$ be a regular prime and $n$ a natural number. Then $B[\sqrt[n]{\pi}]$ is a regular local ring with residue field $\kappa$.*

*Proof.* Since $B$ is a regular and $\pi \in m$ is a regular prime, there exist $\pi_2, \cdots, \pi_d \in m$ such that $m = (\pi, \pi_2, \cdots, \pi_d)$, where $d$ is the dimension of $B$. Let $\tilde{m} = (\sqrt[n]{\pi}, \pi_2, \cdots, \pi_d) \subset B[\sqrt[n]{\pi}]$. Then $\tilde{m}$ is the maximal ideal of $B[\sqrt[n]{\pi}]$ and $B[\sqrt[n]{\pi}]/\tilde{m} \simeq \kappa$. Since the dimension of $B[\sqrt[n]{\pi}]$ is $n$, $B[\sqrt[n]{\pi}]$ is a regular local ring. □

**Corollary 3.3.** *Let $B$ be a regular local ring with field of fractions $K$, residue field $\kappa$ and maximal ideal $m$. Let $n_1, \cdots, n_r$ be natural numbers and $u_1, \cdots, u_r \in B$ units such that $[\kappa(\sqrt[n_1]{\overline{u}_1}, \cdots, \sqrt[n_r]{\overline{u}_r}) : \kappa] = n_1 \cdots n_r$. Let $\pi_1, \cdots, \pi_s \in m$ be a system of regular parameters in $B$ and $d_1, \cdots, d_s$ be natural numbers. Then $B[\sqrt[n_1]{u_1}, \cdots, \sqrt[n_r]{u_r}, \sqrt[d_1]{\pi_1}, \cdots, \sqrt[d_s]{\pi_s}]$ is a regular local ring with residue field $\kappa(\sqrt[n_1]{\overline{u}_1}, \cdots, \sqrt[n_r]{\overline{u}_r})$.*

*Proof.* Proof follows by induction using (3.1) and (3.2). □

**Lemma 3.4.** *(cf. [LPS], 2.4) Let $R$ be a discrete valuation ring with field of fractions $K$. Let $\hat{R}$ be the completion of $R$ at the discrete valuation and $\hat{K}$ the field of fractions of $\hat{R}$. Then a central simple algebra $D$ over $K$ is unramified at $R$ if and only if $D \otimes_K \hat{K}$ is unramified at $\hat{R}$.*

**Proposition 3.5.** *Let $A$ be a complete regular local ring of dimension 2 with field of fractions $F$, residue field $\kappa$ and maximal ideal $m = (\pi, \delta)$. Suppose that char$(F) = 0$ and char$(\kappa) = p > 0$ with $p$-rank$(\kappa) = n$. Let $a_1, \cdots, a_n \in \kappa$ be a $p$-basis of $\kappa$ and $u_1, \cdots, u_n \in A$ be lifts of $a_1, \cdots, a_n$. Suppose that $F$ contains a primitive $p^{\text{th}}$ root of unity. Let $D$ be a central simple algebra over $F$ of period $p$. Suppose that $D$ is ramified on $A$ at most at $(\pi)$ and $(\delta)$. Then $D \otimes_F F(\sqrt[p^2]{u_1}, \cdots, \sqrt[p^2]{u_n}, \sqrt[p]{\pi}, \sqrt[p]{\delta})$ is a matrix algebra. In particular, index$(D)$ divides $p^{2n+2}$.*

*Proof.* Let

$$E = F(\sqrt[p^2]{u_1}, \cdots, \sqrt[p^2]{u_n}, \sqrt[p]{\pi}, \sqrt[p]{\delta})$$

and

$$B = A[\sqrt[p^2]{u_1}, \cdots, \sqrt[p^2]{u_n}, \sqrt[p]{\pi}, \sqrt[p]{\delta}] \subset E.$$

By (3.3), $B$ is a complete regular local ring of dimension 2 with field of fractions $E$ and residue field $\kappa(\sqrt[p^2]{a_1}, \cdots \sqrt[p^2]{a_n})$.

We first show that $D \otimes_F E$ is unramified on $B$. Since $B$ is a regular local ring of dimension 2, it is enough to show that $D \otimes_F E$ is unramified at every height one prime ideal of $B$ ([AG], 7.4). Let $Q$ be a height one prime ideal of $B$ and $P = Q \cap A$. Since $B$ is integral over $A$, the height of $P$ is 1. If $P \neq (\pi)$ or $(\delta)$, then $D$ is unramified at $P$ and hence $D \otimes_F E$ is unramified at $Q$. Suppose that $P = (\pi)$. Then $Q = (\sqrt[p]{\pi})$.

