# PUSH-PULL OPERATORS ON THE FORMAL AFFINE DEMAZURE ALGEBRA AND ITS DUAL

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## 1. Introduction

In a series of papers [KK86], [KK90] Kostant and Kumar introduced and successfully applied the techniques of nil (or 0-) Hecke algebras to study equivariant cohomology and K-theory of flag varieties. In particular, they showed that the dual of the nil Hecke algebra serves as an algebraic model for the $T$-equivariant singular cohomology of $G/B$ (here $G$ is a split semisimple linear algebrac group with a chosen split maximal torus $T$ and $G/B$ is the variety of Borel subgroups). In [HMSZ] and [CZZ], this formalism has been generalized using an arbitrary formal group law associated to an algebraic oriented cohomology theory in the sense of Levine-Morel [LM07], via the Quillen formula. Namely, given a formal group law $F$ and a finite root system with a set of simple roots $\Pi$, one defines the *formal affine Demazure algebra* $\mathbf{D}_F$ and its dual $\mathbf{D}_F^\star$ provides an algebraic model for the $T$-equivariant oriented cohomology $\mathtt{h}_T(G/B)$. Specializing to the additive and the multiplicative formal group laws, one recovers Chow groups (or singular cohomology) and $K$-theory respectively.

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Another motivation for studying the algebra $\mathbf{D}_F$ comes from its close relationship to Hecke algebras. Indeed, for the additive (resp. multiplicative) $F$ it coincides with the completion of the nil (resp. 0-) affine Hecke algebra (see [HMSZ]). Moreover, in section 8, we show that for some elliptic formal group law $F$ and a root system of Dynkin type $A$ the non-affine part of $\mathbf{D}_F$ is isomorphic to the classical Iwahori-Hecke algebra, hence, relating it to equivariant elliptic cohomology.

In the present paper we pursue the 'algebraization program' for oriented cohomology theories started in [CPZ] and continued in [HMSZ] and [CZZ]; the general idea is to match cohomology rings of flag varieties and elements of classical interest in them (such as classes of Schubert varieties) with algebraic and combinatorial objects that can be introduced simply and algebraically, in the spirit of [De73] or [KK86]. This approach is useful to study the structure of these rings, and to perform various computations. We focus here on algebraic constructions pertaining to $T$-equivariant oriented cohomology groups. The precise proofs and details of how our algebraic objects match cohomology groups will be treated in a forthcoming paper; however, for the convenience of the reader, we now give a brief description of the geometric setting.

Given an equivariant oriented cohomology theory $\mathtt{h}$ over a base field whose spectrum is denoted by pt, the formal group algebra $S$ will correspond to $\mathtt{h}_T(\mathrm{pt})$. [1] It is an algebra over $R = \mathtt{h}(\mathrm{pt})$.

The $T$-fixed points of $G/B$ are naturally in bijection with the Weyl group $W$. This gives a pull-back to the fixed locus map $\mathtt{h}_T(G/B) \to \mathtt{h}_T(W) \simeq \bigoplus_{w\in W} \mathtt{h}_T(\mathrm{pt})$. This map happens to be injective. We do not know a direct geometric reason for that, but it follows from our algebraic description, in which it appears as the map $\mathbf{D}_F^\star \to S_W^\star \simeq \bigoplus_{w\in W} S$ of Definition 9.1. It is then convenient to enlarge $S$ to its localization $Q$ at a multiplicative subset generated by Chern class of line bundles corresponding canonically to roots, which gives injections $S \subseteq Q$, $S_W \subseteq Q_W$ and $S_W^\star \subseteq Q_W^*$. Although we do not know good geometric interpretations of $Q$, $Q_W$ or $Q_W^*$, all the formulas and operators we are interested in are easily defined at that localized level, because they involve denominators. The main technical difficulties then lie in proving that these operators actually restrict to $S$, $S_W^\star$, $\mathbf{D}_F^\star$ etc., or so to speak, that the denominators cancel out.

Our central object of study is a push-pull operator on $\mathbf{D}_F^\star$, which is an algebraic version of the composition

$$\mathtt{h}_T(G/P) \xrightarrow{p_*} \mathtt{h}_T(G/Q) \xrightarrow{p^*} \mathtt{h}_T(G/P)$$

of the push-forward followed by the pull-back along the quotient map $p\colon G/P \to G/Q$, where $P \subseteq Q$ are two parabolic subgroups of $G$. Again $p^*$ happens to be injective, and it identifies $\mathtt{h}_T(G/Q)$ to a subring of $\mathtt{h}_T(G/P)$, namely the subring of invariants under the action of the parabolic subgroup $W_Q$ of the Weyl group $W$. This does not seem to be straightforward from the geometry either, and it once more follows from our algebraic description: given subsets $\Xi' \subseteq \Xi$ of a given set of simple roots $\Pi$ (each giving rise to a parabolic subgroup), we define an element $Y_{\Xi/\Xi'}$ in $Q_W$ (see 5.3). We define an action of the Demazure algebra $\mathbf{D}_F$ on its $S$-dual $\mathbf{D}_F^\star$, by precomposition by multiplication on the right. The action of $Y_{\Xi/\Xi'}$ thus defines the desired push-pull operator $A_{\Xi/\Xi'} : (\mathbf{D}_F^\star)^{W_{\Xi'}} \to (\mathbf{D}_F^\star)^{W_\Xi}$. The formula for the

[1] We will require that the cohomology rings are 'complete' in some precise sense, but this is a technical point, that we prefer to hide here for simplicity.

element $Y_{\Xi/\Xi'}$ with $\Xi' = \emptyset$ had already appeared in related contexts, namely, in discussions around the Becker-Gottlieb transfer for topological complex-oriented theories (see [BE90, (2.1)] and [GR12, §4.1]).

Finally, we define the algebraic counterpart of the natural pairing $\mathtt{h}_T(G/B) \otimes \mathtt{h}_T(G/B) \to \mathtt{h}_T(\mathrm{pt})$ obtained by multiplication and push-forward to the point. It is a pairing $\mathbf{D}_F^\star \otimes \mathbf{D}_F^\star \to S$. We show that it is non-degenerate, and that algebraic classes corresponding to (chosen) desingularization of Schubert varieties form a basis of $\mathbf{D}_F^\star$, with a very simple dual basis with respect to the pairing. We provide the same kind of description for $\mathtt{h}_T(G/P)$. This generalizes (to parabolic subgroups and to equivariant cohomology groups) and simplifies several statements from [CPZ, §14], as well as results from [KK86] and [KK90] (to arbitrary oriented cohomology theories).

The paper is organized as follows. In sections 2 and 3, we recall definitions and basic properties from [CPZ, §2,3], [HMSZ, §6] and [CZZ, §4,5]: the formal group algebra $S$, the Demazure and push-pull operators $\Delta_\alpha$ and $C_\alpha$ for every root $\alpha$, the formal twisted group algebra $Q_W$ and its Demazure and push-pull elements $X_\alpha$ and $Y_\alpha$. In section 4, we introduce a left $Q_W$-action '$\bullet$' on the dual $Q_W^*$. It induces both an action of the Weyl group $W$ on $Q_W^*$ (the Weyl-action) and an action of $X_\alpha$ and $Y_\alpha$ on $Q_W^*$ (the Hecke-action). In sections 5 and 6, we introduce and study the more general push-pull elements in $Q_W$ and operators on $Q_W^*$ with respect to given coset representatives of parabolic quotients of the Weyl group. In section 7, we construct a basis of the subring of invariants of $Q_W^*$, which generalizes [KK90, Lemma 2.27].

In section 8, we recall the definition and basic properties of the formal (affine) Demazure algebra $\mathbf{D}_F$ following [HMSZ, §6], [CZZ, §5] and [Zh13]. We show that for the special elliptic formal group law, the formal Demazure algebra is related to the classical Iwahori-Hecke algebra. In section 9, we define the algebraic restriction to the fixed locus map which is used in section 10 to restrict all our push-pull operators and elements to $\mathbf{D}_F$ and its dual $\mathbf{D}_F^\star$ as well as to restrict the non-degenerate pairing on $\mathbf{D}_F^\star$. At last, in section 11, we define and discuss the non-degenerate pairing on the subring of invariants of $\mathbf{D}_F^\star$ under a parabolic subgroup of the Weyl group.

*Acknowledgments* We would like to mention that one of the ingredients of this paper, the push-pull formulas in the context of Weyl group actions, arose in discussions between the first author and Victor Petrov, whose unapparent contribution we therefore gratefully acknowledge.

## 2. Formal Demazure and push-pull operators

In this section we recall definitions of the formal group algebra and of the formal Demazure and push-pull operators, following [CPZ] and [CZZ].

Let $R$ be a commutative ring with unit, and let $F$ be a one-dimensional commutative *formal group law* (FGL) over $R$, i.e. $F(x,y) \in R[\![x,y]\!]$ satisfies

$$F(x,0)=0,\ F(x,y)=F(y,x),\ F(x,F(y,z))=F(F(x,y),z).$$

**Example 2.1.** The *additive* FGL is defined by $F_a(x,y)=x+y$, and a *multiplicative* FGL is defined by $F_m(x,y)=x+y-\beta xy$ with $\beta\in R$. The coefficient ring of the *universal* FGL is generated by the coefficients $a_{ij}$ modulo relations induced by the above properties and is called the *Lazard ring*.

**Example 2.2.** Consider an elliptic curve given in Tate coordinates by

$$(1-\mu_1 t-\mu_2 t^2)s=t^3.$$

The corresponding FGL over the coefficient ring $R=\mathbb{Z}[\mu_1,\mu_2]$ is given by, e.g. see [BB10, Example 63],

$$F_e(x,y):=\tfrac{x+y-\mu_1 xy}{1+\mu_2 xy}$$

and will be called a *special elliptic* FGL. Observe that

$$F_e(x,y)=x+y-xy(\mu_1+\mu_2 F_e(x,y)),$$

and thus that the formal inverse of $F_e$ is identical to the one of $F_m$, i.e. $\frac{x}{\mu_1 x-1}$, and $F_e(x,x)=\frac{2x-\mu_1 x^2}{1+\mu_2 x^2}$.

Let $\Lambda$ be an Abelian group and let $R[\![x_\Lambda]\!]$ be the ring of formal power series with variables $x_\lambda$ for all $\lambda\in\Lambda$. Define the *formal group algebra* $R[\![\Lambda]\!]_F$ to be the quotient of $R[\![x_\Lambda]\!]$ by the closure of the ideal generated by elements $x_0$ and $x_{\lambda_1+\lambda_2}-F(x_{\lambda_1},x_{\lambda_2})$ for any $\lambda_1,\lambda_2\in\Lambda$. Here 0 is the identity element in $\Lambda$. Let $\mathcal{I}_F$ denote the kernel of the augmentation map $\epsilon\colon R[\![\Lambda]\!]_F\to R$, $x_\alpha\mapsto 0$.

Assume that $\Lambda$ is a free Abelian group of finite rank and let $\Sigma$ be a finite subset of $\Lambda$. A *root datum* is an embedding $\Sigma\hookrightarrow\Lambda^\vee$, $\alpha\mapsto\alpha^\vee$ into the dual of $\Lambda$ satisfying certain conditions [SGA, Exp. XXI, Def. 1.1.1]. The *rank* of the root datum is the $\mathbb{Q}$-rank of $\Lambda\otimes_{\mathbb{Z}}\mathbb{Q}$. The *root lattice* $\Lambda_r$ is the subgroup of $\Lambda$ generated by $\Sigma$, and the *weight lattice* $\Lambda_w$ is the Abelian group defined by

$$\Lambda_w:=\{\omega\in\Lambda\otimes_{\mathbb{Z}}\mathbb{Q}\mid \alpha^\vee(\omega)\in\mathbb{Z}\text{ for all }\alpha\in\Sigma\}.$$

We always assume that the root datum is reduced and *semisimple* ($\mathbb{Q}$-ranks of $\Lambda_r$, $\Lambda_w$ and $\Lambda$ are the same and no root is twice another one). We say that a root datum is *simply connected* (resp. *adjoint*) if $\Lambda=\Lambda_w$ (resp. $\Lambda=\Lambda_r$), and then use the notation $\mathcal{D}_n^{sc}$ (resp. $\mathcal{D}_n^{ad}$) for irreducible root data where $\mathcal{D}=A,B,C,D,E,F,G$ is one of the Dynkin types and $n$ is the rank.

The *Weyl group* $W$ of a root datum $(\Lambda,\Sigma)$ is a subgroup of $\mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ generated by simple reflections $s_\alpha$ for all $\alpha\in\Sigma$ defined by

$$s_\alpha(\lambda):=\lambda-\alpha^\vee(\lambda)\alpha,\quad \lambda\in\Lambda.$$

We fix a set of *simple roots* $\Pi=\{\alpha_1,\ldots,\alpha_n\}\subset\Sigma$ that is a basis of the root datum: each element of $\Sigma$ is an integral linear combination of simple roots with either all positive coefficients or negative. This partitions $\Sigma$ into $\Sigma^+$ and $\Sigma^-$ the subsets of *positive roots*, resp. *negative roots*. Let $\ell$ denote the *length function* on $W$ with respect to the set of simple roots $\Pi$. Let $w_0$ be the *longest element* of $W$ with respect to $\ell$ and let $N:=\ell(w_0)$.

Following [CZZ, Def. 4.4] we say that the formal group algebra $R[\![\Lambda]\!]_F$ is $\Sigma$-*regular* if $x_\alpha$ is not a zero-divisor in $R[\![\Lambda]\!]_F$ for all roots $\alpha\in\Sigma$. We will always assume that:

*The formal group algebra $R[\![\Lambda]\!]_F$ is $\Sigma$-regular.*

By [CZZ, Lemma 2.2] this holds if 2 is not a zero-divisor in $R$, or if the root datum does not contain any $C^{sc}$ as an irreducible component.

