THE KAPLANSKY RADICAL OF A QUADRATIC FIELD EXTENSION

KARIM JOHANNES BECHER AND DAVID B. LEEP

Abstract. The radical of a field consists of all nonzero elements that are represented by every binary quadratic form representing 1. Here, the radical is studied in relation to local-global principles, and further in its behaviour under quadratic field extensions. In particular, an example of a quadratic field extension is constructed where the natural analogue to the square-class exact sequence for the radical fails to be exact. This disproves a conjecture of Kijima and Nishi.

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1. Introduction

Let K be a field of characteristic different from 2. Let $K^\times$ denote the multiplicative group of K, $\sum K^{\times 2}$ the subgroup of nonzero sums of squares in K, and $D_K\langle 1,a\rangle$ the subgroup of $K^\times$ consisting of the nonzero elements represented by the binary quadratic form X^2+aY^2, for any $a\in K^\times$. The object of study in this article is the subgroup

$$\mathrm{R}(K) = \bigcap_{a\in K^\times} D_K\langle 1,a\rangle$$

of $K^\times$, called the (*Kaplansky*) *radical* of K. This object was first studied by I. Kaplansky for fields over which there exists a unique quaternion division algebra [7]. It was investigated in more generality by C.M. Cordes [4], who baptized it the *Kaplansky radical* and observed that in several statements about quadratic forms over K one can replace $K^{\times 2}$ by $\mathrm{R}(K)$. We refer to [11, Chap. XII, Sect. 6 & 7] for an introduction to the Kaplansky radical. By [11, Chap.XII, (6.1)] the radical is further characterized as $\mathrm{R}(K)=\{c\in K^\times \mid D_K\langle 1,-c\rangle = K^\times\}$.

In this article we continue the study of the radical. In Section 2 we consider the position of the radical within the inclusions $K^{\times 2}\subseteq \mathrm{R}(K)\subseteq \sum K^{\times 2}$. In Section 3 we study fields satisfying a local-global principle for quadratic forms and derive a determination of the radical as the set of elements that are locally squares. In Section 4 we revisit the behavior of the radical under quadratic field extensions and disprove a conjecture by D. Kijima and M. Nishi discussed in [8], [9], and [6].

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2. Position of the radical

We have the inclusions $K^{\times 2} \subseteq \mathrm{R}(K) \subseteq D_K\langle 1,1\rangle \subseteq \sum K^{\times 2}$. We first consider the two extremal cases for the position of the radical with respect to these inclusions. We say that K is *radical-free* if $\mathrm{R}(K) = K^{\times 2}$.

2.1. **Proposition.** *Assume that $|K^\times/K^{\times 2}| \geq 4$ and there exists $t \in K^\times$ such that $D_K\langle 1,t\rangle = K^{\times 2} \cup tK^{\times 2}$ and $D_K\langle 1,-t\rangle = K^{\times 2} \cup -tK^{\times 2}$. Then K is radical-free.*

Proof. We may choose an element $a \in K^\times \setminus (K^{\times 2} \cup tK^{\times 2})$. Then $a \notin D_K\langle 1,t\rangle$ and thus $-t \notin D_K\langle 1,-a\rangle$, whereby $\mathrm{R}(K) \subseteq D_K\langle 1,-t\rangle \cap D_K\langle 1,-a\rangle = K^{\times 2}$. □

By a $\mathbb{Z}$*-valuation* we mean a valuation with value group $\mathbb{Z}$. For a $\mathbb{Z}$-valuation v on K we denote by K_v the corresponding completion.

2.2. **Corollary.** *Assume that K is henselian with respect to a $\mathbb{Z}$-valuation whose residue field is of characteristic different from 2 and not quadratically closed. Then K is radical-free.*

Proof. It follows from the hypotheses that $|K^\times/K^{\times 2}| \geq 4$. Moreover, any $t \in K^\times$ that has odd value with respect to the given valuation will be such that $D_K\langle 1,t\rangle = K^{\times 2} \cup tK^{\times 2}$ and $D_K\langle 1,-t\rangle = K^{\times 2} \cup -tK^{\times 2}$. Hence, the statement follows from (2.1). □

By [11, Chap. XII, Sect. 6], if K is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ for a prime number p, then K is radical-free; for $p \neq 2$ this can be seen from (2.2).

