SINGULAR AND TOTALLY SINGULAR GENERALISED QUADRATIC FORMS

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Abstract. In this paper we present a decomposition theorem for generalised quadratic forms over a division algebra with involution in characteristic 2. This is a generalisation of a decomposition result on quadratic forms in characteristic 2 from [3] and extends a generalisation of the Witt decomposition theorem for nonsingular forms to cover forms that may be singular.

1. Introduction

Generalised quadratic forms (also known as pseudo-quadratic forms) are an extension of the concept of a quadratic form over a field to the setting of division algebras with involution, first introduced in [8]. Rather than being maps from a vector space over a field to that field, they are defined as being maps on a vector space over a division algebra to that division algebra modulo alternating elements with respect to an involution.

The decomposition theorem of Witt (see [9]) states that any regular quadratic form over a field of characteristic different from 2 uniquely decomposes into an orthogonal sum of an anisotropic part and a hyperbolic part. Our interest is in this theorems generalisation to the theory of generalised quadratic forms over fields of arbitrary characteristic, where it says that every nonsingular generalised quadratic form over a finite dimensional division algebra with involution can be decomposed into an orthogonal sum of an anisotropic part and a hyperbolic part in a unique way.

We consider decompositions of generalised quadratic forms that may be singular. That is, we allow the hermitian form associated to the generalised quadratic form to be degenerate. If the characteristic of the underlying field is different from 2, or the involution is of unitary type, then the hermitian form associated to a generalised quadratic form completely determines the generalised quadratic form (and vice versa), so singular generalised quadratic forms are not interesting in this case. Indeed, forms that are totally singular, that is, whose associated hermitian form is the zero map, are trivial in characteristic different from 2 or in the case of a unitary involution.

However, over fields of characteristic 2 and for involutions of the first kind, singular generalised quadratic forms have a great deal of structure. In particular, in characteristic 2 and when the involution is of the first kind, there can be many different totally singular forms of the same dimension. Singular quadratic forms have been studied over fields, in, for example, [3]. Here it is shown that totally

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singular forms can be studied somewhat independently and with methods quite distinct from the usual theory of quadratic forms (see also, for example, [6]).

In Section 8 we show a decomposition theorem for quadratic forms that are not assumed to be nonsingular. This generalises [3, (2.4)] to our wider setting. In Section 6 we also show some results on totally singular generalised quadratic forms and on the elements they represent. In particular, in (6.3) we show that, excluding the split case, totally singular generalised quadratic forms can be characterised in terms of the elements they represent.

2. Algebras with involution

We refer to [7] as a general reference on finite-dimensional algebras over fields, and for central simple algebras in particular, and to [5] for involutions.

Let A be a finite-dimensional F–algebra. If A is simple (i.e. it has no non-trivial two sided ideals) and E is the centre of A, we can view A as an E–algebra and by Wedderburn's Theorem (see [5, (1.1)]) we have that $A \simeq \mathrm{End}_D(V)$ for an F–division algebra D with centre E and a right D–vector space V. In this case $\dim_E(A)$ is a square, and the positive root of this integer is called the *degree of* A and is denoted $\deg(A)$. If $E = F$, then we call the F–algebra A *central simple*. An F*–quaternion algebra* is a central simple F-algebra of degree 2.

Let Ω be an algebraic closure of F. By Wedderburn's Theorem, under scalar extension to Ω, every central simple F–algebra of degree n becomes isomorphic to $M_n(\Omega)$, the algebra of $n \times n$ matrices over Ω. Therefore if A is a central simple F–algebra we may fix an F-algebra embedding $A \to M_n(\Omega)$ and view every element $a \in A$ as a matrix in $M_n(\Omega)$. The characteristic polynomial of this matrix has coefficients in F and is independent of the embedding of A into $M_n(\Omega)$ (see [7, §16.1]). We call this polynomial the *reduced characteristic polynomial of* A and denote it by

$$\mathrm{Prd}_{A,a} = X^n - s_1(a)X^{n-1} + s_2(a)X^{n-2} - \ldots + (-1)^n s_n(a).$$

The we call $s_1(a)$ the *reduced trace of* a and $s_n(a)$ the *reduced norm of* a and denote them by $\mathrm{Trd}_A(a)$ and $\mathrm{Nrd}_A(a)$ respectively. We also denote $s_2(a)$ by $\mathrm{Srd}_A(a)$.

An F*–involution* on A is an F–linear map $\sigma : A \to A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$ and $\sigma^2 = \mathrm{id}_A$. An F*–algebra with involution* is a pair (A, σ) of a finite-dimensional F–algebra A and an F–involution σ on A such that one has $F = \{x \in Z(A) \mid \sigma(x) = x\}$, and such that either A is simple or A is a product of two simple F–algebras that are mapped to one another by σ. In this situation, there are two possibilities: either $Z(A) = F$, so that A is a central simple F–algebra, or $Z(A)/F$ is a quadratic étale extension with σ restricting to the nontrivial F–automorphism of $Z(A)$. To distinguish these two situations, we speak of algebras with involution of the *first* and *second kind*: we say that the F–algebra with involution (A, σ) is of the *first kind* if $Z(A) = F$ and of the *second kind* otherwise. For more information on involutions of the second kind, also called *unitary involutions*, we refer to [5, Section 2.B].

Let (A, σ) be an F–algebra with involution and E be the centre of A. For $\lambda \in E$ such that $\lambda\sigma(\lambda) = 1$, let

$$\mathrm{Sym}_\lambda(A, \sigma) = \{a \in A \mid \lambda\sigma(a) = a\} \quad \text{and} \quad \mathrm{Alt}_\lambda(A, \sigma) = \{a - \lambda\sigma(a) \mid a \in A\}.$$

These are F–linear subspaces of A and we write $\mathrm{Sym}(A, \sigma) = \mathrm{Sym}_1(A, \sigma)$ and $\mathrm{Alt}(A, \sigma) = \mathrm{Alt}_1(A, \sigma)$.

Example 2.1. Let $(1, u, v, w)$ be basis elements of a 4–dimensional F-vector space. For $a \in F$ be such that $-4a \neq 1$ and $b \in F^\times$, defining a multiplication on this F–vector space through the relations $u^2 = u + a$, $v^2 = b$ and $w = uv = v - vu$ gives the structure of a F–quaternion algebra. We call such basis $(1, u, v, w)$ a *quaternion basis* of this algebra and we denote it by $[a, b)_F$.

