THE LIE ALGEBRA OF TYPE G_2 IS RATIONAL OVER ITS QUOTIENT BY THE ADJOINT ACTION

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Abstract. Let G be a split simple group of type G_2 over a field k, and let $\mathfrak{g}$ be its Lie algebra. Answering a question of J.-L. Colliot-Thélène, B. Kunyavskiĭ, V. L. Popov, and Z. Reichstein, we show that the function field $k(\mathfrak{g})$ is generated by algebraically independent elements over the field of adjoint invariants $k(\mathfrak{g})^G$.

Résumé. Soit G un groupe algébrique simple et déployé de type G_2 sur un corps k. Soit $\mathfrak{g}$ son algèbre de Lie. On démontre que le corps des fonctions $k(\mathfrak{g})$ est transcendant pur sur le corps $k(\mathfrak{g})^G$ des invariants adjoints. Ceci répond par l'affirmative à une question posée par J.-L. Colliot-Thélène, B. Kunyavskiĭ, V. L. Popov et Z. Reichstein.

I. Introduction. Let G be a split connected reductive group over a field k and let $\mathfrak{g}$ be the Lie algebra of G. We will be interested in the following natural question:

Question 1. Is the function field $k(\mathfrak{g})$ *purely transcendental* over the field of invariants $k(\mathfrak{g})^G$ for the adjoint action of G on $\mathfrak{g}$? That is, can $k(\mathfrak{g})$ be generated over $k(\mathfrak{g})^G$ by algebraically independent elements?

In [5], the authors reduce this question to the case where G is simple, and show that in the case of simple groups, the answer is affirmative for split groups of types A_n and C_n, and negative for all other types except possibly for G_2. (The standing assumption in [5] is that $\mathrm{char}(k) = 0$, but here we work in arbitrary characteristic.)

The purpose of this note is to settle Question 1 for the remaining case $G = G_2$.

Theorem 2. *Let k be an arbitrary field and G be the simple split k-group of type G_2. Then $k(\mathfrak{g})$ is purely transcendental over $k(\mathfrak{g})^G$.*

Apart from settling the last case left open in [5], we were motivated by the (still mysterious) connection between Question 1 and the Gelfand-Kirillov (GK) conjecture [9]. Here $\mathrm{char}(k) = 0$. A. Premet [11] recently showed that the GK conjecture fails for simple Lie algebras of any type other than A_n, C_n and G_2. His paper relies on the negative results of [5] and their characteristic p analogues (proved in [11]). It is not known whether a positive answer to Question 1 for $\mathfrak{g}$ implies the GK conjecture for $\mathfrak{g}$. The GK conjecture has been proved for algebras of type A_n (see [9]), but remains open for types C_n and G_2. While Theorem 2 does not settle the GK conjecture for type G_2, it puts the remaining two open cases—for algebras of type C_n and G_2—on equal footing vis-à-vis Question 1.

II. Twisting. Temporarily, let W be a linear algebraic group over k.

Let X be a quasi-projective variety with a (right) W-action defined over k, and let ζ be a (left) W-torsor over k. The diagonal left action of W on $X \times_{\mathrm{Spec}(k)} \zeta$ (by $g.(x, z) = (xg^{-1}, gz)$) makes $X \times_{\mathrm{Spec}(k)} \zeta$ into the total space of a W-torsor $X \times_{\mathrm{Spec}(k)} \zeta \to B$. The base space B of this torsor is usually called the *twist* of X by ζ. We denote it by ${}^{\zeta}X$.

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It is easy to see that if ζ is trivial then ${}^{\zeta}X$ is k-isomorphic to X. Hence, ${}^{\zeta}X$ is a k-form of X, i.e., X and ${}^{\zeta}X$ become isomorphic over an algebraic closure of k.

The twisting construction is functorial in X: a W-equivariant morphism $X \to Y$ (or rational map $X \dashrightarrow Y$) induces a k-morphism ${}^{\zeta}X \to {}^{\zeta}Y$ (resp., rational map ${}^{\zeta}X \dashrightarrow {}^{\zeta}Y$). For details, see [7, Section 3], [8, Section 2], or [5, Section 2].