2.3. **Proposition.** *The following are equivalent:*

(*i*) $\mathrm{R}(K) = \sum K^{\times 2}$;
(*ii*) $\mathrm{R}(K) = D_K\langle 1,1\rangle$;
(*iii*) $I_t^2 K = 0$;
(*iv*) *every torsion 2-fold Pfister form over K is hyperbolic.*

Proof. This follows from [11, Chap. XI, (4.1) and (4.5)] for $n = 2$. □

Condition (*iv*) corresponds to Property (A_2) in the terminology of [5], treated also in [11, Chap. XI, Sect. 4]. Following [9] we say that the field K is *quasi-pythagorean* if it satisfies the equivalent conditions in (2.3). By [11, [Chap. XI, (6.26)] this is further equivalent to having that the u-invariant of K is at most 2. For example, by [11, Chap. XI, (4.10)], any extension of transcendence degree one of a real closed field is quasi-pythagorean.

In [4] Cordes gave an example of a field K with $K^{\times 2} \subsetneq \mathrm{R}(K) \subsetneq \sum K^{\times 2}$ and asked whether one can have such examples where $K^\times/K^{\times 2}$ is finite. M. Kula [10] and L. Berman [2] independently constructed such examples. We give another example where K is a nonreal algebraic extension of $\mathbb{Q}$ having 8 square classes.

2.4. **Example.** The integers $-2, -5$ and 7 are squares in $\mathbb{Q}_3$. Hence, $\mathbb{Q}_3$ contains the field $\mathbb{Q}(\sqrt{-2}, \sqrt{-5})$. Moreover, 7 is not a square in $\mathbb{Q}(\sqrt{-2}, \sqrt{-5})$. Consider the set of subfields of $\mathbb{Q}_3$ that are algebraic extensions of $\mathbb{Q}(\sqrt{-2}, \sqrt{-5})$ and in which 7 is not a square. By Zorn's Lemma, we may choose a maximal element K in this set. Then K is a field whose unique quadratic extension contained in $\mathbb{Q}_3$ is $K(\sqrt{7})$. As the four square classes of $\mathbb{Q}_3$ are represented by $1, 2, 3$ and 6, it follows that the classes of $2, 3, 7$ form an $\mathbb{F}_2$-basis of the square class group $K^\times/K^{\times 2}$. In particular $|K^\times/K^{\times 2}| = 8$.

As $\mathbb{Q}_3^\times = K^\times \mathbb{Q}_3^{\times 2}$ we conclude that $\mathrm{R}(K) \subseteq \mathrm{R}(\mathbb{Q}_3)$. As $\mathbb{Q}_3$ is radical-free, we obtain that $\mathrm{R}(K) \subseteq K^\times \cap \mathbb{Q}_3^{\times 2} = K^{\times 2} \cup 7K^{\times 2}$. Since $2 = 3^2 - 7$, $3 = (\sqrt{-2} \cdot \sqrt{-5})^2 - 7$ and $2 \cdot 3 \cdot 7 = 7^2 - 7$, we see that $D_K\langle 1, -7\rangle = K^\times$. This shows that $\mathrm{R}(K) = K^{\times 2} \cup 7\, K^{\times 2}$.

The number of square classes in (2.4) is minimal for having a nontrivial radical, by the following statement.

2.5. **Proposition.** *If $K^{\times 2} \subsetneq \mathrm{R}(K) \subsetneq \sum K^{\times 2}$ then $|K^\times/K^{\times 2}| \geq 8$.*

Proof. By [11, Chap. XII, (6.10)], if $\mathrm{R}(K)$ has index two in $K^\times$, then K is real and thus $\mathrm{R}(K) = \sum K^{\times 2}$. Hence, if $\mathrm{R}(K) \subsetneq \sum K^{\times 2}$ then $|K^\times/\mathrm{R}(K)| \geq 4$. $\square$

3. The radical as the group of local squares

In certain fields satisfying a local-global principle for isotropy of quadratic forms, the radical consists of the elements that are squares locally.