Let $Q = [a, b)_F$ for $a \in F$ and $b \in F^\times$ as above. Let γ be the F–involution on Q such that $\gamma(u) = 1 - u$ and $\gamma(v) = v$. This is the unique symplectic involution on Q (see [5, (2.21)]) and for all $x \in Q$ we have $\gamma(x) = \mathrm{Trd}_Q(x) - x$ and $\gamma(x)x = \mathrm{Nrd}_Q(x)$. Direct computation then shows that $\mathrm{Alt}_{-1}(Q, \gamma) = F$.

3. Hermitian and Quadratic forms

In this section we recall the basic terminology and results we use from hermitian and quadratic form theory. We refer to [4, Chapter 1] as a general reference on hermitian and quadratic forms.

Let (D, θ) be an F–division algebra with involution with centre E. Further, fix $\lambda \in E$ such that $\lambda\theta(\lambda) = 1$. Note that if (D, θ) is of the first kind one must have that $\lambda = \pm 1$. A *λ–hermitian form* over (D, θ) is a pair (V, h) where V is a finite-dimensional right D-vector space and h is a bi-additive map $h : V \times V \to D$ such that

$$h(x, yd) = h(x, y)d \quad \text{and} \quad h(y, x) = \lambda\theta(h(x, y))$$

holds for all $x, y \in V$ and $d \in D$.

Let $\varphi = (V, h)$ be a λ–hermitian form over (D, θ). We call the set

$$\mathrm{rad}(\varphi) = \{v \in V \mid h(v, w) = 0 \text{ for all } w \in V\}$$

the *radical of* φ. We say φ is *non-degenerate* if $\mathrm{rad}(V, h) = \{0\}$. We say φ represents an element $a \in D$ if $h(x, x) = a$ for some $x \in V \backslash \{0\}$.

Let $\varphi_1 = (V, h_1)$ and $\varphi_2 = (W, h_2)$ be a λ–hermitian forms over (D, θ). The *orthogonal sum* of φ_1 and φ_2 is defined to be the pair $(V \times W, h)$ where the F–linear map $h : (V \times W) \times (V \times W) \to D$ is such that $h((v_1, w_1), (v_2, w_2)) = h_1(v_1, v_2) + h_2(w_1, w_2)$ for any $v_1, v_2 \in V$ and $w_1, w_2 \in W$; we also denote it by $\varphi_1 \perp \varphi_2$.

Lemma 3.1. *Let $\varphi = (V, h)$ be a λ–hermitian form over an F–division algebra with involution (D, θ) such that h is not identically zero on $V \times V$. Then φ represents a non–zero element in D if and only if $(D, \theta) \neq (F, \mathrm{id})$ and $\lambda \neq -1$.*

Proof. Since h is not the trivial map, there exists an F–vector subspace W of V such that $W \oplus \mathrm{rad}(V, h)$. Then $(V, h) = (W, h|_W) \perp (\mathrm{rad}(V, h), h|_{\mathrm{rad}(V,h)})$ as for all $x_1, x_2 \in W$ and $y_1, y_2 \in \mathrm{rad}(V, h)$ we have $h(x_1 + y_1, x_2 + y_2) = h(x_1, x_2) + h(y_1, y_2) = h(x_1, x_2)$. Then by [4, Chapter 1, (6.2.3)], $(W, h|_W)$ represents a non-zero element in D if and only if $(D, \theta) \neq (F, \mathrm{id})$ and $\lambda \neq -1$. □

By a *λ–quadratic form over* (D, θ) we mean a pair (V, q) of a finite-dimensional right D-vector space V and a map $q : V \to D/\mathrm{Alt}_\lambda(D, \theta)$ such that for all $x, y \in V$ and $d \in D$ we have

- $q(xd) = \theta(d)q(x)d$,
- (V, h_q) is a λ–hermitian form over (D, θ), where the map $h_q : V \times V \to D$ is given by $(x, y) \longmapsto q(x + y) - q(x) - q(y) + \mathrm{Alt}_\lambda(D, \theta)$,

Let $\rho = (V, q)$ be a λ–quadratic form (V, q) over (D, θ). We call $\dim_D(V)$ the *dimension of* ρ and write it as $\dim_D(\rho)$. We call the λ–hermitian form (V, h_q) *the polar form of* ρ. We say ρ is *nonsingular* if (V, h_q) is a non-degenerate λ–hermitian form over (D, θ) and *singular* otherwise. We call ρ *totally singular* if h_q is the zero map on $V \times V$. We call the set

$$\mathrm{rad}(\rho) = \{v \in \mathrm{rad}(V, h_q) \mid q(v) = 0\}$$

the *radical of* ρ. We say that ρ is *regular* if $\mathrm{rad}(\rho) = \{0\}$.

If $\mathrm{char}(F) \neq 2$ or if (D, θ) is unitary then any λ–quadratic form is uniquely determined by its polar form (see [4, Chapter 1, (6.6.1)]), and in these cases we can consider the concepts of a λ–quadratic form and of a λ–hermitian form as coinciding. In particular, in this case a λ–quadratic form is regular if and only if it is nonsingular.

Let $\rho = (V, q)$ and $\rho' = (W, q')$ be λ–quadratic forms over (D, θ). By an *isometry* $\phi : \rho \to \rho'$ we mean an isomorphism of D–vector spaces $\phi : V \longrightarrow W$ such that $q = q' \circ \phi$. The *orthogonal sum of* ρ *and* ρ' is defined to be pair $(V \times W, q'')$ where the map $q'' : (V \times W) \to D/\mathrm{Alt}_\lambda(D, \theta)$ is given by $q''((v, w)) = q'(v) + q(w)$ for all $v \in V$ and $w \in W$, and we write $(V \times W, q'') = \rho \perp \rho'$. We say ρ' is a *subform of* ρ if there exists a λ–quadratic form ρ'' over (D, θ) such that $\rho \simeq \rho' \perp \rho''$. For $n \in \mathbb{N}$ be denote the orthogonal sum of n copies of ρ by $n \times \rho$. For $c \in F^\times$ we denote by $c\rho$ the λ–quadratic form (V, cq), where $(cq)(x) = c(q(x))$ for $x \in V$.