III. The split group of type G_2. We fix notation and briefly review the basic facts, referring to [13], [1], or [2] for more details. Over any field k, a simple split group G of type G_2 has a faithful seven-dimensional representation V. Following [2, (3.11)], one can fix a basis $f_1, \dots, f_7$, with dual basis $X_1, \dots, X_7$, so that G preserves the nonsingular quadratic norm $N = X_1X_7 + X_2X_6 + X_3X_5 + X_4^2$. (See [1, §6.1] for the case $\mathrm{char}(k) = 2$. In this case V is not irreducible, since the subspace spanned by f_4 is invariant; the quotient $V/(k \cdot f_4)$ is the minimal irreducible representation. However, irreducibility will not be necessary in our context.) The corresponding embedding $G \hookrightarrow \mathrm{GL}_7$ yields a split maximal torus and Borel subgroup $T \subset B \subset G$, by intersecting with diagonal and upper-triangular matrices. Explicitly, the maximal torus is

$$T = \mathrm{diag}(t_1,\, t_2,\, t_1t_2^{-1},\, 1,\, t_1^{-1}t_2,\, t_2^{-1},\, t_1^{-1}); \tag{1}$$

cf. [2, Lemma 3.13].

The Weyl group $W = N(T)/T$ is isomorphic to the dihedral group with 12 elements, and the surjection $N(T) \to W$ splits. The inclusion $G \hookrightarrow \mathrm{GL}_7$ thus gives rise to an inclusion $N(T) = T \rtimes W \hookrightarrow D \rtimes \mathrm{S}_7$, where $D \subset \mathrm{GL}_7$ is the subgroup of diagonal matrices. On the level of the dual basis $X_1, \dots, X_7$, we obtain an isomorphism $W \cong \mathrm{S}_3 \times \mathrm{S}_2$ realized as follows: S_3 permutes the pairs (X_1, X_7), (X_2, X_6) and (X_3, X_5), and S_2 exchanges the triples (X_1, X_5, X_6) and (X_2, X_3, X_7). The variable X_4 is fixed by W. For details, see [2, §A.3].

The subgroup $P \subset G$ stabilizing the isotropic line spanned by f_1 is a maximal standard parabolic, and the corresponding homogeneous space $P\backslash G$ is isomorphic to the five-dimensional quadric $\mathcal{Q} \subset \mathbb{P}(V)$ defined by the vanishing of the norm, i.e., by the equation

$$X_1X_7 + X_2X_6 + X_3X_5 + X_4^2 = 0. \tag{2}$$

An easy tangent space computation shows that P is smooth regardless of the characteristic of k.

Lemma 3. *The group P is* special, *i.e., $H^1(l, P) = \{1\}$ for every field extension l/k. Moreover, P is rational, as a variety over k.*

Proof. Since the split group of type G_2 is defined over the prime field, we may replace k by the prime field for the purpose of proving this lemma, and in particular, we can assume k is perfect. We begin by briefly recalling a construction of Chevalley [4]. The isotropic line $E_1 \subset V$ stabilized by P is spanned by f_1, and P also preserves an isotropic 3-space E_3 spanned by f_1, f_2, f_3; see, e.g., [2, §2.2]. There is a corresponding map $P \to \mathrm{GL}(E_3/E_1) \cong \mathrm{GL}_2$, which is a split surjection thanks to the block matrix described in [10, p. 13] as the image of "B" in GL_7. The kernel is unipotent, and we have a split exact sequence corresponding to the Levi decomposition:

$$1 \to R_u(P) \to P \to \mathrm{GL}_2 \to 1. \tag{3}$$

Combining the exact sequence in cohomology induced by (3) with the fact that both $R_u(P)$ and GL_2 are special (see [12, pp. 122 and 128]), shows that P is special.

Since P is isomorphic to $R_u(P) \times \mathrm{GL}_2$ as a variety over k, and P is smooth, so is $R_u(P)$. A smooth connected unipotent group over a perfect field is rational [6, IV, §2(3.10)]; therefore $R_u(P)$ is k-rational, and so is P. $\square$

IV. Proof of Theorem 2. Let G be the split simple group of type G_2 over k, $T \subset G$ be a split maximal torus and $W = N(T)/T$ be the Weyl group. We begin by reducing Theorem 2 to a statement about rationality of a twisted quotient of the quadric $\mathcal{Q}$.

Proposition 4. *Consider the following assertions:*

(a) The twisted variety ${}^{\zeta}(G_K/T_K)$ *is rational over* K*, for any* W*-torsor* ζ *over any field* K/k.

(b) The twisted variety ${}^{\zeta}(\mathcal{Q}_K/T_K)$ *is rational over* K*, for any* W*-torsor* ζ *over any field* K/k.

