CODIMENSION 2 CYCLES ON PRODUCTS OF PROJECTIVE HOMOGENEOUS SURFACES

SANGHOON BAEK

Abstract. In the present paper, we provide general bounds for the torsion in the codimension 2 Chow groups of the products of projective homogeneous surfaces. In particular, we determine the torsion for the product of four Pfister quadric surfaces and the maximal torsion for the product of three Severi-Brauer surfaces. We also find an upper bound for the torsion of the product of three quadric surfaces with the same discriminant.

Contents

1. Introduction

Let X be a projective homogenous variety under the action of a semisimple group G over an algebraically closed field F. The Chow group $\mathrm{CH}(X)$ of algebraic cycles modulo the rational equivalence relation is well-understood as well as its ring structure. Namely, the Chow group of X is a free abelian group with the basis of Schubert cycles. For an arbitrary base field F, this is no longer true: the Chow group $\mathrm{CH}(X)$ can have torsion. Indeed, by a transfer argument the problem of determining the Chow group of X over an arbitrary field F reduces to computing its torsion subgroup.

For codimension $d \leq 1$, the Chow group $\mathrm{CH}^d(X)$ is torsion-free. A nontrivial torsion first appears in codimension 2 cycles on X and the exact structure of the torsion subgroup is known in many cases. For a projective quadric X, the torsion subgroup of $\mathrm{CH}^2(X)$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$ [6]. For a Severi-Brauer variety X, it is shown that the torsion subgroup of $\mathrm{CH}^2(X)$ is either 0 or a cyclic group if the corresponding simple algebra satisfies certain conditions [8]. For a simple simply connected group G splits over $F(X)$, it is known that the torsion subgroup of $\mathrm{CH}^2(X)$ is a cyclic group generated by the Rost invariant [2]. However, the only partial results are known for their products. In this case, the structure of the torsion subgroup is more complicated.

As a first step in determining torsion in codimension 2 cycles on the product of flag varieties, we consider in this paper the product of two dimensional flag varieties of the same type. For an integer $n \geq 1$, this can be divided into two classes: the set of all products of n Severi-Brauer surfaces, denoted by SB_n and the set of all products of n quadric surfaces, denoted by Q_n. The latter has a special subclass PQ_n consisting of all product of n Pfister quadric surfaces. Here, we view the product of two conics as the product of two Pfister quadric surfaces since they have the same torsion subgroup in codimension 2 cycles by [3, Corollary 2.5]. To measure the size of the torsion subgroup of codimension 2 cycles, we introduce the following notation: $\mathcal{M}(\mathcal{A}) = \max_{X \in \mathcal{A}} |\,\mathrm{CH}^2(X)_{\mathsf{tors}}|$, where $\mathcal{A}$ is any of SB_n, Q_n, and PQ_n. Hence, the torsion subgroup of any element of $\mathcal{A}$ is an elementary abelian group whose order is a divisor of $\mathcal{M}(\mathcal{A})$. We denote by C_n the set of all products of n conics, thus we have $\mathcal{M}(\mathsf{PQ}_n) = \mathcal{M}(\mathsf{C}_n)$.

It is well-known that $\mathcal{M}(\mathsf{SB}_1) = \mathcal{M}(\mathsf{Q}_1) = \mathcal{M}(\mathsf{C}_2) = 1$. In [10] Peyre proved that $\mathcal{M}(\mathsf{C}_3) = 2$, thus $\mathcal{M}(\mathsf{PQ}_3) = 2$. In [4], [3] and [5] Izhboldin and Karpenko proved that $\mathcal{M}(\mathsf{SB}_2) = 3$ and $\mathcal{M}(\mathsf{Q}_2) = 2$. As it is showed in [11, Theorem 2.1] and [10, Theorem 4.1], the torsion subgroup of codimension 2 cycles is closely related to a relative Galois cohomology group in degree 3 and the above results were used to describe the cohomology groups. Moreover, Izhboldin and Karpenko's result was the key part of their results on isotropy of quadratic forms.

The main goal of this paper is to extend their results to arbitrary n. First, we determine the maximal torsion $\mathcal{M}(\mathsf{C}_n)$ $(= \mathcal{M}(\mathsf{PQ}_n))$ in Corollary 4.4, which gives a general lower bound of $\mathcal{M}(\mathsf{Q}_n)$. In particular, we determine the torsion in the gamma filtration of the product of four conics in Theorem 4.6 as well as its torsion in Chow group. Secondly, we find a general lower bound of $\mathcal{M}(\mathsf{SB}_n)$ in Proposition 5.2. Especially, we show that the bound is sharp for the product of three Severi-Brauer surfaces in Theorem 5.4. The results $\mathcal{M}(\mathsf{PQ}_4) = 2^5$ and $\mathcal{M}(\mathsf{SB}_3) = 3^8$ give the first examples such that the torsion subgroup $\mathrm{CH}^2(X)_{\mathsf{tors}}$ is not cyclic in their classes. In the last part, we provide an elementary proof of Izhboldin and Karpenko's result for the product of two quadric surfaces and find an upper bound for the torsion of the product of three quadric surfaces with the same discriminant in Theorem 6.1, Proposition 6.3, and Theorem 6.5.

As Karpenko showed in [7] and [8], the topological filtration and the gamma filtration on the Grothendieck ring can be used to find the torsion in Chow groups of codimension 2 of projective homogeneous varieties. Moreover, the torsion in the filtrations can be computed by studying the divisibility of certain polynomials produced by the characteristic classes on the Grothendieck ring. We use this general approach to find the torsion in codimension 2 cycles on the product of projective homogeneous surfaces together with some additional combinatorial arguments.

This paper is organized as follows. In Sections 2 and 3, we recall basics of the topological filtration and the gamma filtration on the Grothendieck ring as well as their torsion subgroups of the product of Severi-Brauer varieties. In Section 4, we determine the bound of the torsion in codimension 2 cycles on the product of n conics or the product of n Pfister quadric surfaces. In particular, we determine the torsion of the product of four conics or the product of four Pfister quadric surfaces in terms of the indexes of the corresponding algebras. In the last part of this section, we also present its application to a Galois cohomology group in degree 3. In Section 5, we find a general lower bound of the torsion in codimension 2 cycles on the product of n Severi-Brauer surfaces. Using this bound, we determine the maximal torsion subgroup of the Chow group of codimension 2 cycles on the product of three Severi-Brauer surfaces. In the last section, we recover a result of Izhboldin and Karpenko and extend it to the product of three quadric surfaces with the same discriminant.

In the present paper, A_{tors} denotes the torsion subgroup of an abelian group A and $I_n = \{1, \dots, n\}$ for any integer $n \geq 1$. We denote by $\min\{\cdot\}$ and $\max\{\cdot\}$ the minimum integer of a set and the maximum of a set, respectively.

2. Two filtrations on the Grothendieck ring

In this section, we briefly recall definitions and properties of the topological filtration and the gamma filtration on the Grothendieck ring K of a smooth projective variety (see [1] and [8] for details). We also provide a useful fact concerning the torsion part of these two filtrations on a smooth projective homogeneous variety.

Let X be a smooth projective variety and let $K(X)$ be the Grothendieck ring of X. The topological filtration

$$K(X) = T^0(X) \supset T^1(X) \supset \dots$$

is given by the ideal $T^d(X)$ generated by the class $[\mathcal{O}_Y]$ of the structure sheaf of a closed subvariety Y of codimension at least d. We write $T^{d/d+1}(X)$ for the quotient $T^d(X)/T^{d+1}(X)$.

Let $\Gamma^0(X) = K(X)$ and let $\Gamma^1(X)$ be the kernel of the rank map $K(X) \to \mathbb{Z}$. The gamma filtration

$$K(X) = \Gamma^0(X) \supset \Gamma^1(X) \supset \dots$$

is given by the ideals $\Gamma^d(X)$ generated by the product $\gamma_{d_1}(x_1) \cdots \gamma_{d_i}(x_i)$ with $x_j \in \Gamma^1(X)$ and $d_1 + \cdots + d_i \geq d$, where γ_{d_j} is the gamma operation on $K(X)$. For instance, we have $\gamma_0(x) = 1$ and $\gamma_1 = \mathrm{id}$, where $x \in K(X)$. Indeed, the gamma operation defines the Chern class $c_j(x) := \gamma_j(x - \mathrm{rank}(x))$ with values in K.

For any $d \geq 0$, the gamma filtration $\Gamma^d(X)$ is contained in the topological filtration $T^d(X)$. For small degree $d = 1, 2$, two filtrations coincide. Moreover, the second quotient of the topological filtration can be identified with the codimension 2 cycles so that we have:

$$\Gamma^{2/3}(X) \twoheadrightarrow T^{2/3}(X) = \mathrm{CH}^2(X). \tag{1}$$

Now we assume that X is a smooth projective homogeneous variety over a field F. Let E be a splitting field of X. Then, by [11, Proposition 3.4] we have

$$T^d(X) = \Gamma^d(X) = \Gamma^d(X_E) \cap K(X) \tag{2}$$

for $d = 1, 2$.

Let $\mathcal{F}$ be either the gamma-filtration Γ or the topological filtration T on $K(X)$. Applying the Snake lemma to the commutative diagram involving the exact sequences $0 \to$

$\mathcal{F}^{d+1}(X) \to \mathcal{F}^{d}(X) \to \mathcal{F}^{d/d+1}(X) \to 0$ and the one over a splitting field E, we have the following useful formula [7, Proposition 2]:

$$(3) \qquad |\oplus \mathcal{F}^{d/d+1}(X)_{tors}| \cdot |K(X_E)/K(X)| = \prod_{d=1}^{\dim(X)} |\mathcal{F}^{d/d+1}(X_E)/\operatorname{Im}(\mathrm{res}^{d/d+1})|,$$

where $\mathrm{res}^{d/d+1} : \mathcal{F}^{d/d+1}(X) \to \mathcal{F}^{d/d+1}(X_E)$ is the restriction map.

3. Grothendieck group of the product of Severi-Brauer varieties

We now recall the Grothendieck group of a product of Severi-Brauer varieties. In addition, we state some basic facts about codimension 2 cycles of a product of Severi-Brauer varieties.

Let A_i be a central simple F-algebra of degree d_i for $1 \leq i \leq n$. Consider the restriction map

$$(4) \qquad K(\prod_{i=1}^{n} \mathrm{SB}(A_i)) \to K(\prod_{i=1}^{n} \mathbb{P}_E^{d_i-1}),$$

where the corresponding Severi-Brauer variety $\mathrm{SB}(A_i)$ over a splitting field E is identified with the projective space $\mathbb{P}_E^{d_i-1}$. The latter ring in (4) is isomorphic to the quotient ring $\mathbb{Z}[x_1, \cdots, x_n]/((x_1-1)^{d_1}, \cdots, (x_n-1)^{d_n})$, where x_i is the pullback of the class of the tautological line bundle on $\mathbb{P}_E^{d_i-1}$. Then by [12, §8 Theorem 4.1] (see also [10, Proposition 3.1]) the image of the map (4) coincides with the sublattice with basis

$$(5) \qquad \{\operatorname{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_n^{\otimes i_n}) \cdot x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_j \leq d_j - 1, 1 \leq j \leq n\}.$$

Let X be the product of Severi-Brauer varieties $\mathrm{SB}(A_i)$ as above. For codimension 2 cycles, one can simplify the computation of torsion subgroup by using [4, Proposition 4.7]: if $\langle A_1', \ldots, A_{n'}' \rangle = \langle A_1, \ldots, A_n \rangle$ in the Brauer group $\mathrm{Br}(F)$, then

$$(6) \qquad \mathrm{CH}^2(X)_{tors} \simeq \mathrm{CH}^2(X')_{tors},$$

where $X' = \prod_{i=1}^{n'} \mathrm{SB}(A_i')$.

Now we restrict our attention to p-primary algebras. Let $A_1, \ldots, A_n$ be central simple algebras of p-power degree with given indices $\operatorname{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_n^{\otimes i_n})$ for all nonnegative integers $i_1, \ldots, i_n$. Let X be the product of $\mathrm{SB}(A_1), \ldots, \mathrm{SB}(A_n)$. Then, by [8, Corollary 2.15] the map (1) induces a surjection on torsion subgroups $\Gamma^{2/3}(X)_{tors} \twoheadrightarrow \mathrm{CH}^2(X)_{tors}$. Moreover, by [4, Theorem 4.5] and [9, Proposition III.1] there is a product $\bar{X}$ of Severi-Brauer varieties $\mathrm{SB}(\bar{A}_i)$ of algebras $\bar{A}_i$ with $\operatorname{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_n^{\otimes i_n}) = \operatorname{ind}(\bar{A}_1^{\otimes i_1} \otimes \cdots \otimes \bar{A}_n^{\otimes i_n})$ such that the above surjective map becomes bijective map

$$(7) \qquad \Gamma^{2/3}(\bar{X})_{tors} \xrightarrow{\sim} \mathrm{CH}^2(\bar{X})_{tors}.$$

This variety $\bar{X}$ will be called a generic variety corresponding to X.

4. Product of Conics

In the present section, we provide a general bound for the torsion in codimension 2 cycles of the product of n conics in Corollary 4.4. In case of the product of four conics, we determine its torsion in terms of the indexes of the corresponding algebras in Theorem 4.6. In the last subsection, we present its application to a Galois cohomology group.

The following example was proved in a different way [10, Corollary 3.9].

Example 4.1. Consider the product of two conics $X = \mathrm{SB}(Q_1) \times \mathrm{SB}(Q_2)$, where Q_1 and Q_2 are quaternions. Let $a = \mathrm{ind}(Q_1)$, $b = \mathrm{ind}(Q_2)$, and $\mathrm{ind}(Q_1 \otimes Q_2) = c$. If $ab = 1$, then the Chow group of X is torsion free, thus we may assume that $ab \geq 1$. By (5), we have a basis $\{1, ax_1, bx_2, cx_1x_2\}$ of $K(X)$, where x_1 and x_2 are the pullbacks of the classes of the line bundles on the projective line over a splitting field E. Therefore, we have $|K(X_E)/K(X)| = abc$. By substitution $y_n = x_n - 1$ for $n = 1, 2$, we have a different basis $\{1, ay_1, by_2, c(y_1y_2 + y_1 + y_2)\}$ of $K(X)$. Let $\alpha_n = |T^{n/n+1}(X_E)/\,\mathrm{Im}(\mathrm{res}^{n/n+1})|$. If $c \geq 2$, then by (2) we have ay_1, $by_2 \in T^1(X)$ and $cy_1y_2 \in T^2(X)$. Therefore, $|\oplus T^{n/n+1}(X)_{\mathit{tors}}|$ is trivial. Otherwise, by the diagonal embedding $\mathrm{SB}(Q_1) \hookrightarrow X$ we have $y_1 + y_2 \in \mathrm{Im}(\mathrm{res}^{1/2})$, thus $\alpha_1 \leq \min\{a, b\}$. As $2y_1y_2 \in T^2(X)$, we get $\alpha_2 \leq 2$. Hence, $|\oplus T^{n/n+1}(X)_{\mathit{tors}}|$ is trivial as well. In any case, $\mathrm{CH}^2(X)$ is trivial. In particular, $\mathcal{M}(\mathsf{C}_2) = \mathcal{M}(\mathsf{PQ}_2) = 0$.

We determine the Chow group of codimension 2 of the product of three conics.