Following [CPZ, Def. 3.5 and 3.12] for each $\alpha \in \Sigma$ we define two $R$-linear operators $\Delta_\alpha$ and $C_\alpha$ on $R[\![\Lambda]\!]_F$ as follows:

$$\Delta_\alpha(y) := \tfrac{y - s_\alpha(y)}{x_\alpha}, \quad C_\alpha(y) := \kappa_\alpha y - \Delta_\alpha(y) = \tfrac{y}{x_{-\alpha}} + \tfrac{s_\alpha(y)}{x_\alpha}, \quad y \in R[\![\Lambda]\!]_F, \tag{2.1}$$

where $\kappa_\alpha := \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}$ (note that $\kappa_\alpha \in R[\![\Lambda]\!]_F$). The operator $\Delta_\alpha$ is called the *Demazure* operator and the operator $C_\alpha$ is called the *push-pull* operator or the *BGG* operator.

**Example 2.3.** For the special elliptic formal group law $F_e$ we have $\kappa_\alpha = \mu_1 + \mu_2 F_e(x_{-\alpha}, x_\alpha) = \mu_1$ for each $\alpha \in \Sigma$. If the root datum is of type $A_1^{sc}$, we have $\Sigma = \{\pm\alpha\}$, $\Lambda = \langle\omega\rangle$ with simple root $\alpha = 2\omega$ and

$$C_\alpha(x_\alpha) = \tfrac{x_\alpha}{x_{-\alpha}} + \tfrac{x_{-\alpha}}{x_\alpha} = \mu_1 x_\alpha - 1 + \tfrac{1}{\mu_1 x_\alpha - 1}, \quad C_\alpha(x_\omega) = \tfrac{x_\omega}{x_{-\alpha}} + \tfrac{x_{-\omega}}{x_\alpha} = \mu_1 x_\omega - \tfrac{1+\mu_2 x_\omega^2}{1-\mu_1 x_\omega}.$$

If it is of type $A_2^{sc}$ we have $\Sigma = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$, $\Lambda = \langle\omega_1, \omega_2\rangle$ with simple roots $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = 2\omega_2 - \omega_1$ and $x_{\alpha_1} = \frac{2x_1 - \mu_1 x_1^2 - x_2 - \mu_2 x_1^2 x_2}{1+\mu_2 x_1^2 - \mu_1 x_2 - 2\mu_2 x_1 x_2}$,

$$C_{\alpha_2}(x_1) = \mu_1 x_1, \quad C_{\alpha_1}(x_1) = \mu_1 x_1 - \tfrac{1+\mu_2 x_1^2 - \mu_1 x_2 - 2\mu_2 x_1 x_2}{1-\mu_1 x_1 - \mu_2 x_1 x_2},$$

where $x_1 := x_{\omega_1}$ and $x_2 := x_{\omega_2}$.

According to [CPZ, §3] the operators $\Delta_\alpha$ satisfy the twisted Leibniz rule

$$\Delta_\alpha(xy) = \Delta_\alpha(x)y + s_\alpha(x)\Delta_\alpha(y), \quad x, y \in R[\![\Lambda]\!]_F, \tag{2.2}$$

i.e. $\Delta_\alpha$ is a twisted derivation. Moreover, they are $R[\![\Lambda]\!]_F^{W_\alpha}$-linear, where $W_\alpha = \{e, s_\alpha\}$, and

$$s_\alpha(x) = x \text{ if and only if } \Delta_\alpha(x) = 0. \tag{2.3}$$

**Remark 2.4.** Properties (2.2) and (2.3) suggest that the Demazure operators can be effectively studied using the theory of twisted derivations and the invariant theory of $W$. On the other hand, push-pull operators do not satisfy properties (2.2) and (2.3) but according to [CPZ, Theorem 12.4] they correspond to the push-pull maps between flag varieties and, hence, are of geometric origin.

For the $i$-th simple root $\alpha_i$, let $\Delta_i := \Delta_{\alpha_i}$ and $s_i := s_{\alpha_i}$. Given a sequence $I = (i_1, \ldots, i_m)$ with $i_j \in \{1, \ldots, n\}$, denote $|I| = m$ and define

$$\Delta_I := \Delta_{i_1} \circ \cdots \circ \Delta_{i_m}, \quad C_I := C_{i_1} \circ \cdots \circ C_{i_m} \quad \text{and set } \Delta_\emptyset = C_\emptyset = \mathrm{id}.$$

We say that a sequence $I$ is *reduced* in $W$ if $s_{i_1} s_{i_2} \ldots s_{i_m}$ is a reduced expression in $W$. In this case we also say that $I = I_w$ is a *reduced sequence* of the element $w(I) := s_{i_1} s_{i_2} \ldots s_{i_m}$. We set $I_e = \emptyset$ (here $e$ is the identity element of $W$).

**Remark 2.5.** It is well-known that for a nontrivial root datum, all $w \in W$, $\Delta_{I_w}$ and $C_{I_w}$ are independent of the choice of a reduced sequence $I_w$ of $w \in W$ if and only if $F$ is of the form $F(x,y) = x + y + \beta xy$, $\beta \in R$. The "if" part of the statement is due to Demazure [De73, Th. 1] and the "only if" part to Bressler and Evens [BE90, Theorem 3.7]. So for such $F$ we can define $\Delta_w := \Delta_{I_w}$ and $C_w := C_{I_w}$ for each $w \in W$.

The operators $\Delta_w$ and $C_w$ play a crucial role in Schubert calculus and computations of the singular cohomology ($F = F_a$) and the $K$-theory ($F = F_m$) rings of flag varieties.

For a general $F$, e.g. for $F = F_e$, the situation becomes much more intricate as we have to rely on choices of reduced decomposition $I_w$.

## 3. Two bases of the formal twisted group algebra

We now recall definitions and basic properties of the formal twisted group algebra $Q_W$, Demazure elements $X_\alpha$ and push-pull elements $Y_\alpha$, following [HMSZ] and [CZZ]. For a chosen set of reduced sequences $\{I_w\}_{w\in W}$ we introduce two bases $\{X_{I_w}\}$ and $\{Y_{I_w}\}$ of $Q_W$ and describe the matrices $(a^X_{v,w})$ and $(a^Y_{v,w})$ by expressing them on the canonical basis $\{\delta_w\}$ of $Q_W$. We also relate the coefficients $a^X_{v,w}$ and the corresponding coefficients $a'^X_{v,w}$ of the reversed elements $X_{I_w^{\mathrm{rev}}}$.

For simplicity we write $S := R[\![\Lambda]\!]_F$. Since the formal group algebra $S$ is $\Sigma$-regular, it embeds into the localization $Q = S[\frac{1}{x_\alpha} \mid \alpha\in\Sigma]$. Let $Q_W$ be the *twisted group algebra* of $Q$ and of the group ring $R[W]$ over $R$, i.e. $Q_W = Q\otimes_R R[W]$ as an $R$-module and the product in $Q_W$ is given by

$$(q\otimes\delta_w)(q'\otimes\delta_{w'}) = qw(q')\otimes\delta_{ww'},\quad q,q'\in Q,\ w,w'\in W.$$

where $\delta_w$ is the canonical element corresponding to $w$ in $R[W]$. Note that $Q_W$ is not a $Q$-algebra since the embedding $Q\hookrightarrow Q_W, q\mapsto q\otimes\delta_e$ is not central.

Inside $Q_W$, we use the notation $q := q\otimes\delta_e$ and $\delta_w := 1\otimes\delta_w$, $1 := \delta_e$ and $\delta_\alpha := \delta_{s_\alpha}$ for a root $\alpha\in\Sigma$. Thus $q\delta_w = q\otimes\delta_w$ and $\delta_w q = w(q)\otimes\delta_w$. Clearly, $\{\delta_w\}_{w\in W}$ is a basis of $Q_W$ as a left $Q$-module.

For each $\alpha\in\Sigma$ we define the following elements of $Q_W$ (corresponding to the operators $\Delta_\alpha$ and $C_\alpha$, respectively, by the action of (4.3)):

$$X_\alpha := \tfrac{1}{x_\alpha} - \tfrac{1}{x_\alpha}\delta_\alpha,\quad Y_\alpha := \kappa_\alpha - X_\alpha = \tfrac{1}{x_{-\alpha}} + \tfrac{1}{x_\alpha}\delta_\alpha$$

called the *Demazure elements* and the *push-pull elements*, respectively.

By straightforward computations, for each $\alpha\in\Sigma$ we have

$$X_\alpha^2 = \kappa_\alpha X_\alpha = X_\alpha\kappa_\alpha \quad\text{and}\quad Y_\alpha^2 = \kappa_\alpha Y_\alpha = Y_\alpha\kappa_\alpha, \tag{3.1}$$
$$X_\alpha q = s_\alpha(q)X_\alpha + \Delta_\alpha(q) \quad\text{and}\quad Y_\alpha q = s_\alpha(q)Y_\alpha + \Delta_{-\alpha}(q),\quad q\in Q,$$
$$X_\alpha Y_\alpha = Y_\alpha X_\alpha = 0.$$

Let $\delta_i := \delta_{s_i}$, $X_i := X_{\alpha_i}$ and $Y_i := Y_{\alpha_i}$ be the $i$-th simple root $\alpha_i$. Given a sequence $I = (i_1, i_2,\ldots,i_m)$ with $i_j\in\{1,\ldots,n\}$, let the product $X_{i_1}X_{i_2}\ldots X_{i_m}$ be denoted by $X_I$ and $Y_{i_1}Y_{i_2}\ldots Y_{i_m}$ by $Y_I$. Set $X_\emptyset = Y_\emptyset = 1$.

By [Bo68, Ch. VI, §1, No 6, Cor. 2] if $v\in W$ has a reduced decomposition $v = s_{i_1}s_{i_2}\cdots s_{i_m}$, then

$$v\Sigma^-\cap\Sigma^+ = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}),\ldots,s_{i_1}(s_{i_2}(\cdots s_{i_{m-1}}(\alpha_{i_m})\cdots))\}. \tag{3.2}$$

In particular, $s_i\Sigma^-\cap\Sigma^+ = \{\alpha_i\}$.

**Lemma 3.1.** *Let $I_v$ be a reduced sequence of an element $v\in W$.*

*Then $X_{I_v} = \sum_{w\le v} a^X_{v,w}\delta_w$ for some $a^X_{v,w}\in Q$, where the sum is taken over all elements of $W$ less or equal to $v$ with respect to the Bruhat order and $a^X_{v,v} = (-1)^{\ell(v)}\prod_{\alpha\in v\Sigma^-\cap\Sigma^+} x_\alpha^{-1}$.*

*Moreover, we have $\delta_v = \sum_{w\le v} b^X_{v,w}X_{I_w}$ for some $b^X_{v,w}\in S$ such that $b^X_{v,e} = 1$ and $b^X_{v,v} = (-1)^{\ell(v)}\prod_{\alpha\in v\Sigma^-\cap\Sigma^+} x_\alpha$.*

*Proof.* It follows from [CZZ, Lemma 5.4, Corollary 5.6] and the fact that $\delta_\alpha = 1 - x_\alpha X_\alpha$. □

Similar for $Y_I$'s we have

**Lemma 3.2.** *Let $I_v$ be a reduced sequence for an element $v \in W$.*

*Then $Y_{I_v} = \sum_{w \le v} a^Y_{v,w}\delta_w$ for some $a^Y_{v,w} \in Q$ and $a^Y_{v,v} = (-1)^{\ell(w)} a^X_{v,v}$. Moreover, we have $\delta_v = \sum_{w\le v} b^Y_{v,w} Y_{I_w}$ for some $b^Y_{v,w} \in S$ and $b^Y_{v,v} = (-1)^{\ell(v)} b^X_{v,v}$.*

*Proof.* We follow the proof of [CZZ, Lemma 5.4] replacing $X$ by $Y$. By induction we have

$$Y_{I_v} = (x_{-\beta}^{-1} + x_\beta^{-1}\delta_\beta)\sum_{w\le v'} a^Y_{v',w}\delta_w = x_\beta^{-1} s_\beta(a^Y_{v',v'})\delta_v + \sum_{w<v} a^Y_{v,w}\delta_w,$$

where $I_v = (i_1, \ldots, i_m)$ is a reduced sequence of $v$, $\beta = \alpha_{i_1}$ and $v' = s_\beta v$. This implies the formulas for $Y_{I_v}$ and for $a^Y_{v,v}$. Remaining statements involving $b^Y_{v,w}$ follow by the same arguments as in the proof of [CZZ, Corollary 5.6] using the fact that $\delta_\alpha = x_\alpha Y_\alpha - \frac{x_\alpha}{x_{-\alpha}}$ and $\frac{x_\alpha}{x_{-\alpha}} \in S$. □

As in the proof of [CZZ, Corollary 5.6], Lemmas 3.1 and 3.2 immediately imply:

**Corollary 3.3.** *The elements $\{X_{I_v}\}_{v\in W}$ (resp. $\{Y_{I_v}\}_{v\in W}$) form a basis of $Q_W$ as a left $Q$-module.*

**Example 3.4.** For the root data $A_1^{ad}$ or $A_1^{sc}$ and the formal group law $F_e$ we have $x_\Pi = x_{-\alpha}$ and

$$(a^Y_{v,w})_{v,w\in W} = \begin{pmatrix} 1 & 0 \\ \mu_1 - \frac{1}{x_\alpha} & \frac{1}{x_\alpha}\end{pmatrix}$$

where the first row and column correspond to $e \in W$ and the second to $s_\alpha \in W$.

Given a sequence $I = (i_1, \ldots, i_m)$, let $I^{\mathrm{rev}} := (i_m, \ldots, i_1)$. We set

$$x_\Pi := \prod_{\alpha\in\Sigma^-} x_\alpha \in S.$$

Observe that by (3.2) we have $s_\alpha(x_\Pi)x_{-\alpha} = x_\Pi x_\alpha$ for $\alpha \in \Sigma^+$.