Suppose that $\text{char}(A/P) \neq p$. Since $E/F$ is ramified at $P$ and $\text{char}(\kappa(P)) \neq p$, $D \otimes_F E$ is unramified at $Q$. Suppose that $\text{char}(A/P) = p$. Since $A$ is complete regular local ring with maximal ideal $m = (\pi, \delta)$, $A/(\pi)$ is a complete discrete valuation ring with residue field $\kappa$ and $\text{char}(A/P) = \text{char}(\kappa) = p$. In particular, $A/(\pi) \simeq \kappa[[\overline{\delta}]]$, where $\overline{\delta}$ is the image of $\delta$ in $A/(\pi)$. Let $\kappa(P)$ be the field of fractions of $A/P$. Then $\kappa(P) \simeq \kappa((\overline{\delta}))$. Since $a_1, \cdots, a_n$ is a $p$-basis of $\kappa$ and $u_1, \cdots, u_n \in A$ are lifts of $a_1, \cdots, a_n$, the images of $u_1, \cdots, u_n, \delta$ in $\kappa(P)$ is a $p$-basis of $\kappa(P)$. Let $F_P$ be the completion of $F$ at $P$ and $E_Q$ the completion of $E$ at $Q$. Since $E_Q \simeq F_P(\sqrt[p^2]{u_1}, \cdots, \sqrt[p^2]{u_n}, \sqrt[p]{\delta}, \sqrt[p]{\pi})$ and the residue field of $F_P$ is $\kappa(P)$, by (2.4), $D \otimes_F E_Q$ is split and hence unramified. Thus, by (3.4), $D \otimes_F E$ is unramified at $Q$.

By ([AG], 7.4), there exists an Azumaya $B$-algebra $\mathscr{D}$ such that $\mathscr{D} \otimes_B E \simeq D \otimes_F E$. Since $D \otimes E_Q$ is split and $\hat{B}_Q$ is a discrete valuation ring, $\mathscr{D} \otimes_B \hat{B}_Q$ is zero in the $\text{Br}(\hat{B}_Q)$ ([AG], 7.2). In particular the image $\mathscr{D} \otimes_B \kappa(Q)$ of $\mathscr{D} \otimes_B \hat{B}_Q$ in $\text{Br}(\kappa(Q))$ is zero. Since $\kappa(Q)$ is the field of fractions of regular local ring $B/Q$, by ([AG], 7.2), $\mathscr{D} \otimes_B B/Q$ is zero in $\text{Br}(B/Q)$. Hence $\mathscr{D} \otimes_B B/\tilde{m}$ is zero in $\text{Br}(B/\tilde{m})$, where $\tilde{m}$ is the maximal ideal of $B$. Since $B$ is a complete regular local ring, $\mathscr{D} = 0 \in \text{Br}(B)$ ([C], [KOS]). In particular $\mathscr{D} \otimes_B E \simeq D \otimes_F E$ is zero and index$(D)$ divides $[E : F] = p^{2n+2}$. □

**Theorem 3.6.** *Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that char$(K) = 0$, char$(\kappa) = p > 0$ and $p$-rank$(\kappa) = n$. Let $F$ be a finitely generated field extension of $K$ of transcendence degree 1 and $D$ a central simple algebra over $F$ of period $p$. Then ind$(D)$ divides $p^{2n+2}$.*

*Proof.* As in the proof of (2.4), we assume without loss of generality that $F$ contains a primitive $p^{\text{th}}$ root of unity. Let $K'$ be a finite extension of $K$. Then $K'$ is also a complete discretely valued field with the $p$-rank of the residue field is $n$. Thus, replacing $K$ by a finite extension of $K$, we assume that $F$ is the function field of a geometrically integral smooth projective curve $X$ over $K$.

We choose a proper regular model $\mathscr{X}$ of $F$ over $T$ such that the support of the ramification divisor of $D$ and the components of the reduced special fibre are a union of regular curves with normal crossings on $\mathscr{X}$. Let $Y$ be the special fibre of $\mathscr{X}$.

Let $\eta$ be a generic point of an irreducible component of $Y$ and $F_\eta$ the completion of $F$ at the discrete valuation given by $\eta$. Then the residue field $\kappa(\eta)$ of $F_\eta$ is function field of transcendence degree one over $\kappa$. Since $[\kappa : \kappa^p] = p^n$, we have $[\kappa(\eta) : \kappa(\eta)^p] = p^{n+1}$. By (2.4), $\mathrm{ind}(D \otimes_F F_\eta)$ divides $p^{2n+2}$. By ([HHK2], 5.8 and [KMRT], 1.17), there exists an irreducible open set $U_\eta$ of $Y$ containing $\eta$ such that $\mathrm{ind}(D \otimes_F F_{U_\eta}) = \mathrm{ind}(D \otimes_F F_\eta)$. In particular $\mathrm{ind}(D \otimes_F F_{U_\eta})$ divides $p^{2n+2}$.