Lemma 3.2. *Let (V, q) be a λ–quadratic form over (D, θ). If U is an F–vector subspace of V such that $q|_U$ is nonsingular, then $(U, q|_U)$ is an subform of (V, q).*

Proof. See [4, Chapter 1 (5.4.1)]. □

Lemma 3.3. *Let $\rho = (V, q)$ be a λ–quadratic form over (D, θ). Then for an F–vector subspace U of V such that $V = U \oplus \mathrm{rad}(V, h_q)$ we have*

$$\rho \simeq (U, q|_U) \perp (\mathrm{rad}(V, h_q), q|_{\mathrm{rad}(V, h_q)}).$$

Proof. For all $x \in \mathrm{rad}(\rho)$ and $y \in V$ we have $h_q(x, y) = 0$ and hence $q(x + y) = q(x) + q(y)$. □

We say ρ represents an element $a \in D$ if $q(x) = a + \mathrm{Alt}_\lambda(D, \theta)$ for some $x \in V \backslash \{0\}$. We call ρ *isotropic* if there exists an $x \in V \backslash \{0\}$ such that $q(x) \in \mathrm{Alt}_\lambda(D, \theta)$, and *anisotropic* otherwise. Assume that ρ is nonsingular. Then we call a subspace $W \subset V$ *totally isotropic* (with respect to q) if $q|_W = 0$. We call ρ *hyperbolic* if there exists a totally isotropic subspace $W \subset V$ with $\dim_D(W) = \frac{1}{2}\dim_D(V)$. We denote the λ–quadratic form (D^2, p) over (D, θ) where the map $p : D^2 \to D/\mathrm{Alt}_\lambda(D, \theta)$ is given by $(x, y) \mapsto \theta(x)y$ by $\mathbb{H}_{(\lambda, \theta)}$.

Lemma 3.4. *$\mathbb{H}_{(\lambda, \theta)}$ is nonsingular and hyperbolic.*

Proof. Let $\mathbb{H}_{(\lambda, \theta)} = (D^2, p)$ as above. For all $x_1, x_2, y_1, y_2 \in D$ and for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we have

$$\begin{aligned} p(x + y) - p(x) - p(y) &= \theta(x_1)y_2 + \theta(y_1)x_2 + \mathrm{Alt}_\lambda(D, \theta) \\ &= \theta(x_1)y_2 + \lambda\theta(x_2)y_1 + \mathrm{Alt}_\lambda(D, \theta). \end{aligned}$$

Hence $h_p(x, y) = \theta(x_1)y_2 + \lambda\theta(x_2)y_1$ and (D^2, h_p) is nondegenerate. That $\mathbb{H}_{(\lambda, \theta)}$ is isotropic is clear. That $\mathbb{H}_{(\lambda, \theta)}$ is hyperbolic then follows as $\dim_D(\mathbb{H}_{(\lambda, \theta)}) = 2$. □

Proposition 3.5. *Let $\rho = (V, q)$ be a λ–quadratic form over (D, θ). Then $\rho \simeq n \times \mathbb{H}_{(\lambda,\theta)}$, where $n = \frac{1}{2}\dim_D(\rho)$.*

Proof. By [4, (5.6.1)], a nonsingular hyperbolic λ–quadratic form over (D, θ) is uniquely determined up to isometry by its dimension. The result follows immediately. □

We call a λ–quadratic form isometric to $\mathbb{H}_{(\lambda,\theta)}$ a *hyperbolic plane*.

Proposition 3.6. *Let ρ be a nonsingular λ–quadratic form over (D, θ). Then there exist an anisotropic nonsingular λ–quadratic form ρ' and a nonnegative integer n such that $\rho \simeq \rho' \perp n \times \mathbb{H}_{(\lambda,\theta)}$. Moreover, n is uniquely determined and η' is determined up to isomorphism by ρ.*

Proof. See [4, Chapter 1, (6.5.3)]). □

4. Witt Cancellation

Throughout this section, let (D, θ) be an F–division algebra with involution. The following results taken together are an extension of the 'Witt Cancellation' result for nonsingular generalised quadratic forms from [4, Chapter 1, (6.5.2)] to cover the case where the λ–quadratic forms are potentially singular. The proof of [4, Chapter 1, (6.5.2)] is not explicitly given, but it is noted that the proof of an analogous result for hermitian forms, [4, Chapter 1, (6.4.2)], is straightforward to adapt to this case. In fact, such an adaptation does not require the assumption that the λ–quadratic form is nonsingular.

Let $\mathbb{H}_{(\lambda,\theta)} = (W, p)$. Then by [4, Chapter 1, (5.6.2)] we can find elements $x, y \in W$ such that $p(x) = 0 = p(y)$ and $h_p(x, y) = 1$. For a λ–quadratic form (V, q) over (D, θ), we call a pair $x, y \in V$ such that $q(x) = 0 = q(y)$ and $h_q(x, y) = 1$ a *hyperbolic pair in* (V, q). Let H and H' be hyperbolic planes that are subforms of a λ–quadratic form (V, q) over (D, θ). We say H and H' are *adjacent* if there is a hyperbolic pair $\{x, y\}$ in H and a hyperbolic pair $\{x', y'\}$ in H' with a common element. We say that H and H' are *related* if there is a finite chain of adjacent hyperbolic planes connecting H and H'.

Lemma 4.1. *Let ρ be a λ–quadratic form over (D, θ). Two hyperbolic planes H and H' that are subforms of ρ are always related.*

Proof. The proof follows similarly to the analogous hermitian form result in [4, Chapter 1, (6.4.3)]. □

For a λ–quadratic form (V, q) over (D, θ) and a nonsingular subform (U, q') of (V, q), we denote the set $\{x \in V \mid h_q(x, y) = 0 \text{ for all } y \in U\}$ by $U^\perp$ and the λ–quadratic form $(U^\perp, q|_{U^\perp})$ by $(U, q')^\perp$. This form exists and is a subform of (V, q) by (3.2).