**Lemma 3.5.** *Let $I = (i_1, \ldots, i_m)$ be a sequence in $\{1, \ldots, n\}$. Let*

$$X_I = \sum_{v\in W} a^X_{I,v}\delta_v \quad \text{and} \quad X_{I^{\mathrm{rev}}} = \sum_{v\in W} a'^X_{I,v}\delta_v \qquad \text{for some } a^X_{I,v},\ a'^X_{I,v} \in Q,$$

*then $v(x_\Pi)\, a'^X_{I,v} = v(a^X_{I,v^{-1}})\, x_\Pi$. Similarly, let*

$$Y_I = \sum_{v\in W} a^Y_{I,v}\delta_v \quad \text{and} \quad Y_{I^{\mathrm{rev}}} = \sum_{v\in W} a'^Y_{I,v}\delta_v \qquad \text{for some } a^Y_{I,v},\ a'^Y_{I,v} \in Q,$$

*then $v(x_\Pi)\, a'^Y_{I,v} = v(a^Y_{I,v^{-1}})\, x_\Pi$.*

*Proof.* If $I = (i)$, then $Y_I = Y_{I^{\mathrm{rev}}}$. So $a'_{I,v} = a_{I,v}$ for all $v \in W$ (we omit the superscripts '$Y$'). For $v \notin \{e, s_i\}$ we have $a'_{I,v} = a_{I,v} = 0$. For $v = e$ we have $x_\Pi a'_{I,e} = x_\Pi a_{I,e}$. For $v = s_i$ we have $s_i(x_\Pi)a'_{I,s_i} = x_\Pi s_i(a_{I,s_i})$, since $a_{I,s_i} = \frac{1}{x_{\alpha_i}}$. The proof is analogous for $X_i$.

Now the conclusion immediately follows by induction on the length $|I|$ using Lemma 3.6 below with $x = x_\Pi$, $f = Y_J$, $f' = Y_{J^{\mathrm{rev}}}$, $g = Y_K$ and $g' = Y_{K^{\mathrm{rev}}}$ for any splitting of $I$ in smaller sequences $J$ followed by $K$. □

**Lemma 3.6.** *Given $x \in Q$, assume that elements*

$$f = \sum_{v\in W} a_v\delta_v \quad \textit{and} \quad f' = \sum_{v\in W} a'_v\delta_v \quad \textit{with } a_v, a'_v \in Q$$

$$g = \sum_{v\in W} b_v\delta_v \quad \textit{and} \quad g' = \sum_{v\in W} b'_v\delta_v \quad \textit{with } b_v, b'_v \in Q$$

*of $Q_W$ satisfy $v(x)a'_v = v(a_{v^{-1}})\,x$ and $v(x)b'_v = v(b_{v^{-1}})\,x$ for all $v \in W$. Then the product $fg = \sum c_v\delta_v$ and the element $(fg)' := g'f' = \sum c'_v\delta_v$ bear the same relation: $v(x)c'_v = v(c_{v^{-1}})\,x$.*

*Proof.* By definition of the product, we have

$$c'_v = \sum_{v_1v_2=v} b'_{v_1}v_1(a'_{v_2}) \quad \text{and} \quad c_{v^{-1}} = \sum_{v_1v_2=v} a_{v_2^{-1}}v_2^{-1}(b_{v_1^{-1}}).$$

We then compute (all sums still being over pairs $(v_1, v_2) \in W^2$ such that $v_1v_2 = v$)

$$\begin{aligned} v(c_{v^{-1}})\,x &= \sum v_1v_2\big(a_{v_2^{-1}}v_2^{-1}(b_{v_1^{-1}})\big)\,x = \sum v_1v_2(a_{v_2^{-1}})v_1(b_{v_1^{-1}})\,x \\ &= \sum v_1v_2(a_{v_2^{-1}})v_1(x)\,b'_{v_1} = \sum v_1\big(v_2(a_{v_2^{-1}})x\big)\,b'_{v_1} = \sum v_1\big(v_2(x)a'_{v_2}\big)\,b'_{v_1} \\ &= v(x)\sum b'_{v_1}v_1(a'_{v_2}) = v(x)c'_v. \end{aligned}$$

□

## 4. The Weyl and the Hecke actions

In the present section we recall several basic facts concerning the $Q$-linear dual $Q_W^*$ following [HMSZ] and [CZZ]. We introduce a left $Q_W$-action '$\bullet$' on $Q_W^*$. The latter induces an action of the Weyl group $W$ on $Q_W^*$ (the Weyl-action) and the action by means of $X_\alpha$ and $Y_\alpha$ on $Q_W^*$ (the Hecke-action). These two actions will play an important role in the sequel.

Let $Q_W^* := \mathrm{Hom}_Q(Q_W, Q)$ denote the $Q$-linear dual of the left $Q$-module $Q_W$. By definition, $Q_W^*$ is a left $Q$-module via $(qf)(z) := qf(z)$ for any $z \in Q_W$, $f \in Q_W^*$ and $q \in Q$. Moreover, there is a $Q$-basis $\{f_w\}_{w\in W}$ of $Q_W^*$ dual to the basis $\{\delta_w\}_{w\in W}$ defined by $f_w(\delta_v) := \delta_{w,v}$ (the Kronecker symbol), $w, v \in W$.

**Definition 4.1.** We define a left action of $Q_W$ on $Q_W^*$ as follows:

$$(z \bullet f)(z') := f(z'z), \quad z, z' \in Q_W,\ f \in Q_W^*.$$

By definition, this action is left $Q$-linear, i.e. $z \bullet (qf) = q(z \bullet f)$ and it induces a different left $Q$-module structure on $Q_W^*$ via the embedding $q \mapsto q\delta_e$, i.e.

$$(q \bullet f)(z) := f(zq).$$

It also induces a $Q$-linear action of $W$ on $Q_W^*$ via $w(f) := \delta_w \bullet f$.

**Lemma 4.2.** *We have $q \bullet f_w = w(q)f_w$ and $w(f_v) = f_{vw^{-1}}$ for any $q \in Q$ and $w, v \in W$.*

*Proof.* We have $(q \bullet f_w)(\delta_v) = f_w(v(q)\delta_v) = v(q)\delta_{w,v}$ which shows that $q \bullet f_v = v(q)f_v$. As for the second we have $[w(f_v)](\delta_u) = f_v(\delta_u\delta_w) = \delta_{v,uw}$, so $w(f_v) = f_{vw^{-1}}$. □

There is a coproduct on $Q_W$ defined by [CZZ, Def. 8.9]:

$$\triangle : Q_W \to Q_W \otimes_Q Q_W, \quad q\delta_w \mapsto q\delta_w \otimes \delta_w.$$

Here $\otimes_Q$ is the tensor product of left $Q$-modules. It is cocommutative with co-unit $\varepsilon : Q_W \to Q$, $q\delta_w \mapsto q$ [CZZ, Prop. 8.10]. The coproduct structure on $Q_W$ induces a product structure on $Q_W^*$, which is $Q$-bilinear for the natural action of $Q$ on $Q_W^*$ (not the one using $\bullet$). In terms of the basis $\{f_w\}_{w\in W}$ the product is given by component-wise multiplication:

$$(\sum_{v\in W} q_v f_v)(\sum_{w\in W} q'_w f_w) = \sum_{w\in W} q_w q'_w f_w, \quad q_w, q'_w \in Q. \tag{4.1}$$

In other words, if we identify the dual $Q_W^*$ with the $Q$-module of maps $\mathrm{Hom}(W, Q)$ via

$$Q_W^* \to \mathrm{Hom}(W, Q), \quad f \mapsto f', \quad f'(w) := f(\delta_w),$$

then the product is the classical multiplication of functions with values in a ring.

The multiplicative identity $\mathbf{1}$ of this product corresponds to the counit $\varepsilon$ and equals $\mathbf{1} = \sum_{w\in W} f_w$. We also have

$$q \bullet (ff') = (q \bullet f)f' = f(q \bullet f') \text{ for } q \in Q \text{ and } f, f' \in Q_W^*. \tag{4.2}$$

**Lemma 4.3.** *For any $\alpha \in \Sigma$ and $f, f' \in Q_W^*$ we have $s_\alpha(ff') = s_\alpha(f)s_\alpha(f')$, i.e. $W$ acts on the algebra $Q_W^*$ by $Q$-linear automorphisms.*

*Proof.* By $Q$-linearity of the action of $W$ and of the product, it suffices to check the formula on basis elements $f = f_w$ and $f' = f_v$, for which it is straightforward. □

Observe that the ring $Q$ can be viewed as a left $Q_W$-module via the following action:

$$(q\delta_w) \cdot q' := qw(q'), \quad q, q' \in Q, \ w \in W. \tag{4.3}$$

Then by definition we have

$$(q \bullet \mathbf{1})(z) = z \cdot q, \quad z \in Q_W. \tag{4.4}$$

**Definition 4.4.** For $\alpha \in \Sigma$ we define two $Q$-linear operators on $Q_W^*$ by

$$A_\alpha(f) := Y_\alpha \bullet f \quad \text{and} \quad B_\alpha(f) := X_\alpha \bullet f, \quad f \in Q_W^*.$$

An action by means of $A_\alpha$ or $B_\alpha$ will be called a *Hecke-action* on $Q_W^*$.

**Remark 4.5.** If $F = F_m$ (resp. $F = F_a$) one obtains actions introduced by Kostant–Kumar in [KK90, $I_{18}$] (resp. in [KK86, $I_{51}$]).

As in (2.2) and (2.3) we have

$$B_\alpha(ff') = B_\alpha(f)f' + s_\alpha(f)B_\alpha(f') \text{ and } B_\alpha \circ s_\alpha = -B_\alpha, \quad \text{for } f, f' \in Q_W^*, \tag{4.5}$$

$$B_\alpha(f) = 0 \text{ if and only if } f \in (Q_W^*)^{W_\alpha}. \tag{4.6}$$

Indeed, using (4.2) and Lemma 4.3 we obtain

$$\begin{aligned} B_\alpha(f)f' + s_\alpha(f)B_\alpha(f') &= [\tfrac{1}{x_\alpha}(1-\delta_\alpha) \bullet f]f' + s_\alpha(f)[\tfrac{1}{x_\alpha}(1-\delta_\alpha) \bullet f'] \\ &= [\tfrac{1}{x_\alpha} \bullet (f - s_\alpha(f))]f' + s_\alpha(f)[\tfrac{1}{x_\alpha} \bullet (f' - s_\alpha(f'))] \\ &= \tfrac{1}{x_\alpha} \bullet (ff' - s_\alpha(f)s_\alpha(f')) = B_\alpha(ff') \end{aligned}$$

and $B_\alpha(s_\alpha(f)) = \frac{1}{x_\alpha}(1-\delta_\alpha) \bullet s_\alpha(f) = \frac{1}{x_\alpha} \bullet (s_\alpha(f) - f) = -B_\alpha(f)$. As for (4.6) we have $0 = B_\alpha(f) = X_\alpha \bullet f = \frac{1}{x_\alpha} \bullet [(1-\delta_\alpha) \bullet f]$ which is equivalent to $f = s_\alpha(f)$.

And similarly to (3.1) we obtain

$$A_\alpha^{\circ 2}(f) = \kappa_\alpha \bullet A_\alpha(f) = A_\alpha(\kappa_\alpha \bullet f), \quad B_\alpha^{\circ 2}(f) = \kappa_\alpha \bullet B_\alpha(f) = B_\alpha(\kappa_\alpha \bullet f),$$
$$A_\alpha \circ B_\alpha = B_\alpha \circ A_\alpha = 0. \tag{4.7}$$

We set $A_i = A_{\alpha_i}$ and $B_i := B_{\alpha_i}$ for the $i$-th simple root $\alpha_i$. We set $A_I = A_{i_1} \circ \ldots \circ A_{i_m}$ and $B_I = B_{i_1} \circ \ldots \circ B_{i_m}$ for a sequence $I = (i_1, \ldots, i_m)$ with $i_j \in \{1, \ldots, n\}$. The operators $A_I$ and $B_I$ are key ingredients in the proof that the natural pairing of Theorem 10.7 on the dual of the formal affine Demazure algebra is non-degenerate.

**Lemma 4.6.** *For any sequence $I$, we have*

$$A_{I^{\mathrm{rev}}}(x_\Pi f_e) = \sum_{v \in W} v(x_\Pi) a_{I,v}^Y f_v \quad \textit{and} \quad B_{I^{\mathrm{rev}}}(x_\Pi f_e) = \sum_{v \in W} v(x_\Pi) a_{I,v}^X f_v.$$

*Proof.* We prove the first formula only. The second one is obtained using similar arguments. Let $Y_{I^{\mathrm{rev}}} = \sum_{v \in W} a'^Y_{I,v} \delta_v$ and $Y_I = \sum_{v \in W} a^Y_{I,v} \delta_v$ as in Lemma 3.5. Then

$$A_{I^{\mathrm{rev}}}(x_\Pi f_e) = Y_{I^{\mathrm{rev}}} \bullet x_\Pi f_e = \sum_{v \in W} x_\Pi (a'^Y_{I,v} \delta_v \bullet f_e) =$$
$$\sum_{v \in W} x_\Pi (a'^Y_{I,v} \bullet f_{v^{-1}}) = \sum_{v \in W} x_\Pi v^{-1}(a'^Y_{I,v}) f_{v^{-1}} = \sum_{v \in W} x_\Pi v(a'^Y_{I,v^{-1}}) f_v.$$

The formula then follows by Lemma 3.5. $\square$

## 5. Push-pull operators and elements

Let us now introduce and study a key notion of the present paper, the notion of push-pull operators (resp. elements) on $Q$ (resp. in $Q_W$) with respect to given coset representatives in parabolic quotients of the Weyl group.

Let $(\Sigma, \Lambda)$ be a root datum with a chosen set of simple roots $\Pi$. Let $\Xi \subseteq \Pi$ and let $W_\Xi$ denote the subgroup of the Weyl group $W$ of the root datum generated by simple reflections $s_\alpha$, $\alpha \in \Xi$. We thus have $W_\emptyset = \{e\}$ and $W_\Pi = W$. Let $\Sigma_\Xi := \{\alpha \in \Sigma \mid s_\alpha \in W_\Xi\}$ and let $\Sigma_\Xi^+ := \Sigma_\Xi \cap \Sigma^+$, $\Sigma_\Xi^- := \Sigma_\Xi \cap \Sigma^-$ be subsets of positive and negative roots respectively.