Let $S_0$ be a finite set of closed points of $\mathscr{X}$ containing all the points of intersection of the components of $Y$ and the support of the ramification divisor of $D$. Let $S$ be a finite set of closed points of $\mathscr{X}$ containing $S_0$ and $Y \setminus (\cup\, U_\eta)$, where $\eta$ varies over generic points of $Y$. Then, by ([HHK1], 5.1),

$$\mathrm{ind}(D) = l.c.m\{\mathrm{ind}(D \otimes F_\zeta\},$$

where $\zeta$ running over $S$ and irreducible components of $Y \setminus S$.

Suppose $\zeta = U$ for some irreducible component $U$ of $Y \setminus S$. Let $\eta$ be the generic point of $U$. Then $U \subset U_\eta$ and $R_{U_\eta} \subset R_U$. Since $F_{U_\eta} \subset F_U$, $\mathrm{ind}(D \otimes_F F_U)$ divides $p^{2d+2}$.

Suppose $\zeta = P \in S$. Let $A_P$ be the regular local ring at $P$. Then, by the choice of $\mathscr{X}$, the maximal ideal $m_P$ of $A_P$ is generated by $\pi$ and $\delta$ such that $A$ is ramified on $A_P$ at most possibly at $(\pi)$ and $(\delta)$. Since the residue field $\kappa(P)$ at $P$ is a finite extension of $\kappa$, we have $p\text{-rank}(\kappa(P)) = p\text{-rank}(\kappa) = p^n$. Thus, by (3.5), $\mathrm{ind}(D \otimes_F F_P)$ divides $p^{2n+2}$. Hence $\mathrm{ind}(D)$ divides $p^{2n+2}$. □

**Corollary 3.7.** *Let $F$ and $n$ be as in (3.6). Then $Br_p dim(F) \leq 2n + 2$.*

*Proof.* Let $F'$ be a finite extension of $F$ and $D$ a central simple algebra of period $p$. Since the transcendence degree of $F'$ over $K$ is 1, by (3.6), $\mathrm{ind}(D)$ divides $p^{2n+2}$. Corollary follows from (1.6). □

**Corollary 3.8.** *Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that $\kappa$ is finitely generated field of transcendence degree $n$ over a perfect field of characteristic $p > 0$. If $F$ is a function field of a curve over $K$, then the Brauer $p$-dimension of $F$ is at most $2n + 2$.*

*Proof.* Since $\kappa$ is a finitely generated field of transcendence degree $d$ over a perfect field, we have $[\kappa : \kappa^p] = p^n$ ([B], A.V.135, Corollary 3). Hence the result follows from (3.6). □

Let $K$ be a $p$-adic field and $F$ the function field of curve over $K$. Let $A$ be a central simple algebra over $F$ . If the period of $A$ is coprime to $p$, then a theorem of Saltman ([S1]) asserts that $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^2$. If the period of $A$ is a power of $p$, then it is proved in ([LPS]) that the $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^3$. We have the following

**Corollary 3.9.** *Let $F$ be the function field of a curve over a $p$-adic field $K$. Then for every central simple algebra over $F$, the index divides the square of the period.*

*Proof.* Let $A$ be a central simple algebra over $F$ of period a power of $p$. Since the residue field $\kappa$ of $K$ is a finite field, $[\kappa : \kappa^p] = 1$. Thus, by (3.6), $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^2$. □

## 4. $u$-INVARIANT

Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ the function field of a curve over $K$. In this section we compute the $u$-invariant of $F$ when $\kappa$ is a perfect field of characteristic 2 and $\mathrm{char}(K) = 0$.

For any field $L$ of characteristic not equal to 2, let $W(L)$ be the Witt ring of quadratic forms over $L$ and $I^n(L)$ be the $n^{\text{th}}$ power of the fundamental ideal $I(L)$ of $W(L)$.

Let $R$ be an integral domain with field of fractions $F$. A quadratic form $q$ over $R$ is *non-singular* if the associated quadric is smooth over $R$. We say that a quadratic form $q$ over $F$ is *defined over* $R$ if there exists a non-singular quadratic form $q'$ over $R$ such that $q' \otimes_R F \simeq q$.

In the rest of this section, until (4.7), $A$ denotes a complete regular local ring of dimension two with field of fractions $F$ and residue field $\kappa$. Suppose that $\mathrm{char}(F) = 0$, $\mathrm{char}(\kappa) = 2$ and $\kappa$ is a perfect field. Suppose that the maximal ideal $m = (\pi, \delta)$ and $2 = u_0\pi^i\delta^j$ for some $u_0 \in A^*$ and $i, j \geq 0$.