Lemma 4.2. *Let ρ be a λ–quadratic form over (D, θ). If H and H' are two adjacent hyperbolic planes that are subforms of ρ, then the λ–quadratic forms $H^\perp$ and $H^\perp$ are isometric.*

Proof. The proof follows similarly to the analogous hermitian form result in [4, Chapter 1, (6.4.4)]. □

Corollary 4.3. *Let ρ, ρ_1 and ρ_2 be λ–quadratic forms over (D, θ) such that ρ is nonsingular. If $\rho_1 \perp \rho \simeq \rho_2 \perp \rho$ then $\rho_1 \simeq \rho_2$.*

Proof. As $\rho\perp(-\rho)$ is hyperbolic by [4, Chapter 1, (5.5.2)] and $\rho_1\perp\rho\perp(-\rho) \simeq \rho_2\perp\rho\perp(-\rho)$, the result follows from (4.2). □

5. Diagonalisability of generalised quadratic forms

Throughout this section, let (D,θ) be an F–division algebra with involution. For $a_1,\ldots,a_n \in D$ we denote the λ–quadratic form (D^n, q) with $q : D^n \to D/\mathrm{Alt}_\lambda(D,\theta)$ given by

$$(x_1,\ldots,x_n) \mapsto \sum_{i=1}^{n} \theta(x_i)a_ix_i + \mathrm{Alt}_\lambda(D,\theta)$$

by $\langle a_1,\ldots,a_n\rangle_{(\lambda,\theta)}$. We call such a form a *diagonal form*, and we call a λ–quadratic form *diagonalisable* if it is isometric to a diagonal form. Similarly, we denote the λ–hermitian form (D^n, h) with $h : D^n \times D^n \to D$ given by

$$(x_1,\ldots,x_n) \times (y_1,\ldots,y_n) \mapsto \sum_{i=1}^{n} \theta(x_i)a_iy_i$$

by $\langle a_1,\ldots,a_n\rangle^{her}_{(\lambda,\theta)}$.

Lemma 5.1. *Take $a_1,\ldots,a_n \in D$. Then the polar form of $\langle a_1,\ldots,a_n\rangle_{(\lambda,\theta)}$ is $\langle a_1+\lambda\theta(a_1),\ldots,a_n+\lambda\theta(a_n)\rangle^{her}_{(\lambda,\theta)}$.*

Proof. It suffices to prove the result for $n = 1$. Let $a = a_1$ and $(D,q) = \langle a\rangle_{(\lambda,\theta)}$. Then for all $x, y \in D$ we have

$$\begin{aligned} q(x+y) - q(x) - q(y) &= \theta(x)ay + \theta(y)ax + \mathrm{Alt}_\lambda(D,\theta) \\ &= \theta(x)ay + \theta(y)ax - (\theta(y)ax - \lambda\theta(x)\theta(a)y) + \mathrm{Alt}_\lambda(D,\theta) \\ &= \theta(x)(a + \lambda\theta(a)) + \mathrm{Alt}_\lambda(D,\theta), \end{aligned}$$

as required. □

Proposition 5.2. *Let ρ be a λ–quadratic form over (D,θ). Then ρ is diagonalisable except if $(D,\theta) = (F,\mathrm{id})$, $\lambda = -1$ and if $\mathrm{char}(F) = 2$, ρ is not totally singular.*

Proof. Assume first that $(D,\theta) = (F,\mathrm{id})$. If $\mathrm{char}(F) \neq 2$, that ρ is not diagonalisable if $\lambda = -1$ follows from [1, (1.8)]. If $\mathrm{char}(F) = 2$ then by [1, (7.24)], ρ is diagonalisable if and only if ρ is totally singular.

Assume now that $(D,\theta) \neq (F,\mathrm{id})$ or $\lambda \neq -1$. Let $\rho = (V,q)$. By (3.3), we can find a nonsingular λ–quadratic form $\rho_1 = (V_1,q_1)$ and a totally singular λ–quadratic form $\rho_2 = (V_2,q_2)$ over (D,θ) such that $\rho \simeq \rho_1\perp\rho_2$. That ρ_2 is diagonalisable is clear as for all $x, y \in V_2$ we have $h_{q_2}(x,y) = 0$, and hence $q_2(x+y) = q_2(x) + q_2(y)$, so any decomposition of V_2 into 1–dimensional summands gives an orthogonal decomposition of ρ_2. Therefore we may assume that ρ is nonsingular.

By (3.6), we have that $\rho \simeq \rho'\perp n \times \mathbb{H}_{(\lambda,\theta)}$ for some anisotropic λ–quadratic form ρ' and an nonnegative integer n. Let $a \in D\backslash\mathrm{Sym}(D,\theta)$. By (5.1), the polar form of $\langle a,-a\rangle_{(\lambda,\theta)}$ is $\langle a+\lambda\theta(a), -(a+\lambda\theta(a))\rangle^{her}_{(\lambda,\theta)}$. In particular, $\langle a,-a\rangle_{(\lambda,\theta)}$ is nonsingular and clearly $\langle a,-a\rangle_{(\lambda,\theta)}$ isotropic. Hence by (3.5), $\langle a,-a\rangle_{(\lambda,\theta)} \simeq \mathbb{H}_{(\lambda,\theta)}$. Therefore we may assume that ρ is anisotropic.

Since $\lambda \neq -1$ or $(D,\theta) \neq (F,\mathrm{id})$, there exists an $z \in V$ such that $h_q(z,z) \neq 0$ by (3.1). The proof of [4, Chapter 1 (6.5.3)] then shows that we can write ρ as an orthogonal sum of a nonsingular 1–dimensional λ–quadratic form on zD and a nonsingular λ–quadratic form of smaller dimension. The result follows from induction on the dimension of V. $\square$

6. Totally Singular Generalised quadratic forms

In this section we consider totally singular generalised quadratic forms. Since totally singular forms over (D,θ) are only of interest if $\mathrm{char}(F)=2$ and (D,θ) is an F–division algebra with involution of the first kind, we assume we are in this case throughout the section. In particular, since we always have $\lambda=1$ in this case, we drop it from our notation.