Given subsets $\Xi' \subseteq \Xi$ of $\Pi$, let $\Sigma^+_{\Xi/\Xi'} := \Sigma^+_\Xi \setminus \Sigma^+_{\Xi'}$ and $\Sigma^-_{\Xi/\Xi'} := \Sigma^-_\Xi \setminus \Sigma^-_{\Xi'}$. We define

$$x_{\Xi/\Xi'} := \prod_{\alpha \in \Sigma^-_{\Xi/\Xi'}} x_\alpha \quad \text{and set } x_\Xi := x_{\Xi/\emptyset}.$$

**Lemma 5.1.** *Given subsets $\Xi' \subseteq \Xi$ of $\Pi$ we have*

$$v(\Sigma^-_{\Xi/\Xi'}) = \Sigma^-_{\Xi/\Xi'} \textit{ and } v(\Sigma^+_{\Xi/\Xi'}) = \Sigma^+_{\Xi/\Xi'} \textit{ for any } v \in W_{\Xi'}.$$

*Proof.* We prove the first statement only, the second one can be proven similarly.

Since $v$ acts faithfully on $\Sigma_\Xi$, it suffices to show that for any $\alpha \in \Sigma^-_{\Xi/\Xi'}$, the root $\beta := v(\alpha) \notin \Sigma_{\Xi'}$ and is negative. Indeed, if $\beta \in \Sigma_{\Xi'}$, then so is $\alpha = v^{-1}(\beta)$ (as $v^{-1} \in W_{\Xi'}$), which is impossible. On the other hand, if $\beta$ is positive, then

$$\beta = v(\alpha) \in v\Sigma^-_\Xi \cap \Sigma^+_\Xi = v\Sigma^-_{\Xi'} \cap \Sigma^+_{\Xi'},$$

where the latter equality follows from (3.2) and the fact that $v \in W_{\Xi'}$. So $\alpha = v^{-1}(\beta) \in \Sigma_{\Xi'}$, a contradiction. $\square$

**Corollary 5.2.** *For any $v\in W_{\Xi'}$, we have $v(x_{\Xi/\Xi'})=x_{\Xi/\Xi'}$.*

**Definition 5.3.** Given a set of left coset representatives $W_{\Xi/\Xi'}$ of $W_\Xi/W_{\Xi'}$ we define a *push-pull operator* on $Q$ with respect to $W_{\Xi/\Xi'}$ by

$$C_{\Xi/\Xi'}(q):=\sum_{w\in W_{\Xi/\Xi'}} w\big(\tfrac{q}{x_{\Xi/\Xi'}}\big).$$

and a *push-pull element* with respect to $W_{\Xi/\Xi'}$ by

$$Y_{\Xi/\Xi'}:=\Big(\sum_{w\in W_{\Xi/\Xi'}}\delta_w\Big)\tfrac{1}{x_{\Xi/\Xi'}}.$$

We set $C_\Xi:=C_{\Xi/\emptyset}$ and $Y_\Xi:=Y_{\Xi/\emptyset}$ (so they do not depend on the choice of $W_{\Xi/\emptyset}=W_\Xi$ in these two special cases).

By definition, we have $C_{\Xi/\Xi'}(q)=Y_{\Xi/\Xi'}\cdot q$, where $Y_{\Xi/\Xi'}$ acts on $q\in Q$ by (4.3). Also in the trivial case where $\Xi=\Xi'$, then $x_{\Xi/\Xi}=1$, while $C_{\Xi/\Xi}=\mathrm{id}_Q$ and $Y_{\Xi/\Xi}=1$ if we choose $e$ as representative of the only coset. Observe that for $\Xi=\{\alpha_i\}$ we have $W_\Xi=\{e,s_i\}$ and $C_\Xi=C_i$ (resp. $Y_\Xi=Y_i$) is the push-pull operator (resp. element) introduced before and preserves $S$.

**Example 5.4.** For the formal group law $F_e$ and the root datum $A_2$, we have $x_\Pi=x_{-\alpha_1}x_{-\alpha_2}x_{-\alpha_1-\alpha_2}$ and

$$C_\Pi(1)=\sum_{w\in W} w\big(\tfrac{1}{x_\Pi}\big)=\mu_1\big(\tfrac{1}{x_{-\alpha_2}x_{-\alpha_1-\alpha_2}}+\tfrac{1}{x_{-\alpha_1}x_{\alpha_2}}+\tfrac{1}{x_{\alpha_1}x_{\alpha_1+\alpha_2}}\big)=\mu_1^3+\mu_1\mu_2.$$

**Lemma 5.5.** *The operator $C_{\Xi/\Xi'}$ restricted to $Q^{W_{\Xi'}}$ is independent of the choices of representatives $W_{\Xi/\Xi'}$ and it maps $Q^{W_{\Xi'}}$ to $Q^{W_\Xi}$.*

*Proof.* Since $1/x_{\Xi/\Xi'}\in Q^{W_{\Xi'}}$ by corollary 5.2, the independence statement is clear. The second part follows, since for any $v\in W_\Xi$, and for any set of coset representatives $W_{\Xi/\Xi'}$, the set $vW_{\Xi/\Xi'}$ is again a set of coset representatives. □

Actually, we will see in Corollary 10.4 that the operator $C_\Xi$ sends $S$ to $S^{W_\Xi}$.

**Remark 5.6.** The formula for the operator $C_\Xi$ (with $\Xi'=\emptyset$) had appeared before in related contexts, namely, in discussions around the Becker-Gottlieb transfer for topological complex-oriented theories (see [BE90, (2.1)] and [GR12, §4.1]). The definition of the element $Y_{\Xi/\Xi'}$ can be viewed as a generalized algebraic analogue of this formula.

**Lemma 5.7** (Composition rule)**.** *Given subsets $\Xi''\subseteq\Xi'\subseteq\Xi$ of $\Pi$ and given sets of representatives $W_{\Xi/\Xi'}$ and $W_{\Xi'/\Xi''}$, take $W_{\Xi/\Xi''}:=\{wv\mid w\in W_{\Xi/\Xi'},\ v\in W_{\Xi'/\Xi''}\}$ as the set of representatives of $W_\Xi/W_{\Xi''}$. Then*

$$C_{\Xi/\Xi'}\circ C_{\Xi'/\Xi''}=C_{\Xi/\Xi''}\ \text{and}\ Y_{\Xi/\Xi'}Y_{\Xi'/\Xi''}=Y_{\Xi/\Xi''}.$$

*Proof.* We prove the formula for $Y$'s, the one for $C$'s follows since $C$ acts as $Y$, and the composition of actions corresponds to multiplication. We have $Y_{\Xi/\Xi'}Y_{\Xi'/\Xi''}=$

$$\Big(\sum_{w\in W_{\Xi/\Xi'}}\delta_w\tfrac{1}{x_{\Xi/\Xi'}}\Big)\Big(\sum_{v\in W_{\Xi'/\Xi''}}\delta_v\tfrac{1}{x_{\Xi'/\Xi''}}\Big)=\sum_{w\in W_{\Xi/\Xi'},\,v\in W_{\Xi'/\Xi''}}\delta_{wv}\tfrac{1}{v^{-1}(x_{\Xi/\Xi'})x_{\Xi'/\Xi''}}.$$

By Corollary 5.2, we have $v^{-1}(x_{\Xi/\Xi'})=x_{\Xi/\Xi'}$. Therefore, $v^{-1}(x_{\Xi/\Xi'})x_{\Xi'/\Xi''}=x_{\Xi/\Xi'}x_{\Xi'/\Xi''}=x_{\Xi/\Xi''}$. We conclude by definition of $W_{\Xi/\Xi''}$. □

The following lemma is clear from the defining formula of $C_{\Xi/\Xi'}$.

**Lemma 5.8** (Projection formula)**.** *We have*

$$C_{\Xi/\Xi'}(qq') = q\,C_{\Xi/\Xi'}(q') \quad \text{for any } q \in (Q_W)^{W_\Xi} \text{ and } q' \in (Q_W)^{W_{\Xi'}}.$$

**Lemma 5.9.** *Given a subset $\Xi$ of $\Pi$ and $\alpha \in \Xi$ we have*

*(a) $Y_\Xi = Y'Y_\alpha = Y_\alpha Y''$ for some $Y'$ and $Y'' \in Q_W$,*
*(b) $Y_\Xi X_\alpha = X_\alpha Y_\Xi = 0$, $Y_\alpha Y_\Xi = \kappa_\alpha Y_\Xi$ and $Y_\Xi Y_\alpha = Y_\Xi \kappa_\alpha$.*

*Proof.* (a) The first identity follows from Lemma 5.7 applied to $\Xi' = \{\alpha\}$ (in this case $Y' = Y_{\Xi/\Xi'}$).

For the second identity, let ${}^\alpha W_\Xi$ be set of right coset representatives of $W_\alpha \backslash W_\Xi$, thus each $w \in W_\Xi$ can be written uniquely either as $w = s_\alpha u$ or as $w = u$ with $u \in {}^\alpha W_\Xi$. Then

$$\begin{aligned} Y_\Xi &= \sum_{u \in {}^\alpha W_\Xi} (1+\delta_\alpha)\delta_u \tfrac{1}{x_\Xi} = \sum_{u \in {}^\alpha W_\Xi} (1+\delta_\alpha)\tfrac{1}{x_{-\alpha}} x_{-\alpha}\delta_u \tfrac{1}{x_\Xi} \\ &= \sum_{u \in {}^\alpha W_\Xi} Y_\alpha x_{-\alpha}\delta_u \tfrac{1}{x_\Xi} = Y_\alpha \sum_{u \in {}^\alpha W_\Xi} \delta_w \tfrac{w^{-1}(x_{-\alpha})}{x_\Xi}. \end{aligned}$$

(b) then follows from (a) and (3.1). $\square$

## 6. The push-pull operators on the dual

We now introduce and study the push-pull operators on the dual of the twisted formal group algebra $Q_W^*$.

For $w \in W$, we define $f_w^\Xi := \sum_{v \in wW_\Xi} f_v$. Observe that $f_w^\Xi = f_{w'}^\Xi$ if and only if $wW_\Xi = w'W_\Xi$. Consider the subring of invariants $(Q_W^*)^{W_\Xi}$ by means of the '$\bullet$'-action of $W_\Xi$ on $Q_W^*$ and fix a set of representatives $W_{\Pi/\Xi}$. By Lemma 4.2, we then have the following lemma:

**Lemma 6.1.** *The set $\{f_w^\Xi\}_{w \in W_{\Pi/\Xi}}$ forms a basis of $(Q_W^*)^{W_\Xi}$ as a left $Q$-module, and $f_w^\Xi f_v^\Xi = \delta_{w,v} f_v^\Xi$ for any $w, v \in W_{\Pi/\Xi}$.*

In other words, $\{f_w\}_{w \in W_{\Pi/\Xi}}$ is a set of pairwise orthogonal projectors, and the direct sum of their images is $(Q_W^*)^{W_\Xi}$.

**Definition 6.2.** Given subsets $\Xi' \subseteq \Xi$ of $\Pi$ and a set of representatives $W_{\Xi/\Xi'}$ we define a $Q$-linear operator on $Q_W^*$ by

$$A_{\Xi/\Xi'}(f) := Y_{\Xi/\Xi'} \bullet f, \quad f \in Q_W^*,$$

and call it the *push-pull operator* with respect to $W_{\Xi/\Xi'}$. It is $Q$-linear since so is the '$\bullet$'-action. We set $A_\Xi = A_{\Xi/\emptyset}$.

Lemma 5.7 immediately implies:

**Lemma 6.3** (Composition rule)**.** *Given subsets $\Xi'' \subseteq \Xi' \subseteq \Xi$ of $\Pi$ and sets of representatives $W_{\Xi/\Xi'}$ and $W_{\Xi'/\Xi''}$, let $W_{\Xi/\Xi''} = \{wv \mid w \in W_{\Xi/\Xi'},\, v \in W_{\Xi'/\Xi''}\}$, then we have $A_{\Xi/\Xi'} \circ A_{\Xi'/\Xi''} = A_{\Xi/\Xi''}$.*

Here is an analogue of Lemma 5.5.

**Lemma 6.4.** *The operator $A_{\Xi/\Xi'}$ restricted to $(Q_W^*)^{W_{\Xi'}}$ is independent of the choices of representatives $W_{\Xi/\Xi'}$ and it maps $(Q_W^*)^{W_{\Xi'}}$ to $(Q_W^*)^{W_\Xi}$.*

*Proof.* Let $f \in (Q_W^*)^{W_{\Xi'}}$. For any $w \in W$ and $v \in W_{\Xi'}$, by Corollary 5.2, we have

$$(\delta_{wv}\frac{1}{x_{\Xi/\Xi'}}) \bullet f = (\delta_w \frac{1}{x_{\Xi/\Xi'}}\delta_v) \bullet f = (\delta_w \frac{1}{x_{\Xi/\Xi'}}) \bullet \delta_v \bullet f = (\delta_w \frac{1}{x_{\Xi/\Xi'}}) \bullet f.$$

which proves that the action on $f$ of any factor $\delta_w(\frac{1}{x_{\Xi/\Xi'}})$ in $Y_{\Xi/\Xi'}$ is independent of the choice of the coset representative $w$.

Now if $v \in W_\Xi$, we have

$$v(A_{\Xi/\Xi'}(f)) = \delta_v \bullet Y_{\Xi/\Xi'} \bullet f = (\delta_v Y_{\Xi/\Xi'}) \bullet f = A_{\Xi/\Xi'}(f)$$

where the last equality holds since $\delta_v Y_{\Xi/\Xi'}$ is again an operator $Y_{\Xi/\Xi'}$ corresponding to the set of coset representatives $vW_{\Xi/\Xi'}$ instead of $W_{\Xi/\Xi'}$. This proves the second claim. □

**Lemma 6.5.** *We have*

$$A_{\Xi/\Xi'}(f_v) = \frac{1}{v(x_{\Xi/\Xi'})} \sum_{w \in W_{\Xi/\Xi'}} f_{vw^{-1}} = \frac{1}{v(x_{\Xi/\Xi'})} \sum_{w \in W_{\Xi/\Xi'}} w(f_v).$$

*In particular, we have* $A_{\Xi/\Xi'}(f_v^{\Xi'}) = \frac{1}{v(x_{\Xi/\Xi'})} f_v^{\Xi}$ *and* $A_{\Pi/\Xi}(f_v^{\Xi}) = \frac{1}{v(x_{\Pi/\Xi})}\mathbf{1}$.