Proposition 4.2. (*cf.* [10, Proposition 6.1, Proposition 6.3]) *Let Q_1, Q_2, Q_3 be quaternions and $X = \mathrm{SB}(Q_1) \times \mathrm{SB}(Q_2) \times \mathrm{SB}(Q_3)$. Then, we have $\mathcal{M}(\mathsf{C}_3) = \mathcal{M}(\mathsf{PQ}_3) = 2$. Moreover, $\mathrm{CH}^2(X)_{\mathit{tors}}$ is trivial except the cases where the division algebras Q_i satisfying $\mathrm{ind}(Q_i \otimes Q_j) = \mathrm{ind}(Q_1 \otimes Q_2 \otimes Q_3) = 2$ or 4 for all $1 \leq i \neq j \leq 3$ and $\mathrm{CH}^2(\bar{X})_{\mathit{tors}} = \mathbb{Z}/2\mathbb{Z}$ in these cases, where $\bar{X}$ is the corresponding generic variety.*

Proof. Let $d = \mathrm{ind}(Q_1 \otimes Q_2 \otimes Q_3)$ and $e_{ij} = \mathrm{ind}(Q_i \otimes Q_j)$ for $1 \leq i \neq j \leq 3$. If one of $\mathrm{ind}(Q_i)$, e_{ij}, and d is 1, then by (6) and Example 4.1 $\mathrm{CH}^2(X)_{\mathit{tors}}$ is trivial. Therefore, by (5) we have the following basis

$$\{1, 2y_i, e_{ij}y_iy_j, d(y_1y_2y_3 + y_1y_2 + y_1y_3 + y_2y_3)\}$$

of $K(X)$, where $y_i = x_i - 1$, x_i is the pullback of the tautological line bundle on the projective line over a splitting field E, and $e_{ij}, d \geq 2$.

If either $d = 8$ or $d = 4$ and $e_{ij} = 2$ for some i, j, then the set $K(X) \cap T^3(X_E)$ has only one element $dy_1y_2y_3$ of the basis. As $e_{ij}y_iy_j \in T^2(X)$ and $2y_k \in T^1(X)$ for $k \neq i, j$, we have $dy_1y_2y_3 = 2e_{ij}y_1y_2y_3 = 2y_k(e_{ij}y_iy_j) \in T^3(X)$, which implies that $T^{2/3}(X)_{\mathit{tors}} = \mathrm{CH}^2(X)_{\mathit{tors}}$ is trivial.

Now we assume that $d = e_{ij} = 2$ or 4 for all $1 \leq i \neq j \leq 3$. Then, we have $|K(X_E)/K(X)| = 2^3e_{ij}^3d$. Let $\alpha_n = |\Gamma^{n/n+1}(X_E)/\,\mathrm{Im}(\mathrm{res}^{n/n+1})|$. Then, we obtain $\alpha_1 \leq 2^3$, $\alpha_2 \leq e_{ij}^3$, and $\alpha_3 \leq 2d$ as $2dy_1y_2y_3 = 2y_1(dy_2y_3) \in \Gamma^3(X)$. Hence, $|\oplus \Gamma^{n/n+1}(X)_{\mathit{tors}}| \leq 2$. As $c_2(dx_1x_2x_3) = \binom{d}{2}(6y_1y_2y_3 + 2y_1y_2 + 2y_1y_3 + 2y_2y_3) \in \Gamma^2(X)$, we have $dy_1y_2y_3 \in \Gamma^2(X)$. Observe that $\Gamma^3(X)$ is generated by $\Gamma^1(X) \cdot \Gamma^2(X)$ and any element of $\Gamma^1(X) \cdot \Gamma^2(X)$ is divisible by $2d$. Therefore, the class of $dy_1y_2y_3$ gives a torsion of order 2 in $\Gamma^{2/3}(X)$. Hence, we have $\Gamma^{2/3}(X)_{\mathit{tors}} = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{CH}^2(\bar{X})_{\mathit{tors}} = \mathbb{Z}/2\mathbb{Z}$ by (7). $\square$

We determine the maximal torsion in Chow group of codimension 2 of the product of n conics or the product of n Pfister quadric surfaces.

Proposition 4.3. *Let $n \geq 2$ and $1 \leq i \leq n$ be integers, Q_i quaternion algebras satisfying $\operatorname{ind}(\otimes_{k=1}^{m} Q_{i_k}) = 2$ for any $1 \leq m \leq n$ and all distinct i_k, and $X = \prod_{i=1}^{n} \mathrm{SB}(Q_i)$. Then, the torsion subgroup $\mathrm{CH}^2(\bar{X})_{tors}$ of the corresponding generic variety $\bar{X}$ is*

$$(\mathbb{Z}/2\mathbb{Z})^{\oplus N}, \text{ where } N = 2^n - (\binom{n}{2} + n + 1).$$

In particular, $\mathcal{M}(C_n) = \mathcal{M}(PQ_n) \geq 2^N$.

Proof. Let $1 \leq i \leq n$ and let Q_i be a quaternion such that $\operatorname{ind}(\otimes_{k=1}^{m} Q_{i_k}) = 2$ for any $1 \leq m \leq n$ and all distinct i_k, and $X = \prod_{i=1}^{n} \mathrm{SB}(Q_i)$. Then, by (5) we have a basis $\{1, 2x_{i_1} \cdots x_{i_m}\}$ of $K(X)$, where x_i is the pullback of the class of the tautological line bundle on the projective line over a splitting field E. Let $y_i = x_i - 1$. Consider another basis $\{1, 2y_{i_1} \cdots y_{i_m}\}$ of $K(X)$.

Let $j \geq 1$. As any element of $\Gamma^{2j+1}(X)$ is divisible by 2^{j+1}, we have

$$2^j y_{i_1} \cdots y_{i_k} \in \Gamma^{2j}(X) \backslash \Gamma^{2j+1}(X) \tag{8}$$

for any $2j+1 \leq k \leq n$. Moreover, if $2j+1 \leq k \leq 2j+2$, then we obtain

$$2^{j+1} y_{i_1} \cdots y_{i_k} \in \Gamma^{2j+2}(X). \tag{9}$$

Then, it follows from (8) and (9) that for any $j \geq 1$ the class of $2^j y_{i_1} \cdots y_{i_k}$ generates a subgroup of $\Gamma^{2j/2j+1}(X)_{tors}$ of order 2. By the divisibility of an element of $\Gamma^{2j+1}(X)$, any two subgroups generated by different classes of the elements $2^j y_{i_1} \cdots y_{i_k}$ and $2^j y_{i_1} \cdots y_{i_{k'}}$ have trivial intersection. Hence, we have

$$(\mathbb{Z}/2\mathbb{Z})^{\oplus N_j} \subseteq \Gamma^{2j/2j+1}(X)_{tors}, \tag{10}$$

where $N_j = \sum_{k=2j+1}^{n} \binom{n}{k}$.

Let $\beta_i = |\Gamma^{i/i+1}(X_E)/\operatorname{Im}(\mathrm{res}^{i/i+1})|/|K^i(X_E)/K^i(X)|$, where $K^i(X_E)$ (resp. $K^i(X)$) is the codimension i part of $K(X_E)$ (resp. $K(X)$). Then, It follows from the base of $K(X)$ that $\beta_1, \beta_2 \leq 1$ and

$$\beta_i \leq 2^{([i+1/2]-1)\binom{n}{i}} = 2^{[i+1/2]\binom{n}{i}}/2^{\binom{n}{i}}$$

for any $3 \leq i \leq n$. Hence, by (3) we have

$$|\oplus \Gamma^{i/i+1}(X)_{tors}| \leq 2^{\sum_{i=3}^{n}([i+1/2]-1)\binom{n}{i}}. \tag{11}$$

Therefore, it follows from (10) and (11) that for any $j \geq 1$

$$\Gamma^{2j/2j+1}(X)_{tors} = (\mathbb{Z}/2\mathbb{Z})^{N_j}.$$

As $N_1 = N$, the result follows from (7). $\square$

Corollary 4.4. *For any $n \geq 2$, we have $\mathcal{M}(C_n) = \mathcal{M}(PQ_n) = 2^N$.*

Proof. Let $X = \prod_{i=1}^{n} \mathrm{SB}(Q_i) \in \mathcal{C}_n$ such that $\mathrm{ind}(Q_i) = 2$ for all i. For $1 \leq m \leq n$, set $e_{i_1\dots i_m} = \max\{\mathrm{ind}(Q_{i_1}^{\otimes j_1} \otimes \cdots \otimes Q_{i_m}^{\otimes j_m})\}$, where the maximum ranges over $0 \leq j_1, \dots, j_m \leq 1$. If $m \geq 3$, then $e_{i_1\dots i_m} y_{i_1} \cdots y_{i_{m-1}} \in T^2(X)$. As $2y_{i_m} \in T^1(X)$, we have

$$2e_{i_1\dots i_m} y_{i_1} \cdots y_{i_m} \in T^3(X). \tag{12}$$

Since the subgroup $T^3(X_E) \cap K(X)$ is generated by $e_{i_1\dots i_m} y_{i_1} \cdots y_{i_m}$ for $m \geq 3$, it follows from (1), (2), and (12) that $|\,\mathrm{CH}^2(X)_{tors}| \leq 2^N$. If $\mathrm{ind}(Q_i) = 1$ for some i, then by (6) the computation of the upper bound for the torsion subgroup reduces to the case of product of less than n conics. Hence, we have $\mathcal{M}(\mathcal{C}_n) \leq 2^N$, thus by Proposition 4.3 we obtain the result. □

4.1. **Four Conics.** Let Q_i be a quaternion algebra over a field F for $1 \leq i \leq 4$. Consider the product X of the corresponding conics $\mathrm{SB}(Q_i)$. If $\mathrm{ind}(Q_i) = 1$ for some i, then by (6) the problem to find torsion in $\mathrm{CH}^2(X)$ is reduced to the case of product of three conics (Proposition 4.2). Hence, we may assume that $\mathrm{ind}(Q_i) = 2$ for all i. Let $g_{ij} = \mathrm{ind}(Q_i \otimes Q_j)$, $h_i = \mathrm{ind}(Q_j \otimes Q_k \otimes Q_l)$, and $d = \mathrm{ind}(Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4)$ for all integers i, j, k, l such that $\{i, j, k, l\} = I_4$. For the same reason, we may assume that g_{ij}, h_i, $d \geq 2$. By (5), we have the following basis of $K(X)$

$$\{1,\, 2x_i, g_{ij} x_i x_j,\, h_i x_j x_k x_l,\, d x_1 x_2 x_3 x_4\}, \tag{13}$$

where x_i, x_j, x_k, x_l are the pullbacks of the classes of the tautological line bundles on the projective lines. By substitution $y_i = x_i - 1$ for all $1 \leq i \leq 4$, it follows from (13) that we have another basis $K(X)$

$$\{1,\, 2y_i, g_{ij} y_i y_j,\, h_i(y_j y_k y_l + y_j y_k + y_j y_l + y_k y_l),\, d(y_1 y_2 y_3 y_4 + \sum y_i y_j y_k + \sum y_i y_j\}. \tag{14}$$

Now we will set up some notations, which will be used in this subsection. For any integer $m \in \{2, 4, 8\}$, we set

$$H_m = \{1 \leq i \leq 4 \,|\, h_i = m\}.$$

Let J be the set of all indices $\{12,\ 13,\ 14,\ 23,\ 24,\ 34\}$ of g_{ij}. We consider the decompositions of J:

$$J = J_1 \cup J_2 \cup J_3 = K_i \cup L_i,$$

where $J_1 = \{12, 34\}$, $J_2 = \{13, 24\}$, $J_3 = \{14, 23\}$, $K_i = \{jk,\ jl,\ kl\}$, and $L_i = J \backslash K_i$ for $j < k < l$. We set

$$G = \{ij \in J \,|\, g_{ij} = 2\}.$$

We will use the following lemma to find bounds for the torsion in $\mathrm{CH}^2(X)$.

Lemma 4.5. *Let $1 \leq m \leq 3$ and $1 \leq p < q < r \leq 4$ be integers and let i, j, k, l be integers such that $\{i, j, k, l\} = I_4$. Then, in codimension 2, 3, and 4 respectively, we have*

$$\text{(1) } \mathrm{Im}(\mathrm{res}^{2/3}) \ni \begin{cases} 2\sum y_p y_q & \text{if } d = 2, \\ 2(y_i y_j + y_i y_k + y_j y_k) & \text{if } h_l = 2, \end{cases}$$

$$(2)\ \mathrm{Im}(\mathrm{res}^{3/4}) \ni \begin{cases} 4y_iy_jy_k,\ 4y_iy_jy_l & \textit{if } g_{ij}=2,\\ 4y_iy_jy_k, -4y_iy_jy_k+4\sum y_py_qy_r & \textit{if } h_l=2,\\ 4y_iy_jy_k,\ 4y_iy_jy_l,\ 4y_iy_ky_l & \textit{if } g_{ij}=g_{ik}=2,\\ \forall\ 4y_py_qy_r & \textit{if } d=2 \textit{ or } |G|\geq 4 \textit{ or } |G\cap J_m|=2\\ & \textit{or } |G\cap K_i|=3, \end{cases}$$

$$(3)\ \Gamma^4(X) \ni \begin{cases} 8y_1y_2y_3y_4 & \textit{if } d\in\{2,4\} \textit{ or } |G|\geq 1 \textit{ or } |H_2|\geq 1,\\ 4y_1y_2y_3y_4 & \textit{if } |G\cap J_m|=2. \end{cases}$$

Proof. If $d=2$, then we have

$$c_2(2x_1x_2x_3x_4) = 2(y_1y_2y_3y_4 + \sum y_py_qy_r + \sum y_py_q) \in \Gamma^2(X),$$

thus, we obtain $c_1(2x_i)c_1(2x_j)c_2(2x_1x_2x_3x_4) = 8y_1y_2y_3y_4 \in \Gamma^4(X)$, which imply the results in Lemma 4.5 (1), (3), respectively. Since $8\sum y_py_qy_r \in \Gamma^3(X)$, the result in Lemma 4.5 (2) follows from

$$c_1(2x_i)c_2(2x_1x_2x_3x_4) = 12y_1y_2y_3y_4 + 4y_i(y_jy_k + y_jy_l + y_ky_l) \in \Gamma^3(X) \text{ and}$$

$$c_1(2x_1x_2x_3x_4)c_2(2x_1x_2x_3x_4) = 72y_1y_2y_3y_4 + 12\sum y_py_qy_r \in \Gamma^3(X).$$

If $h_l=2$, then it follows from (2) and (14) that

$$2(y_iy_jy_k + y_iy_j + y_iy_k + y_jy_k) \in \Gamma^2(X), \tag{15}$$

which completes the proof of Lemma 4.5 (1). Multiplying (15) by $c_1(2x_k)$ and $c_1(2x_l)$, respectively, we obtain the results in Lemma 4.5 (2). Multiplying $4y_iy_jy_k$ by $c_1(2x_l)$, we have the result in Lemma 4.5 (3).

If $|G\cap J_m|=2$ for some m, then we have $g_{ij}=g_{kl}=2$ for some i,j,k,l, thus the results in Lemma 4.5 (2)(3) follow from (14) and (2).

If $g_{ij}=2$, then it follows from (14) and (2) that

$$2y_iy_jc_1(2y_k),\ 2y_iy_jc_1(2y_l) \in \Gamma^3(X). \tag{16}$$

Multiplying the first element in (16) by $c_1(2x_l)$, we obtain the result in Lemma 4.5 (3). If $d=4$, then we have $c_4(4x_1x_2x_3x_4) = 24y_1y_2y_3y_4 \in \Gamma^4(X)$. As $16y_1y_2y_3y_4 \in \Gamma^4(X)$, we complete the proof of Lemma 4.5 (3).

If $|G|\geq 4$, then we have $|G\cap J_m|=2$ for some m, hence the result in Lemma 4.5 (2) follows from the case of $|G\cap J_m|=2$. If $g_{ij}=g_{ik}=2$, then it follows from (14) and (2) that

$$2y_iy_jc_1(2x_k),\ 2y_iy_jc_1(2x_l), 2y_iy_kc_1(2x_l) \in \Gamma^3(X).$$

If $|G\cap K_i|=3$, then we get $g_{jk}=g_{jl}=g_{kl}=2$, thus, by the same argument we complete the proof of Lemma 4.5 (2). □

We determine the codimension 2 cycles of the product of four conics as in Proposition 4.2.

Theorem 4.6. *The torsion in Chow group of codimension* 2 *of the product* X *of four conics is trivial if it satisfies one of the conditions* (17), (22), (23), (25), (31), (32), (33), $d = 16$. *Otherwise, the torsion subgroup admits all elementary abelian group whose order is a divisor of* 2^5.