**Lemma 4.1.** *Let $A$, $F$, $\kappa$, $m = (\pi, \delta)$ as above. Let $\alpha \in H^2(F, \mu_2)$. If $\alpha$ is unramified on $A$ except possibly at $(\pi)$ and $(\delta)$. Then $\alpha = (uc, \pi) + (vc\pi^\epsilon, \delta)$ for some units $u, v \in A$, $c \in A$ not divisible by $\pi$, $\delta$ and $\epsilon = 0$ or 1.*

*Proof.* Since $\alpha$ is unramified except at $(\pi)$ and $(\delta)$ and $\kappa$ is perfect, by (3.5), $\alpha \otimes F(\sqrt{\pi}, \sqrt{\delta})$ is zero. In particular, by a theorem of Albert, $\alpha = (a, \pi) + (b, \delta)$ for some $a, b \in F^*$. Without loss of generality we assume that $a, b \in A$ and square free. Since $(-d, d) = 0$ for any $d \in F^*$, we assume that $\pi$ does not divide $a$ and $\delta$ does not divide $b$. Since $A$ is a regular local ring, it is a unique factorisation domain ([AB]). We write $a = ca_1\delta^{\epsilon_1}$ and $b = cb_1\pi^{\epsilon_2}$ with $c, a_1, b_1 \in A$ square free, $a_1$ and $b_1$ are coprime, $\pi$ and $\delta$ do not divide $ca_1b_1$ and $0 \leq \epsilon_1, \epsilon_2 \leq 1$.

Let $\theta$ be a prime in $A$ which divides $a_1$. Write $a_1 = \theta a_2$. Then $\theta$ does not divide $cb_1\pi\delta$. In particular, the characteristic of the residue field $\kappa(\theta)$ at $\theta$ is not equal to 2 and $\alpha$ is unramified at $\theta$. Since the residue of $\alpha$ at $\theta$ is the image $\overline{\pi}$ of $\pi$ in $\kappa(\theta)/\kappa(\theta)^{*2}$, $\overline{\pi}$ is a square in $\kappa(\theta)$. Let $L = F[\sqrt{\pi}]$ and $B = A[\sqrt{\pi}]$. Then $B$ is a regular local ring of dimension 2 (cf. (3.2)) and hence a unique factorisation domain ([AB]). Since $\overline{\pi}$ is a square in $\kappa(\theta)$ and $\mathrm{char}(\kappa(\theta)) \neq 2$, we have $\theta B = Q_1Q_2$ with $Q_1$ and $Q_2$ two distinct prime ideals of $B$. In particular $N_{L/F}(Q_1) = \theta A$. Since $B$ is a unique factorisation domain, $Q_1 = (\eta)$ for some $\eta \in B$ and hence there exists a unit $u \in A$ such that $N_{L/F}(\eta) = u\theta$. We have

$$\begin{aligned}(a, \pi) &= (au\theta, \pi)\\ &= (ca_1\delta^{\epsilon_1}u\theta, \pi)\\ &= (c\theta a_2\delta^{\epsilon_1}u\theta, \pi)\\ &= (ca_2\delta^{\epsilon_1}u, \pi).\end{aligned}$$

Thus by induction on the number of primes dividing $a_1$, we conclude that $(a, \pi) = (uc\delta^{\epsilon_1}, \pi)$ for some unit $u \in A$. Similarly $(b, \delta) = (vc\pi^{\epsilon_2}, \delta)$ for some unit $v \in A$. Thus we have $\alpha = (uc\delta^{\epsilon_1}, \pi) + (vc\pi^{\epsilon_2}, \delta)$. Suppose that $\epsilon_1 = 1$. Then

$$\begin{aligned}\alpha &= (uc\delta, \pi) + (vc\pi^{\epsilon_2}, \delta)\\ &= (uc, \pi) + (\delta, \pi) + (vc\pi^{\epsilon_2}, \delta)\\ &= (uc, \pi) + (vc\pi^{\epsilon_2+1}, \delta)\\ &= (uc, \pi) + (vc\pi^{\epsilon}, \delta),\end{aligned}$$

where $\epsilon = \epsilon_2 + 1 \pmod 2$. □

For any field $K$ and $i \geq 1$, let $H^i_2(F)$ be the Kato cohomology groups ([K2], §0). If $\mathrm{char}(K) \neq 2$, we have $H^i_2(F) = H^i(F, \mu_2)$. If $\mathrm{char}(K) = 2$, we have $H^2_2(K) = {}_2\mathrm{Br}(K)$ and $H^1_2(K) = H^1(K, \mathbf{Z}/2\mathbf{Z})$. For $a \in K^*$, let $[a) \in H^1_2(K)$ be the element defined by $K[X]/(X^2 + X + a)$. Note that $[a)$ is $K \times K$ or a separable extension of $K$ of degree 2. Let $b \in K$. Let $[a) \cdot (b)$ be the quaternion algebra over $K$ generated by $i$ and $j$ with $i^2 + i + a = 0$, $j^2 = b$ and $ji = -(1 + i)j$.