Proposition 6.1. *Let ρ be a totally singular quadratic form over (D,θ). Then there exists an anisotropic totally singular quadratic form ρ' and a nonnegative integer j such that $\rho\simeq\rho'\perp(j\times\langle 0\rangle_\theta)$.*

Proof. If $\rho=(V,q)$ is anisotropic, then we are done. Otherwise, let take $x\in V\backslash\{0\}$ such that $q(x)\in\mathrm{Alt}(D,\theta)$. Then $V\simeq xD\oplus U$ for an F–vector subspace U of V. As h_q is the zero map, it follows that $q(x+y)=q(x)+q(y)$ for all $y\in U$. Hence $\rho\simeq(U,q|_U)\perp\langle 0\rangle_\theta$. The result follows by induction on the dimension of V. $\square$

Lemma 6.2. *Let (V,q) be a quadratic form over (D,θ). For all $x\in V$ we have $h_q(x,x)=\kappa+\theta(\kappa)$, where κ is any representative $\kappa\in D$ of $q(x)\in D/\mathrm{Alt}(D,\theta)$. In particular, $h_q(x,x)\in\mathrm{Alt}(D,\theta)$ for all $x\in V$.*

Proof. See [2, (1.1)]. $\square$

Proposition 6.3. *Assume that $(D,\theta)\neq(F,\mathrm{id})$. Let ρ be a quadratic form over (D,θ). Then ρ is totally singular if and only if every element represented by ρ is in $\mathrm{Sym}(D,\theta)/\mathrm{Alt}(D,\theta)$.*

Proof. Suppose $\rho=(V,q)$ only represents elements in $\mathrm{Sym}(D,\theta)$. By (6.2) this implies that $h_q(x,x)=0$ for all $x\in V$. Therefore (V,h_q) is the zero map on V by (3.1). Conversely, if there exists an $x\in V\backslash\{0\}$ such that $q(x)\notin\mathrm{Sym}(D,\theta)/\mathrm{Alt}(D,\theta)$ then by (6.2) $h_q(x,x)\neq 0$ and hence ρ is not totally singular. $\square$

Remark 6.4. (6.3) can also be shown using (5.2) and (5.1).

Corollary 6.5. *Assume that $(D,\theta)\neq(F,\mathrm{id})$. Let ρ be an anisotropic n–dimensional quadratic form over (D,θ). Then ρ is totally singular if and only if there exist $b_1,\dots,b_n\in\mathrm{Sym}(D,\theta)$ such that $\rho\simeq\langle b_1,\dots,b_n\rangle_\theta$.*

Proof. The result follows directly from (5.2) and (6.3). $\square$

Remark 6.6. The description of totally singular forms in $\mathrm{char}(F)=2$ from (6.5) holds even if $(D,\theta)=(F,\mathrm{id})$ by [1, (7.24)], as then $\mathrm{Sym}(D,\theta)=F$.

Corollary 6.7. *Assume that $(D,\theta)\neq(F,\mathrm{id})$. Take $a\in D$. Then $\langle a\rangle_\theta$ is totally singular if and only if $a\in\mathrm{Sym}(D,\theta)$. Otherwise $\langle a\rangle_\theta$ is nonsingular and does not represent any elements in $\mathrm{Sym}(D,\theta)/\mathrm{Alt}(D,\theta)$.*

Proof. All elements represented by $\langle a\rangle_\theta$ are of the form $\theta(x)ax+\mathrm{Alt}(D,\theta)$ for $x\in D$. If $x\neq 0$, then $\theta(x)ax\in\mathrm{Sym}(D,\theta)/\mathrm{Alt}(D,\theta)$ if and only if $a\in\mathrm{Sym}(D,\theta)$. Hence $\langle a\rangle_\theta$ represents a non-zero element in $\mathrm{Sym}(D,\theta)/\mathrm{Alt}(D,\theta)$ if and only if all elements represented by $\langle a\rangle_\theta$ are elements in $\mathrm{Sym}(D,\theta)/\mathrm{Alt}(D,\theta)$. That is, if and only if $\langle a\rangle_\theta$ is totally singular by (6.3). That $\langle a\rangle_\theta$ is nonsingular if $\langle a\rangle_\theta$ is not totally singular is clear as $\dim_D(\langle a\rangle_\theta)=1$. $\square$

7. Nonsingular forms representing symmetric elements

In (6.3) we showed that, in characteristic 2, totally singular generalised quadratic forms only represent symmetric elements. This means that the isotropy of an orthogonal sum of an anisotropic nonsingular generalised quadratic form and an anisotropic totally singular generalised quadratic form depends on whether the nonsingular form represents a symmetric element. In this section, we investigate when this can occur.

Again, throughout we assume that $\mathrm{char}(F) = 2$ and (D, θ) is an F–division algebra with involution of the first kind, and as we always have $\lambda = 1$ we drop it from our notation.

Lemma 7.1. *For all elements $a \in D$ and $b \in \mathrm{Sym}(D, \theta)$ we have $\langle a, b\rangle_\theta \simeq \langle a + b, b\rangle_\theta$.*

Proof. Let $(V, q) = \langle a, b\rangle_\theta$. Let α be the map $\alpha : V \to V$ given by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then for $(x, y) \in V$ we have

$$\begin{aligned} q(\alpha(x, y)) &= q((x, x + y)) + \mathrm{Alt}(D, \theta) \\ &= \theta(x)ax + \theta(x)bx + \theta(y)by + \mathrm{Alt}(D, \theta) \\ &= \theta(x)ax + \theta(x)bx + \theta(y)by + (\theta(x)by + \theta(y)bx) + \mathrm{Alt}(D, \theta) \\ &= \theta(x)(a + b)x + \theta(x)bx + \mathrm{Alt}(D, \theta). \end{aligned}$$

Hence α gives an isometry $\langle a, b\rangle_\theta \simeq \langle a + b, b\rangle_\theta$. □

Lemma 7.2. *Assume $(D, \theta) \neq (F, \mathrm{id})$. Let ρ be a nonsingular anisotropic quadratic form over (D, θ). Take $b \in \mathrm{Sym}(D, \theta)$. Then ρ represents b if and only if there exists an $a \in D \setminus \mathrm{Sym}(D, \theta)$ and an anisotropic quadratic form ρ' such that $\rho \simeq \langle a, a + b\rangle_\theta \perp \rho'$.*

Proof. That $\langle a, a + b\rangle_\theta$ represents b, and hence the 'if' implication in the statement, is clear. Assume now that ρ represents b and let $\rho = (V, q)$. If $\dim_D(V) = 1$ then that ρ represents an element in $\mathrm{Sym}(D, \theta)/\mathrm{Alt}(D, \theta)$, contradicts the nonsingularity of ρ by (6.7). Therefore we may assume that $\dim_D(V) > 1$.

If ρ only represents elements in $\mathrm{Sym}(D, \theta)/\mathrm{Alt}(D, \theta)$, then ρ is totally singular by (6.3). Hence there exists an element $a \in D \setminus \mathrm{Sym}(D, \theta)$ represented by ρ. Since $\langle a\rangle_\theta$ is nonsingular by (6.7), it follows from (3.2) that $\rho \simeq \langle a\rangle_\theta \perp \rho'$ for some anisotropic λ–quadratic form $\rho' = (W, q')$.