Proof. Let m be an integer in I_3, Q_i a quaternion division algebra over F and X the product of the corresponding conics $\mathrm{SB}(Q_i)$ for $1 \leq i \leq 4$. Set

$$\beta_i = |\Gamma^{i/i+1}(X_E)/\operatorname{Im}(\mathrm{res}^{i/i+1})|/|K^i(X_E)/K^i(X)|,$$

where E is a splitting field of X and $K^i(X_E)$ (resp. $K^i(X)$) is the codimension i part of $K(X_E)$ (resp. $K(X)$). We find upper bounds of β_i using case by case analysis. Note that we have $\beta_1 \leq 1$.

We begin with some observations on d: if $d = 2^4$, then we have $h_i = d/2$ for all i, which implies that $g_{ij} = d/4$ for all $1 \leq i \neq j \leq 4$. Therefore, we obtain $\beta_i \leq 1$ for all i, thus, $|\oplus \Gamma^{i/i+1}(X)_{tors}| \leq 1$, i.e., $\mathrm{CH}^2(X)_{tors}$ is trivial. Hence, we may assume that $2 \leq d \leq 2^3$. Note also that we have

$$d \leq 4 \text{ if } |H_2| \geq 1, \text{ and } |H_8| = 0 \text{ if } d = 2,$$

thus we only consider the cases where $d = 4, 8$ (resp. $d = 2, 4$) in the first 3 cases (resp. the last 4 cases) of the following. In case where $|H_2| = 4$, we have $d \in \{2, 4, 8\}$.

Case: $|H_8| \geq 3$. In this case, we have $g_{ij} = 4$ for all i, j. It follows from Lemma 4.5 (1) that $\beta_2 \leq (4^5 \cdot 2^{2-|H_2|})/4^6$. By Lemma 4.5 (2), we obtain

$$\beta_3 \leq \begin{cases} (8^2 \cdot 4^2)/8^3 \cdot 2 & \text{if } |H_2| = 1, \\ 2^{|H_4|} & \text{otherwise.} \end{cases}$$

It follows by Lemma 4.5 (3) that $\beta_4 \leq 2$. Hence, by (3) the order of the group $\oplus\Gamma^{i/i+1}(X)_{tors}$ is nontrivial except the case where

$$|H_2| = 1, |H_8| = 3. \tag{17}$$

If $|H_8| = 4$, $d = 8$, then by the divisibility of elements in $\Gamma^3(X)$ the class of $8y_1y_2y_3y_4$ gives a torsion element of $\Gamma^{2/3}(X)_{tors}$ of order 2. As $\beta_1\beta_2\beta_3\beta_4 \leq 2$, we have

$$\Gamma^{2/3}(X)_{tors} = \mathbb{Z}/2\mathbb{Z}. \tag{18}$$

Similarly, if $|H_8| = 3$, $h_i = 4$ and $d = 4$, then we have

$$\Gamma^{2/3}(X)_{tors} = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \tag{19}$$

generated by the classes of $4y_jy_ky_l$ and $4(y_iy_jy_k + y_iy_jy_l + y_iy_ky_l + y_1y_2y_3y_4)$.

Case: $|H_8| = 2$. A simple calculation of index implies that $0 \leq |G| \leq 1$. Hence, it follows from Lemma 4.5 (1) that

$$\beta_2 \leq (4^{6-|G|-|H_2|} \cdot 2^{|G|+|H_2|})/4^{6-|G|} \cdot 2^{|G|}. \tag{20}$$

If $|G| = |H_2| = 0$, then we have $\beta_3 \leq 8^4/(8^2 \cdot 4^2)$. Otherwise, by Lemma 4.5 (2) we obtain

$$\beta_3 \leq (8^{3-|G|-|H_2|} \cdot 4^{1+|G|+|H_2|})/8^2 \cdot 4^{2-|H_2|} \cdot 2^{|H_2|}.$$

In summary, we have

$$\beta_3 \leq \begin{cases} 1 & \text{if } |G| = 1, |H_2| = 0 \text{ or } |G| = |H_2| = 1, \\ 2 & \text{if } |G| = 0, |H_2| = 1 \text{ or } |H_2| = 2, \\ 2^2 & \text{otherwise.} \end{cases}$$

It follows from Lemma 4.5 (3) that

$$\beta_4 \leq \begin{cases} 1 & \text{if } |G| \geq 1, |H_2| = 0, d = 8, \\ 2 & \text{otherwise.} \end{cases} \tag{21}$$

Therefore, it follows by the same argument as in the previous case that $|\oplus \Gamma^{i/i+1}(X)_{tors}|$ is nontrivial except the case where

$$|H_2| = 2 \text{ or } |G| = |H_2| = 1 \text{ or } |G| = 1, |H_2| = 0, d = 8. \tag{22}$$

Case: $|H_8| = 1$. By this assumption, we have $0 \leq |G| \leq 3$. It follows by the same argument as in the previous case that we have the same upper bounds in (20) and (21) for β_2 and β_4, respectively. Let $G_n = G \cap (\cap_{i \in H_n} K_i)$ for $n = 2, 4$. If $|G| = |G_2| = |H_2| = 1$ (resp. $|G| = |G_4| = 1, |H_2| = 2$), then we have $h_i = 2$, $h_l = 8$, $h_j = h_k = 4$ (resp. $h_j = 2$, $h_k = 4$), and $g_{il} = 2$ (resp. $g_{kl} = 2$) for $\{i, j, k, l\} = I_4$, which implies that $\beta_3 \leq 4^4/(8 \cdot 4^2 \cdot 2)$ (resp. $\beta_3 \leq 8 \cdot 4^3/(8 \cdot 4 \cdot 2^2)$) by Lemma 4.5 (2). If $|H_2| = 0$, $|G| \geq 2$ (resp. $|H_2| = 0, |G| = 0$), then by Lemma 4.5 (2) we obtain $\beta_3 \leq 8 \cdot 4^3/(8 \cdot 4^3)$ (resp. $\beta_3 \leq 8^4/(8 \cdot 4^3)$). Otherwise, by Lemma 4.5 (2) we have

$$\beta_3 \leq \min\{1, 8^{3-|G|-|H_2|}\} \cdot \min\{4^4, 4^{1+|G|+|H_2|}\}/(8 \cdot 4^{3-|H_2|} \cdot 2^{|H_2|}).$$

Therefore, we conclude that

$$\beta_3 \leq \begin{cases} 1 & \text{if } 0 \leq |H_2| \leq 1, |G| \geq 2 \text{ or } |H_2| = |G_2| = |G| = 1, \\ 2 & \text{otherwise,} \\ 2^2 & \text{if } 1 \leq |H_2| \leq 2, |G| = 0 \text{ or } |H_2| = 2, |G| = |G_4| = 1, \\ 2^3 & \text{if } |H_2| = |G| = 0. \end{cases}$$

By the same argument, the order of the torsion $|\oplus \Gamma^{i/i+1}(X)_{tors}|$ is nontrivial except the following cases:

$$|G| \geq 2, |H_2| = 0, d = 8 \text{ or } |G| \geq |H_2||H_4| = 2 \text{ or } |H_2||H_4| = 2, |G| = |G_2| = 1 \text{ or } |H_2| = 3. \tag{23}$$

If $d = 8$, $|G| = 6$, and $h_i = 8$, then by the same argument as above we obtain

$$\Gamma^{2/3}(X)_{tors} = (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \text{ and } \Gamma^{3/4}(X)_{tors} = \mathbb{Z}/2\mathbb{Z} \tag{24}$$

generated by $4y_iy_ky_l$, $4y_iy_jy_l$, $4y_iy_jy_k$, and $8y_1y_2y_3y_4$, respectively.

Case: $|H_4| = 4$. By Lemma 4.5 (1), we have

$$\beta_2 \leq \begin{cases} (4^{5-|G|} \cdot 2^{|G|+1})/4^{6-|G|} \cdot 2^{|G|} & \text{if } d = 2, |G| \neq 6, \\ (4^{6-|G|} \cdot 2^{|G|})/4^{6-|G|} \cdot 2^{|G|} & \text{otherwise.} \end{cases}$$

It follows from Lemma 4.5 (2) that

$$4^4\cdot\beta_3\leq\begin{cases}4^4 & \text{otherwise,}\\ 8\cdot 4^3 & \text{if } d\in\{4,8\}, 2\leq |G|=|G\cap L_i|\leq 3 \text{ for some } i,\\ 8^2\cdot 4^2 & \text{if } d\in\{4,8\}, |G|=1,\\ 8^4 & \text{if } d\in\{4,8\}, |G|=0.\end{cases}$$

By Lemma 4.5 (3) and (14), we obtain

$$\beta_4\leq\begin{cases}1 & \text{if } d=8, |G|\neq 0 \text{ or } d=4, |G\cap J_m|=2 \text{ for some } m,\\ 2 & \text{otherwise,}\\ 2^2 & \text{if } d=2, |G|\neq 6, |G\cap J_m|\neq 2\ \forall m.\end{cases}$$

Applying the same argument, we have that the order of $\oplus\Gamma^{i/i+1}(X)_{\mathit{tors}}$ is nontrivial except the following cases: for some m and i

$$d=2, |G|\neq 6, |G\cap J_m|=2 \text{ or } d=4, |G\cap J_m|=2 \text{ or } d=8, |G\cap K_i|=3. \tag{25}$$

If $|G|=0$ and $d=4$, then by (14) and (2) one has $h_iy_jy_ky_l$, $4y_1y_2y_3y_4\in\Gamma^2(X)\backslash\Gamma^3(X)$ for all $\{i,j,k,l\}=I_4$ since any element of $\Gamma^3(X)$ is divisible by 2^3. Therefore, by (14) and Lemma 4.5 (3) the classes of $h_iy_jy_ky_l$ and $4y_1y_2y_3y_4$ give torsion elements of $\Gamma^{2/3}(X)$ of order 2. Moreover, the subgroups generated by these classes have trivial intersection by the divisibility of an element of $\Gamma^3(X)$. Hence, it follows from Corollary 4.4 that

$$\Gamma^{2/3}(X)_{\mathit{tors}}=(\mathbb{Z}/2\mathbb{Z})^{\oplus 5},\ \text{thus}\ \mathrm{CH}^2(\bar{X})_{\mathit{tors}}=(\mathbb{Z}/2\mathbb{Z})^{\oplus 5}, \tag{26}$$

where $\bar{X}$ is the corresponding generic variety. If $|G|=0$ and $d=8$, then by the same argument as above the classes of $h_iy_jy_ky_l$ generate different subgroups of $\Gamma^{2/3}(X)_{\mathit{tors}}$ of order 2 and the class of $8y_1y_2y_3y_4=2y_1(4y_2y_3y_4)$ generates a subgroup of $\Gamma^{3/4}(X)_{\mathit{tors}}$ of order 2. Therefore, it follows Corollary 4.4 that

$$\Gamma^{2/3}(X)_{\mathit{tors}}=(\mathbb{Z}/2\mathbb{Z})^{\oplus 4} \text{ and } \Gamma^{3/4}(X)_{\mathit{tors}}=\mathbb{Z}/2\mathbb{Z}. \tag{27}$$

Case: $|H_2|=4$. By (14) and Lemma 4.5 (1), one has

$$\beta_2\leq\begin{cases}d/2^5 & \text{otherwise,}\\ d/2^4 & \text{if } |G|=d=4 \text{ or } |G|=3, d=2,\\ d/2^3 & \text{if } |G|=4, d=2 \text{ or } |G|=5, d=4,\\ d/2^2 & \text{if } |G|=6, d=4 \text{ or } |G|=5, d=2,\\ d/2 & \text{if } |G|=6, d=2.\end{cases} \tag{28}$$

It follows from (14) that $\beta_3\leq 2^4=8^4/4^4$. By Lemma 4.5 (3), we have

$$\beta_4\leq\begin{cases}2^2/d & \text{if } |G|\geq 2, |G\cap J_m|=2 \text{ for some } m,\\ 2^3/d & \text{otherwise.}\end{cases} \tag{29}$$

Applying the same argument together with the upper bounds β_i, we have $\Gamma^{2/3}(X)_{\mathit{tors}}\neq 0$ for any case. In particular, if $|G|=6$ and $d=2$, then by Proposition 4.3 we have

$$\Gamma^{2/3}(X)_{\mathit{tors}}=(\mathbb{Z}/2\mathbb{Z})^{\oplus 5},\ \text{thus}\ \mathrm{CH}^2(\bar{X})_{\mathit{tors}}=(\mathbb{Z}/2\mathbb{Z})^{\oplus 5}. \tag{30}$$

Case: $|H_2| = 3$, $|H_4| = 1$. Let $H'_2 = \{i \in H_2 \,|\, |K_i \cap G| = 3\}$. Then, by the same argument as in the previous case we have the same upper bound (28) for β_2 if we add one case $|G| = 3$, $d = 4$, $|H'_2| = 1$ to the second line of (28). It follows by Lemma 4.5 (2) that $\beta_3 \leq 2^3$. By applying the same argument as in the previous case we have the same upper bound (29) for β_4. it follows by the same argument as in the previous case that $|\oplus\, \Gamma^{i/i+1}(X)_{tors}|$ is nontrivial except the case:

$$|G| = 2, |G \cap J_m| = 2 \text{ or } |G| = 3, |G \cap J_m| = 2, d = 4 \text{ for some } m. \tag{31}$$

Case: $|H_2| = 2$, $|H_4| = 2$. By Lemma 4.5 (1), we have

$$\beta_2 \leq \begin{cases} d/2^4 & \text{otherwise,} \\ d/2^3 & \text{if } 3 \leq |G| \leq 4, |H'_2| = 1 \text{ or } |G| = 4, |H'_2| = 0, d = 2 \text{ or } |G| = 5, |H'_2| \leq 1, d = 4, \\ d/2^2 & \text{if } |G| = 5, |H'_2| = 2 \text{ or } |G| = 5, |H'_2| \leq 1, d = 2 \text{ or } |G| = 6, d = 4, \\ d/2 & \text{if } |G| = 6, d = 2. \end{cases}$$

It follows from Lemma 4.5 (2) that

$$\beta_3 \leq \begin{cases} 2^2 & \text{otherwise,} \\ 2^3 & \text{if } |G| = 0, d = 4 \text{ or } |G| = |G_2| = 1, d = 4. \end{cases}$$

Applying the same argument as in the previous case, one has the same upper bound (29) for β_4. Therefore, by the same argument, the order of the torsion $|\oplus\Gamma^{i/i+1}(X)_{tors}|$ is nontrivial except the following cases: for some m and i

$$|G| = |G \cap J_m| = 2 \text{ or } |G| = |G \cap K_i| = 3 \text{ or } |G| = 4, |H'_2| = 0, d = 4. \tag{32}$$

Case: $|H_2| = 1$, $|H_4| = 3$. By Lemma 4.5 (1), we obtain

$$\beta_2 \leq \begin{cases} d/2^3 & \text{otherwise,} \\ d/2^2 & \text{if } 3 \leq |G| \leq 5, |H'_2| = 1 \text{ or } |G| = 5, |H'_2| = 0, d = 2 \text{ or } |G| = 6, d = 4, \\ d/2 & \text{if } |G| = 6, d = 2. \end{cases}$$

In codimension 3, it follows from Lemma 4.5 (2) that

$$\beta_3 \leq \begin{cases} 2 & \text{otherwise,} \\ 2^2 & \text{if } |G| = |G_2| = 1, d = 4, \\ 2^3 & \text{if } |G| = 0, d = 4. \end{cases}$$

In codimension 4, it follows by Lemma 4.5 (2) that

$$\beta_4 \leq \begin{cases} 1 & \text{if } 2 \leq |G| \leq 3, |G \cap J_m| = 2 \text{ for some } m, d = 4 \text{ or } |G| \geq 4, d = 4, \\ 2 & \text{otherwise,} \\ 2^2 & \text{if } 0 \leq |G| \leq 3, |G \cap J_m| \neq 2\ \forall m, d = 2. \end{cases}$$

Hence, by the same argument as above the order of $\oplus\Gamma^{i/i+1}(X)_{tors}$ is nontrivial except the following cases:

$$2 \leq |G| \leq 3, |G \cap J_m| = 2 \text{ for some } m \text{ or } |G| = 4, |H'_2| = 0 \text{ or } |G| = 5, |H'_2| = 0, d = 4. \tag{33}$$

Finally, the second statement of the theorem follows from (18), (19), (24), (26), and (27). □

Remark 4.7. Indeed, one can easily show that the upper bounds $\beta_1\beta_2\beta_3\beta_4$ for each case of the proof of Theorem 4.6 are sharp.