3.1. **Proposition.** *Let $(K_\wp)_{\wp \in \mathcal{P}}$ be a family of extension fields of K such that $K_\wp^\times = K^\times K_\wp^{\times 2}$ for every $\wp \in \mathcal{P}$. Then*

$$\mathrm{R}(K) \subseteq \bigcap_{\wp \in \mathcal{P}} (K^\times \cap \mathrm{R}(K_\wp))\,.$$

This inclusion is an equality if every 3-dimensional anisotropic quadratic form φ over K stays anisotropic over $K_\wp$ for some $\wp \in \mathcal{P}$.

Proof. For $c \in \mathrm{R}(K)$ and $\wp \in \mathcal{P}$, one has $K_\wp^\times = K^\times K_\wp^{\times 2} = D_K\langle 1, -c\rangle K_\wp^{\times 2}$ and thus $c \in \mathrm{R}(K_\wp)$. This shows that $\mathrm{R}(K) \subseteq \bigcap_{\wp \in \mathcal{P}}(K^\times \cap \mathrm{R}(K_\wp))$.

Consider now $c \in K^\times \setminus \mathrm{R}(K)$. As $D_K\langle 1, -c\rangle \subsetneq K^\times$ there exists $b \in K^\times$ such that the form $\langle 1, -c, -b\rangle$ over K is anisotropic. If $\wp \in \mathcal{P}$ is such that $\langle 1, -c, -b\rangle$ stays anisotropic over $K_\wp$, then we conclude that $c \notin \mathrm{R}(K_\wp)$. Therefore, if every 3-dimensional anisotropic quadratic form φ over K stays anisotropic over $K_\wp$ for some $\wp \in \mathcal{P}$, we obtain that $\mathrm{R}(K) = \bigcap_{\wp \in \mathcal{P}}(K^\times \cap \mathrm{R}(K_\wp))$. $\square$

3.2. **Proposition.** *Let Ω be a set of $\mathbb{Z}$-valuations of K whose residue fields are of characteristic different from 2 and not quadratically closed. The following hold:*

(*a*) *One has $\mathrm{R}(K) \subseteq \bigcap_{v \in \Omega}(K^\times \cap K_v^{\times 2})$.*

(*b*) *If $\bigcap_{v \in \Omega}(K^\times \cap K_v^{\times 2}) = K^{\times 2}$, then K is radical-free.*

(*c*) *If for every* 3*-dimensional anisotropic quadratic form* φ *over* K *there exists* $v \in \Omega$ *such that* φ *stays anisotropic over* K_v*, then* $\mathrm{R}(K) = \bigcap_{v\in\Omega}(K^\times \cap K_v^{\times 2})$.

Proof. For $v \in \Omega$, we have $K_v^\times = K^\times K_v^{\times 2}$ as well as $\mathrm{R}(K_v) = K_v^{\times 2}$ by (2.2). Therefore (3.1) applies and yields (*a*) and (*c*). Moreover (*a*) implies (*b*). □

Using (3.2) we retrieve the well-known fact that any number field is radical-free:

3.3. **Example.** Let K be a global field of characteristic different from 2 and let Ω denote the set of all non-dyadic $\mathbb{Z}$-valuations of K. As K has only finitely many archimedean and non-archimedean dyadic places, the Global-Square-Theorem (cf. [13, (65:15)]) implies that $\bigcap_{v\in\Omega}(K^\times \cap K_v^{\times 2}) = K^{\times 2}$. Hence, (3.2) yields that K is radical-free.