We have

$$0 \neq b = \theta(d)ad + q'(y) + \mathrm{Alt}(D, \theta)$$

for some $y \in W$ and $d \in D$. If $d = 0$, then (W', q') represents b and the result follows from induction on $\dim_D(V)$.

If $d \neq 0$, then we may scale b, x and y in order to assume that $d = 1$. We must have that $y \neq 0$ as otherwise $\langle a\rangle_\theta$ represents $b \in \mathrm{Sym}(D, \theta)/\mathrm{Alt}(D, \theta)$, which cannot occur by (6.7). Hence ρ' represents $a + b$. Therefore, again by (3.2), we have that $\rho' \simeq \langle a + b\rangle_\theta \perp \rho''$ for an anisotropic λ-quadratic form ρ''. Hence $\rho \simeq \langle a, a + b\rangle_\theta \perp \rho''$. □

Proposition 7.3. *Let ρ_1 and ρ_2 be anisotropic quadratic forms over (D, θ) such that ρ_1 is nonsingular and ρ_2 is totally singular. If $\rho \simeq \rho_1 \perp \rho_2$ is isotropic, then there exists a nonsingular anisotropic quadratic form ρ_1' such that $\rho \simeq \rho_1' \perp \mathbb{H}_\theta \perp \rho_2$.*

Proof. See the remarks preceding [3, (2.4)] for the case of $(D,\theta) = (F,\mathrm{id})$.

Now assume that $(D,\theta) \neq (F,\mathrm{id})$. The isotropy of ρ and the anisotropy of $\rho_1 = (W_1, q_1)$ and $\rho_2 = (W_2, q_2)$ imply that there exist elements $x_1 \in W_1\backslash\{0\}$ and $x_2 \in W_2\backslash\{0\}$ such that $q_1(x_1), q_2(x_2) \notin \mathrm{Alt}(D,\theta)$ and $q_1(x_1) + q_2(x_2) \in \mathrm{Alt}(D,\theta)$. By (6.3) we have that

$$q(x_1) = q(x_2) = b \in \mathrm{Sym}(D,\theta)/\mathrm{Alt}(D,\theta).$$

Then by (7.2) there exists anisotropic λ–hermitian forms ρ_1' and ρ_2' with ρ_1' is nonsingular and ρ_2' is totally singular such that $\rho \simeq \langle a, a+b\rangle_\theta \perp \rho_1' \perp \langle b\rangle_\theta \perp \rho_2'$. Finally, (7.1) gives

$$\langle a, a+b\rangle_\theta \perp \rho_1' \perp \langle b\rangle_\theta \perp \rho_2' \simeq \mathbb{H}_\theta \perp \rho_1' \perp \rho$$

as required. □

8. Witt Decomposition of singular quadratic forms

In this section we give our generalisation of [3, (2.4)] to the setting of generalised quadratic forms. Throughout this section, let (D,θ) be an F–division algebra with involution. The following is a generalisation of [3, (2.6)]. The proof of [3, (2.6)] is easily adapted to our setting, but we include it for connivence.

Lemma 8.1. *Let ρ_1 and ρ_2 be regular λ–quadratic forms over (D,θ). If $\rho_1 \perp (j \times \langle 0\rangle_{(\lambda,\theta)}) \simeq \rho_2 \perp (j \times \langle 0\rangle_{(\lambda,\theta)})$ for a nonnegative integer j, then $\rho_1 \simeq \rho_2$.*

Proof. By (3.3), (3.6) and (7.3) we can write $\rho_1 \simeq m \times \mathbb{H}_{(\lambda,\theta)} \perp \rho_1'$ and $\rho_2 \simeq n \times \mathbb{H}_{(\lambda,\theta)} \perp \rho_2'$ where m, n are nonnegative integers and ρ_1' and ρ_2' are anisotropic. We may assume that $m \geqslant n$. By (4.3) we have $(m-n) \times \mathbb{H}_{(\lambda,\theta)} \perp \rho_1' \simeq \rho_2'$, and hence we must have that $m = n$. Therefore it suffices to prove that for $\pi_1 = \rho_1' \perp (j \times \langle 0\rangle_{(\lambda,\theta)})$ and $\pi_2 = \rho_2' \perp (j \times \langle 0\rangle_{(\lambda,\theta)})$, we have $\pi_1 \simeq \pi_2$ implies that $\rho_1' \simeq \rho_2'$.

Let $\rho_i' = (W_i, q_i)$ and $\pi_i = (V_i, p_i)$ for $i = 1, 2$. Further, for $i = 1, 2$, let U_i be such that $V_i = W_i \oplus U_i$ and $(U_i, p_i|_{U_i}) = j \times \langle 0\rangle_{(\lambda,\theta)}$. Now let $\phi : V_1 \to V_2$ be an F–vector space isomorphism that is an isometry of π_1 and π_2. Let $\sigma : V_2 = W_2 \oplus U_2 \to W_2$ be a projection onto W_2 and define $\tau : W_1 \to W_2$ by $\tau = \phi \circ \sigma|_{W_1}$.

If $w \in W_1$ and $\sigma(w) = w' + u'$, for some $w' \in W_2$ and $u' \in U_2$, then $\tau(w) = w'$ and thus

$$q_2(\tau(w)) = p_2(\tau(w)) = p_2(w') = p_2(w' + u') = p_2(\sigma(w)) = p(w) = q_1(w),$$

where the third equality holds as $u' \in U_2 \subset V_2^\perp$ and $\varphi_2(u') = 0$. To show that τ is an isometry, it suffices to show that τ is bijective. If $0 \neq w \in W_1$ then $q_1(w) \notin \mathrm{Alt}(D,\theta)$ as ρ_1' is anisotropic. Hence $0 \neq q_2(\tau(w)) = q_1(w) \notin \mathrm{Alt}(D,\theta)$ and in particular $w \neq 0$. Therefore τ is injective, and bijectivity follows as $\dim_D(W_1) = \dim_D(W_2)$. □

Theorem 8.2. *Assume* $\mathrm{char}(F) = 2$ *and (D,θ) is of the first kind. Let ρ be a 1–quadratic form over (D,θ). Then there exists a nonsingular 1–quadratic form ρ_1, a totally singular 1–quadratic form ρ_2 and nonnegative integers i and j such that $\rho_1 \perp \rho_2$ is anisotropic and*

$$\rho \simeq \rho_1 \perp (i \times \mathbb{H}_{(1,\theta)}) \perp \rho_2 \perp (j \times \langle 0\rangle_{(1,\theta)}).$$

The integers n and m are uniquely determined, and the 1–quadratic forms $\rho_1 \perp \rho_2$ and ρ_2 are uniquely determined up to isometry.