4.2. **Galois cohomology and torsion groups.** As mentioned in Section 1, the torsion subgroup of the Chow group of codimension 2 cycles can be used to measure how far is a relative Galois cohomology group from being a decomposable subgroup (generated by the class of A_i below) [10, Theorem 4.1]. Namely, for an F-variety $X = \prod_i \mathrm{SB}(A_i) \in \mathsf{C}_n$ or SB_n we have

$$\mathrm{CH}^2(X)_{\mathit{tors}} \simeq H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))/ \oplus_i H^1(F, \mathbb{Q}/\mathbb{Z}(1)) \cup [A_i], \tag{34}$$

where $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$ denotes the kernel of $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z})$ of Galois cohomology groups with coefficient in $\mathbb{Q}/\mathbb{Z}(2)$ and $[A_i]$ denotes the class in the Brauer group $\mathrm{Br}(F) = H^2(F, \mathbb{Q}/\mathbb{Z}(1))$. Therefore, our main results (Corollary 4.4, Proposition 5.2, and Theorem 5.4) tell us how large indecomposable subgroups we can have.

Moreover, by [10, Remark 4.1] there is a canonical injection from a Galois cohomology group with the finite coefficient $\boldsymbol{\mu}_n^{\otimes 2}$ to the torsion subgroup:

$$H^3(F(X)/F, \boldsymbol{\mu}_n^{\otimes 2})/ \oplus_i H^1(F, \boldsymbol{\mu}_n) \cup [A_i] \hookrightarrow \mathrm{CH}^2(X)_{\mathit{tors}}. \tag{35}$$

Therefore, if the torsion subgroup $\mathrm{CH}^2(X)_{\mathit{tors}}$ is trivial, then one can write the relative Galois cohomology group in terms of decomposable subgroups with the finite coefficient $\boldsymbol{\mu}_n$. For instance, if $X \in C_4$ satisfying one of the conditions (17), (22), (23), (25), (31), (32), (33), $d = 16$ in Theorem 4.6, then we obtain $|\,\mathrm{CH}^2(X)_{\mathit{tors}}| = 1$, thus by (35) we have

$$H^3(F(X)/F, \mathbb{Z}/2\mathbb{Z}) = \oplus_{i=1}^4 H^1(F, \mathbb{Z}/2\mathbb{Z}) \cup [Q_i].$$

5. Product of Severi-Brauer surfaces

In this section, we find a general lower bound of the torsion in Chow group of codimension 2 of the product of n Severi-Brauer surfaces in Proposition 5.2 and prove that the lower bound is sharp for $n = 3$ in Theorem 5.4.

To prove Proposition 5.2 we shall need the following lemma.

Lemma 5.1. *Let p be an odd prime and let $n \geq 2$ and m_i be integers such that $1 \leq m_i \leq p-1$ for all $1 \leq i \leq n$. Let Φ be the polynomial $(\prod_{i=1}^n (s_i+1)^{m_i} - 1)^p$ in $\mathbb{Z}[s_1, \dots, s_n]$. Then, the alternating sum $\sum_{j_1=\cdots=j_n=1}^{p-1} (-1)^{j_1+\cdots+j_n} C_{j_1\cdots j_n}$ in the quotient $\mathbb{Z}[s_1, \dots, s_n]/(s_1^p, \dots, s_n^p)$ is divisible by p^2, where $C_{j_1\cdots j_n}$ is the coefficient of the monomial $s_1^{j_1} \cdots s_n^{j_n}$ in Φ.*

Proof. Let $t_n = (s_n+1)^{m_n}$. For any $1 \leq j_1, \dots, j_{n-1} \leq p-1$, we write $\sum_{k=1}^p d_{j_1\cdots j_{n-1}k} t^k$ for the coefficient of $s_1^{j_1} \cdots s_{n-1}^{j_{n-1}}$ in Φ. Let $e_{j_1\cdots j_{n-1}}$ be the coefficient of $s_1^{j_1} \cdots s_{n-1}^{j_{n-1}}$ in $\Psi_n := (s_1+1)^{pm_1} \cdots (s_{n-1}+1)^{pm_{n-1}}$. By expanding each factor $(s_i+1)^{pm_i} = ((s_i+1)^p)^{m_i}$ of Ψ_n, we have

$$e_{j_1\cdots j_{n-1}} = (\binom{p}{j_1} m_1 + \alpha_1) \cdots (\binom{p}{j_{n-1}} m_{n-1} + \alpha_{n-1}), \tag{36}$$

where $p^2 \mid \alpha_i$ for all $1 \leq i \leq n-1$.

We prove by induction on n. Assume $n = 2$. First, observe that $C_{j_1 0} = 0$ in the quotient $\mathbb{Z}[s_1, s_2]/(s_1^p, s_2^p)$. Hence, we have

$$\sum_{j_2=1}^{p-1} (-1)^{j_2} C_{j_1 j_2} = \sum_{j_2=0}^{p-1} (-1)^{j_2} C_{j_1 j_2} = \sum_{k=1}^{p} d_{j_1 k} \sum_{i=0}^{p-1} (-1)^i \binom{m_2 k}{i} = \sum_{k=1}^{p} d_{j_1 k} \binom{m_2 k - 1}{p-1},$$

which implies that

$$\sum_{j_1=j_2=1}^{p-1} (-1)^{j_1+j_2} C_{j_1 j_2} = \sum_{j_1=1}^{p-1} \sum_{k=1}^{p} d_{j_1 k} \binom{m_2 k - 1}{p-1} (-1)^{j_1}. \tag{37}$$

For each $1 \le k \le p-1$, we have

$$p \mid \binom{m_2 k - 1}{p-1}. \tag{38}$$

As

$$\Phi = \sum_{i=0}^{p} \binom{p}{i} (s_1+1)^{m_1 i} t_2^i (-1)^{p-i},$$

for any $1 \le k \le p$ we obtain

$$p \mid d_{j_1 k}. \tag{39}$$

From (38) and (39), it suffices to show that

$$p^2 \mid \sum_{j_1=1}^{p-1} d_{j_1 p} \binom{m_2 p - 1}{p-1} (-1)^{j_1}. \tag{40}$$

Since $d_{j_1 p} = e_{j_1}$ and $\sum_{j_1=1}^{p-1} \binom{p}{j_1} (-1)^{j_1} = 0$, the divisibility in (40) follows from (36).

Now we assume that the result holds for $n-1$. By the induction hypothesis, it is enough to show that $\sum_{j_1=\cdots=j_{n-1}=1, j_n=0}^{p-1} (-1)^{j_1+\cdots+j_n} C_{j_1\cdots j_n}$ is divisible by p^2. Applying the same argument as in the case $n=2$, we have

$$\sum_{j_1=\cdots=j_{n-1}=1, j_n=0}^{p-1} (-1)^{j_1+\cdots+j_n} C_{j_1\cdots j_n} = \sum_{j_1=\cdots=j_{n-1}=1}^{p-1} \sum_{k=1}^{p} d_{j_1\cdots j_{n-1}k} \binom{m_n k - 1}{p-1} (-1)^{j_1+\cdots+j_{n-1}}. \tag{41}$$

We have $p \mid \binom{m_n k-1}{p-1}$ for each $1 \le k \le p-1$. By the same argument with t_n, we also have $p \mid d_{j_1\cdots j_{n-1}k}$ for each $1 \le k \le p$. For $k = p$, we have $p^2 \mid d_{j_1\cdots j_{n-1}p}$ by (36). Therefore, the result immediately follows. □

We will provide lower bounds of the torsion in Chow group of codimension 2 of the product of n Severi-Brauer surfaces, which generalize the case of $n = 2$ in [4, Proposition 6.3].

Proposition 5.2. *Let $n \ge 2$ and $1 \le i \le n$ be integers, A_i a central simple algebra satisfying* $\operatorname{ind}(A_1^{\otimes j_1} \otimes \cdots \otimes A_n^{\otimes j_n}) = 3$ *for any integers $0 \le j_1, \ldots, j_n \le 2$, not all equal to 0,* *and $X = \prod_{i=1}^{n} \mathrm{SB}(A_i)$. Then, the torsion subgroup $\mathrm{CH}^2(\bar{X})_{tors}$ of the corresponding generic variety $\bar{X}$ contains*

$$(\mathbb{Z}/3\mathbb{Z})^{\oplus N}, \text{ where } N = 2^n + 4\binom{n}{3} - (n+1).$$

In particular, $\mathcal{M}(\mathsf{SB}_n) \geq 3^N$.

Proof. Let $1 \leq i \leq n$ and let A_i be a central simple algebra such that $\mathrm{ind}(\otimes_{k=1}^{n} A_k^{\otimes j_k}) = 3$ for any integers $0 \leq j_1, \ldots, j_n \leq 2$, and $X = \prod_{i=1}^{n} \mathrm{SB}(A_i)$. Then, by (5) we have a basis $\{1, 3x_{i_1}^{j_1} \cdots x_{i_m}^{j_m}\}$ of $K(X)$, where x_i is the pullback of the class of the tautological line bundle on the projective plane and $1 \leq m \leq n$.

Let $y_i = x_i - 1$. Consider another basis $\{1, 3y_{i_1}^{j_1} \cdots y_{i_m}^{j_m}\}$ of $K(X)$. Let $B_1 = \{b_{pq} := 3y_p^2 y_q^2\}$, $B_2 = \{b_{pqr} := 3y_p y_q y_r\}$, $B_3 = \{b'_{pqr} := 3y_p^2 y_q y_r\}$, and $D_s = \{d_{i_1 \cdots i_s} := 3(y_{i_1} \cdots y_{i_s})^2\}$ for all distinct $1 \leq p, q, r \leq n$ and $3 \leq s \leq n$. Then we have $y_i^3 = 0$, $|B_1| = \binom{n}{2}$, $|B_2| = \binom{n}{3}$, $|B_3| = 3\binom{n}{3}$, and $|D_s| = \binom{n}{m}$. Set

$$N = |B_1| + |B_2| + |B_3| + \sum_{s=3}^{n} |D_s| = 2^n + 4\binom{n}{3} - (n+1).$$

It follows from (2) that each element of B_1, B_2, B_3, D_s is in $\Gamma^2(X)$. If any element of B_1, B_2, B_3, D_s is in $\Gamma^3(X)$, then by applying Lemma 5.1 with $y_i = -1$ for all $1 \leq i \leq n$ we have 3 is divisible by 9, which is impossible. Therefore, any element of B_1, B_2, B_3, D_s is in $\Gamma^2(X) \backslash \Gamma^3(X)$. Since $3b_{pq} = 3y_p^2(3y_q^2) \in \Gamma^4(X)$, $3b_{pqr} = 3y_p y_q (3y_r) \in \Gamma^3(X)$, $3b'_{par} = 3y_p^2(3y_q y_r) \in \Gamma^3(X)$, $3d_{i_1 \cdots i_s} = 3y_{i_1}^2(3y_{i_2}^2 \cdots y_{i_m}^2) \in \Gamma^4(X)$, any element of B_1, B_2, B_3, D_s gives a torsion of $\Gamma^{2/3}(X)$ of order 3.

We show that any two subgroups generated by any two elements of B_1, B_2, B_3, D_s have trivial intersection. Let b_{pq} and b_{tu} be two different elements of B_1. If $b_{pq} \pm b_{tu} \in \Gamma^3(X)$, then by applying Lemma 5.1 with $y_p = y_q = -1, y_l = 0$ for all $1 \leq l \neq p, q \leq n$ we obtain 3 is divisible by 9, which is a contradiction. Hence, the subgroups generated by b_{pq} and b_{tu} have trivial intersection. Moreover, by the same argument the subgroup generated by b_{pq} has trivial intersection with any subgroup generated by any element of B_2, B_3, and D_s.

Let z be either 1 or a product of $x_1, \cdots, x_n$ which does not contain any of x_p, x_q, and x_r. Consider the sequence β'_{pqr} consisting of the coefficient of $b'_{pqr}/3$ in

$$\begin{aligned} &c_3(3x_p^2 x_q x_r z), c_3(3x_p x_q^2 x_r z), c_3(3x_p x_q x_r^2 z), c_3(3x_p^2 x_q^2 x_r z), c_3(3x_p^2 x_q x_r^2), \\ &c_3(3x_p x_q^2 x_r^2 z), c_3(3x_p^2 x_q^2 x_r^2 z), c_3(3x_p x_q x_r z), \text{ respectively.} \end{aligned} \tag{42}$$

Then, by a direct calculation, we have $\beta'_{pqr} = (66, 30, 30, 132, 132, 60, 264, 15)$. Hence, each element of $\beta'_{pqr} - \beta'_{qpr}$, $\beta'_{pqr} - \beta'_{rpq}$, and $\beta'_{qpr} - \beta'_{rpq}$ is divisible by 9, i.e.,

$$9 \mid \beta'_{pqr} - \beta'_{qpr},\ \beta'_{pqr} - \beta'_{rpq},\ \beta'_{qpr} - \beta'_{rpq}. \tag{43}$$

Consider another sequence β_{pqr} consisting of the coefficient of $b_{pqr}/3$ in (42). Then, we have $\beta_{pqr} = (12, 12, 12, 24, 24, 24, 48, 6)$. Therefore, we have

$$9 \mid \beta_{pqr} - \beta'_{pqr},\ \beta_{pqr} - \beta'_{qpr},\ \beta_{pqr} - \beta'_{rpq}. \tag{44}$$

Let b'_{pqr} and b'_{tuv} be two different elements of B_3. If $b'_{pqr} \pm b'_{tuv} \in \Gamma^3(X)$, then by applying Lemma 5.1 with $y_p = y_q = y_r = -1, y_l = 0$ for all $1 \leq l \neq p, q, r \leq n$ we obtain 3 is divisible by 9, which is impossible. Therefore, the subgroups generated by b'_{pqr} and b'_{tuv} have trivial intersection. Assume that $b'_{pqr} \pm b_{tuv} \in \Gamma^3(X)$ or $b'_{pqr} \pm d_{i_1 \cdots i_s} \in \Gamma^3(X)$. Then, this contradicts the divisibility in (43). Hence, the subgroup generated by b'_{pqr} has trivial intersection with any subgroup generated by any element of B_2 and D_s.

Let b_{pqr} and b_{tuv} be two different elements of B_2. Suppose that $b_{pqr} \pm b_{tuv} \in \Gamma^3(X)$. Then, by applying Lemma 5.1 with $y_p = y_q = y_r = -1, y_l = 0$ for all $1 \leq l \neq p, q, r \leq n$ we obtain -3 is divisible by 9, which is a contradiction. Therefore, the subgroups generated by b_{pqr} and b_{tuv} have trivial intersection. If $b_{pqr} \pm d_{i_1 \cdots i_s} \in \Gamma^3(X)$, then we obtain a contradiction by the divisibility in (44). Hence, the subgroup generated by b_{pqr} has trivial intersection with any subgroup generated by any element of D_s.

