WEDDERBURN'S THEOREM FOR REGULAR LOCAL RINGS

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In [Pa] Ivan Panin proved the following theorem.

Theorem 1. *Let R be a regular local ring, K its field of fractions and (V,Φ) a quadratic space over R. Suppose R contains a field of characteristic zero. If $(V,\Phi)\otimes_R K$ is isotropic over K, then (V,Φ) is isotropic over R.*

The proof rests on a series of lemmas which can be summarized in the following one.

Lemma 2. *Let k be a field of characteristic zero, u a closed point of a smooth k-variety and $R = \mathcal{O}_{U,u}$ the local ring of U at u. Let further $\mathcal{X}$ be a projective R-scheme, smooth over R. Let K be the field of fractions of R and suppose that $\mathcal{X}$ has a K-point. Then, for every prime number p there exist an integral R-etale algebra S of degree prime to p and an S-point of $\mathcal{X}$.*

Proof. See [Pa], Lemma 3, Lemma 4 and proof of Theorem 1..

I want to show that the argument used for proving Theorem 1 also yields the following extension of Wedderburn's theorem to a large class of regular local rings.

Theorem 3. *Let R be a regular local ring, K its field of fractions and A an Azumaya algebra over R. Suppose R contains a field k of characteristic zero. If $A\otimes_R K$ is isomorphic to $M_n(D)$ where D is a central division algebra over K, then A is isomorphic to $M_n(\Delta)$ where Δ is a maximal (unramified) R-order of D. In other words, every class of the Brauer group of R is represented by a maximal order in a division K-algebra.*

Proof. Let d^2 be the dimension of D over K. It suffices to show that A contains a right ideal I such that A/I is free of rank $(n^2-n)d^2$ over R. In fact, since any A-module is projective over A if and only if it is projective over R, the projection $A \to A/I$ splits, I is a direct factor of the right A-module A, and $\Delta := End_A(I)$ is an Azumaya algebra equivalent to A. Clearly $\Delta\otimes_R K = D$ and by Morita theory

$$A = End_\Delta(Hom_A(I,A)) = M_n(\Delta).$$

In order to find a right ideal I of the right rank we consider the set $\mathcal{I}$ of all such ideals or, more precisely, we consider the functor $\mathcal{I}$ that associates to any R-algebra S the set of such ideals in $A\otimes_R S$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Lemma 4. *$\mathcal{I}$ is a smooth closed subscheme of the Grassmannian scheme $\mathcal{G}$ consisting of all the free R-submodules of A which are direct factors of A and have rank nd^2.*

Proof. We denote by m the maximal ideal of R. To show that $\mathcal{I}$ is closed we first remark that A, as an R-module, is generated by the set A^* of all invertible elements of A. In fact for any $a \in A$ and any $\lambda \in k$ the reduced norm of $\lambda + a$ is a polynomial

$$P(\lambda) = \lambda^{nd} + c_1\lambda^{n-1} + \cdots + c_{nd}$$

whose coefficients are in R and only depend on a. Choosing λ in k^* such that $P(\lambda)$ is not 0 in R/m insures that $\lambda + a$ is invertible and allows to write $a = (\lambda + a) - \lambda$. So an R- submodule M of A is an ideal if $aM = M$ for every unit a. In other words, we must show that the set of fixed points of $\mathcal{G}$ under the action of A^* is closed. This is well-known.

The second point is the smoothness of $\mathcal{I}$. This means that for any R-algebra S and any ideal I of S, any S/I-point of $\mathcal{X}$ can be lifted to an S/I^2-point. But points correspond to right ideals generated by an idempotent and it is well-known that idempotents can be lifted.

Note that it suffices to treat the case when A is of prime power order in the Brauer group $Br(R)$ of R. In fact the class of A is a product of classes $[A_i]$ of order $p_i^{e_i}$ for some distinct primes $p_1, \ldots, p_r$. If each of them is represented by an order Δ_i in $D_i = \Delta_i \otimes_R K$ then A is Brauer equivalent to $\Delta_1 \otimes_R \cdots \otimes_R \Delta_r$ which is an order in $D = D_1 \otimes_K \cdots \otimes_K D_r$ and we know that D is a division algebra.

We now assume that R is of geometric type, in other words R is the local ring of a closed point u of a smooth k-variety. The general case then follows from this special case by a standard application of Popescu's theorem, saying that a regular ring containing a field is an inductive limit of smooth algebras.

Suppose now that A is of prime power exponent in $Br(R)$ and that the degree of D is p^e for some prime number p. Since $A \otimes_R K = M_n(D)$ the scheme $\mathcal{I}$ has a K-point and according to Lemma 2 it also has an S-point, where S is an integral etale algebra whose degree d is prime to p. This means that $A \otimes_R S = M_n(B)$ for some maximal order B in $D \otimes_K L$, L being the field of fractions of S. Note that $D \otimes_K L$ remains a division algebra because the degree of L over K is prime to p. So the Brauer class $[A]_S$ of $A \otimes_R S$ in $Br(S)$ is represented by a degree p^e algebra. In [Ga] (see also [AdJ], Proposition 2.6.1) Gabber proved that any class $\alpha \in Br(R)$ which is represented by a degree m algebra when extended to a finite faithfully flat R-algebra S of degree d can be represented by an R-algebra of degree dn. We can thus find an Azumaya algebra A_1 of degree dp^e in the same class as A. On the other hand, we dispose of Ferrand's [Fe] norm functor $N_{S/R}$ from S-algebras to R-algebras. Applying it to B we find that $N_{S/R}(B) = A_2$ is an Azumaya R-algebra equivalent to $A^{\otimes d}$ ([Fe], section 7.3), of degree p^{ed} ([Fe], Thorme 4.3.4). If the integer c is an inverse of d modulo p^e, the algebra $A_3 = A_2^{\otimes c}$ is Brauer equivalent to A and its degree is p^{cde}. Recall now that DeMeyer [DM] proved that every class in $Br(R)$ is represented by a unique "minimal" Azumaya algebra Δ with the property that every algebra in the same class is isomorphic to some matrix algebra over Δ. What is the degree m of this Δ in our case? We must have $A_1 \simeq M_{s_1}(\Delta)$ and $A_3 \simeq M_{s_3}(\Delta)$, hence $s_1 m = dp^e$ and $s_3 m = p^{cde}$. Since d is prime to p, this implies that m divides p^e and extending the scalars to K shows that $m = p^e$. The theorem is proved.

Easy and well-known examples (the simplest one being the usual quaternion algebra extended to $\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2)$) show that we cannot replace regularity by, say, normality.

In a very interesting, recent article, Benjamin Antieau and Ben Williams [AB] show that Theorem 3 fails for nonlocal regular rings.

References

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