*Proof.* By Lemma 4.2 we obtain $A_{\Xi/\Xi'}(f_v) =$

$$= (\sum_{w \in W_{\Xi/\Xi'}} \delta_w \tfrac{1}{x_{\Xi/\Xi'}}) \bullet f_v = \sum_{w \in W_{\Xi/\Xi'}} \delta_w \bullet (\tfrac{1}{v(x_{\Xi/\Xi'})} f_v) = \tfrac{1}{v(x_{\Xi/\Xi'})} \sum_{w \in W_{\Xi/\Xi'}} f_{vw^{-1}}.$$

In particular

$$A_{\Xi/\Xi'}(f_v^{\Xi'}) = \sum_{w \in W_{\Xi'}} \tfrac{1}{vw(x_{\Xi/\Xi'})} \sum_{u \in W_{\Xi/\Xi'}} f_{vwu^{-1}} = \tfrac{1}{v(x_{\Xi/\Xi'})} \sum_{w \in W_{\Xi'}} \sum_{u \in W_{\Xi/\Xi'}} f_{vwu^{-1}}$$
$$= \tfrac{1}{v(x_{\Xi/\Xi'})} \sum_{w \in vW_\Xi} f_w = \tfrac{1}{v(x_{\Xi/\Xi'})} f_v^{\Xi}$$

where the second equality follows from Corollary 5.2. □

Together with Lemma 6.1 we therefore obtain:

**Corollary 6.6.** *We have* $A_{\Xi/\Xi'}((Q_W^*)^{W_{\Xi'}}) = (Q_W^*)^{W_\Xi}$.

**Lemma 6.7** (Projection formula)**.** *We have*

$$A_{\Xi/\Xi'}(ff') = fA_{\Xi/\Xi'}(f') \quad \textit{for any } f \in (Q_W^*)^{W_\Xi} \textit{ and } f' \in (Q_W^*)^{W_{\Xi'}}.$$

*Proof.* Using (4.2) and Lemma 4.3, we compute

$$A_{\Xi/\Xi'}(ff') = Y_{\Xi/\Xi'} \bullet (ff') = (\sum_{w \in W_{\Xi/\Xi'}} \delta_w \tfrac{1}{x_{\Xi/\Xi'}}) \bullet (ff') = \sum_{w \in W_{\Xi/\Xi'}} \delta_w \bullet \tfrac{1}{x_{\Xi/\Xi'}} \bullet (ff')$$
$$= \sum_{w \in W_{\Xi/\Xi'}} \delta_w \bullet (f(\tfrac{1}{x_{\Xi/\Xi'}} \bullet f')) = \sum_{w \in W_{\Xi/\Xi'}} (\delta_w \bullet f)(\delta_w \bullet \tfrac{1}{x_{\Xi/\Xi'}} \bullet f')$$
$$= f \sum_{w \in W_{\Xi/\Xi'}} \delta_w \bullet \tfrac{1}{x_{\Xi/\Xi'}} \bullet f' = fA_{\Xi/\Xi'}(f') \quad \square$$

**Lemma 6.8.** *Given a sequence $I$ in $\{1, \dots, n\}$, for any $x, y \in S$ and $f, f' \in Q_W^*$ we have*

$$C_\Pi(\Delta_I(x)y) = C_\Pi(x\Delta_{I^{\mathrm{rev}}}(y)) \quad \textit{and} \quad A_\Pi(B_I(f)f') = A_\Pi(fB_{I^{\mathrm{rev}}}(f')).$$

*Proof.* We prove the second formula only. The first one is obtained similarly. By Lemma 5.9.(b) we have $Y_\Pi X_\alpha = 0$ for any $\alpha \in \Pi$. By (4.5) we obtain

$$0 = A_\Pi\big(B_\alpha(s_\alpha(f)f')\big) = A_\Pi\big(fB_\alpha(f') - B_\alpha(f)f'\big).$$

Hence, $A_\Pi(B_\alpha(f)f') = A_\Pi(fB_\alpha(f'))$. The formula then follows by iteration. □

Let $\{X^*_{I_w}\}_{w\in W}$ and $\{Y^*_{I_w}\}_{w\in W}$ be the $Q$-linear basis of $Q^*_W$ dual to $\{X_{I_w}\}_{w\in W}$ and $\{Y_{I_w}\}_{w\in W}$, respectively, i.e. $X^*_{I_w}(X_{I_v}) = \delta_{w,v}$ for $w, v \in W$. We have $X^*_{I_e} = \mathbf{1}$. Indeed, from Lemma 3.1, we have $\delta_v = \sum_{w\le v} b^X_{v,w}X_{I_w}$ with $b^X_{v,e} = 1$. So for each $v \in W$ we have $X^*_{I_e}(\delta_v) = b^X_{v,e} = 1 = \mathbf{1}(\delta_v)$.

**Lemma 6.9.** *Let $w_0$ be the longest element in $W$, of length $N$. We have*

$$A_\Pi(X^*_{I_{w_0}}) = (-1)^N\mathbf{1} \quad \text{and} \quad A_\Pi(Y^*_{I_{w_0}}) = \mathbf{1}.$$

*Proof.* Consider the first formula. By Lemma 3.1 $\delta_v = \sum_{w\le v} b^X_{v,w}X_{I_w}$, therefore $X^*_{I_w} = \sum_{v\ge w} b^X_{v,w} f_v$. Lemma 6.5 yields

$$A_\Pi(X^*_{I_w}) = \sum_{v\ge w} \tfrac{b^X_{v,w}}{v(x_\Pi)}\mathbf{1}.$$

If $w = w_0$ is the longest element, then $A_\Pi(X^*_{I_{w_0}}) = \frac{b^X_{w_0,w_0}}{w_0(x_\Pi)}\mathbf{1}$. By Lemma 3.1 we have $b^X_{w_0,w_0} = (-1)^N \prod_{\alpha\in\Sigma^+} x_\alpha$ which by (3.2) equals to $(-1)^N w_0(x_\Pi)$.

The second formula is obtained similarly using Lemma 3.2 instead. □

**Definition 6.10.** We define the *characteristic map* $c\colon Q \to Q^*_W$ by $q \mapsto q \bullet \mathbf{1}$.

By the definition of the '$\bullet$' action, $c$ is an $R$-algebra homomorphism given by $c(q) = \sum_{w\in W} w(q)f_w$. Note that $c$ is $Q_W$-equivariant with respect to the action (4.3) and the '$\bullet$'-action. Indeed, $c(z\cdot q) = (z\cdot q)\bullet\mathbf{1} = z\bullet(q\bullet\mathbf{1}) = z\bullet c(q)$. In particular, it is $W$-equivariant.

The following lemma provides an analogue of the push-pull formula of [CPZ, Theorem. 12.4].

**Lemma 6.11.** *Given subsets $\Xi' \subseteq \Xi$ of $\Pi$, we have $A_{\Xi/\Xi'}\circ c = c\circ C_{\Xi/\Xi'}$.*

*Proof.* By definition, we have

$$A_{\Xi/\Xi'}(c(q)) = Y_{\Xi/\Xi'}\bullet c(q) = c(Y_{\Xi/\Xi'}\cdot q) = c(C_{\Xi/\Xi'}(q)).$$ □

## 7. Another basis of the $W_\Xi$-invariant subring

Recall that $\{f^\Xi_w\}_{w\in W_{\Pi/\Xi}}$ is a basis of the invariant subring $(Q^*_W)^{W_\Xi}$. In the present section we construct another basis $\{X^*_{I_u}\}_{u\in W^\Xi}$ of the subring $(Q^*_W)^{W_\Xi}$, which generalizes [KK90, Lemma 2.27].

Given a subset $\Xi$ of $\Pi$ we define

$$W^\Xi = \{w\in W \mid \ell(ws_\alpha) > \ell(w) \text{ for any } \alpha\in\Xi\}.$$

Note that $W^\Xi$ is a set of left coset representatives of $W/W_\Xi$ such that each $w \in W^\Xi$ is the unique representative of minimal length.

We will extensively use the following fact [Hu90, §1.10]:

$$\text{For any } w\in W \text{ there exist unique } u\in W^\Xi \text{ and } v\in W_\Xi \text{ such that } w = uv \text{ and } \ell(w) = \ell(u)+\ell(v). \tag{7.1}$$

**Definition 7.1.** Let $\Xi$ be a subset of $\Pi$. We say that the set of reduced sequences $\{I_w\}_{w\in W}$ is *$\Xi$-compatible* if for each $w \in W$ and the unique factorization $w = uv$ with $u \in W^\Xi$ and $v \in W_\Xi$, $\ell(w) = \ell(u) + \ell(v)$ of (7.1) we have $I_w = I_u \cup I_v$, i.e. $I_w$ starts with $I_u$ and ends by $I_v$.

Observe that there always exists a $\Xi$-compatible set of reduced sequences. Indeed, we can take any reduced sequence for $w \in W^\Xi \cup W_\Xi$ and then define $I_w := I_u \cup I_v$ for $w = uv$ with $u \in W^\Xi$ and $v \in W_\Xi$.

**Lemma 7.2.** *For any $z \in Q_W$, we have $X^*_{I_e}(z) = z \cdot 1$, where $e$ is the neutral element of $W$ and $z \cdot 1$ is defined in (4.3). In particular, for any sequence $I$ with $|I| \geq 1$, the coefficient of $X^*_{I_e}(X_I) = 0$.*

*Proof.* Since $X_\alpha \cdot 1 = 0$, for any sequence $I$ with $|I| \geq 1$, we have $X_I \cdot 1 = 0$, and of course $X_{I_e} \cdot 1 = 1 \cdot 1 = 1$. Thus

$$z \cdot 1 = (\sum_{w\in W} X^*_{I_w}(z) X_{I_w}) \cdot 1 = X^*_{I_e}(z). \qquad \square$$

It follows that if $|I| \geq 1$ and we express $X_I = \sum_{v\in W} q_v X_{I_v}$, then $q_e = 0$.

**Lemma 7.3.** *For any reduced sequence $I$ of an element $w$ and $q \in Q$ we have*

$$X_I q = \sum_{v\leq w} \phi_{I,v}(q) X_{I_v} \quad \textit{for some } \phi_{I,I_v}(q) \in Q.$$

*Proof.* For any subsequence $J$ of $I$ (not necessarily reduced), we have $w(J) \leq w$ by [De77, Th. 1.1]. Thus, by developing all $X_i = \frac{1}{x_{\alpha_i}}(1 - \delta_{\alpha_i})$, moving all coefficients to the left, and then using Lemma 3.1 and transitivity of the Bruhat order,

$$X_I q = \sum_{w\leq v} \tilde{\phi}_{I,w}(q)\delta_w = \sum_{w\leq v} \phi_{I,w}(q) X_{I_w}$$

for some coefficients $\tilde{\phi}_{I,w}(q)$ and $\phi_{I,w}(q) \in Q$. $\square$

**Theorem 7.4.** *Assume that the set of reduced sequences $\{I_w\}_{w\in W}$ is $\Xi$-compatible. For any $u \in W^\Xi$, and for any sequence $I$ of length at least $1$ and in $W_\Xi$ (i.e. $\alpha_i \in \Xi$ for any $i$ appearing in the sequence $I$), we have*

$$X^*_{I_u}(zX_I) = 0 \textit{ for all } z \in Q_W.$$

*Proof.* Since $\{X_{I_w}\}_{w\in W}$ is a basis of $Q_W$, we may assume that $z = X_{I_w}$ for some $w \in W$. We proceed by induction on the length of $w$. First decompose $X_I = \sum_{e<v\in W_\Xi} q_v X_{I_v}$ with $q_v \in Q$ by Lemma 7.2.

When $\ell(w) = 0$, we have $X_{I_w} = X_{I_e} = 1$. Since $W_\Xi \cap W^\Xi = \{e\}$, for any $v \in W_\Xi$, $v \neq e$, we have $X^*_{I_u}(X_{I_v}) = 0$ so this case is clear.

Then, the induction step goes as follows: since the sequences are $\Xi$-compatible, we have $X_{I_w} X_I = X_{I_{w'}} X_{I_{v'}} X_I = X_{I_{w'}} X_{I'}$ with $w' \in W^\Xi$, $v' \in W_\xi$ and $I' \in W_\Xi$ with $\ell(I') \geq \ell(I) \geq 1$. We can thus assume that $w \in W^\Xi$. Then, by Lemma 7.3,

$$X_{I_w} X_I = \sum_{v\neq e} X_{I_w} q_v X_{I_v} = \sum_{w'\leq w, v\neq e} \phi_{I_{w,w'}}(q_v) X_{I_{w'}} X_{I_v}.$$

Now $X^*_{I_u}(X_{I_w} X_{I_v}) = X^*_{I_u}(X_{I_{wv}}) = 0$ since $wv$ is not minimal ($v \neq e$) so $wv \neq u$. Applying $X^*_{I_u}$ to other terms in the above summation gives zero by induction. $\square$

**Remark 7.5.** The proof will not work if we replace $X$'s by $Y$'s, because constant terms appear.

**Corollary 7.6.** *Assume that the set of reduced sequences* $\{I_w\}_{w\in W}$ *is* $\Xi$*-compatible. The elements* $\{X_{I_u}^*\}_{u\in W^\Xi}$ *form a* $Q$*-module basis of* $(Q_W^*)^{W_\Xi}$*.*

*Proof.* For every $\alpha_i\in\Xi$ we have

$$(\delta_i\bullet X_{I_u}^*)(z)=X_{I_u}^*(z\delta_i)=X_{I_u}^*(z(1-x_iX_i))=X_{I_u}^*(z),\quad z\in Q_W,$$

where the last equality follows by Theorem 7.4. Therefore, $X_{I_u}^*$ is $W_\Xi$-invariant.