Let $3 \leq s, s' \leq n$ and let $d_{i_1 \cdots i_s}$ and $d_{i'_1 \cdots i'_{s'}}$ be two different elements of D_s and $D_{s'}$, respectively. If $d_{i_1 \cdots i_s} \pm d_{i'_1 \cdots i'_{s'}} \in \Gamma^3(X)$, then by applying Lemma 5.1 with $y_{i_1} = \cdots = y_{i_s} = -1$ and $y_l = 0$ for all $1 \leq l \neq i_1, \ldots, i_s \leq n$ we have 3 is divisible by 9, which is a contradiction. It follows that the torsion subgroup $\Gamma^{2/3}(X)_{\mathit{tors}}$ contains $(\mathbb{Z}/3\mathbb{Z})^{\oplus N}$, so does $\mathrm{CH}^2(\bar{X})_{\mathit{tors}}$. □

5.1. **Three Severi-Brauer surfaces.** We consider the product of three Severi-Brauer surfaces. Let A_1, A_2, A_3 be a central simple algebras of degree 3 over F and let X be the product of the corresponding Severi-Brauer surfaces $\mathrm{SB}(A_1), \mathrm{SB}(A_2), \mathrm{SB}(A_3)$. If one of A_1, A_2, A_3 is split, then by (6) the problem to compute torsion in Chow group of codimension 2 is reduced to the case of product of two Severi-Brauer varieties, which was done in [4, Theorem 5.1]. Therefore, we may assume that $\mathrm{ind}(A_m) = 3$ for all $1 \leq m \leq 3$. Let $e_i = \mathrm{ind}(A_j \otimes A_k)$, $f_i = \mathrm{ind}(A_j^{\otimes 2} \otimes A_k)$, $d = \mathrm{ind}(A_1 \otimes A_2 \otimes A_3)$, and $g_i = \mathrm{ind}(A_i^{\otimes 2} \otimes A_j \otimes A_k)$ for all i, j, k such that $\{i, j, k\} = I_3$. For the same reason, we may assume that $e_i, f_i, d, g_i \geq 3$. By (5), we have the following basis of $K(X)$

$$\{1,\, 3x_m,\, 3x_m^2,\, e_i x_j x_k,\, f_i x_j^2 x_k,\, d x_1 x_2 x_3,\, e_i x_j^2 x_k^2,\, g_i x_i^2 x_j x_k,\, g_i x_i x_j^2 x_k^2,\, d x_1^2 x_2^2 x_3^2\}, \tag{45}$$

where x_m is the pullback of the class of the tautological line bundle on the projective plane. We will need the following lemma to find upper bounds of the torsion.

Lemma 5.3. *Let i, j, k be integers such that $\{i, j, k\} = I_3$ and let $y_m = x_m - 1$ for all $1 \leq m \leq 3$. Then, we have*

(1) $3y_m^2 \in \Gamma^2(X)$ *in any case and* $3y_j y_k \in \Gamma^2(X)$, $3y_j^2 y_k + 3y_j y_k^2 \in \Gamma^{3/4}(X)$ *if* $e_i = 3$,

(2) $3y_j^2 y_k - 3y_j y_k^2 \in \Gamma^{3/4}(X)$ *if* $f_i = 3$,

(3) $3\sum_{m,l=1}^{3} y_m y_l \in \mathrm{Im}(\mathrm{res}^{2/3})$, $6y_1 y_2 y_3 + 3\sum_{m,l=1}^{3} y_m^2 y_l \in \Gamma^{3/4}(X)$ *and* $9y_1^2 y_2^2 y_3^2 \in \Gamma^6(X)$ *if* $d = 3$,

(4) $3\sum_{m,l=1}^{3} y_m^2 y_l + 3(y_i y_j^2 + y_i^2 y_j) + 12 y_1 y_2 y_3 \in \Gamma^{3/4}(X)$, $3(y_j y_k - y_i y_k - y_i y_j) \in \mathrm{Im}(\mathrm{res}^{2/3})$ *if* $g_i = 3$,

(5) $3y_1^2 y_2^2 y_3 - 3y_1^2 y_2 y_3^2 \in \Gamma^3(X)$ *if* $f_m = g_m = e_m = 3$ *for all* m.

(6) $3y_1^2 y_2^2 y_3^2 - 6y_1^2 y_2^2 y_3 \in \Gamma^3(X)$ *if* $f_m = g_m = e_m = d = 3$ *for all* m.

Proof. (1) By (45), we have $\gamma_2(3x_m - 3) = 3y_m^2 \in \Gamma^2(X)$. Since $3y_jy_k \in K(X)$, this element is in $\Gamma^2(X)$. The rest of them follow from direct computation of $c_3(3x_jx_k)$.

(2) By (45) and Lemma 5.3 (1), the elements $9y_j^2y_k = 3y_j^2(3y_k)$ and $9y_jy_k^2 = 3y_j(3y_k^2)$ are in $\Gamma^3(X)$, the result follows from the calculation of $c_3(3x_j^2x_k)$.

(3) By (45), we obtain $3(y_1y_2y_3 + \sum_{m,l=1}^3 y_my_l) \in \Gamma^2(X)$. Thus, the first inclusion immediately follows. As $27(y_1y_2y_3)^2 \in \Gamma^6(X)$, the rest of them follow from the computations of $c_3(3x_1x_2x_3)$ and $c_6(6x_1x_2x_3)$.

(4) By the calculation of $c_3(3x_i^2x_jx_k)$, the first inclusion follows. By (45) we get $3x_i^2x_jx_k \in \Gamma^2(X)$. Hence, the second inclusion follows by expanding the element $3(y_i+1)^2(y_j+1)(y_k+1)$.

(5) By direct calculation, we have

$$\begin{aligned}150(y_1^2y_2^2y_3 - y_1^2y_2y_3^2) = &-6(c_3(3x_1x_2^2x_3) + c_3(3x_1x_2^2)) + 6(c_3(3x_1x_2x_3^2) + c_3(3x_1x_3^2))\\ &+3(c_3(3x_1^2x_2^2x_3) + c_3(3x_2^2x_3)) - 3(c_3(3x_1^2x_2x_3^2) + c_3(3x_2x_3^2))\\ &-9c_3(3x_1^2x_2) + 9c_3(3x_1^2x_3) + 42c_3(3x_1x_2) - 42c_3(3x_1x_3).\end{aligned}$$

Since $3y_1^2(3y_2^2y_3) - 3y_1^2(3y_2y_3^2) \in \Gamma^4(X)$, the result follows.

(6) It follows by a direct computation that

$$\begin{aligned}150y_1^2y_2^2y_3^2 - 300y_1^2y_2^2y_3 = &\ 4(c_3(3x_1x_2x_3^2) - c_3(3x_1^2x_2^2x_3)) + 6(c_3(3x_1^2x_2) + c_3(3x_1x_2^2))\\ &+2(c_3(3x_1x_3^2) + c_3(3x_2x_3^2)) - 2(c_3(3x_1^2x_2x_3^2) + c_3(3x_1x_2^2x_3^2))\\ &+8(c_3(3x_1^2x_2x_3) + c_3(3x_1x_2^2x_3)) - 8(c_3(3x_1x_3) + c_3(3x_2x_3))\\ &+c_3(3x_1^2x_2^2x_3^2) - 16c_3(3x_1x_2x_3) - 36c_3(3x_1x_2).\end{aligned}$$

As $9y_1^2y_2^2y_3^2 \in \Gamma^4(X)$ by Lemma 5.3 (3) and $3y_1^2(3y_2^2y_3) \in \Gamma^4(X)$, the result follows. □

Applying Proposition 5.2, we prove the main result of this section.

Theorem 5.4. *The maximal torsion in Chow group of codimension* 2 *of the product of three Severi-Brauer surfaces is* $(\mathbb{Z}/3\mathbb{Z})^{\oplus 8}$. *In other words,* $\mathcal{M}(\mathsf{SB}_3) = 3^8$.

Proof. Let A_1, A_2, A_3 be division algebras of degree 3 over F and let X be the product of the corresponding Severi-Brauer surfaces $\mathrm{SB}(A_1), \mathrm{SB}(A_2), \mathrm{SB}(A_3)$. Set

$$\beta_n = |\Gamma^{n/n+1}(X_E)/\operatorname{Im}(\mathrm{res}^{n/n+1})|/|K^n(X_E)/K^n(X)|,$$

where E is a splitting field of X, $1 \le n \le 6$, and $K^n(X_E)$ (resp. $K^n(X)$) is the codimension n part of $K(X_E)$ (resp. $K(X)$).

We shall find upper bounds of β_n for $1 \le n \le 6$. First of all, by (45) we have $\beta_1 \le 1$. For the rest of them, we will find upper bounds using case by case analysis. Let i, j, k be integers such that $\{i, j, k\} = I_3$, $f = f_1f_2f_3$, $g = g_1g_2g_3$, $e = e_1e_2e_3$, $G = \{1 \le m \le 3 \,|\, g_m = 3\}$ and $H = \{1 \le m \le 3 \,|\, f_m = 9\}$. Observe that if one of $\{e_1, e_2, e_3\}$ is 3 (say, $e_i = 3$), then $d, g_m \le 9$ for $1 \le m \le 3$ and $9y_1y_2y_3 = 3y_jy_k(3y_i) \in \Gamma^3(X)$ by Lemma 5.3 (1).

Case: $e_1 = e_2 = e_3 = 3$. By Lemma 5.3 (1), we have $\beta_2 \le 1$. If $d = 3$, then by Lemma 5.3 (1) and (3) we get $3y_1y_2y_3 \in \mathrm{Im}(\mathrm{res}^{3/4})$. Hence, by Lemma 5.3 (1), (2), and (4) we have

$$f^2 \cdot \beta_3 \le \begin{cases} 3^6 & \text{if } |H| = 0, \\ 3^3 f_i f_j \min\{g_i, g_j\} & \text{if } |H| = 1, f_k = 9, \\ 3^3 f_i \max\{3^{4-|G|}, 3^2\} & \text{if } |H| = 2, f_i = 3, \\ 3^3 \max\{3^{6-|G|}, 3^4\} & \text{if } |H| = 3. \end{cases}$$

It follows from 5.3 (1) that $9y_iy_jy_k^2, 9y_i^2y_j^2 \in \mathrm{Im}(\mathrm{res}^{4/5})$, thus we obtain $\beta_4 \le 3^{12}/3^3g$. Again, by Lemma 5.3 (1), we have $3y_i^2(3y_j^2y_k + 3j_jy_k^2) \in \mathrm{Im}(\mathrm{res}^{4/5})$. Therefore, we have $\beta_5 \le 3^6/g$. It follows from Lemma 5.3 (3) that we have $\beta_6 \le 3^3/3^2$. Finally, by (3) we conclude that

$$|\oplus \Gamma^{n/n+1}(X)_{tors}| \le \begin{cases} 3^{16}/g^2 & \text{if } |H| = 0, \\ 3^{13} \min\{g_i, g_j\}/g^2 & \text{if } |H| = 1, f_k = 9, \\ 3^{10} \max\{3^{4-|G|}, 3^2\}/g^2 & \text{if } |H| = 2, \\ 3^7 \max\{3^{6-|G|}, 3^4\}/g^2 & \text{if } |H| = 3. \end{cases} \tag{46}$$

The maximum upper bound of (46) is 3^{10} when $g_m = f_m = 3$ for all $1 \le m \le 3$ and $d \in \{3, 9\}$. If $g_m = f_m = d = 3$, then by Lemma 5.3 (5) and (6) we have $a := 3y_1^2y_2^2y_3 - 3y_1^2y_2y_3^2, b := 3y_1^2y_2^2y_3^2 - 6y_1^2y_2^2y_3 \in \Gamma^3(X)\backslash\Gamma^4(X)$ as any element of $\Gamma^4(X)$ is divisible by 9. Hence, the classes of a and b give torsion elements of $\Gamma^{3/4}(X)$ of order 3 since $3a = 3y_1^2(3y_2^2y_3) - 3y_1^2(3y_2y_3^2), 3b = 9y_1^2y_2^2y_3^2 - 3y_1^2(3y_2^2y_3) \in \Gamma^4(X)$. Moreover, we have $a - b$, $a + b \notin \Gamma^4$ as any element of $\Gamma^4(X)$ is divisible by 9, thus the subgroups generated by a and b have trivial intersection. By Proposition 5.2, we have

$$\Gamma^{2/3}(X)_{tors} = (\mathbb{Z}/3\mathbb{Z})^{\oplus 8} \text{ and } \Gamma^{3/4}(X)_{tors} = (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}.$$

In this case, by (7) we have

$$\mathrm{CH}^2(\bar{X})_{tors} = (\mathbb{Z}/3\mathbb{Z})^{\oplus 8}, \tag{47}$$

where $\bar{X}$ is the corresponding generic variety. If $g_m = f_m = 3$ and $d = 9$, then by the same argument the class a is a torsion of $\Gamma^{3/4}(X)$. Moreover, we have either $9y_1^2y_2^2y_3^2$ is a torsion of order 3 in $\Gamma^{4/5}(X) \oplus \Gamma^{5/6}(X)$ or $\beta_6 \le 1$. Therefore, we obtain

$$|\oplus \Gamma^{n/n+1}(X)_{tors}| \le 3^8. \tag{48}$$

Case: $e_i = e_j = 3$ and $e_k = 9$. By Lemma 5.3 (1), (3) and (4) we have

$$\beta_2 \le (3^5 \min\{d, g_1, g_2, g_3\})/3^3e.$$

If $d = 9$, then by Lemma 5.3 (1), (2), and (4) we have

$$f^2 \cdot \beta_3 \le \begin{cases} 3^2 f \min\{9, g_k\} & \text{if } |H| = 0, \\ 3^2 f_t f_k g_t g_k & \text{if } |H| = 1, f_s = 9, \\ 3^2 f_i f_j g_k \min\{g_i, g_j\} & \text{if } |H| = 1, f_k = 9, \\ 3^{|H|+1} g & \text{if } |H| \ge 2, \end{cases}$$

where s and t are integers such that $\{s,t\}=\{i,j\}$. Similarly, if $d=3$, then it follows by Lemma 5.3 (1), (2), (3), and (4) that

$$f^2\cdot\beta_3\leq\begin{cases}3^2f\min\{9,g_m\} & \text{if } |H|=0,\\ 3^2f_tf_kg_s\min\{g_t,g_k\} & \text{if } |H|=1, f_s=9,\\ 3^2f_if_j\max\{3^{4-|G|},3^2\} & \text{if } |H|=1, f_k=9,\\ 3^{|H|+1}g & \text{if } |H|\geq 2,\end{cases}$$

where the minimum of the first inequality ranges over $1\leq m\leq 3$. By Lemma 5.3 (1), we have $9y_iy_jy_k^2, 9y_i^2y_j^2\in\mathrm{Im}(\mathrm{res}^{4/5})$. Hence, $\beta_4\leq 3^{12}/(3^4g)$.

By Lemma 5.3 (1), we have $3y_i^2(3y_j^2y_k+3y_jy_k^2), 3y_j^2(3y_i^2y_k+3y_iy_k^2)\in\mathrm{Im}(\mathrm{res}^{5/6})$. If $f_i=3$ (resp. $f_j=3$), then by Lemma 5.3 (2) we obtain $3y_i^2(3y_j^2y_k-3y_jy_k^2)\in\mathrm{Im}(\mathrm{res}^{5/6})$ (resp. $3y_j^2(3y_i^2y_k-3y_iy_k^2)\in\mathrm{Im}(\mathrm{res}^{5/6})$). Moreover, if $d=3$, then by Lemma 5.3 (3) $3y_k^2(-3y_1y_2y_3+3\sum_{m,l=1}^{3}y_m^2y_l)=9y_i^2y_jy_k^2+9y_iy_j^2y_k^2\in\mathrm{Im}(\mathrm{res}^{5/6})$. Therefore, we conclude that $\beta_5\leq 3^6/g$ (resp. $3^7/g$) if one of $\{f_i,f_j,d\}$ is 3 (resp. otherwise). By Lemma 5.3 (3), we have $\beta_6\leq 3$. In conclusion, we have

$$|\oplus\Gamma^{n/n+1}(X)_{tors}|\leq 3^8 \text{ for all cases.} \tag{49}$$

Case: $e_i=3$ and $e_j=e_k=9$. By Lemma 5.3 (1), (3) and (4) we obtain

$$\beta_2\leq(3^4\min\{g_j,g_k\}\min\{d,g_i\})/3^3e.$$

In codimension 3, by Lemma 5.3 (1), (2), (3), and (4) we have

$$f^2\cdot\beta_3\leq\begin{cases}3fg_jg_k & \text{if } |H|=0, d=9,\\ 3fg_i\min\{g_j,g_k\} & \text{if } |H|=0, d=3,\\ 3^3f_jf_kg_jg_k & \text{if } |H|=1, f_i=9, d=9,\\ 3^3f_jf_k\max\{3^{4-|G|},9\} & \text{if } |H|=1, f_i=9, d=3,\\ 3^3g & \text{if } |H|=1, f_i\neq 9,\\ 3^{|H|+2}g & \text{if } |H|\geq 2.\end{cases}$$

By Lemma 5.3 (1), we get $3y_i^2(3y_jy_k), 3y_i(3y_j^2y_k+3y_jy_k^2)\in\mathrm{Im}(\mathrm{res}^{4/5})$. Hence, it follows from Lemma 5.3 (2) that $\beta_4\leq 3^8/g$ (resp. $3^7/g$) if $|H|=3$ (resp. otherwise).