Let $A$, $F$, $\kappa$ be as above. Let $\theta \in A$ be a prime. Suppose $\theta$ does not divide $2 = u_0\pi^i\delta^j$. Then the characteristic of the residue field $\kappa(\theta)$ at $\theta$ is 0. Suppose $\theta$ divides 2. Then $(\theta) = (\pi)$ or $(\theta) = (\delta)$ and $A/(\theta)$ is a complete discrete valuation ring. Since the residue field $\kappa$ of $A$ is a perfect field, we have $[\kappa(\theta) : \kappa(\theta)^2] = 2$. By ([K2], §1), we have residue homomorphisms $\partial_\theta : H^3(F, \mu_2) \to H^2_2(\kappa(\theta)) \simeq {}_2\mathrm{Br}(\kappa(\theta))$ and $\partial : H^2_2(\kappa(\theta)) \to H^1_2(\kappa)$.

**Lemma 4.2.** *(cf. [Su], 1.1) Let $A$, $F$, $\kappa$, $m = (\pi, \delta)$ be as above. Then, for any unit $u \in A^*$, $\partial_\pi([u) \cdot (\delta) \cdot (\pi)) = [\overline{u}) \cdot (\overline{\delta})$ and $\partial([\overline{u}) \cdot (\overline{\delta})) = [\overline{\overline{u}})$, where for any $a \in A$, $\overline{a}$ denotes the image modulo $\pi$ and $\overline{\overline{a}}$ denotes the image modulo $m$.*

*Proof.* Suppose that $[\overline{\overline{u}})$ is trivial in $H^1_2(\kappa)$. Since $u$ is a unit in $A$ and $A$ is complete, $[u)$ is trivial in $H^1(F, \mu_2)$. In particular $[u) \cdot (\pi) \cdot (\delta)$ and $[\overline{u}) \cdot (\overline{\delta})$ are trivial.

Suppose that $[\overline{\overline{u}})$ is non-trivial. Let $\kappa' = \kappa[X]/(X^2 + x + \overline{\overline{u}})$. Then $\kappa'$ is a separable quadratic extension of $\kappa$ and $[\overline{\overline{u}})$ is the only non-trivial element of the kernel of the restriction homomorphism from $H^1_2(\kappa)$ to $H^1_2(\kappa')$.

Let $\kappa(\pi)' = \kappa(\pi)[X]/(X^2 + X + \overline{u})$. Then $\kappa(\pi)'$ is a complete discretely valued field with residue field $\kappa'$ and $\overline{\delta}$ as a parameter. Thus $\kappa(\pi)'/\kappa(\pi)$ is unramified and $\overline{\delta} \in \kappa(\pi)$ is not a norm from $\kappa(\pi)'$ and hence $[\overline{u}) \cdot (\overline{\delta})$ is non-trivial. Since $\partial$ is an isomorphism ([K2], Lemma 1.4(3)), $\partial([\overline{u}) \cdot (\overline{\delta}))$ is non-trivial in $H^1_2(\kappa)$. Since $[\overline{u}) \cdot (\overline{\delta})$ is trivial over $\kappa(\pi)'$, by the functoriality of $\partial$, the image of $\partial([\overline{u}) \cdot (\overline{\delta}))$ in $H^1_2(\kappa')$ is trivial. Since the only non-trivial element of the kernel of the restriction homomorphism from $H^1_2(\kappa)$ to $H^1_2(\kappa')$ is $[\overline{\overline{u}})$, $\partial([\overline{u}) \cdot (\overline{\delta})) = [\overline{\overline{u}})$.