Proof. Since $\lambda = 1$ throughout this proof, we drop it from the notation. By (3.3), (3.6), (6.1) and (7.3) we need only prove the uniqueness of the decomposition.

Suppose

$$\rho \simeq \rho_1 \perp (i \times \mathbb{H}_\theta) \perp \rho_2 \perp (j \times \langle 0 \rangle_\theta) \simeq \rho_2' \perp (i' \times \mathbb{H}_\theta) \perp \rho_2' \perp (j' \times \langle 0 \rangle_\theta)$$

for nonnegative integers i, i', j and j', nonsingular quadratic forms ρ_1 and ρ_1' over (D, θ) and totally singular quadratic forms ρ_2 and ρ_2' over (D, θ). We must have that $j = j'$ as j and j' is the dimension of the radical of the respective quadratic forms, and any isometry maps the radical of a quadratic form to the radical of the other quadratic form. It then follows from (8.1) that

$$\rho_1 \perp (i \times \mathbb{H}_\theta) \perp \rho_2 \simeq \rho_1' \perp (i' \times \mathbb{H}_\theta) \perp \rho_2.$$

That $\rho_1 \perp \rho_2 \simeq \rho_1' \perp \rho_2'$ and $\rho_2 \simeq \rho_2'$ then follows from (4.3). □

Note that, in the situation of (8.2), ρ_1 is generally not uniquely determined up to isometry, as we show in (9.6).

9. EXPLICIT EXAMPLES

Throughout this section we assume that $\mathrm{char}(F) = 2$ and that (D, θ) is an F–division algebra with involution of the first kind. As we again have $\lambda = 1$ throughout, we drop it from the notion. The importance of (7.2) in the proof of (8.2) suggests the following question.

Question 9.1. Assume $(D, \theta) \neq (F, \mathrm{id})$. For which $a \in D \setminus \mathrm{Sym}(D, \theta)$ and $b \in \mathrm{Sym}(D, \theta)$ is $\langle a, a + b \rangle_\theta$ isotropic?

We now give an example of an F–division algebra with involution over which there exist both anisotropic and isotropic quadratic forms of the type in Question 9.1. This also provides an example showing that totally singular generalised quadratic form cannot be cancelled in general.

For $a \in F$ with $-4a \neq 1$, let $q : F^2 \to F$ be the map given by $q(x, y) = x^2 + xy + ay^2$. Then (F^2, q) is a nonsingular quadratic form over (F, id). We denote (F^2, q) by $[1, a]$.

Lemma 9.2. *Let $\mathbb{F}_2$ be the field of two elements and let $F = \mathbb{F}_2(X)$, where X is an indeterminate. Let $Q = [X, 1 + X)_F$. Then Q is an F–division algebra.*

Proof. By [1, (12.5)], Q is division if and only if $\pi = [1, X] \perp (1 + X) \cdot [1, X]$ is anisotropic. By [1, (23.11)], π is anisotropic if and only if $\rho = [1, X] \perp \langle 1 + X \rangle_{(1,\mathrm{id})}$ is anisotropic. It is clear that ρ is isotropic if and only if either $[1, X]$ represents $1 + X$ or $[1, X]$ is isotropic. First we show that $[1, X]$ does not represent $1 + X$.

By [1, (17.3)], $[1, X]$ represents $1 + X$ if and only if there exist elements $a, b \in \mathbb{F}_2[X]$, not both zero, such that $a^2 + ab + a^2 X = 1 + X$. Assume such elements exist and write $a = \sum_{i=0}^{n} a_i X^i$ and $b = \sum_{i=0}^{m} b_i X^i$ for some $n, m \in \mathbb{N}$ and $a_0, \dots, a_n, b_0, \dots, b_m \in \mathbb{F}_2$ and such that $a_n \neq 0$ and $b_m \neq 0$.

Assume first that $m \geqslant n > 0$. Then we get that $1 + X = b_m^2 X^{2m+1} + c$ where $\deg_X(c) < 2m + 1$. This contradicts $b_m \neq 0$. Now assume that $n > m > 0$. Then we have that $1 + X = a_n X^{2n} + c'$ where $\deg_X(c') < 2n$, contradicting $a_n \neq 0$. Therefore we must have that $a, b \in \mathbb{F}_2$. In particular, we must have that $b = 1$ and $a^2 + a = 1$. However, $x^2 + x = 1$ has no solution in $\mathbb{F}_2$, therefore $[1, X]$ does not represent $1 + X$. That $[1, X]$ is anisotropic can be shown in a similar way. □

We denote the central simple F–algebra of $n \times n$ matrices over D by $M_n(D)$ and the $n \times n$ identity matrix by I_n. For a matrix $M \in M_n(D)$, let M^t denote the transpose of M and let M^* denote the image of M under the F–involution on $M_n(D)$ given by

$$((a_{ij})_{1\leqslant i,j\leqslant n})^* = (\theta(a_{ij}))^t_{1\leqslant i,j\leqslant n}\,.$$

Note that as $I_n \in \mathrm{Alt}(M_n(D),*)$, this involution is symplectic by [5, (2.6)].

Let (V,q) be an n–dimensional quadratic form over (D,θ). By [4, Chapter 1, (5.1.1)] one can find a matrix $M \in M_n(D)$ such that $q : V \to D/\mathrm{Alt}(D,\theta)$ is given by

$$(x_1,\dots,x_n) \mapsto (\theta(x_1),\dots,\theta(x_n))M(x_1,\dots,x_n)^t + \mathrm{Alt}(D,\theta)\,.$$

We call M the *matrix associated to* (V,q). Then if (V,h) is the polar form of (V,q), the map $h : V \to D$ is given by

$$(x_1,\dots,x_n)\times(y_1,\dots,y_n) \mapsto (\theta(x_1),\dots,\theta(x_n))(M+M^*)(y_1,\dots,y_n)^t\,,$$

(see [4, Chapter 1, (5.3)]), and we say $M+M^*$ is the *matrix associated to the polar of* (V,q).