3.4. **Proposition.** *Assume that* K *is a rational function field in one variable over a field* k*. Let* Ω *denote the set of* $\mathbb{Z}$*-valuations on* K *that are trivial on* k*. Then*

$$\bigcap_{v\in\Omega}(K^\times \cap K_v^{\times 2}) = K^{\times 2}\,.$$

Moreover, if $k(\sqrt{-1})$ *is not quadratically closed then* K *is radical-free.*

Proof. Let $T \in K$ be such that $K = k(T)$. Any square class of K is given by a square-free polynomial $f \in k[T]$. Note that $v(f)$ is 0 or 1 for every $v \in \Omega$ corresponding to an irreducible monic polynomial in $k[T]$. If $v(f) = 1$ for one such v, then $f \notin K_v^{\times 2}$. If $v(f) = 0$ for all such v, then $f \in k$. Finally, if $f \in k^\times \setminus k^{\times 2}$, then $f \notin K_v^{\times 2}$ where v is the valuation given by T. This together yields the claimed equality.

Assume now that $k(\sqrt{-1})$ is not quadratically closed. It follows that no finite extension of k is quadratically closed. In fact, if there were a finite field extension k'/k such that k' is quadratically closed, then k' would contain $k(\sqrt{-1})$ and [11, Chap. VIII, (5.11)] would imply that $k(\sqrt{-1})$ is quadratically closed. In particular, for $v \in \Omega$, the residue field of v is not quadratically closed. Thus K is radical-free, by (3.2). □

3.5. **Corollary.** *Assume that* K *contains a subfield* k *such that* K/k *is purely transcendental of transcendence degree at least two. Then* K *is radical-free.*

Proof. Let $\mathcal{X}$ be a transcendence basis of K/k with $K = k(\mathcal{X})$. Choose $x \in \mathcal{X}$ and put $\mathcal{X}' = \mathcal{X} \setminus \{x\}$ and $K_0 = k(\mathcal{X}')$. Then $K = K_0(x)$. As $\mathcal{X}' \neq \emptyset$ by the hypothesis, we have that $K_0(\sqrt{-1}) = k(\sqrt{-1})(\mathcal{X}')$ is not quadratically closed. Hence, we conclude from (3.4) that $\mathrm{R}(K) = K^{\times 2}$. □

3.6. **Question.** *Assume that* K *is a finitely generated field extension of transcendence degree at least two of another field* k*. Is then* K *radical-free? Is every non-square in* K *a non-square in the completion of a* $\mathbb{Z}$*-valuation on* K *that is trivial on* k *and whose residue-field is an algebraic function field over* k*?*

3.7. **Theorem.** *Assume that $K = k((X_1, \ldots, X_n))$ for a field k of characteristic different from 2. Let Ω denote the set of $\mathbb{Z}$-valuations on K corresponding to the localizations of $k[[X_1, \ldots, X_n]]$ at its height one prime ideals. Then*

$$\bigcap_{v\in\Omega}(K^\times \cap K_v^{\times 2}) = K^{\times 2}\,.$$

In particular, K is radical-free unless k is quadratically closed and $n = 1$.

Proof. Let $A = k[[X_1, \ldots, X_n]]$. Note that A is a unique factorization domain by [12, (20.3) and (20.8)], and noetherian by [1, (10.27)]. In particular, by [12, (20.1)] any height one prime ideal in A is principal.

Consider an arbitrary element $a \in K^\times$. We may write $a = u \cdot p_1 \ldots p_r \cdot x^2$ where $u \in A^\times$, $x \in K^\times$, $r \geq 0$, and where $p_1, \ldots, p_r$ are pairwise non-associate prime elements of A. Let c denote the constant term of u as a power series. Then $c^{-1}u$ is a 1-unit in A, and therefore a square in A. Note that, for $v \in \Omega$, we have that $v(a)$ is odd if v is associated to one of the prime elements $p_1, \ldots, p_r$, and $v(a)$ is even otherwise. Assume now that $a \in \bigcap_{v\in\Omega}(K^\times \cap K_v^{\times 2})$. Then $v(a)$ is even for every $v \in \Omega$, whereby $r = 0$, $a = ux^2$, and $aK^{\times 2} = cK^{\times 2}$. Let w be the $\mathbb{Z}$-valuation associated to the irreducible element X_n in A. Note that $K_w = k((X_1, \ldots, X_{n-1}))((X_n))$. It follows that $c \in k^\times \cap K_w^{\times 2} = k^{\times 2}$, whereby $a \in K^{\times 2}$. This argument shows that $\bigcap_{v\in\Omega}(K^\times \cap K_v^{\times 2}) = K^{\times 2}$.