Then (b) $\Longrightarrow$ *(a)* $\Longrightarrow$ *Theorem 2.*

A dominant rational map $\mathcal{Q} \dashrightarrow Y$ induced by the inclusion of fields $k(\mathcal{Q})^T \hookrightarrow k(\mathcal{Q})$ is called the *rational quotient map* for the T-action on $\mathcal{Q}$. After replacing Y by a dense open subset, we may assume that the W-action on $\mathcal{Q}$ descends to Y. The resulting variety Y is unique up to a W-equivariant birational isomorphism; this is the W-variety $\mathcal{Q}/T$ in the statement of part (b) (and similarly for $\mathcal{Q}_K/T_K$). We will construct an explicit birational model for $\mathcal{Q}/T$ below.

Proof. (a) $\Longrightarrow$ Theorem 2: Let $\mathfrak{g}_{\rm reg}$ and $\mathfrak{t}_{\rm reg}$ denote the open subsets of regular semisimple elements in the Lie algebras of G and T, respectively. The following diagram commutes:

$$\begin{tikzcd} G/T \times_{\mathrm{Spec}(k)} \mathfrak{t}_{\rm reg} \arrow[r] \arrow[d] & \mathfrak{g}_{\rm reg} \arrow[d, "\pi"] \\ \mathfrak{t}_{\rm reg} \arrow[r, "\pi"] & \mathfrak{g}_{\rm reg}/\!/G. \end{tikzcd}$$

Here π is the categorical quotient map, and the top horizontal map, given by $(g, t) \mapsto \mathrm{ad}(g) \cdot t$, is G-equivariant. The Weyl group acts on $\mathfrak{t}_{\rm reg}$ and G/T (on the right), and diagonally on $G/T \times \mathfrak{t}_{\rm reg}$. The horizontal maps are W-torsors; see [5, Proposition 2.9]. Thus we have the following diagram of inclusions of fields:

$$\begin{tikzcd} k(G/T \times_{\mathrm{Spec}(k)} \mathfrak{t}_{\rm reg})^W \arrow[d, dash] & k(\mathfrak{g}) \arrow[l, "\sim"'] \arrow[d, dash] \\ k(\mathfrak{t})^W & k(\mathfrak{g})^G. \arrow[l, "\sim"'] \end{tikzcd}$$

Setting $L = k(\mathfrak{t})$ and $K = k(\mathfrak{t})^W$ and noting that

$$k(G/T \times_{\mathrm{Spec}(k)} \mathfrak{t}_{\rm reg})^W = K((G/T)_K \times_{\mathrm{Spec}(K)} \mathrm{Spec}\, L),$$

the field extension on the left can be rewritten as $K({}^{\zeta}(G/T))/K$, where ζ is the W-torsor $\mathrm{Spec}\, L$. By part (a) this field extension is purely transcendental. Hence, so is the vertical extension on the right side of the diagram, i.e., Theorem 2 holds.

(b) $\Longrightarrow$ (a): For the purpose of proving this implication, we may we may view K as a new base field and replace it with k.

We claim that the left action of P on G/T is generically free. By the (characteristic-free) argument at the beginning of the proof of [5, Lemma 9.1], in order to establish this claim it suffices to show that the right T-action on $\mathcal{Q} = P \backslash G$ is generically free. The latter action, given by restricting the linear action (1) of T on $\mathbb{P}^6$ to the quadric $\mathcal{Q}$ given by (2), is clearly generically free.

The W-equivariant rational map $G/T \dashrightarrow \mathcal{Q}/T$ induced by the projection $G \to P\backslash G = \mathcal{Q}$ is the rational quotient map for the left P-action on G/T; cf. [5, p. 458]. Since the P-action is generically free, this map is a P-torsor over the generic point of $\mathcal{Q}/T$; see [3, Theorem 4.7]. By the functoriality of the twisting operation, after twisting by a W-torsor ζ, we obtain a rational map ${}^{\zeta}(G/T) \dashrightarrow {}^{\zeta}\mathcal{Q}/T$, which is a P-torsor over the generic point of ${}^{\zeta}\mathcal{Q}/T$. This torsor has a rational section, since P is special; in particular, ${}^{\zeta}(G/T)$ is k-birationally isomorphic to $P \times {}^{\zeta}(\mathcal{Q}/T)$. Since P is k-rational, ${}^{\zeta}(G/T)$ is rational over ${}^{\zeta}(\mathcal{Q}/T)$, and we conclude that $(b) \Longrightarrow$ (a), as desired. $\square$

It remains to show that the assertion of Proposition 4(b) always holds. As before, we may replace the field K with k. The following lemma completes the proof of Theorem 2.