Let ρ be an n–dimensional nonsingular quadratic form over (D,θ). Let $M \in M_n(D)$ be the matrix associated to ρ and let N be the matrix associated to the polar of ρ. As ρ is nonsingular we have that N is invertible and that at least one of $\deg(D)$ or $\dim_D(\rho)$ is even. Let $2m = \deg(D)\cdot\dim_D(\rho)$. We denote the set $\{a^2+a \mid a \in F\}$ by $\wp(F)$. The *Arf invariant of* ρ is then defined as the class in $F/\wp F$ given by

$$\mathrm{Srd}_{M_n(D)}(N^{-1}\cdot M) + \frac{m(m-1)}{2} + \wp(F).$$

We denote this class by $\Delta(\rho)$. By [8, Corollaire 4], $\Delta(\rho)$ depends only on the isometry class of ρ and not on the choice of M.

Lemma 9.3. *Let ρ_1 and ρ_2 be nonsingular quadratic forms over (D,θ). Then $\Delta(\rho_1\perp\rho_2) = \Delta(\rho_1)+\Delta(\rho_2)$.*

Proof. For $i = 1,2$, let $n_i = \dim_D(\rho_i)$, $M_i \in M_{n_i}(D)$ be the matrix associated with ρ_i and $N_i \in M_{n_i}(D)$ be the matrix associated with the polar of ρ_i. Then matrix associated with $\rho_1\perp\rho_2$ in $M_{n_1+n_2}(D)$ is $M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2\end{pmatrix}$ and the matrix associated with its polar form is $N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2\end{pmatrix}$. By [5, (0.2)] we have

$$\begin{aligned}\mathrm{Srd}_{M_{n_1+n_2}(D)}(N^{-1}M) &= \mathrm{Srd}_{M_{n_1}(D)}(N_1^{-1}M_1) + \mathrm{Srd}_{M_{n_2}(D)}(N_2^{-1}M_2)\\ &\quad + \mathrm{Trd}_{M_{n_1}(D)}(N_1^{-1}M_1)\cdot\mathrm{Trd}_{M_{n_2}(D)}(N_2^{-1}M_2)\,.\end{aligned}$$

For $i=1,2$ we have that $N_i^* = N$ and that N, and hence N_i^{-1}, commutes with M_i. Hence, we have that $N^{-1}M + (N^{-1}M)^* = I_{n_i}$. It therefore follows from [5, (2.13) and (2.12)] that $\mathrm{Trd}_{M_{n_i}}(N_i^{-1}M_i) = \frac{1}{2}\deg(D)\cdot n_i$ for $i=1,2$. The formula for $\Delta(\rho_1\perp\rho_2)$ is then easily checked. □

Lemma 9.4. *Let ρ be a hyperbolic nonsingular quadratic form over (D,θ). Then $\Delta(\rho) = 0$.*

Proof. Let $m = \deg(D)$. By (3.5), we have that $\rho \simeq n \times \mathbb{H}_\theta$ for some $n \in \mathbb{N}$. Hence by (9.3), it suffices to show the result for the case $\rho = \mathbb{H}_\theta$. The matrix in $M_2(D)$ associated with ρ is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. As shown in the proof of (3.4), the matrix associated with the polar form of ρ in $M_2(D)$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then, as e is idempotent in $M_2(D)$, we have

$$\mathrm{Prd}_{M_2(D),e} = (X-1)^m \text{ and hence } \mathrm{Srd}_{M_2(D)}(e) = \frac{m(m-1)}{2}.$$

Therefore $\Delta(\rho) = 0$. □

The following gives an example of an anisotropic and a hyperbolic generalised quadratic form of the type considered in Question 9.1.

Example 9.5. Let $\mathbb{F}_2$ be the field of two elements and let $F = \mathbb{F}_2(X)$, where X is an indeterminate. Let $Q = [X, 1+X)_F$. Then Q is an F–division algebra by (9.2). Let γ be the unique symplectic involution on Q. Then there exist elements $u, v \in Q$ such that $u^2 = u + X$, $\gamma(u) = 1 + u$, $v^2 = 1 + X$ and $\gamma(v) = v$. In particular, $u \notin \mathrm{Sym}(D, \theta)$ and $v, v + uv \in \mathrm{Sym}(D, \theta)$.

Let

$$\rho = \langle u \rangle_\gamma, \rho_1 = \langle u + v \rangle_\gamma \text{ and } \rho_2 = \langle u + v + uv \rangle_\gamma.$$

By (5.1), $\langle 1 \rangle_\gamma^{her}$ is the polar form of each of ρ, ρ_1 and ρ_2 and therefore ρ, ρ_1 and ρ_2 are all nonsingular. It is also clear that ρ, ρ_1 and ρ_2 are all anisotropic as they are 1-dimensional. Note that as Q is a quaternion algebra, we have that $\mathrm{Srd}_Q = \mathrm{Nrd}_Q$. Hence $\Delta(\rho) = \mathrm{Nrd}_Q(1 \cdot u) = X$. Similarly, $\Delta(\rho_1) = 1$ and $\Delta(\rho_2) = X^2 = X \bmod \wp(F)$.

Note that $1 + X \notin \wp(F)$ holds if and only if $[1, 1+X]$ is anisotropic over F, and that $[1, 1+X]$ is anisotropic follows from an argument similar to the proof of (9.2). Therefore it follows from (9.3) that $\Delta(\rho \perp \rho_1) = 1 + X \neq 0 \in F/\wp(F)$. Hence $\rho \perp \rho_1$ is not hyperbolic by (9.4) and therefore anisotropic as $\dim(\rho \perp \rho_1) = 2$.

Finally as

$$\gamma(u+v)u(u+v) = u + v + uv + (1+X)$$

and $1 + X \in \mathrm{Alt}(Q, \gamma)$, we have that $\rho \perp \rho_2$ is isotropic and hence hyperbolic as $\dim_D(\rho \perp \rho_2) = 2$.

The following example shows that totally singular generalised quadratic forms cannot be cancelled in general.

Example 9.6. Let F, (Q, γ), ρ and ρ_1 all be as in (9.5). Then $\langle v \rangle_\gamma$ is totally singular by (6.7). Further, $\rho \perp \langle v \rangle_\gamma \simeq \rho_1 \perp \langle v \rangle_\gamma$ by (7.1). If $\rho \simeq \rho_1$, then $\rho \perp \rho_1$ is hyperbolic, but $\rho \perp \rho_1$ is anisotropic by (9.5). Hence $\rho \not\simeq \rho_1$.

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