Let $F_\pi$ be the completion of $F$ at $\pi$. Since $u$ and $\delta$ are units at $\pi$, $[u) \cdot (\delta)$ is a quaternion algebra defined over $A_{(\pi)}$. If $\pi$ is a reduced norm from $[u) \cdot (\delta)$ over $F_\pi$, $[\overline{u}) \cdot (\overline{\delta})$ is a split algebra over $\kappa(\pi)$, contradicting the non-triviality of $[\overline{u}) \cdot (\overline{\delta})$ in $H^2_2(\kappa(\pi))$. Hence $\pi$ is not a reduced norm form of the quaternion algebra $([u) \cdot (\delta)) \otimes_F F_\pi$ and $[u) \cdot (\delta) \cdot (\pi)$ is non-trivial in $H^3(F_\pi, \mu_2)$. Let $L = F[X]/(X^2 + X + u)$. Let $B$ be the integral closure of $A$ in $L$. Then $B$ is a complete regular local ring with maximal ideal $(\pi, \delta)$ and residue field $\kappa'$. Since the image of $[u) \cdot (\delta) \cdot (\pi)$ in $H^3(L, \mu_2)$ is zero, by the functoriality of the residue homomorphisms, the image of $\partial(\partial_\pi([u) \cdot (\delta) \cdot (\pi)))$ in $H^1_2(\kappa')$ is zero. Since $[u) \cdot (\delta) \cdot (\pi)$ is non-trivial in $H^3(F_\pi, \mu_2)$ and $\partial_\pi : H^3(F_\pi, \mu_2) \to H^2_2(\kappa(\pi))$ and $\partial : H^2_2(\kappa(\pi)) \to H^1_2(\kappa)$ are isomorphisms ( [K2], Lemma 1.4(3)), $\partial(\partial_\pi([u) \cdot (\delta) \cdot (\pi))$ is non-trivial and hence equal to $[\overline{\overline{u}})$. Since $\partial([\overline{u}) \cdot (\overline{\delta}) = [\overline{\overline{u}})$ and $\partial$ is an isomorphism, we have $\partial_\pi([u) \cdot (\delta) \cdot (\pi)) = [\overline{u}) \cdot (\overline{\delta})$. $\square$

The following is a result of Kato ([K2], 1.7)

**Proposition 4.3.** *Let $A$, $F$ and $\kappa$ be as above. Then*

$$H^3(F, \mu_2) \stackrel{\oplus \partial_\gamma}{\to} \oplus_{\gamma \in \mathrm{Spec}(A)^{(1)}} H^2_2(\kappa(\gamma)) \stackrel{\sum \partial_\gamma}{\to} H^1_2(\kappa)$$

*is a complex.*

We define $H^3_{nr}(F/A,\mu_2)$ to be the kernel of residue homomorphism

$$H^3(F,\mu_2) \stackrel{\oplus\partial_\gamma}{\to} \oplus_{\gamma\in Spec(A)^{(1)}} H^2_2(\kappa(\gamma))).$$

**Proposition 4.4.** *Let $A$, $F$ and $\kappa$ be as above. Then $H^3_{nr}(F/A,\mu_2)=0$.*

*Proof.* Since $\mathrm{cd}_2(F)\le 3$ ([GO]), $I^4(F)=0$ ([AEJ], Cor.2. p.653) and $e_3: I^3(F)\to H^3(F,\mu_2)$ is an isomorphism ([AEJ], Thm. 2. p.653). Let $\zeta\in H^3_{nr}(F/A,\mu_2)$. Suppose that $\zeta\neq 0$. Let $q$ be an anisotropic quadratic form over $F$ such that $q\in I^3(F)$ and $e_3(q)=\zeta$. Let $\theta\in A$ be a prime and $F_\theta$ be the completion of $F$ at $\theta$. Since $A/(\theta)$ is a complete local ring of dimension one with residue field perfect of characteristic 2, $H^3_2(\kappa(\theta))=0$ ([GO]). Suppose that $\mathrm{char}(\kappa(\theta))\neq 2$. Since $H^3(\kappa(\theta),\mu_2)=H^3_2(\kappa(\theta))=0$, $\partial_\theta: H^3(F_\theta,\mu_2)\to H^2_2(\kappa(\theta))$ is an isomorphism. Suppose that $\mathrm{char}(\kappa(\theta))=2$. Since the 2-rank of $\kappa(\theta)$ is 1, by ([K2], Lemma 1.4(3)), $\partial: H^3(F_\theta,\mu_2)\to H^2_2(\kappa(\theta))$ is an isomorphism. Since $\zeta\in H^3_{nr}(F/A,\mu_2)$, the image of $\zeta$ in $H^3(F_\theta,\mu_2)$ is zero. In particular, $q$ is hyperbolic over $F_\theta$. Thus $q$ comes from a non-singular quadratic form over the localisation $A_{(\theta)}$ of $A$ at the prime ideal $(\theta)$ (cf. [O], Thm,8). Since $A$ is a two dimensional regular ring, there exists a non-singular quadratic forn $q'$ over $A$ such that $q'\otimes_A F\simeq q$ ([CTS], Cor.2.5, cf. [APS], 4.2 ).