By Lemma 5.3 (1), we obtain $3y_i^2(3y_j^2y_k+3y_jy_k^2)\in\mathrm{Im}(\mathrm{res}^{5/6})$. Therefore, by Lemma 5.3 (2) and (3) we have

$$g\cdot\beta_5\leq\begin{cases}9^3 & \text{if } |H|\leq 1 \text{ or } d=3,\\ 9^2\cdot 27 & \text{if } |H|=2, d\neq 3,\\ 9\cdot 27^2 & \text{if } |H|=3, d\neq 3.\end{cases}$$

Finally, by Lemma 5.3 (3) $\beta_6\leq 3$. Therefore, for all cases we have

$$|\oplus\Gamma^{n/n+1}(X)_{tors}|\leq 3^6. \tag{50}$$

Case: $e_1=e_2=e_3=9$. By Lemma 5.3 (1), (3) and (4) we have

$$\beta_2\leq 3^3\max\{1/3^3, 1/3^{|G|+[3/d]}\}/3^3.$$

In codimension 3, by Lemma 5.3 (2), (3) and (4) we obtain $\beta_3 \le (3^3 fg)/df^2$. It follows from Lemma 5.3 (4) that

$$eg \cdot \beta_4 \le \begin{cases} 9^5 \cdot 27 & \text{if } |H| \le 1, \\ 9^4 \cdot 27^2 & \text{if } |H| = 2, \\ 9^3 \cdot 27^3 & \text{if } |H| = 3. \end{cases}$$

In codimension 5, by Lemma 5.3 (2) and (3) we obtain

$$g \cdot \beta_5 \le \begin{cases} 9^3 & \text{if } d = 3, \\ 9^2 \cdot 27 & \text{if } d \ne 3, |H| \le 1, \\ 9 \cdot 27^2 & \text{if } d \ne 3, |H| = 2, \\ 27^3 & \text{if } d \ne 3, |H| = 3. \end{cases}$$

Finally, by Lemma 5.3 (3) we have $\beta_6 \le 3^3/\max\{9, d\}$. Therefore, for all cases we obtain

$$|\oplus \Gamma^{n/n+1}(X)_{tors}| \le 3^7. \tag{51}$$

In conclusion, the result follows from (47), (48), (49), (50), and (51). □

6. Product of Quadric surfaces

In this section, we obtain upper bounds for the torsion in Chow group of codimension 2 of the product of two quadric surfaces in Theorem 6.1 and the product of three quadric surfaces with the same discriminant in Theorem 6.5. In the case of the product of two quadric surfaces, we also provide a sharp lower bound in the gamma filtration in Proposition 6.3.

Let F be a field of characteristic different from 2 and let $q = \langle c, -a, -b, ab\rangle$ be a non-degenerate quadratic form over F of rank 4 for $c, a, b \in F^\times$. If the discriminant is trivial, then the quadric surface corresponding to the form q is birational to $\mathbb{P}^1 \times \mathrm{SB}(Q)$, where Q is the quaternion F-algebra determined by a and b. Otherwise, the quadric is isomorphic to $R_{L/F}(\mathrm{SB}(Q))$, where $R_{L/F}$ is the Weil restriction over a quadratic field $L = F(\sqrt{c})$. We shall write $\operatorname{disc} Q$ for the discriminant c.

Consider two quadric surfaces with the corresponding quaternions Q_1, Q_2 and quadratic extensions L_1, L_2 as above. We set

$$X = \begin{cases} \mathrm{SB}(Q_1) \times \mathrm{SB}(Q_2) & \text{if } \operatorname{disc} Q_i = 1, \\ \mathrm{SB}(Q_1) \times R_{L_2/F}(\mathrm{SB}(Q_2)) & \text{if } \operatorname{disc} Q_1 = 1 \ne \operatorname{disc} Q_2, \\ R_{L_1/F}(\mathrm{SB}(Q_1)) \times R_{L_2/F}(\mathrm{SB}(Q_2)) & \text{if } \operatorname{disc} Q_i \ne 1. \end{cases} \tag{52}$$

Then, by [3, Corollary 2.5] the torsion in codimension 2 cycles of X of the first and second cases of (52) is isomorphic to that of the product of two quadric surfaces. Therefore, it suffices to consider X for the torsion in codimension 2 cycles of the product of two quadric surfaces. We call X the variety associated to the product of two quadric surfaces.

Consider the last case of (52). If $\operatorname{ind}(Q_1)_{L_1} = \operatorname{ind}(Q_1)_{L_1} = 1$, then the associated variety X has torsion-free Chow groups. Thus, we may assume that $\operatorname{ind}(Q_1)_{L_1} = 2$. We choose a splitting field E of X as follows. If $\operatorname{ind}(Q_2)_{L_2} = 1$, then we take a maximal subfield ($\ne L_2$) of Q_1 for E. Otherwise, we take for E a common maximal subfield ($\ne L_1, L_2$) of Q_1 and

Q_2 if $\mathrm{ind}(Q_1 \otimes Q_2) \leq 2$ or the tensor product of maximal subfields $E_1(\neq L_2)$ of Q_1 and $E_2(\neq L_1)$ of Q_2 if $\mathrm{ind}(Q_1 \otimes Q_2) = 4$. Hence, $d := [E : F] = 4$ if $\mathrm{ind}(Q_1 \otimes Q_2) = 4$ and $d = 2$ otherwise. For the second case (52), we choose a splitting field E in the same way.

The theorem below was proven in [3]. Here, we give an elementary proof which does not use any cohomological method and K theory of quadrics. Moreover, we find upper bound of the total torsion in the topological filtration of the product of two quadric surfaces with nontrivial discriminants.

Theorem 6.1. [3, Theorems 5.1, 5.7, 5.8, 5.9] *Let X be the variety associated to the product of two quadric surfaces. Then, the torsion subgroup $\mathrm{CH}^2(X)_{tors}$ is either trivial or $\mathbb{Z}/2\mathbb{Z}$, i.e., $\mathcal{M}(Q_2) \leq 2$.*

Proof. Let Q_i be a quaternion algebra for $i = 1, 2$, X the variety associated to the product of two quadric surfaces of Q_i, and E the splitting field of X. If $\operatorname{disc} Q_i = 1$ for all i, then the variety X is torsion free. From now on we only consider the other cases. To apply (3) we shall find upper bounds of $\alpha_n := |T^{n/n+1}(X_E)/\operatorname{Im}(\mathrm{res}^{n/n+1})|$ for each of the following 3 cases.

Case: $L_1L_2 := L_1 \otimes L_2$ is a biquadratic field extension. Let $L = L_1L_2$. Then, we have $X_E = R_{EL_1/E}(\mathbb{P}^1) \times R_{EL_2/E}(\mathbb{P}^1)$ and $X_{EL} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which have torsion-free Chow groups. For each $1 \leq k \leq 4$, let x_k be the pullback of the class of tautological line bundle on the projective space in $K(X_{EL})$. We set $x_{i_1 \cdots i_k} = x_{i_1} \cdots x_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq 4$. By the action of the Galois group of EL/E, we have the following bases of $K(X_E)$ and $K(X)$, respectively:

$$\{1, x_{2i-1} + x_{2i}, x_{12}, x_{34}, (x_1 + x_2)(x_3 + x_4), x_{12}(x_3 + x_4), x_{34}(x_1 + x_2), x_{1234}\} \text{ and} \tag{53}$$

$$\{1, e_i(x_{2i-1} + x_{2i}), x_{12}, x_{34}, f(x_1 + x_2)(x_3 + x_4), e_2x_{12}(x_3 + x_4), e_1x_{34}(x_1 + x_2), x_{1234}\}, \tag{54}$$

where $e_i = \mathrm{ind}(Q_i)_L$ and $f = \mathrm{ind}(Q_1 \otimes Q_2)_L$. Then, we have $|K(X_E)/K(X)| = e_1^2e_2^2f$. If $e_1 = e_2 = 1$, then $f = 1$. Hence, we may assume that $e_1e_2 \geq 2$.

Let $y_k = x_k - 1$. Set $y_{i_1 \cdots i_k} = y_{i_1} \cdots y_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq 4$. We will use other bases for $K(X_E)$ the basis (53) by replacing x_k by y_k and for $K(X)$

$$\{1, e_i(y_{2i-1} + y_{2i}), z_{12}, z_{34}, f(y_1 + y_2)(y_3 + y_4), e_2z_{12}(y_3 + y_4), e_1z_{34}(y_1 + y_2), z_{12}z_{34}\}, \tag{55}$$

where $z_{12} = y_1 + y_2 + y_1y_2$ and $z_{34} = y_3 + y_4 + y_3y_4$.

It follows from $z_{12}, z_{34} \in K(X)$ that $y_1 + y_2, y_3 + y_4 \in \operatorname{Im}(\mathrm{res}^{1/2})$, thus we have $\alpha_1 = 1$. By the same argument, we get $(y_1 + y_2)(y_3 + y_4) \in \operatorname{Im}(\mathrm{res}^{2/3})$. In addition, it follows from the basis and (2) that $e_1y_{12}, e_2y_{34} \in T^2(X)$. Hence, $\alpha_2 \leq e_1e_2$.

If $\mathrm{ind}(Q_1 \otimes Q_2) = 1$, then it follows from closed embeddings $\mathrm{SB}(Q_1) \times R_{L_2/F}(\mathrm{SB}(Q_1)) \hookrightarrow X$ and $R_{L_1/F}(\mathrm{SB}(Q_1)) \times \mathrm{SB}(Q_1) \hookrightarrow X$ that we have $y_{34}(y_1 + y_2), y_1y_2(y_3 + y_4) \in \operatorname{Im}(\mathrm{res}^{3/4})$, respectively. Otherwise, it follows from $e_1y_{12} \cdot (y_{34} + y_3 + y_4)$ and $e_2y_{34} \cdot (y_{12} + y_1 + y_2)$ that $e_1y_{12}(y_3 + y_4), e_2y_{34}(y_1 + y_2) \in \operatorname{Im}(\mathrm{res}^{3/4})$. Therefore, $\alpha_3 \leq 1$ if $\mathrm{ind}(Q_1 \otimes Q_2) = 1$ and $\alpha_3 \leq e_1e_2$ otherwise.

By a transfer argument, we have $\alpha_4 \leq d$. Hence, we obtain

$$|\oplus T^{n/n+1}(X)_{tors}| \leq \begin{cases} 1 & \text{if } \mathrm{ind}(Q_1 \otimes Q_2) = 1, \\ d/f & \text{otherwise.} \end{cases}$$

If $f=1$, then $\mathrm{ind}(Q_1\otimes Q_2)\leq 2$, thus $d=2$. Therefore, $|T^{2/3}(X)_{tors}|\leq 2$. Moreover, the order of the group $\oplus T^{n/n+1}(X)_{tors}$ is trivial if $f=4$ or $f=d=2$ or $\mathrm{ind}(Q_1\otimes Q_2)=1$.

Case: $L_1=L_2$. Let $L=L_1=L_2$. Then, $X_E=R_{EL/E}(\mathbb{P}^1)\times R_{EL/E}(\mathbb{P}^1)$. Applying the same argument with the previous case, we have the basis (53) (resp. (54)) replacing $(x_1+x_2)(x_3+x_4)$ (resp. $f(x_1+x_2)(x_3+x_4)$) with two elements $x_{13}+x_{24}$ and $x_{14}+x_{23}$ (resp. $f(x_{13}+x_{24})$ and $f(x_{14}+x_{23})$ for $K(X_E)$ (resp. $K(X)$). As $e_1=2$, we have $|K(X_E)/K(X)|=4e_2^2f^2$.

Similarly, we use other bases for $K(X_E)$ the basis by replacing x_k with y_k and for $K(X)$ the basis replacing $f(y_1+y_2)(y_3+y_4)$ with two elements $f(y_{13}+y_{24}+\sum_{k=1}^4 y_k)$ and $f(y_{14}+y_{23}+\sum_{k=1}^4 y_k)$.

By the same argument used in the previous case, we have $\alpha_1=1$. In codimension 2, we have $2y_{12}, e_2y_{34}, (y_1+y_2)(y_3+y_4)\in\mathrm{Im}(\mathrm{res}^{2/3})$. If $f=1$, then $e_2=d=2$, and $X=R_{L/F}(\mathrm{SB}(Q_1))\times R_{L/F}(\mathrm{SB}(Q_1))$, thus by the diagonal embedding $R_{L/F}(\mathrm{SB}(Q_1))\hookrightarrow X$ the sum of all elements of codimension 2 in the basis is contained in the group $\mathrm{Im}(\mathrm{res}^{2/3})$. Moreover, if $f\neq 1$, then $f(y_{13}+y_{24}), f(y_{14}+y_{23})\in T^2(X)$. Therefore, we obtain $\alpha_2\leq 2e_2f$.

If $f=1$, then we have $z_{12}(y_{12}+y_{34}+y_{13}+y_{24}+y_{14}+y_{23})\in\mathrm{Im}(\mathrm{res}^{3/4})$. Otherwise, we obtain $2y_{12}(y_3+y_4), e_2y_{34}(y_1+y_2)\in\mathrm{Im}(\mathrm{res}^{3/4})$. Hence, we have $\alpha_3\leq 2$ if $f=1$ and $\alpha_3\leq 2e_2$ otherwise. Finally, we get $\alpha_4\leq d$, thus

$$|\oplus T^{n/n+1}(X)_{tors}|\leq\begin{cases}1 & \text{if } f=1,\\ d/f & \text{otherwise.}\end{cases}\tag{56}$$

Hence, the group $\oplus T^{n/n+1}(X)_{tors}$ is trivial except the case where $f=2$ and $d=4$. In the latter case, we can further reduce the upper bound of α_4 to 2 if $2y_{1234}\in T^4(X)$. Hence, the group $\oplus T^{n/n+1}(X)_{tors}$ is trivial in this case. If $2y_{1234}\notin T^4(X)$, then the class of $2y_{1234}=2y_{12}\cdot z_{34}-z_{12}\cdot 2(y_{13}+y_{24})\in T^3(X)$ gives a torsion element of order 2 in $T^{3/4}(X)_{tors}$. Therefore, the group $T^{2/3}(X)_{tors}$ is trivial in all cases.

Case: one of $\operatorname{disc} Q_i$ is trivial. Let $\operatorname{disc} Q_1=1$. Then, we have $X_E=\mathbb{P}^1\times R_{L_2E/E}(\mathbb{P}^1)$. We have the following bases of $K(X_E)$ and $K(X)$, respectively:

$$\{1,x_1,x_2+x_3,x_{23},x_1(x_2+x_3),x_{123}\}\text{ and }\{1,e_1x_1,e_2(x_2+x_3),x_{23},fx_1(x_2+x_3),e_1x_{123}\},$$

where $e_1=\mathrm{ind}(Q_1)_{L_2}, e_2=\mathrm{ind}(Q_2)_{L_2}, f=\mathrm{ind}(Q_1\otimes Q_2)_{L_2}$. It follows that we obtain $|K(X_E)/K(X)|=e_1^2e_2f$. We will use other bases for $K(X_E)$ the above basis by replacing x_k by y_k and for $K(X)$

$$\{1,e_1y_1,e_2(y_2+y_3),z_{23},f(y_1+1)(y_2+y_3),e_1y_1z_{23}\},\tag{57}$$

where $z_{23}=y_2+y_3+y_{23}$.