Furthermore, if $n = 1$, then $K = k((X_1))$ and it follows by (2.2) that K is radical-free unless k is quadratically closed. Assume now that $n \geq 2$. The residue field of any valuation $v \in \Omega$ is k-isomorphic to a finite extension of $k((X_1, \ldots, X_{n-1}))$ and therefore is not quadratically closed. Using (3.2) the proven equality yields that K is radical-free. □

4. The radical complex for a quadratic extension

We consider a finite field extension L/K and ask about the relations between $\mathrm{R}(K)$ and $\mathrm{R}(L)$. After a first general result, we shall focus on the case of a quadratic extension. Let $N_{L/K} : L^\times \longrightarrow K^\times$ be the group homomorphism given by the norm map.

4.1. **Proposition.** *We have $N_{L/K}(\mathrm{R}(L)) \subseteq \mathrm{R}(K)$.*

Proof. For $a \in K^\times$, as $\mathrm{R}(L) \subseteq D_L\langle 1, a\rangle$ we have that $N_{L/K}(\mathrm{R}(L)) \subseteq D_K\langle 1, a\rangle$, by [11, Chap. VII, (4.3)]. Hence, $N_{L/K}(\mathrm{R}(L)) \subseteq \bigcap_{a\in K^\times} D_K\langle 1, a\rangle$. □

For the remainder of this section we consider the case where L/K is a quadratic field extension. We denote by $\iota_{L/K}$ the inclusion homomorphism $K^\times \longrightarrow L^\times$.

4.2. **Proposition.** *Assume that $L \simeq K(\sqrt{a})$ where $a \in K^\times$. For any $b \in K^\times$ we have that*

$$D_L\langle 1, -b\rangle \cap K^\times = D_K\langle 1, -b\rangle \cdot D_K\langle 1, -ab\rangle\,.$$

Proof. See e.g. [3, (2.4)]. □

The following was shown in [4, Cor. of Prop. 3; Prop. 5] as a partial analogue to the square-class exact sequence in [11, Chap. VII, (3.8)].

4.3. **Proposition.** *We have* $\mathrm{R}(K) \subseteq \mathrm{R}(L)$ *and* $N_{L/K}(K^\times \mathrm{R}(L)) \subseteq \mathrm{R}(K)$. *In particular, the maps* $\iota_{L/K}$ *and* $N_{L/K}$ *induce a complex*

$$K^\times/\mathrm{R}(K) \longrightarrow L^\times/\mathrm{R}(L) \longrightarrow K^\times/\mathrm{R}(K)\,,$$

which is exact if and only if $K^\times\mathrm{R}(L) = N_{L/K}^{-1}(\mathrm{R}(K))$.

Proof. Consider $b \in K^\times$. By the Norm Principle [11, Chap. VII (5.10)] we have that

$$N_{L/K}^{-1}(D_K\langle 1,-b\rangle) \;=\; K^\times D_L\langle 1,-b\rangle\,.$$

Hence, if $D_K\langle 1,-b\rangle = K^\times$, then $D_L\langle 1,-b\rangle = L^\times$. This shows that $\mathrm{R}(K) \subseteq \mathrm{R}(L)$.

Since $N_{L/K}(\mathrm{R}(L)) \subseteq \mathrm{R}(K)$ by (4.1) and $N_{L/K}(K^\times) \subseteq K^{\times 2}$, it follows that $N_{L/K}(K^\times\mathrm{R}(L)) \subseteq \mathrm{R}(K)$. The statement follows from this. □

There are examples of quadratic field extensions L/K where K is radical-free whereas L is not. For example, in [2, Section 2], for any positive integer n a real pythagorean field K is constructed such that $L = K(\sqrt{-1})$ satisfies $|L^\times/\mathrm{R}(L)| = 4$ and $|\mathrm{R}(L)/L^{\times 2}| = 2^n$.