Let $\sigma\in(Q_W^*)^{W_\Xi}$, i.e. for each $\alpha_i\in\Xi$ we have $\sigma=s_i(\sigma)=\delta_i\bullet\sigma$. Then

$$\sigma(zX_i)=\sigma(z\tfrac{1}{x_{\alpha_i}}(1-\delta_{\alpha_i}))=\sigma(z\tfrac{1}{x_{\alpha_i}})-(\delta_i\bullet\sigma)(z\tfrac{1}{x_i})=(\sigma-\delta_i\bullet\sigma)(z\tfrac{1}{x_i})=0$$

for any $z\in Q_W$. Write $\sigma=\sum_{w\in W}c_wX_{I_w}^*$ for some $c_w\in S$. If $w\notin W^\Xi$, then $I_w$ ends by some $i$ such that $\alpha_i\in\Xi$ which implies that

$$c_w=\sigma(X_{I_w})=\sigma(X_{I_w\setminus i}X_i)=0,$$

where $I_w\setminus i$ is the sequence obtained by deleting the last $i$ in $I_w$. So $\sigma$ is a linear combination of $\{X_{I_u}^*\}_{u\in W^\Xi}$. □

**Corollary 7.7.** *Assume that the set of reduced sequences* $\{I_w\}_{w\in W}$ *is* $\Xi$*-compatible. Then we have* $b_{wv,u}^X=b_{w,u}^X$ *for any* $v\in W_\Xi$*,* $u\in W^\Xi$ *and* $w\in W$*, where* $b_{wv,u}^X$ *are the coefficients of Lemma 3.1.*

*Proof.* From Lemma 3.1 we have $X_{I_u}^*=\sum_{w\geq u}b_{w,u}^Xf_w$. By Lemma 4.2 we obtain that $v(X_{I_u}^*)=\sum_{w\geq u}b_{w,u}^Xf_{wv^{-1}}$ for any $v\in W_\Xi$. Since $X_{I_u}^*$ is $W_\Xi$-invariant by Corollary 7.7 and $\{f_w\}$ is a basis of $Q_W^*$, this implies that $b_{wv^{-1},u}^X=b_{w,u}^X$. □

## 8. The formal affine Demazure algebra

In the present section we recall the definition and basic properties of the formal (affine) Demazure algebra $\mathbf{D}_F$ following [HMSZ], [CZZ] and [Zh13]. We show that for the special elliptic formal group law $F_e$, the formal Demazure algebra is related to the classical Iwahori-Hecke algebra.

Following [HMSZ], we define the *formal affine Demazure algebra* $\mathbf{D}_F$ to be the $R$-subalgebra of the twisted formal group algebra $Q_W$ generated by elements of $S$ and the Demazure elements $X_i$ for all $i\in\{1,\dots,n\}$. By [CZZ, Lemma 5.8], $\mathbf{D}_F$ is also generated by $S$ and all $X_\alpha$ for all $\alpha\in\Sigma$. Since $\kappa_\alpha\in S$, the algebra $\mathbf{D}_F$ is also generated by $Y_\alpha$'s and elements of $S$. Finally, since $\delta_\alpha=1-x_\alpha X_\alpha$, all elements $\delta_w$ are in $\mathbf{D}_F$, and $\mathbf{D}_F$ is a sub-$S_W$-module of $Q_W$, both on the left and on the right.

**Remark 8.1.** Since $\{X_{I_w}\}_{w\in W}$ is a $Q$-linear basis of $Q_W$, restricting the action (4.3) of $Q_W$ onto $\mathbf{D}_F$ we obtain an isomorphism between the algebra $\mathbf{D}_F$ and the $R$-subalgebra $\mathcal{D}(\Lambda)_F$ of $\mathrm{End}_R(S)$ generated by operators $\Delta_\alpha$ (resp. $C_\alpha$) for all $\alpha\in\Sigma$, and multiplications by elements from $S$. This isomorphism maps $X_\alpha\mapsto\Delta_\alpha$ and $Y_\alpha\mapsto C_\alpha$. Therefore, for any identity or statement involving elements $X_\alpha$ or $Y_\alpha$ there is an equivalent identity or statement involving operators $\Delta_\alpha$ or $C_\alpha$.

According to [HMSZ, Theorem 6.14] (or [CZZ, 7.9] when the ring $R$ is not necessarily a domain), in type $A_n$, the algebra $\mathbf{D}_F$ is generated by the Demazure elements $X_i$, $i\in\{1,\dots,n\}$, and multiplications by elements from $S$ subject to the folowing relations:

(a) $X_i^2 = X_i$
(b) $X_iX_j = X_jX_i$ for $|i-j|>1$,
(c) $X_iX_jX_i - X_jX_iX_j = \kappa_{ij}(X_j - X_i)$ for $|i-j|=1$ and
(d) $X_iq = s_i(q)X_i + \Delta_i(q)$,

Recall that the Iwahori-Hecke algebra $\mathcal{H}$ of the symmetric group $S_{n+1}$ is an $\mathbb{Z}[t,t^{-1}]$-algebra with generators $T_i$, $i \in \{1,\dots,n\}$, subject to the following relations:

(A) $(T_i+t)(T_i-t^{-1}) = 0$ or, equivalently, $T_i^2 = (t^{-1}-t)T_i + 1$,
(B) $T_iT_j = T_jT_i$ for $|i-j|>1$ and
(C) $T_iT_jT_i = T_jT_iT_j$ for $|i-j|=1$.

(The $T_i$'s appearing in the definition of the Iwahori-Hecke algebra in [CG10, Def. 7.1.1] correspond to $tT_i$ in our notation, where $t = q^{-1/2}$.)

Following [HMSZ, Def. 6.3] let $\mathrm{D}_F$ denote the $R$-subalgebra of $\mathbf{D}_F$ generated by the elements $X_i$, $i \in \{1,\dots,n\}$, only. By [HMSZ, Prop. 7.1], over $R=\mathbb{C}$, if $F=F_a$ (resp. $F=F_m$), then $\mathrm{D}_F$ is isomorphic to the completion of the nil-Hecke algebra (resp. the 0-Hecke algebra) of Kostant-Kumar. The following observation provides another motivation for the study of formal (affine) Demazure algebras.

Let us consider the special elliptic formal group law of example 2.2 with coefficient $\mu_1 = 1$. Then its formal inverse is $x/(x-1)$, and since $(1+\mu_2 x_ix_j)x_{i+j} = x_i + x_j - x_ix_j$, the coefficient $\kappa_{ij}$ of relation (c) is simply $\mu_2$:

$$\kappa_{ij} = \tfrac{1}{x_{i+j}x_j} - \tfrac{1}{x_{i+j}x_{-i}} - \tfrac{1}{x_ix_j} = \tfrac{x_i+x_j-x_ix_j-x_{i+j}}{x_ix_jx_{i+j}} = \tfrac{(1+\mu_2x_ix_j)x_{i+j}-x_{i+j}}{x_ix_jx_{i+j}} = \mu_2 \tag{8.1}$$

**Proposition 8.2.** *Let $F_e$ be a normalized (i.e. $\mu_1 = 1$) special elliptic formal group law over an integral domain $R$ containing $\mathbb{Z}[t,t^{-1}]$, and let $a,b\in R$. Then the following are equivalent*

*(1) The assignment $T_i \mapsto aX_i + b$, $i \in \{1,\dots,n\}$, defines an isomorphism of $R$-algebras $\mathcal{H}\otimes_{\mathbb{Z}[t,t^{-1}]} R \to \mathrm{D}_F$.*
*(2) We have $a = t+t^{-1}$ or $-t-t^{-1}$ and $b=-t$ or $t^{-1}$ respectively. Furthermore $\mu_2(t+t^{-1})^2 = -1$; in particular, the element $t+t^{-1}$ is invertible in $R$.*

*Proof.* Assume there is an isomorphism of $R$-algebras given by $T_i \mapsto aX_i + b$. Then relations (b) and (B) are equivalent and relation (A) implies that

$$0 = (aX_i+b)^2 + (t-t^{-1})(aX_i+b) - 1 = [a^2+2ab+a(t-t^{-1})]X_i + b^2 + b(t-t^{-1}) - 1.$$

Therefore $b=-t$ or $t^{-1}$ and $a = t^{-1}-t-2b = t+t^{-1}$ or $-t-t^{-1}$ respectively, since 1 and $X_i$ are $S$-linearly independent in $\mathrm{D}_F \subseteq \mathbf{D}_F$.

Relations (C) and (a) then imply

$$\begin{aligned} 0 &= (aX_i+b)(aX_j+b)(aX_i+b) - (aX_j+b)(aX_i+b)(aX_j+b) \\ &= a^3(X_iX_jX_i - X_jX_iX_j) + (a^2b+ab^2)(X_i - X_j). \end{aligned}$$

Therefore, by relation (c) and (8.1), we have $a^3\mu_2 - a^2b - ab^2 = 0$ which implies that $0 = a^2\mu_2 - ab - b^2 = (t+t^{-1})^2\mu_2 + 1$.

Conversely, by substituting the values of $a$ and $b$, it is easy to check that the assignment is well defined, essentially by the same computations. It is an isomorphism since $a = \pm(t+t^{-1})$ is invertible in $R$. ∎

**Remark 8.3.** The isomorphism of Theorem 8.2 provides a presentation of the Iwahori-Hecke algebra with $t+t^{-1}$ inverted in terms of the Demazure operators on the formal group algebra $R[\![\Lambda]\!]_{F_e}$.

## 9. The algebraic restriction to the fixed locus

In the present section we define the algebraic counterpart of the restriction to $T$-fixed points of $G/B$.

Let $S_W$ be the twisted group algebra of $S$ and the group ring $R[W]$, i.e. $S_W = S \otimes_R R[W]$ as a $R$-module and the multiplication is defined by

$$(x\delta_w)(x'\delta_{w'}) = xw(x')\delta_{ww'}, \quad x, x' \in S, \quad w, w' \in W.$$

The algebra $S_W$ is a free $S$-module with basis $\{\delta_w\}_{w\in W}$. Since $S$ is $\Sigma$-regular, it injects into its localization $Q$. Therefore, $S_W$ injects into $Q_W$ via $\delta_w \mapsto \delta_w$.

Since $\delta_\alpha = 1 - x_\alpha X_\alpha$ for each $\alpha \in \Sigma$, there is a natural inclusion of $S$-modules $\eta\colon S_W \hookrightarrow \mathbf{D}_F$. By [CZZ, Prop. 7.7] the elements $\{X_{I_w}\}_{w\in W}$ and, hence, $\{Y_{I_w}\}_{w\in W}$ form a basis of $\mathbf{D}_F$ as a left $S$-module. Tensoring $\eta$ by $Q$ we obtain an isomorphism $\eta_Q\colon Q_W \xrightarrow{\simeq} Q \otimes_S \mathbf{D}_F$, because both are free $Q$-modules and their bases $\{X_{I_w}\}_{w\in W}$ are mapped to each other. Observe that by definition $\mathbf{D}_F$ injects into $Q \otimes_S \mathbf{D}_F \simeq Q_W$.

Consider the $S$-linear dual $S_W^\star = \mathrm{Hom}_S(S_W, S)$. Since $\{\delta_w\}_{w\in W}$ is a basis for both $S_W$ and $Q_W$, $S_W^\star$ can be identified with the free $S$-submodule of $Q_W^*$ with basis $\{f_w\}_{w\in W}$ or, equivalently, with the subset $\{f \in Q_W^* \mid f(S_W) \subseteq S\}$. Consider the $S$-linear dual $\mathbf{D}_F^\star = \mathrm{Hom}_S(\mathbf{D}_F, S)$.

**Definition 9.1.** The induced map $\eta^\star\colon \mathbf{D}_F^\star \to S_W^\star$ (composition with $\eta$) will be called the *algebraic restriction to the fixed locus*, because of its geometric interpretation, given in the introduction.

**Lemma 9.2.** *The map $\eta^\star$ is injective and its image in $S_W^\star \subseteq Q_W^* = Q \otimes_S S_W^\star$ coincides with the subset*

$$\{f \in Q_W^* \mid f(\mathbf{D}_F) \subseteq S\}.$$

*Proof.* There is a commutative diagram

$$\begin{tikzcd} \mathbf{D}_F^\star \arrow[r, "\eta^\star"] \arrow[d] & S_W^\star \arrow[d] \\ Q \otimes_S \mathbf{D}_F^\star \arrow[r, "\eta_Q^\star", "\simeq"'] & Q \otimes_S S_W^\star \end{tikzcd}$$

where the vertical maps are injective by freeness of the modules and because $S$ injects into $Q$. The description for the image then follows from the fact that $\{X_{I_w}\}_{w\in W}$ is a basis for both $\mathbf{D}_F$ and $Q_W$. □

**Remark 9.3.** Observe that $\eta^\star$ is not surjective, unless the root datum is trivial. Indeed, since $X_\alpha = \frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_\alpha$, we have $f_{s_\alpha}(X_\alpha) = -\frac{1}{x_\alpha} \notin S$, so $f_{s_\alpha}$ is not in the image of $\eta^*$.

Since $\mathbf{D}_F$ is a subring of $Q_W$, the '$\bullet$'-action of $Q_W$ on $Q_W^*$ restricts to an $S$-linear action of $\mathbf{D}_F$ on $\mathbf{D}_F^\star$ given by the same formula: $(z' \bullet \sigma)(z) = \sigma(zz')$ for $\sigma \in \mathbf{D}_F^\star$ and $z, z'$ in $\mathbf{D}_F$. Thus, the action of $W$ on $Q_W^*$ restricts to an action on $\mathbf{D}_F$.

By [CZZ, Theorem 9.2] the coproduct $\triangle$ on $Q_W$ restricts to a coproduct on $\mathbf{D}_F$. Hence, the dual $\mathbf{D}_F^\star$ becomes a subring of $Q_W^*$.