Since $q\in I^3(F)$ and $q$ is anisotropic, the rank of $q$, and hence the rank of $q'$, is at least 8. Since $\kappa$ is a perfect field, $q'\otimes_A\kappa$ is isotropic ([MMW], Corollary 1). Since $A$ is a complete regular local ring and $q'$ is a non-singular quadratic form over $A$ with $q'\otimes_A\kappa$ is isotropic, $q'$ is isotropic ([Gr], Theorem 18.5.17). Thus $q$ is isotropic, leading to a contradiction. □

**Lemma 4.5.** *Let $A$, $F$, $\kappa$ and $m=(\pi,\delta)$ be as be above. Let $\zeta\in H^3(F,\mu_2)$. Suppose that $\zeta$ is ramified at most along $(\pi)$ and $(\delta)$. Then $\zeta=[u)\cdot(\pi)\cdot(\delta)$ for some unit $u$ in $A$.*

*Proof.* Let $\alpha=\partial_\pi(\zeta)\in H^2_2(\kappa(\pi))$ and $\beta=\partial_\delta(\zeta)\in H^2_2(\kappa(\delta))$. Then, by (4.3), $\partial(\alpha)=\partial(\beta)\in H^1_2(\kappa)$. Let $a\in\kappa^*$ be such that $[a)=\partial(\alpha)=\partial(\beta)\in H^1_2(\kappa)$. Let $u\in A^*$ be a lift of $a$. Since $\partial(\alpha)=[a)=\partial([u)\cdot(\overline{\delta}))$ (cf. 4.2) and $\partial$ is an isomorphism, we have $\alpha=[\overline{u})\cdot(\overline{\delta})$. and $\beta=[\overline{u})\cdot(\overline{\pi})$. Let $\zeta'=[u)\cdot(\pi)\cdot(\delta)\in H^3(F,\mu_2)$. Then $\zeta'$ is unramified on $A$ except at $\pi$ and $\delta$. By (4.2), $\partial_\pi(\zeta')=\partial_\pi(\zeta)$ and $\partial_\delta(\zeta')=\partial_\delta(\zeta)$. Since $\zeta$ is unramified on $A$ except at $\pi$ and $\delta$, $\zeta-\zeta'\in H^3_{nr}(F,\mu_2)$. Since $H^3_{nr}(F,\mu_2)=0$ by (4.4), we have $\zeta=\zeta'=[u)\cdot(\pi)\cdot(\delta)$. □

**Proposition 4.6.** *Let $A$, $F$, $\kappa$ and $m=(\pi,\delta)$ be as above. Let $q=<a_1,\cdots,a_9>$ be a quadratic form over $F$ of rank 9 with only prime factors of $a_1a_2\cdots a_9$ are at most $\pi$ and $\delta$. Then $q$ is isotropic.*

*Proof.* Let $c(q)\in H^2(F,\mu_2)$ be the Clifford invariant of $q$. Since the prime factors of $a_1a_2\cdots a_9$ are at most $\pi$ and $\delta$, $c(q)$ is unramified on $A$ except possibly at $(\pi)$ and $(\delta)$. By (4.1), we have $c(q)=(uc,\pi)+(vc\pi^\epsilon,\delta)$ for some units $u,v\in A$, $c\in A$ not divisible by $\pi$ and $\delta$, and $\epsilon=0$ or 1. Let $q_1=<1,uc\pi,-\pi,-uc\delta,uv\pi^\epsilon\delta>$. Since $-ucq_1$ is a rank five subform of the Albert form associated to $c(q)=(uc,\pi)+(vc\pi^\epsilon,\delta)$, $c(q_1)=c(q)$ (cf. [L], p. 118). Since $q$ is isotropic if and only if $\lambda q$ is isotropic for any $\lambda\in F^*$, by scaling $q$ we assue that $d(q)=d(q_1)$. We note that we only need to scale by $\lambda\in A$ with prime factors at most $\pi$ and $\delta$. Hence, after scaling, we still have $q=<a_1,\cdots,a_9>$ with only prime factors of $a_1a_2\cdots a_9$ at most $\pi$ and $\delta$. Since the dimension of $q$ is odd, we have $c(\lambda q)=c(q)$. Thus, after scaling, we have $c(q)=c(q_1)$

and $d(q) = d(q_1)$. Since the rank of $q \perp -q_1$ is 14, it follows that $q - q_1 \in I^3(F)$ ([M]).