Lemma 5. *The twisted variety ${}^{\zeta}(\mathcal{Q}/T)$ is rational over k, for any W-torsor ζ over k.*

Proof. We begin by constructing an explicit birational model for the W-variety $\mathcal{Q}/T$. The affine open subset $\mathcal{Q}^{\text{aff}} = \{x_1x_7 + x_2x_6 + x_3x_5 + 1 = 0\} \subset \mathbb{A}^6$ (where $X_4 \neq 0$) is $N(T)$-invariant. Here the affine coordinates on $\mathbb{A}^6$ are $x_i := X_i/X_4$, for $i \neq 4$. The field of invariant rational functions for the T-action on $\mathcal{Q}^{\text{aff}}$ is $k(y_1, y_2, y_3, z_1, z_2)$, where the variables

$$y_1 = x_1x_7, \quad y_2 = x_2x_6, \quad y_3 = x_3x_5, \quad z_1 = x_1x_5x_6, \text{ and } z_2 = x_2x_3x_7$$

are subject to the relations $y_1 + y_2 + y_3 + 1 = 0$ and $y_1y_2y_3 = z_1z_2$. In other words, the rational quotient $\mathcal{Q}^{\text{aff}}/T$ (or equivalently, $\mathcal{Q}/T$) is W-equivariantly birationally isomorphic to the affine subvariety Λ_1 of $\mathbb{A}^5$ given by these two equations, where $W = \mathrm{S}_2 \times \mathrm{S}_3$ acts on the coordinates as follows: S_2 permutes z_1, z_2, and S_3 permutes y_1, y_2, y_3. We claim that Λ_1 is W-equivariantly birationally isomorphic to

$$\begin{aligned}
\Lambda_2 &= \{(Y_1 : Y_2 : Y_3 : Z_0 : Z_1 : Z_2) \,:\, Y_1 + Y_2 + Y_3 + Z_0 = 0 \text{ and } Y_1Y_2Y_3 = Z_1Z_2Z_0\} \subset \mathbb{P}^5,\\
\Lambda_3 &= \{(Y_1 : Y_2 : Y_3 : Z_1 : Z_2) \,:\, Y_1Y_2Y_3 + (Y_1 + Y_2 + Y_3)Z_1Z_2 = 0\} \subset \mathbb{P}^4, \text{ and}\\
\Lambda_4 &= \{(Y_1 : Y_2 : Y_3 : Z_1 : Z_2) \,:\, Z_1Z_2 + Y_2Y_3 + Y_1Y_3 + Y_1Y_2 = 0\} \subset \mathbb{P}^4,
\end{aligned}$$

where W acts on the projective coordinates $Y_1, Y_2, Y_3, Z_1, Z_2, Z_0$ as follows: S_2 permutes Z_1, Z_2, S_3 permutes Y_1, Y_2, Y_3, and every element of W fixes Z_0. Note that $\Lambda_2 \subset \mathbb{P}^5$ is the projective closure of $\Lambda_1 \subset \mathbb{A}^5$; hence, using $\simeq$ to denote W-equivariant birational equivalence, we have $\Lambda_1 \simeq \Lambda_2$. The isomorphism $\Lambda_2 \simeq \Lambda_3$ is obtained by eliminating Z_0 from the system of equations defining Λ_2. Finally, the isomorphism $\Lambda_3 \simeq \Lambda_4$ comes from the Cremona transformation $\mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ given by $Y_i \to 1/Y_i$ and $Z_j \to 1/Z_j$ for $i = 1, 2, 3$ and $j = 1, 2$.

The resulting W-equivariant birational isomorphism $\mathcal{Q}/T \simeq \Lambda_4$ gives rise to a birational isomorphism ${}^{\zeta}(\mathcal{Q}/T) \simeq {}^{\zeta}\Lambda_4$ of k-varieties, for any W-torsor ζ over k. Since Λ_4 is a W-equivariant quadric hypersurface in $\mathbb{P}^4$, and the W-action on $\mathbb{P}^4$ is induced by a linear representation $W \to \mathrm{GL}_5$, Hilbert's Theorem 90 tells us that ${}^{\zeta}\mathbb{P}_4$ is k-isomorphic to $\mathbb{P}^4$, and ${}^{\zeta}\Lambda_4$ is isomorphic to a quadric hypersurface in $\mathbb{P}^4$ defined over k; see [7, Lemma 10.1]. It is easily checked that Λ_4 is smooth over k, and therefore so is ${}^{\zeta}\Lambda_4$. The zero-cycle of degree 3 given by $(1:0:0:0:0) + (0:1:0:0:0) + (0:0:1:0:0)$ in Λ_4 is W-invariant, so it defines a zero-cycle of degree 3 in ${}^{\zeta}\Lambda_4$. By Springer's theorem, the smooth quadric ${}^{\zeta}\Lambda_4$ has a k-rational point, hence is k-rational. $\square$

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