## 10. The push-pull operators on $\mathbf{D}_F^\star$

In this section we restrict the push-pull operators onto the dual of the formal affine Demazure algebra $\mathbf{D}_F^\star$, and define a non-degenerate pairing on it.

**Lemma 10.1.** *For any subset $\Xi$ of $\Pi$ we have $Y_\Xi \in \mathbf{D}_F$.*

*Proof.* The ring $Q_W$ is functorial in the root datum (i.e. along morphisms of lattices that send roots to roots) and in the formal group law. This functoriality sends elements $X_\alpha$ (or $Y_\alpha$) to themselves, so it restricts to a functoriality the subring $\mathbf{D}_F$. We can therefore assume that the root datum is adjoint, and that the formal group law is the universal one over the Lazard ring, in which all integers are regular, since it is a polynomial ring over $\mathbb{Z}$.

Consider the involution $\iota$ on $Q_W$ given by $q\delta_w \mapsto (-1)^{\ell(w)} w^{-1}(q)\delta_{w^{-1}}$. It satisfies $\iota(zz') = \iota(z')\iota(z)$. Since $\iota(X_\alpha) = Y_{-\alpha}$, it restricts to an anti-automorphism on $\mathbf{D}_F$. Hence, it suffices to show that $\iota(Y_\Xi) \in \mathbf{D}_F$. By definition we have

$$\iota(Y_\Xi) = \sum_{w \in W_\Xi} (-1)^{\ell(w)} \tfrac{1}{x_\Xi} \delta_{w^{-1}} = \tfrac{1}{x_\Xi} \sum_{w \in W_\Xi} (-1)^{\ell(w)} \delta_w.$$

Since the root datum is adjoint, $\mathbf{D}_F = \{z \in Q_W \mid z \cdot S \subseteq S\}$ by [CZZ, Remark 7.8]. It therefore suffices to show that $\iota(Y_\Xi) \cdot x \in S$ for any $x \in S$. We have

$$\iota(Y_\Xi) \cdot x = \tfrac{1}{x_\Xi} \sum_{w \in W_\Xi} (-1)^{\ell(w)} w(x).$$

For any $\alpha \in \Sigma_\Xi^-$ let ${}^\alpha W_\Xi = \{w \in W_\Xi \mid \ell(s_\alpha w) > \ell(w)\}$. Then $W_\Xi = {}^\alpha W_\Xi \amalg s_\alpha {}^\alpha W_\Xi$ and the sum

$$\sum_{w \in W_\Xi} (-1)^{\ell(w)} w(x) = \sum_{w \in {}^\alpha W_\Xi} (-1)^{\ell(w)} (w(x) - s_\alpha(w(x))) = x_\alpha \sum_{w \in {}^\alpha W_\Xi} (-1)^{\ell(w)} \Delta_\alpha(w(x))$$

is divisible by $x_\alpha$. By using the next lemma recursively, we then conclude that $x_\Xi$ divides the sum and thus that $(\iota(Y_\Xi)) \cdot x \in \mathbf{D}_F$. $\square$

**Lemma 10.2.** *Assume that all integers are regular in $R$, and that the root datum is adjoint. Let $\alpha$ and $\beta$ be roots such that $\alpha \neq \pm\beta$ and let $x' \in S$. Then if $x_\alpha \mid x_\beta x'$, we have $x_\alpha \mid x'$.*

*Proof.* By [CZZ, Lemma 2.1], the root $\alpha$ can be completed as a basis $(e_1 = \alpha, e_2, \ldots, e_n)$ of the lattice, giving an $R$-algebra isomorphism $\phi : S \to R[\![x_1, \ldots, x_n[\![$, sending $x_\alpha$ to $x_1$, by [CPZ, Cor 2.13]. Since $\beta \neq \pm\alpha$, we have $\beta = \sum_i n_i e_i$ with $n_i \neq 0$ for some $i \neq 1$ and $\phi(x_\beta) = \sum_i n_i x_i + z$, where $z \in I^2$, the square of the augmentation ideal (generated by the variables). Since $R[\![x_1, \ldots, x_n]\!]/(x_1) \simeq R[\![x_2, \ldots, x_n]\!]$, the result follows from the regularity of the class of $x_\beta$ in that quotient, which in turn follows from [CZZ, Lemma 12.3]. $\square$

**Corollary 10.3.** *The operator $Y_\Xi$ (resp. $A_\Xi$) restricted to $S$ (resp. to $\mathbf{D}_F^\star$) defines an operator on $S$ (resp. on $\mathbf{D}_F^\star$). Moreover, we have*
$$Y_\Xi(S) \subseteq S^{W_\Xi} \quad \text{and} \quad A_\Xi(\mathbf{D}_F^\star) \subseteq (\mathbf{D}_F^\star)^{W_\Xi}.$$

*Proof.* Here $Y_\Xi$ acts on $S \subseteq Q$ via (4.3). Since $Y_\Xi \in \mathbf{D}_F \subseteq \{z \in Q_W \mid z \cdot S \subseteq S\}$ by [CZZ, Remark 7.8] and $Y_\Xi \cdot Q \subseteq (Q)^{W_\Xi}$, the result follows.

As for $A_\Xi$, by Lemma 9.2 any $f \in \mathbf{D}_F^\star$ has the property that $f(\mathbf{D}_F) \subseteq S$. Therefore, $(A_\Xi(f))(\mathbf{D}_F) = (Y_\Xi \bullet f)(\mathbf{D}_F) = f(\mathbf{D}_F Y_\Xi) \subseteq S$, so $A_\Xi(f) \in \mathbf{D}_F^\star$. The result then follows by Lemma 6.7. □

**Corollary 10.4.** *Suppose that the root datum has no irreducible component of type $C_n^{sc}$ or that 2 is invertible in $R$. Then if $|W_{\Xi'}|$ is regular in $R$, for any $\Xi' \subseteq \Xi \subseteq \Pi$, we have*
$$Y_{\Xi/\Xi'}(S^{W_{\Xi'}}) \subseteq S^{W_\Xi}.$$

*Proof.* Let $x \in (S)^{W_{\Xi'}}$, then $|W_{\Xi'}| \cdot x = \sum_{w \in W_{\Xi'}} w(x)$. So we have
$$|W_{\Xi'}| \cdot Y_{\Xi/\Xi'}(x) = Y_{\Xi/\Xi'}(|W_{\Xi'}| \cdot x) = \sum_{u \in W_{\Xi/\Xi'}} u(\frac{|W_{\Xi'}| \cdot x}{x_{\Xi/\Xi'}})$$
$$= \sum_{u \in W_{\Xi/\Xi'}} \sum_{v \in W_{\Xi'}} uv(\frac{x}{x_{\Xi/\Xi'}}) = \sum_{w \in W_\Xi} w(\frac{x x_{\Xi'}}{x_\Xi}) \in S^{W_\Xi}.$$
Thus $|W_{\Xi'}| \cdot Y_{\Xi/\Xi'}(x) \in S$, which implies that $Y_{\Xi/\Xi'}(x) \in S$ by [CZZ, Lemma 3.5]. Besides, it is fixed by $W_\Xi$ by 5.5. □

For each $v \in W$ define
$$\tilde{f}_v := x_\Pi f_v \in S_W^\star, \quad \text{i.e. } \tilde{f}_v(\sum_{w \in W} q_w \delta_w) = x_\Pi q_v.$$

**Lemma 10.5.** *For any $v \in W$ we have $\tilde{f}_v \in \mathbf{D}_F^\star$.*

*Proof.* We first show that $\tilde{f}_{w_0} \in \mathbf{D}_F^\star$. By Lemma 3.1, we have
$$X_{I_v} = \sum_{w \le v} a_{v,w}^X \delta_w, \text{ where } a_{v,v}^X = \prod_{\alpha \in (v\Sigma^-) \cap \Sigma^+} (-\tfrac{1}{x_\alpha}).$$
Since $w_0 \Sigma^- = \Sigma^+$, we have
$$\tilde{f}_{w_0}(X_{I_v}) = \tilde{f}_{w_0}(\sum_{w \le v} a_{v,w}^X \delta_w) = (x_\Pi a_{w_0,w_0}^X) \delta_{v,w_0} = \prod_{\alpha \in \Sigma^+} (-\tfrac{x_{-\alpha}}{x_\alpha}) \delta_{v,w_0} \in S.$$
By Lemma 9.2, we have $\tilde{f}_{w_0} \in \mathbf{D}_F^\star$. For an arbitrary $v \in W$, by Lemma 4.2, we obtain
$$\tilde{f}_v = x_\Pi f_{w_0 w_0^{-1} v} = v^{-1} w_0 (x_\Pi f_{w_0}) = v^{-1} w_0 (\tilde{f}_{w_0}) \in \mathbf{D}_F^\star. \qquad \square$$

**Corollary 10.6.** *For any $X \in \mathbf{D}_F$ we have $x_\Pi X \in S_W$.*

*Proof.* It suffices to show that for any sequence $I_v$, $x_\Pi X_{I_v} \in S_W$. And indeed,
$$x_\Pi X_{I_v} = x_\Pi (\sum_{w \le v} a_{v,w}^X \delta_w) = \sum_{w \le v} (x_\Pi a_{v,w}^X) \delta_w = \sum_{w \le v} \tilde{f}_w(X_{I_v}) \delta_w \in S_W. \qquad \square$$

**Theorem 10.7.** *For any $v, w \in W$, we have*

$$A_\Pi(Y^*_{I_v} A_{I_w^{\mathrm{rev}}}(x_\Pi f_e)) = \delta_{w,v}\mathbf{1} = A_\Pi(X^*_{I_v} B_{I_w^{\mathrm{rev}}}(x_\Pi f_e)).$$

*Consequently, the pairing*

$$A_\Pi\colon \mathbf{D}_F^\star \times \mathbf{D}_F^\star \to (\mathbf{D}_F^\star)^W \cong S, \quad (\sigma,\sigma') \mapsto A_\Pi(\sigma\sigma')$$

*is non-degenerate and satisfies that $\{A_{I_w^{\mathrm{rev}}}(x_\Pi f_e)\}_{w\in W}$ is dual to the basis $\{Y^*_{I_v}\}_{v\in W}$, and $\{B_{I_w^{\mathrm{rev}}}(x_\Pi f_e)\}_{w\in W}$ is dual to the basis $\{X^*_{I_v}\}_{v\in W}$.*

*Proof.* We prove the first identity. The second identity is obtained similarly.

Let $Y_{I_w^{\mathrm{rev}}} = \sum_{v\in W} a'_{w,v}\delta_v$ and $Y_{I_w} = \sum_{v\in W} a_{w,v}\delta_v$. Let $\delta_w = \sum_{v\in W} b_{w,v} Y_{I_v}$ so that $\sum_{v\in W} a_{w,v} b_{v,u} = \delta_{w,u}$ and $Y^*_{I_u} = \sum_{v\in W} b_{v,u} f_v$.

Combining the formula of Lemma 4.6 with the formula $A_\Pi(f_v) = \frac{1}{v(x_\Pi)}\mathbf{1}$ of Lemma 6.5, we obtain

$$A_\Pi(Y^*_{I_u} A_{I_w^{\mathrm{rev}}}(x_\Pi f_e)) = \sum_{v\in W} b_{v,u} v(x_\Pi) a_{w,v} A_\Pi(f_v) = \sum_{v\in W} b_{v,u} a_{w,v}\mathbf{1} = \delta_{w,u}\mathbf{1}. \qquad \square$$

The characteristic map $c$ introduced in 6.10 restricts to a homomorphism of $R$-algebras $c_S\colon S \to \mathbf{D}_F^\star$ which maps $x \in S$ to the evaluation at $x$ using the action of $\mathbf{D}_F$ on $S$, that is $c_S(x)(z) = z\cdot x$ for $z \in \mathbf{D}_F$. In particular, we have

$$c_S(x)(X_I) = \Delta_I(x) \text{ and } c_S(x)(\delta_w) = w(x), \quad w \in W.$$

**Lemma 10.8.** *For any sequence $I$ and $x \in S$, we have*

$$A_\Pi(c_S(x) A_{I^{\mathrm{rev}}}(x_\Pi f_e)) = C_I(x)\mathbf{1} \textit{ and } A_\Pi(c_S(x) B_{I^{\mathrm{rev}}}(x_\Pi f_e)) = \Delta_I(x)\mathbf{1}.$$

*Proof.* We prove the first formula only. The second formula is obtained similarly.

Let $Y_I = \sum_{v\in W} a^Y_{I,v}\delta_v$. Since $c_S(x) = \sum_{v\in W} v(x) f_v$, by Lemma 4.6, we get

$$A_\Pi(c_S(x) A_{I^{\mathrm{rev}}}(x_\Pi f_e)) = A_\Pi(\sum_{v\in W} v(x) v(x_\Pi) a^Y_{I,v} f_v) = \sum_{v\in W} a^Y_{I,v} v(x)\mathbf{1} = C_I(x)\mathbf{1}. \quad \square$$

## 11. The non-degenerate pairing on the $W_\Xi$-invariant subring

In this last section, we construct a non-degenerate pairing on the subring of invariants $(\mathbf{D}_F^\star)^{W_\Xi}$. Using this pairing we provide several $S$-module bases of $(\mathbf{D}_F^\star)^{W_\Xi}$.

**Lemma 11.1.** *Suppose that the set of reduced sequences $\{I_w\}_{w\in W}$ is $\Xi$-compatible, then the set $\{X^*_{I_u}\}_{u\in W^\Xi}$ is a basis of $(\mathbf{D}_F^\star)^{W_\Xi}$.*

*Proof.* This follows immediately from Lemma 6.1 and the identity $(Q_W^\star)^{W_\Xi} \cap \mathbf{D}_F^\star = (\mathbf{D}_F^\star)^{W_\Xi}$. $\square$

Given representatives $u, u' \in W^\Xi$ we set

$$d^Y_{u,u'} := u'(x_{\Pi/\Xi}) \sum_{v\in W_\Xi} a^Y_{u,u'v}, \quad d^X_{u,u'} := u'(x_{\Pi/\Xi}) \sum_{v\in W_\Xi} a^X_{u,u'v}, \quad \eta_\Xi = \prod_{w\in W^\Xi} w(x_{\Pi/\Xi})$$

where $a^X_{w,v}$ and $a^Y_{w,v}$ are the coefficients introduced in Lemma 3.1 and 3.2.