Let $\zeta = e_3(q - q_1) \in H^3(F, \mu_2)$. Let $\theta \in A$ be a prime. Suppose that $\theta$ does not divide $\pi\delta$. Then $\mathrm{char}(\kappa(\theta))$ is 0. Hence we have the second residue homomorphism $\partial^2_\theta : W(F) \to W(\kappa(\theta))$ with $\partial^2_\theta(I^3(F)) \subset I^2(\kappa(\theta))$. Since $q =< a_1, \cdots, a_9 >$ with $a_1 a_2 \cdots a_9$ having only $\pi$ and $\delta$ as possible prime factors and $\theta$ does not divide $\pi\delta$, $\partial^2_\theta(q) = 0$. Since $q_1 =< 1, uc\pi, -\pi, -uc\delta, uv\pi^\epsilon\delta >$, the rank of $\partial^2_\theta(q_1)$ is at most two. Since $\partial^2_\theta(q - q_1) \in I^2(\kappa(\theta))$ and is of rank at most 2, $\partial^2_\theta(q - q_1) = 0$. In particular $q - q_1$ is unramified at $\theta$ and hence $\zeta = e_3(q - q_1)$ is unramified at $\theta$. Thus, by (4.5), we have $\zeta = [w) \cdot (\pi) \cdot (\delta)$ for some unit $w \in A$. Since $[w) \cdot (w')$ is unramified on $A$ for any unit $w' \in A$, we have $[w) \cdot (w') = 0$. In particular, we have $\zeta = [w) \cdot (\pi) \cdot (w'\delta)$ for any unit $w'$ in $A$.

Suppose that $\epsilon = 0$. Since $uv$ is a unit, we have $\zeta = [w) \cdot (\pi) \cdot (-uv\pi^\epsilon\delta)$. Suppose that $\epsilon = 1$. Since $\zeta = [w) \cdot (\pi) \cdot (-uv\delta)$ and $(\pi) \cdot (-\pi) = 1$, we have $\zeta = [w) \cdot (\pi) \cdot (-uv\pi\delta)$. Thus in either case, we have $\zeta = e_3(q - q_1) = [w) \cdot (\pi) \cdot (-uv\pi^\epsilon\delta)$.

Since $\mathrm{char}(F) = 0$, we have $[w) = (w')$ for some unit $w' \in A$. Let $q_2 =< 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta >\in I^3(F)$. Then $e_3(-q_2) = e_3(q_2) = (w') \cdot (\pi) \cdot (-uv\pi^\epsilon\delta) = [w) \cdot (\pi) \cdot (-uv\pi^\epsilon\delta) = e_3(q - q_1)$. Since $H^4(F, \mu_2) = 0$ ([AEJ], Cor.2. p.653), we have $I^4(F, \mu_2) = 0$ and $e_3$ is an isomorphism ([AEJ], Thm. 2. p.653). Hence

$$q - q_1 = - < 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta > .$$

In particular,

$$q = q_1 - < 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta > .$$

Since $< 1, -\pi, uv\pi^\epsilon\delta >$ is a subform of both $q_1$ and $< 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta >$, the anisotropic rank of $q_1 - < 1, -w' >< 1, -\pi >< 1, uv\pi^\epsilon\delta >$ is at most 7. Since the rank of $q$ is 9, $q$ is isotropic. □

**Theorem 4.7.** *Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ a function field of a curve over $K$. If $\mathrm{char}(K) = 0$ and $\kappa$ is a perfect field of characteristic 2, then $u(F) \leq 8$.*

*Proof.* Let $q =< a_1, \cdots, a_9 >$ be a quadratic form over $F$ rank 9. Let $\mathscr{X}$ be a regular proper scheme over the valuation ring of $K$ with function field $F$ and the support of the principle divisor $(2a_1 \cdots a_9)$ on $\mathscr{X}$ is a union of regular curves with normal crossings. Let $C_1, \cdots, C_r$ be the irreducible components of the special fibre of $\mathscr{X}$ and let $\nu_1, \cdots, \nu_r$ be the corresponding discrete valuations on $F$. Let $F_{\nu_i}$ be the completion $F$ at $\nu_i$ and the residue field $\kappa(\nu_i)$. Then $\mathrm{char}(\kappa(\nu_i)) = 2$ and 2-rank$(\kappa(\nu_i)) = 1$. Hence, by ([MMW], Corollar 1), $u(\kappa(\nu_i)) \leq 4$ and by ([Sp]), $u(F_{\nu_i}) \leq 8$. In particular $q$ is isotropic over $F_{\nu_i}$. By ([HHK2], 5.8), there exists an affine open subset $U_i$ of $C_i$ such that $U_i$ does not intersect $C_j$ for $j \neq i$ and $q$ is isotropic over $F_{U_i}$.

Let $\mathscr{P}$ be a finite set of closed points of $\mathscr{X}$ containing all those points which are not in $U_i$ for any $i$. Let $P \in \mathscr{P}$. Then $\hat{A}_P$ is a complete two dimensional local ring with residue field perfect of characteristic 2. By the choice of $\mathscr{X}$ and (4.6), $q$ is isotropic over $F_P$. By ([HHK1], 4.2), $q$ is isotropic over $F$ and $u(F) \leq 8$. □

**Corollary 4.8.** *([Le]) Let $K$ be a 2-adic field and $F$ the function field of a curve over $K$. Then $u(K) = 8$.*

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