Obviously, we have $\alpha_1\leq e_1$. In codimension 2, we have $e_2y_{23}, e_1y_1(y_2+y_3)\in T^2(X)$. If $f=1$, then $(y_{12}+y_{13}+y_2+y_3)+z_{23}-2(y_2+y_3)=y_{12}+y_{13}+y_{23}\in T^2(X)$. Hence, we get $\alpha_2\leq\min\{e_1,e_2\}$ if $f=1$ and $\alpha_3\leq e_1e_2$ otherwise. Finally, we have $\alpha_3\leq d$, thus the same upper bound (56) is obtained for the order of the group $\oplus T^{n/n+1}(X)_{tors}$. Hence, the group $\oplus T^{n/n+1}(X)_{tors}$ is trivial if $f=1$ or $f=d=2$ or $f=4$. Assume that $f=2$ and $d=4$. Then it follows from (57) that $2y_{123}\in T^2(X)$. If $2y_{123}\in T^3(X)$, then we have $\alpha_3\leq 2$. Therefore, the group $\oplus T^{n/n+1}(X)_{tors}$ is trivial in this case. Otherwise, the class of

$2y_{123}$ gives a torsion element of order 2 in $T^{2/3}(X)$. In any case, we have $|T^{2/3}(X)_{tors}| \leq 2$. In conclusion, the result follows from (1). $\square$

Remark 6.2. Observe that the proof of Theorem 6.1 indicates when the torsion in Chow group of codimension 2 of X is trivial. Moreover, the proof still works if we replace the topological filtration by the gamma filtration.

Now we provide a nontrivial torsion subgroup in the gamma filtration:

Proposition 6.3. *With the above notations, we have $\Gamma^{2/3}(X)_{tors} = \mathbb{Z}/2\mathbb{Z}$ if L is a biquadratic extension, $f = 2$, and $d = 4$.*

Proof. It follows from the basis (55) that $2y_{1234} \in K(X)$. Hence, by (2) we have $2y_{1234} \in \Gamma^2(X)$. As $d = 4$, $4y_{1234} \in \Gamma^3(X)$. We show that this element is not contained in $\Gamma^3(X)$ by computing the Chern classes of the elements in the basis (54).

Consider the basis (54) of $K(X)$ with $f = e_i = 2$, where $i = 1, 2$. It follows from Whitney formula that we have $c_1(2(x_{2i-1} + x_{2i})) = 2(y_{2i-1} + y_{2i})$, $c_1(x_{2i-1} + x_{2i}) = y_{2i-1} + y_{2i}$, and $c_2(x_{2i-1} + x_{2i}) = y_{2i-1}y_{2i}$. Therefore, we have

$$c_2(2(x_{2i-1} + x_{2i})) = 4y_{2i-1}y_{2i},\ c_1(x_{2i-1}x_{2i})^2 = 2y_{2i-1}y_{2i} \text{ and } c_1(x_{2i-1}x_{2i})^3 = 0. \tag{58}$$

Similarly, we obtain $c_j(2(x_{2i-1} + x_{2i})) = 0$ for $3 \leq j \leq 4$.

Let $z = (x_1 + x_2)(x_3 + x_4)$, $z' = (y_1 + y_2)(y_3 + y_4)$, $u = y_{123} + y_{124}$, and $v = y_{134} + y_{234}$. Then, by a direct computation, we have

$$c_j(z) = \begin{cases} 2(\sum_{k=1}^{4} y_k) + z' & \text{for } j = 1, \\ 2y_{1234} + 3z' + 4(u + v + y_{12} + y_{34}) & \text{for } j = 2, \\ 4(u + v + 3y_{1234}) & \text{for } j = 3, \\ 2y_{1234} & \text{for } j = 4. \end{cases} \tag{59}$$

Since $c_2(2z) = c_1(z)^2 + 2c_2(z)$, it follows from (59) that

$$c_2(2z) = 8y_{1234} + 14z' + 16(u + v + y_{12} + y_{34}). \tag{60}$$

As $c_3(2z) = 2(c_1(z)c_2(z) + c_3(z))$ and $c_4(2z) = 2(c_1(z)c_3(z) + c_4(z)) + c_2(z)^2$, it follows from (59) that

$$c_j(2z) \equiv 0 \mod 4 \text{ for } j = 3, 4. \tag{61}$$

Let $w = x_{12}(x_3 + x_4)$. Then, we obtain $c_1(w) = 2z_{12} + y_3 + y_4 + z' + u$ and $c_2(w) = 4y_{1234} + 3u + 2(y_{12} + v) + y_{13} + y_{14} + y_{23} + y_{24} + y_{34}$. Therefore, we have

$$c_j(2w) = \begin{cases} 2(8y_{1234} + 9u + 4v + 6y_{12} + 3y_{13} + 3y_{23} + 3y_{14} + 3y_{24} + 2y_{34}) & \text{for } j = 2, \\ 4(10y_{1234} + 3u + 2v) & \text{for } j = 3, \\ 8y_{1234} & \text{for } j = 4. \end{cases} \tag{62}$$

Let $w' = x_{34}(x_1 + x_2)$. Then, we have the Chern classes (62) for $2w'$ by replacing 1, 2, 3, 4 with 3, 4, 1, 2, respectively.

It follows from (58) that $c_1(x_{12})^2c_1(x_{34}) = c_1(x_{12})^2c_1(x_{1234}) = 2(u+y_{1234})$, $c_1(x_{34})^2c_1(x_{12}) = c_1(x_{34})^2c_1(x_{1234}) = 2(v + y_{1234})$. Since $c_1(x)$ is divisible by 2 for any element $x \in \{2z, 2w, 2w', 2(x_{2i-1} + x_{2i})\}$, one can see easily that the subgroup generated by the products of three of the first Chern classes of any element of the basis (54) is generated by

$2(u+y_{1234})$ and $2(v+y_{1234})$ modulo 4. Similarly, using (58), (60), (62) one sees that the subgroup generated by the products of the first and second Chern classes of any element of the basis (54) is also generated by $2(u+y_{1234})$ and $2(v+y_{1234})$ modulo 4. It follows from (61) and (62) that the third and fourth Chern classes of $2z, 2w, 2w'$ is divisible by 4. Therefore, the subgroup $\Gamma^3(X)$ is generated by $2(u+y_{1234})$ and $2(v+y_{1234})$ modulo 4. Hence, $2y_{1234}$ is not contained in $\Gamma^3(X)$ and this element gives a torsion of $\Gamma^{2/3}(X)$ of order 2. The result immediately follows from Remark 6.2. □

Remark 6.4. If $\bar{X}$ is a corresponding generic variety to X in Proposition 6.3, then we obtain $\mathrm{CH}^2(\bar{X}) = \mathbb{Z}/2\mathbb{Z}$, which recovers a theorem of Izhboldin and Karpenko [5, Theorem 14.1]. Indeed, it is possible to find such a variety by showing that the gamma filtration for the variety $R_{L_1/F}(\mathrm{SB}(Q_1')) \times R_{L_2/F}(\mathrm{SB}(Q_2')) \times R_{L/F}(\mathrm{SB}(2, Q_1' \otimes Q_2'))$ is torsion free, where $L = L_1L_2$ is a biquadratic extension and $\mathrm{SB}(2, Q_1' \otimes Q_2')$ is the generalized Severi-Brauer variety of rank 2 left ideals in the biquaternion algebra $(Q_1' \otimes Q_2')_L$.

6.1. **Three quadric surfaces with the same discriminant.** In this subsection we consider the product of three quadric surfaces with the same discriminant. Let Q_1, Q_2, Q_3 be three quaternion F-algebras and let L be the quadratic extension over F corresponding to three quadratic surfaces with the same discriminant as above.

We set

$$X = \begin{cases} \mathrm{SB}(Q_1) \times \mathrm{SB}(Q_2) \times \mathrm{SB}(Q_3) & \text{if } \operatorname{disc} Q_i = 1, \\ R_{L/F}(\mathrm{SB}(Q_1)) \times R_{L/F}(\mathrm{SB}(Q_2)) \times R_{L/F}(\mathrm{SB}(Q_3)) & \text{otherwise} \end{cases} \tag{63}$$

and call it the variety associated to the product of three quadric surfaces with the same discriminant. Then, by the same argument as in the case of two quadric surfaces, the torsion in codimension 2 cycles of X is isomorphic to that of the product of three quadric surfaces with the same discriminant.

Let $h = \operatorname{ind}(Q_1 \otimes Q_2 \otimes Q_3)$, $J = \{\{1,2\},\{3,4\},\{5,6\}\}$, and $f_{pq} = \operatorname{ind}(Q_{\max\{r,s\}/2} \otimes Q_{\max\{t,u\}/2})_L$ for any $\{\{p,q\},\{r,s\},\{t,u\}\} = J$. Set

$$F_m = \{\{p,q\} \in J \mid f_{pq} = m\} \text{ for } m = 1, 2, 4.$$

Consider the second case of (63). If $\operatorname{ind}(Q_i)_L = 1$ for all $1 \le i \le 3$, then the variety X has torsion-free Chow groups, thus we may assume that $\operatorname{ind}(Q_1)_L \neq 1$. We choose a splitting field E of X as follows. If $\operatorname{ind}(Q_2)_L = \operatorname{ind}(Q_3)_L = 1$, then we take a maximal subfield for E. If $\operatorname{ind}(Q_2)_L = 2$ and $\operatorname{ind}(Q_3)_L = 1$, then we take for E a common maximal subfield of Q_1 and Q_2 if $\operatorname{ind}(Q_1 \otimes Q_2) \le 2$ or the tensor product of maximal subfields of Q_1 and Q_2 if $\operatorname{ind}(Q_1 \otimes Q_2) = 4$.

Now we may assume that $\operatorname{ind}(Q_i)_L = 2$ for all $1 \le i \le 3$. If $h = 8$, then we take for E the tensor product of maximal subfields of Q_i. If $h = 1$, then $|F_2| = 3$, thus we take the tensor product of a common maximal subfield of Q_1 and Q_2 and a maximal subfield of Q_3 for E. If $|F_1| \ge 2$, then we have $(Q_1)_L \simeq (Q_2)_L \simeq (Q_3)_L$, thus we take for E a maximal subfield of Q_1. If $h \in \{2,4\}$, $|F_1| \le 1$, and $|F_4| \ge 1$, then there exist Q_i and Q_j such that $\operatorname{ind}(Q_i \otimes Q_j) = 4$ for some $1 \le i \neq j \le 3$, thus we take for E the tensor product of maximal subfields of Q_i and Q_j which also splits the remaining quaternion algebra. If $h \in \{2,4\}$, $|F_1| = 1$, and $|F_4| = 0$, then there exist Q_i and Q_j such that $(Q_i)_L \simeq (Q_j)_L$ for some $1 \leq i \neq j \leq 3$, thus we take for E the tensor product of a maximal subfield of Q_i and a maximal subfield of the remaining quaternion algebra. Hence, we have

$$d := [E : F] = \begin{cases} 2 & \text{if } |F_1| \geq 2, \\ 4 & \text{if } h = 1 \text{ or } h \in \{2, 4\}, |F_1| \leq 1, \\ 8 & \text{if } h = 8. \end{cases}$$

Theorem 6.5. *The torsion subgroup in the codimension* 2 *Chow group of the product of three quadric surfaces with the same discriminant is contained in* $(\mathbb{Z}/2\mathbb{Z})^{\oplus 7}$.

Proof. Let Q_i be a quaternion F-algebra for $1 \leq i \leq 3$ such that the corresponding qaudrics have the same discriminant. Let X be the associated variety to the product of three quadric surfaces of Q_i and E be the splitting field of X as above. If the discriminant is trivial, the result follows from Proposition 4.2. Hence, we may assume that the discriminant is non-trivial, thus we have $X_E = R_{EL/E}(\mathbb{P}^1) \times R_{EL/E}(\mathbb{P}^1) \times R_{EL/E}(\mathbb{P}^1)$ and $X_{EL} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

For each $1 \leq k \leq 6$, let x_k be the pullback of the class of tautological bundle on the projective line in $K(X_{EL})$. Set $x_{i_1 \cdots i_k} = x_{i_1} \cdots x_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq 6$. It follows from the action of the Galois group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ of LE/E that we have the bases of $K(X_E)$ and $K(X)$, repectively:

$\{1,\, x_p + x_q,\, x_{pq},\, x_{pr} + x_{qs},\, x_{rs}(x_p + x_q),\, x_{prt} + x_{qsu},\, x_{pqrs},\, x_{pq}(x_{rt} + x_{su}),\, x_{pqrs}(x_t + x_u),$
$x_{pqrstu}\}$ and

$\{1,\, e_{pq}(x_p + x_q),\, x_{pq},\, f_{tu}(x_{pr} + x_{qs}),\, e_{pq}x_{rs}(x_p + x_q),\, g(x_{prt} + x_{qsu}),\, x_{pqrs},\, f_{pq}x_{pq}(x_{rt} + x_{su}),$
$e_{tu}x_{pqrs}(x_t + x_u),\, x_{pqrstu}\}$,

where $e_{pq} = \operatorname{ind}(Q_{\max\{p,q\}/2})_L$, $g = \operatorname{ind}(Q_1 \otimes Q_2 \otimes Q_3)_L$, and p, q, r, s, t, u range over $\{\{p, q\}, \{r, s\}, \{t, u\}\} = J$. Then, we have $|K(X_E)/K(X)| = (e_{12}e_{34}e_{56}f_{12}f_{34}f_{56}g)^4$.

Let $y_k = x_k - 1$ and $y_{i_1 \cdots i_k} = y_{i_1} \cdots y_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq 6$. To simplify the computation, we shall use other bases for $K(X_E)$ the above basis replacing x_k with y_k and for $K(X)$

$\{1,\, e_{pq}(y_p + y_q),\, z_{pq}, f_{tu}(z_{pr} + z_{qs}),\, e_{pq}z_{rs}(y_p + y_q),\, g(z_{prt} + z_{qsu}),\, z_{pq}z_{rs}, f_{pq}z_{pq}(z_{rt} + z_{su}),$
$z_{pq}z_{rs}z_{tu}\}$,

where $z_{pq} = y_p y_q + y_p + y_q$, $z_{pr} = y_p y_r + y_p + y_r$, and $z_{prt} = y_{prt} + z_{pr} + y_{pt} + y_{rt} + y_t$.

Let $\alpha_n = |T^{n/n+1}(X_E)/\operatorname{Im}(\operatorname{res}^{n/n+1})|$. We will find upper bounds of α_n for $1 \leq n \leq 6$. Observe that any basis element of $K(X_E)$ multiplied by d is contained in the image of the restriction map. Since $z_{pq} \in K(X)$, we get $y_p + y_q \in \operatorname{Im}(\operatorname{res}^{1/2})$. Hence, we have $\alpha_1 = 1$. We divide the proof into three cases.