D. Kijima and M. Nishi [8] raised the question whether the complex in (4.3) is always exact. We will show that the answer is negative by providing a construction that produces counter-examples. To simplify the discussion of the problem, we say that the quadratic field extension L/K is *radical-exact* if the equality $K^\times\mathrm{R}(L) = N_{L/K}^{-1}(\mathrm{R}(K))$ holds, that is, if the complex in (4.3) is exact.

4.4. **Corollary.** *Let* L/K *be a quadratic field extension such that* $N_{L/K}$ *is surjective. Then* $\mathrm{R}(K) = K^\times \cap \mathrm{R}(L)$ *and the maps* $\iota_{L/K}$ *and* $N_{L/K}$ *induce a complex*

$$1 \longrightarrow K^\times/\mathrm{R}(K) \longrightarrow L^\times/\mathrm{R}(L) \longrightarrow K^\times/\mathrm{R}(K) \longrightarrow 1\,,$$

which is exact on the left and on the right. In particular, this is an exact sequence provided that L/K *is radical-exact.*

Proof. Let $a \in K^\times$ be such that $L = K(\sqrt{a})$. As $N_{L/K}$ is surjective, the norm form $\langle 1,-a\rangle$ of L/K is universal over K, whereby $a \in \mathrm{R}(K)$. This further shows that the complex is exact on the right.

Consider an arbitrary element $b \in K^\times$. As $a \in \mathrm{R}(K)$, by [11, Chap. XII, (6.3)] we have that $D_K\langle 1,-b\rangle = D_K\langle 1,-ab\rangle$. Using (4.2) we thus obtain that $D_K\langle 1,-b\rangle = K^\times \cap D_L\langle 1,-b\rangle$. Therefore, if the form $\langle 1,-b\rangle$ is universal over L, it is also universal over K. This shows that $K^\times \cap \mathrm{R}(L) \subseteq \mathrm{R}(K)$. Since by (4.3) the opposite inclusion also holds, we obtain that $\mathrm{R}(K) = K^\times \cap \mathrm{R}(L)$. In particular, the complex is exact on the left.

The rest follows from (4.3). □

The following recovers [9, (2.13)].

4.5. **Proposition.** *Assume that $L \simeq K(\sqrt{d})$ with $d \in \sum K^{\times 2}$. Then K is quasi-pythagorean if and only if L quasi-pythagorean, and in this case L/K is radical-exact.*

Proof. This claimed equivalence is [5, (4.10); (4.5)] for $n = 2$. Assume now that K and L are quasi-pythagorean. Using the Norm Principle [11, Chap. VII, (5.10)], we obtain that

$$N_{L/K}^{-1}(\mathrm{R}(K)) = N_{L/K}^{-1}(D_K\langle 1,1\rangle) = K^\times D_L\langle 1,1\rangle = K^\times \mathrm{R}(L)\,,$$

showing that L/K is radical-exact. □

4.6. **Proposition.** *Let L/K be a quadratic field extension with $\mathrm{R}(L) = L^{\times 2}$. Then $\mathrm{R}(K) \subseteq K^\times \cap L^{\times 2}$ and $K^\times \mathrm{R}(L) \subseteq N_{L/K}^{-1}(\mathrm{R}(K))$, and exactly one of the two inclusions is strict.*

Proof. Since $\mathrm{R}(L) = L^{\times 2}$ we have that

$$K^\times \mathrm{R}(L) = K^\times L^{\times 2} = N_{L/K}^{-1}(K^{\times 2})\,.$$

By (4.3) we obtain that $\mathrm{R}(K) \subseteq K^\times \cap \mathrm{R}(L) = K^\times \cap L^{\times 2}$. Let $a \in K^\times$ be such that $L = K(\sqrt{a})$. Then $K^\times \cap L^{\times 2} = K^{\times 2} \cup aK^{\times 2}$. Hence, either $\mathrm{R}(K) = K^{\times 2}$ or $\mathrm{R}(K) = K^{\times 2} \cup aK^{\times 2}$.

If $\mathrm{R}(K) = K^{\times 2}$, then $\mathrm{R}(K) \subsetneq K^\times \cap L^{\times 2}$ and from the above we obtain that $K^\times \mathrm{R}(L) = N_{L/K}^{-1}(\mathrm{R}(K))$. Assume now that $\mathrm{R}(K) = K^{\times 2} \cup aK^{\times 2}$. Then in particular $a \in D_K\langle 1, -a\rangle = N_{L/K}(L^\times)$. Hence, we obtain that

$$K^\times \mathrm{R}(L) = N_{L/K}^{-1}(K^{\times 2}) \subsetneq N_{L/K}^{-1}(K^{\times 2} \cup aK^{\times 2}) = N_{L/K}^{-1}(\mathrm{R}(K))\,.$$

□

4.7. **Lemma.** *Let $a \in K^\times \setminus K^{\times 2}$. Let $\mathcal{C}$ be the set of isomorphism classes of smooth conics over K having a $K(\sqrt{a})$-rational point. For $C \in \mathcal{C}$ let $K(C)$ denote the corresponding function field, determined by C up to K-isomorphism. Let K' be a field composite of all $K(C)$ with $C \in \mathcal{C}$. Then $K^\times \subseteq D_{K'}\langle 1, -a\rangle$ and the extension $K'(\sqrt{a})/K(\sqrt{a})$ is purely transcendental.*

Proof. The field $K'(\sqrt{a})$ is the compositum of the function fields $K(\sqrt{a})(C)$ for all $C \in \mathcal{C}$. Since every $C \in \mathcal{C}$ is rational over $K(\sqrt{a})$, the field $K'(\sqrt{a})$ is a compositum of rational function fields in one variable over $K(\sqrt{a})$, thus a purely transcendental extension of $K(\sqrt{a})$.

By construction, every smooth conic over K that has a $K(\sqrt{a})$-rational point has a K'-rational point. Hence, for any $b \in K^\times$ the ternary quadratic form $\langle 1, -a, -b\rangle$ becomes isotropic over K'. Thus $K^\times \subseteq D_{K'}\langle 1, -a\rangle$. □

4.8. **Theorem.** *Let L/K be a quadratic field extension. There exists a field extension K'/K that is linearly disjoint to L/K and such that LK' is radical-free, $\mathrm{R}(K') = K'^\times \cap (LK')^{\times 2}$, and LK'/K' not radical-exact.*

Proof. Let $L = K(\sqrt{a})$ with $a \in K^\times$. We define a tower of extension fields $(K_i)_{i\in\mathbb{N}}$ of K by letting $K_0 = K$ and, K_{i+1} the field composite over K_i of all $K_i(C)$ where C runs over the isomorphism classes of conics over K_i having a $K_i(\sqrt{a})$-rational point. Let K' denote the direct limit of the tower of fields $(K_i)_{i\in\mathbb{N}}$. For $i \in \mathbb{N}$, then K'/K_i is linearly disjoint to any algebraic extension of K_i and, moreover, by (4.7) $K_{i+1}(\sqrt{a})/K_i(\sqrt{a})$ is purely transcendental and every element of $K_i^\times$ is represented over K_{i+1} by the form $\langle 1, -a\rangle$. It follows that K'/K is linearly disjoint to L/K, that the form $\langle 1, -a\rangle$ is universal over K', and that $K'L = K'(\sqrt{a})$ is a purely transcendental extension of $K(\sqrt{a})$, whereby $K'L$ is radical-free by (3.5). Note that $K'^\times \cap (K'L)^{\times 2} = K'^{\times 2} \cup aK'^{\times 2} \subseteq \mathrm{R}(K')$. Using (4.6) we conclude that $\mathrm{R}(K') = K'^\times \cap (K'L)^{\times 2}$ and that $K'L/K'$ is not radical-exact. $\square$

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Departement Wiskunde–Informatica, Universiteit Antwerpen, Belgium / Zukunftskolleg, Universität Konstanz, Germany.

E-mail address: `becher@maths.ucd.ie`

Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA.

E-mail address: `leep@email.uky.edu`