Case: $f_{pq} \neq 1$ for all $\{p, q\} \in J$, i.e., $|F_1| = 0$. It follows from the basis of $K(X)$ that $e_{pq}y_{pq} \in T^2(X)$ and $(y_p + y_q)(y_r + y_s) \in \operatorname{Im}(\operatorname{res}^{2/3})$ for any $\{p, q\} \neq \{r, s\} \in J$. Since $f_{tu} \neq 1$, we obtain $f_{tu}(y_{pr} + y_{qs}) \in T^2(X)$ for any $\{t, u\} \in J$. Therefore, we have

$$\alpha_2 \leq e_{12}e_{34}e_{56}f_{12}f_{34}f_{56}. \tag{64}$$

Since $z_{pq} \in T^1(X)$, we have $e_{rs}y_{rs} \cdot (y_p + y_q) \in \operatorname{Im}(\operatorname{res}^{3/4})$. Moreover, as $z_{12}z_{34}z_{56} \in T^3(X)$ we obtain $\sum y_{prt} + y_{qsu} \in \operatorname{Im}(\operatorname{res}^{3/4})$, where the sum ranges over all such elements in the basis. As $f_{tu} \neq 1$, the element $f_{tu}(y_{pr}+y_{qs})\cdot(y_t+y_u)$ is contained in the group $\mathrm{Im}(\mathrm{res}^{3/4})$. Hence, we have

$$\alpha_3 \leq (e_{12}e_{34}e_{56})^2 f_{12}f_{34}f_{56}d/\max\{f_{12},f_{34},f_{56}\}. \tag{65}$$

As $e_{pq}y_{pq},\, f_{pq}(y_{rt}+y_{su}) \in T^2(X)$, we obtain $e_{pq}e_{rs}y_{pqrs},\, e_{pq}y_{pq}\cdot f_{pq}(y_{rt}+y_{su}) \in T^4(X)$. As $e_{pq}y_{pq}\cdot z_{rs}\cdot z_{tu} \in \mathrm{Im}(\mathrm{res}^{4/5})$, we have $\alpha_4 \leq \prod_{pq\in J} e_{pq}\min\{d, e_{12}e_{34}e_{56}/e_{pq}\}\min\{d, f_{pq}e_{pq}\}$.

It follows from $e_{pq}y_{pq} \in T^2(X)$ that we have $\alpha_5 \leq \prod_{pq\in J}\min\{d, e_{12}e_{34}e_{56}/e_{pq}\}$. Obviously, $\alpha_6 \leq d$. Hence, we obtain

$$|\oplus T^{n/n+1}(X)_{\mathit{tors}}| \leq \frac{d^2}{g^4\max\{f_{12},f_{34},f_{56}\}}\prod_{pq\in J}\frac{\min\{d, e_{12}e_{34}e_{56}/e_{pq}\}\min\{d, f_{pq}e_{pq}\}}{f_{pq}^2}. \tag{66}$$

Let B be the right-hand side of the inequality of (66). We first consider the case where $G := g = h$. Then, one can compute B for each subcase of $g = h$. Consider a subcase where $G = 4$ and $|F_2| = 1$. Then, $f_{pq} = 2$ for some $pq \in J$. If $2y_{rstu} \in T^4(X)$, then $2y_{rstu}(y_p+y_q) \in \mathrm{Im}(\mathrm{res}^{5/6})$, thus we can reduce the upper bound $B(=2^2)$ to 1. Hence, we have $|\oplus T^{n/n+1}(X)_{\mathit{tors}}| \leq 1$. Otherwise, by Lemma 6.6 (1) below we obtain either $|\oplus T^{n/n+1}(X)_{\mathit{tors}}| \leq 2^2$, $T^{3/4}(X)_{\mathit{tors}} = T^{4/5}(X)_{\mathit{tors}} = \mathbb{Z}/2\mathbb{Z}$ or $|\oplus T^{n/n+1}(X)_{\mathit{tors}}| \leq 2$, $T^{3/4}(X)_{\mathit{tors}} = \mathbb{Z}/2\mathbb{Z}$. Therefore, $T^{2/3}(X)_{\mathit{tors}} = 0$ in any case.

Let $E_2 = \{\{p,q\} \in J \,|\, e_{pq} = 2\}$. Then, we may assume that $|E_2| \geq 1$. If $G = 1$, then we have $|F_2| = |E_2| = 3$ and $d = 4$. Therefore, by Lemma 6.6 (2) we can reduce the upper bound 2^6 (resp. 2^{10}) obtained by (64) (resp. 65) to 2^5 (resp. 2^8).

Applying the same argument, together with Lemma 6.6 to each subcase of $g = h$, we obtain

$$|T^{2/3}(X)_{\mathit{tors}}| \leq \begin{cases} 1 & \text{if } G=4,\ 0\leq|F_2|\leq 2 \text{ or } G=|E_2|=d=2, |F_2|=3,\\ 2\,(\text{resp. } 2^5) & \text{if } G=8 \text{ or } G=4\,(\text{resp. } 2), |F_2|=3, |E_2|=3,\\ 2^4 & \text{if } G=|E_2|=2, |F_2|=3, d=4 \text{ or } G=2,\ 0\leq|F_2|\leq 2,\\ 2^6 & \text{if } G=1. \end{cases} \tag{67}$$

Now we consider the case where $H := g = h/2$. By the same argument, we have the same upper bound (67) for $|T^{2/3}(X)_{\mathit{tors}}|$ if $G = H = 1, 2$. Similarly, if $H = 4$, then we obtain

$$|T^{2/3}(X)_{\mathit{tors}}| \leq \begin{cases} 1 & \text{if } H=4, |F_2|=2, |E_2|=2,\\ 2^2 & \text{if } H=4, 2\leq|F_2|\leq 3, |E_2|=3,\\ 2^5\ (\text{resp. } 2^3) & \text{if } H=4, |F_2|=0\,(\text{resp. } |F_2|=1). \end{cases} \tag{68}$$

Case: $|F_1| = 1$. Let $f_{tu} = 1$, $e = e_{pq} = e_{rs}$, and $e' = e_{tu}$. For any number t, we write $m(t)$ for $\min\{d,t\}$. Then, Lemma 6.6 (3) and the above argument implies that

$$\begin{aligned}|\oplus T^{n/n+1}(X)_{\mathit{tors}}| \leq &\big(e^2e'f_{pq}f_{rs}\big)\big(e^3e'^2m(f_{pq})m(f_{rs})d\big)\big(edm(e^2)m(ee')^2m(f_{pq}e)m(f_{rs}e)\big)\cdot\\ &\big(m(e^2)m(ee')^2\big)\big(d\big)/\big(e^2e'f_{pq}f_{rs}g\big)^4,\end{aligned}$$

where each term inside the parentheses of the numerator is the upper bound of α_n for $2 \leq n \leq 6$.

By the same argument used above, Lemma 6.6 (1) and (2) yield

$$|T^{2/3}(X)_{tors}| \leq \begin{cases} 2 & \text{if } G\,(\text{or } H) = 2, |F_2| = 2, |E_2| = 2, \\ 2^2 & \text{if } G\,(\text{or } H) = 2, |F_2| = 2, |E_2| = 1 \text{ or } G\,(\text{or } H) = 1, \\ 2^7 & \text{if } G\,(\text{or } H) = 2,\ 0 \leq |F_2| \leq 2,\ |E_2| = 3. \end{cases} \tag{69}$$

Case: $|F_1| \geq 2$. Then, we have $|F_1| = |E_2| = 3$, $d = 2$, and $g \in \{1, 2\}$. It follows from Lemma 6.6 (3) that $y_{pr} + y_{qs} - (y_{pq} + y_{rs}) \in T^2(X)$ for all $\{p, q\} \neq \{r, s\} \in J$. Since we have $2y_{pq} \in T^2(X)$ and $(y_p + y_q)(y_r + y_s) \in \mathrm{Im}(\mathrm{res}^{2/3})$ for all $\{p, q\} \neq \{r, s\} \in J$, we obtain $\alpha_2 \leq 2^3$.

As $d = 2$ and $\sum y_{prt} + y_{qsu} \in \mathrm{Im}(\mathrm{res}^{3/4})$, where the sum ranges over all such elements in the basis, it follows from Lemma 6.6 (3) that $\alpha_3 \leq 2^6$. Similarly, by Lemma 6.6 (3) we have $\alpha_4 \leq 2^5$. As $d = 2$, we obtain $\alpha_5 \leq 2^3$ and $\alpha_6 \leq 2$. In conclusion, we have

$$|\oplus T^{n/n+1}(X)_{tors}| \leq 2^7/g^4 \tag{70}$$

for $1 \leq g \leq 2$.

The result follows from (67), (68), (69), and (70). □

Lemma 6.6. *With the above notation, the followings hold:*

(1) *If* $f_{pq} = e_{rs}$ *(resp.* $f_{pq} = e_{tu}$*), then* $\begin{cases} e_{rs}y_{rstu}\,(\textit{resp. } e_{tu}y_{rstu}) \in T^3(X), \\ e_{rs}y_{rstu}z_{pq}\,(\textit{resp. } e_{tu}y_{rstu}z_{pq}) \in T^4(X), \\ e_{pq}e_{rs} \cdot y_{123456}(\textit{resp. } e_{pq}e_{tu}y_{123456}) \in T^5(X). \end{cases}$

Moreover, if in addition $e_{rs}y_{rstu}\,(\textit{resp. } e_{tu}y_{rstu}) \notin T^4(X)$*, then we have a subgroup* $\langle e_{rs}y_{rstu}\,(\textit{resp. } e_{tu}y_{rstu})\rangle \subseteq T^{3/4}(X)_{tors}$ *of order* e_{tu} *(resp.* e_{rs}*). If in addition* $e_{rs}y_{rstu}z_{pq}$ *(resp.* $e_{tu}y_{rstu}z_{pq}$*)* $\notin T^5(X)$ *and* $e_{tu}e_{rs}y_{123456} \in T^5(X)$*, then we obtain a subgroup* $\langle e_{rs}y_{rstu} \cdot (y_p + y_q)\,(\textit{resp. } e_{tu}y_{rstu}(y_p + y_q))\rangle \subseteq T^{4/5}(X)_{tors}$ *of order* e_{tu} *(resp.* e_{rs}*). If in addition* $e_{pq}e_{rs}y_{123456}$ *(resp.* $e_{pq}e_{tu}y_{123456}$*)* $\notin T^6(X)$*, then we have a subgroup* $\langle e_{pq}e_{rs}y_{123456}\,(\textit{resp. } e_{pq}e_{tu}y_{123456})\rangle \subseteq T^{5/6}(X)_{tors}$ *of order* e_{tu} *(resp.* e_{rs}*).*

(2) *If* $g = 1$*, then* $\begin{cases} (y_{pr} + y_{qs}) + (y_{pt} + y_{qu}) + (y_{rt} + y_{su}) - (y_{pq} + y_{rs} + y_{tu}) \in \mathrm{Im}(\mathrm{res}^{2/3}), \\ (y_r + y_s)(y_{pt} + y_{qu} - y_{pq} - y_{tu}) + y_{rs}(y_p + y_q + y_t + y_u) \in \mathrm{Im}(\mathrm{res}^{3/4}), \\ (y_p + y_q)(y_{rt} + y_{su} - y_{rs} - y_{tu}) + y_{pq}(y_r + y_s + y_t + y_u) \in \mathrm{Im}(\mathrm{res}^{3/4}). \end{cases}$

(3) *If* $f_{tu} = 1$*, then* $\begin{cases} y_{pr} + y_{qs} - (y_{pq} + y_{rs}) \in T^2(X), \\ y_{pq}(y_r + y_s) - y_{rs}(y_p + y_q) \in \mathrm{Im}(\mathrm{res}^{3/4}), \\ y_{pq}(y_{rt} + y_{su} + y_{ru} + y_{st}) - y_{rs}(y_{pt} + y_{qu} + y_{pu} + y_{qt}) \in \mathrm{Im}(\mathrm{res}^{4/5}), \\ y_{pqtu} + y_{rstu} + y_{tu}(y_{pr} + y_{qs}) + y_{tu}(y_{ps} + y_{qr}) \in \mathrm{Im}(\mathrm{res}^{4/5}). \end{cases}$

Proof. (1) For simplicity, we give the proof for the case of $f_{pq} = e_{rs}$. In this case, we have

$$e_{rs}y_{rs} \cdot (y_{tu} + y_t + y_u) - (y_{rs} + y_r + y_s) \cdot f_{pq}(y_{rt} + y_{su}) = e_{rs}y_{rstu} \in T^3(X).$$

It follows from the previous result that $e_{rs}y_{rstu} \cdot z_{pq} \in T^4(X)$ and $e_{pq}y_{pq} \cdot e_{rs}y_{rstu} \in T^5(X)$.

Since we have $e_{tu}y_{tu} \cdot e_{rs}y_{rs} \in T^4(X)$, $e_{rs}y_{rs} \cdot e_{tu}y_{tu} \cdot z_{pq} - e_{tu}e_{rs}y_{123456} \in T^5(X)$ and $e_{pq}y_{pq} \cdot e_{rs}y_{rs} \cdot e_{tu}y_{tu} \in T^6(X)$, the second statement immediately follows.

(2) If $g = 1$, then $z_{prt} + z_{qsu} - (z_{pq} + z_{rs} + z_{tu}) \in T^2(X)$, which implies the first result. As $y_r + y_s$, $y_p + y_q \in \mathrm{Im}(\mathrm{res}^{1/2})$, the remaining results follow by multiplication the first result by these elements.

(3) Since $f_{tu} = 1$, we have $y_{pr} + y_{qs} - (y_{pq} + y_{rs}) = z_{pr} + z_{qs} - (z_{pq} + z_{rs}) \in T^2(X)$. As $y_p + y_q \in \mathrm{Im}(\mathrm{res}^{1/2})$, the second and third results follow from $(y_p + y_q)(y_{pr} + y_{qs} - y_{pq} - y_{rs}) \in \mathrm{Im}(\mathrm{res}^{3/4})$ and $(y_t + y_u)[y_{pq}(y_r + y_s) - y_{rs}(y_p + y_q)] \in \mathrm{Im}(\mathrm{res}^{4/5})$, respectively.

The assumption implies that $(Q_{\max\{p,q\}/2})_L \simeq (Q_{\max\{r,s\}/2})_L =: Q_L$. Hence, the last one follows from the closed embedding $R_{L/F}(\mathrm{SB}(Q)) \times R_{L/F}(\mathrm{SB}(Q_{\max\{t,u\}/2})) \hookrightarrow R_{L/F}(\mathrm{SB}(Q)) \times R_{L/F}(\mathrm{SB}(Q)) \times R_{L/F}(\mathrm{SB}(Q_{\max\{t,u\}/2}))$. □

Acknowledgments. The author would like to thank Nikita Karpenko for helpful discussion. This work was partially supported by an internal fund from KAIST, TJ Park Junior Faculty Fellowship of POSCO TJ Park Foundation, and National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2013R1A1A1010171).

References

[1] W. Fulton and S. Lang, *Riemann-Roch algebra*, Springer-Verlag, New York, (1985).

[2] S. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological invariants in Galois cohomology*, American Mathematical Society, Providence, RI, (2003).

[3] O. T. Izhboldin and N. A. Karpenko, *Isotropy of six-dimensional quadratic forms over function fields of quadrics*, J. Algebra **1** (1998), no. 1, 65–93.

[4] O. T. Izhboldin and N. A. Karpenko, *Generic splitting fields of central simple algebras: Galois cohomology and nonexcellence*, Algebr. Represent. Theory **2** (1999), no. 1, 19–59.

[5] O. T. Izhboldin and N. A. Karpenko, *Some new examples in the theory of quadratic forms*, Math. Z. **234** (2000), no. 4, 647–695.

[6] N. A. Karpenko, *Algebro-geometric invariants of quadratic forms*, Leningrad (St. Petersburg) Math. J. **2** (1991), no. 1, 119–138.

[7] N. Karpenko, *On topological filtration for Severi-Brauer varieties*, Proc. Sympos. Pure Math. **58** (1995), 275–277.

[8] N. Karpenko, *Codimension* 2 *cycles on Severi-Brauer varieties*, K-Theory **13** (1998), no. 4, 305–330.

[9] N. Karpenko, *Three theorems on common splitting fields of central simple algebras*, Israel J. Math. **111** (1999), 125–141.

[10] E. Peyre, *Products of Severi-Brauer varieties and Galois cohomology*, Proc. Sympos. Pure Math. **58** (1995), 369–401.

[11] E. Peyre, *Galois cohomology in degree three and homogeneous varieties*, K-Theory **15** (1995), no. 2, 99–145.

[12] D. Quillen, *Higher algebraic K-theory. I*, Lecture Notes in Math. **341**, Springer (1973), 85–147.

Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 305-701, Republic of Korea

E-mail address: sanghoonbaek@kaist.ac.kr