**Lemma 11.2.** *For any $u \in W^\Xi$ we have*

$$A_\Xi(A_{I_u^{\mathrm{rev}}}(x_\Pi f_e)) = \sum_{u'\in W^\Xi} d^Y_{u,u'} f^\Xi_{u'}, \quad A_\Xi(B_{I_u^{\mathrm{rev}}}(x_\Pi f_e)) = \sum_{u'\in W^\Xi} d^X_{u,u'} f^\Xi_{u'}.$$

*Proof.* We prove the first formula only. The second formula follows similarly. By Lemma 4.6 and 6.5 we obtain

$$A_\Xi(A_{I_u^{\mathrm{rev}}}(x_\Pi f_e)) = A_\Xi(\sum_{w\in W} w(x_\Pi)a^Y_{u,w}f_w) = \sum_{w\in W} w(x_{\Pi/\Xi})a^Y_{u,w}f^\Xi_w =$$

by (7.1), representing $w = u'v$, and Lemma 5.1 we get

$$= \sum_{u'\in W^\Xi,\, v\in W_\Xi} u'v(x_{\Pi/\Xi})a^Y_{u,u'v}f^\Xi_{u'v} = \sum_{u'\in W^\Xi,\, v\in W_\Xi} u'(x_{\Pi/\Xi})a^Y_{u,u'v}f^\Xi_{u'}. \qquad \square$$

**Theorem 11.3.** *Assume that the set of reduced sequences $\{I_w\}_{w\in W}$ is $\Xi$-compatible, then the sets $\{A_\Xi(A_{I_u^{\mathrm{rev}}}(x_\Pi f_e))\}_{u\in W^\Xi}$ and $\{A_\Xi(B_{I_u^{\mathrm{rev}}}(x_\Pi f_e))\}_{u\in W^\Xi}$ are $S$-module bases of $(\mathbf{D}_F^\star)^{W_\Xi}$.*

*Proof.* Observe that by Corollary 10.3 our sets are in the $S$-module $(\mathbf{D}_F^\star)^{W_\Xi}$. To show that they are bases, it suffices to show that the respective matrices $M^Y_\Xi$ and $M^X_\Xi$ expressing them on the basis $\{X^*_{I_u}\}_{u\in W^\Xi}$ of Lemma 11.1 have invertible determinants (in $S$).

If $u'\in W^\Xi$ and $v\in W_\Xi$, we have $u'\le u'v$ where the equality holds if and only if $v=e$. By Lemma 3.2, we get $a^Y_{u,u'v}=0$ unless $u'\le u$ and $a^Y_{u,uv}=0$ if $v\neq e$. This implies that $d^Y_{u,u'}=0$ unless $u'\le u$, and that

$$d^Y_{u,u} = u(x_{\Pi/\Xi})\sum_{v\in W_\Xi} a^Y_{u,uv} = u(x_{\Pi/\Xi})a^Y_{u,u} = u(x_{\Pi/\Xi})\tfrac{1}{b^Y_{u,u}}.$$

Hence, the matrix $D^Y_\Xi := (d^Y_{u,u'})_{u,u'\in W^\Xi}$ is lower triangular with determinant $\eta_\Xi\prod_{u\in W^\Xi}\frac{1}{b^Y_{u,u}}$. Similarly, the matrix $D^X_\Xi := (d^X_{u,u'})_{u,u'\in W^\Xi}$ is lower triangular with determinant $\eta_\Xi\prod_{u\in W^\Xi}\frac{1}{b^X_{u,u}}$.

On the other hand, for $u\in W^\Xi$, we have

$$X^*_{I_u} = \sum_{w\in W} b^X_{w,u}f_w = \sum_{u'\in W^\Xi}\sum_{v\in W_\Xi} b^X_{u'v,u}f_{u'v}.$$

By Corollary 7.7, and because $X^*_{I_u}$ is fixed by $W_\Xi$, we have $b^X_{u'v,u}=b^X_{u',u}$. Therefore,

$$X^*_{I_u} = \sum_{u'\in W^\Xi} b^X_{u',u}\sum_v f_{u'v} = \sum_{u'\in W^\Xi} b^X_{u',u}f^\Xi_{u'}.$$

By Lemma 3.1, $b^X_{u',u}=0$ unless $u'\ge u$, so the matrix $E^X_\Xi := \{b^X_{u',u}\}_{u',u\in W^\Xi}$ is lower triangular with determinant $\prod_{u\in W^\Xi} b^X_{u,u} = \prod_{u\in W^\Xi}(-1)^{\ell(u)}b^Y_{u,u}$.

The matrix $M^X_\Xi = (E^X_\Xi)^{-1}D^X_\Xi$ has determinant

$$\eta_\Xi\prod_{u\in W^\Xi}\tfrac{1}{(b^X_{u,u})^2} = \eta_\Xi\prod_{u\in W^\Xi}\tfrac{1}{(b^Y_{u,u})^2}$$

which is invertible in $S$ by Lemma 11.5 below. Since the determinant of $M^Y_\Xi = (E^X_\Xi)^{-1}D^Y_\Xi$ differs by sign only, it is invertible as well. $\square$

Recall the definition of $\Sigma_\Xi$ from the beginning of section 5, and let $w_0^\Xi$ be the longest element of $W_\Xi$.

**Lemma 11.4.** *For any $w\in W_\Xi$, we have $b^Y_{w,w}b^Y_{ww_0^\Xi,ww_0^\Xi} = w_0^\Xi(x_\Xi)$. In particular, if $\Xi=\Pi$ we have $b^Y_{w,w}b^Y_{ww_0,ww_0}=w_0(x_\Pi)$.*

*Proof.* Recall from Lemma 3.2 that $b^Y_{w,w} = \prod_{w\Sigma^-\cap\Sigma^+} x_\alpha$. By (3.2), it also equals $\prod_{w\Sigma^-_\Xi\cap\Sigma^+_\Xi} x_\alpha$. Since $w_0^\Xi\Sigma^-_\Xi = \Sigma^+_\Xi$, we have $ww_0^\Xi\Sigma^-_\Xi\cap\Sigma^+_\Xi = w\Sigma^+_\Xi\cap\Sigma^+_\Xi$. Moreover,

$$(w\Sigma^-_\Xi\cap\Sigma^+_\Xi)\cap(w\Sigma^+_\Xi\cap\Sigma^+_\Xi)\subset w\Sigma^-_\Xi\cap w\Sigma^+_\Xi = w(\Sigma^-_\Xi\cap\Sigma^+_\Xi)=\emptyset$$

and their union is $\Sigma^+_\Xi$. □

**Lemma 11.5.** *For any $\Xi\subset\Pi$ the product $\eta_\Xi\prod_{u\in W^\Xi}\frac{1}{(b^Y_{u,u})^2}$ is an invertible element in $S$.*

*Proof.* We already know that this product is in $S$, since it is the determinant of the matrix $M^X_\Xi$ whose coefficients are in $S$. Consider the $R$-linear involution $u\mapsto\bar u$ on $S=R[\![\Lambda]\!]_F$ induced by $\lambda\mapsto-\lambda$, $\lambda\in\Lambda$. Observe that it is $W$-equivariant.

Set $b_v := b^Y_{v,v}$. For any $\alpha\in\Xi$, we have

$$x_\Xi = s_\alpha(x_\Xi)x_{-\alpha}x_\alpha^{-1} = s_\alpha(x_\Xi)\bar b_{s_\alpha}b^{-1}_{s_\alpha}$$

and, therefore, by induction $x_\Xi = w(x_\Xi)\bar b_v b_v^{-1}$ for any $v\in W_\Xi$. In particular, $x_\Pi = w(x_\Pi)\bar b_w b_w^{-1}$ for any $w\in W$. Then

$$x_\Xi^{|W_\Xi|} = \prod_{v\in W_\Xi} v(x_\Xi)\bar b_v b_v^{-1}\quad\text{and}\quad x_\Pi^{|W|} = \prod_{w\in W} w(x_\Pi)\bar b_w b_w^{-1}.$$

If $w=uv$ with $\ell(w)=\ell(u)+\ell(v)$, by (3.2), we see that

$$w\Sigma^-\cap\Sigma^+ = (u\Sigma^-\cap\Sigma^+)\sqcup u(v\Sigma^-\cap\Sigma^+),$$

so $b_{uv}=b_u u(b_v)$ and $\bar b_{uv} = \bar b_u u(\bar b_v)$. Hence

$$\begin{aligned} x_\Pi^{|W|} &= \prod_{w\in W} w(x_\Pi)\bar b_w b_w^{-1} = \prod_{u\in W^\Xi}\prod_{v\in W_\Xi} uv(x_{\Pi/\Xi}x_\Xi)\bar b_{uv}b_{uv}^{-1}\\ &\overset{5.2}{=} \prod_{u\in W^\Xi} u(x_{\Pi/\Xi}^{|W_\Xi|})\prod_{v\in W_\Xi} uv(x_\Xi)\bar b_u u(\bar b_v)b_u^{-1}u(b_v^{-1})\\ &= \eta_\Xi^{|W_\Xi|}\prod_{u\in W^\Xi}\big(\bar b_u b_u^{-1}\big)^{|W_\Xi|}u\Big(\prod_{v\in W_\Xi} v(x_\Xi)\bar b_v b_v^{-1}\Big)\\ &= \eta_\Xi^{|W_\Xi|}\prod_{u\in W^\Xi}\big(\bar b_u b_u^{-1}\big)^{|W_\Xi|}u(x_\Xi)^{|W_\Xi|} \end{aligned}\tag{11.1}$$

On the other hand, by Lemma 11.4,

$$\bar x_\Xi^{|W_\Xi|} = w_0^\Xi(x_\Xi)^{|W_\Xi|} = \prod_{v\in W_\Xi} b_v b_{vw_0^\Xi} = \prod_{v\in W_\Xi} b_v^2$$

and, in particular, $\bar x_\Xi^{|W|} = \prod_{w\in W} b_w^2$. So, we obtain

$$\begin{aligned} \bar x_\Pi^{|W|} &= \prod_{w\in W} b_w^2 = \prod_{u\in W^\Xi}\prod_{v\in W_\Xi} b_{uv}^2 = \prod_{u\in W^\Xi}\prod_{v\in W_\Xi} b_u^2 u(b_v^2)\\ &= \Big(\prod_{u\in W^\Xi} b_u^{2|W_\Xi|}\Big)\Big(\prod_{u\in W^\Xi} u\big(\prod_{v\in W_\Xi} b_v^2\big)\Big) = \Big(\prod_{u\in W^\Xi} b_u^2\Big)^{|W_\Xi|}\Big(\prod_{u\in W^\Xi} u(\bar x_\Xi)\Big)^{|W_\Xi|}. \end{aligned}$$

Combining this with equation (11.1), we obtain

$$\Big(\eta_\Xi^{-1}\prod_{u\in W^\Xi} b_u^2\Big)^{|W_\Xi|} = \bar x_\Pi^{|W|}x_\Pi^{-|W|}\Big(\prod_{u\in W^\Xi} u(\bar x_\Xi^{-1}x_\Xi)\bar b_u b_u^{-1}\Big)^{|W_\Xi|}$$

which is an element of $S$, since it is a product of elements of the form $x_\alpha x_{-\alpha}^{-1} \in S$. Therefore $\eta_\Xi \prod_{u\in W^\Xi} \frac{1}{b_u^2}$ is invertible, since so is its $|W_\Xi|$-th power. □

**Corollary 11.6.** *Given $\Xi' \subseteq \Xi \subseteq \Pi$ we have $A_\Xi(\mathbf{D}_F^\star) = (\mathbf{D}_F^\star)^{W_\Xi}$. For any set of coset representatives $W_{\Xi/\Xi'}$ the operator $A_{\Xi/\Xi'}$ induces a surjection $(\mathbf{D}_F^\star)^{W'_\Xi} \to (\mathbf{D}_F^\star)^{W_\Xi}$ (independent of the choices of $W_{\Xi/\Xi'}$ by Lemma 6.4).*

*Proof.* By Corollary 10.3 and Theorem 11.3, we obtain the first part. To prove the second part, let $\sigma \in (\mathbf{D}_F^\star)^{W_{\Xi'}}$. By the first part, there exists $\sigma' \in \mathbf{D}_F^\star$ such that $\sigma = A_{\Xi'}(\sigma')$, so by Lemma 6.3 we have

$$A_{\Xi/\Xi'}(\sigma) = A_{\Xi/\Xi'}(A_{\Xi'}(\sigma')) = A_\Xi(\sigma') \in (\mathbf{D}_F^\star)^{W_\Xi}.$$

Hence, $A_{\Xi/\Xi'}$ restricts to $A_{\Xi/\Xi'} \colon (\mathbf{D}_F^\star)^{W_{\Xi'}} \to (\mathbf{D}_F^\star)^{W_\Xi}$. Since $A_\Xi(\mathbf{D}_F^\star) = (\mathbf{D}_F^\star)^{W_\Xi}$, we also have $A_{\Xi/\Xi'}((\mathbf{D}_F^\star)^{W_{\Xi'}}) = (\mathbf{D}_F^\star)^{W_\Xi}$. □

**Theorem 11.7.** *Assume that the set of reduced sequences $\{I_w\}_{w\in W}$ is $\Xi$-compatible. If $u \in W^\Xi$, then $A_{\Pi/\Xi}(X_{I_u}^* A_\Xi(B_{I_w}^{\mathrm{rev}}(x_\Pi f_e))) = \delta_{w,u}\mathbf{1}$. Consequently, the pairing*

$$(\mathbf{D}_F^\star)^{W_\Xi} \times (\mathbf{D}_F^\star)^{W_\Xi} \to (\mathbf{D}_F^\star)^W \cong S, \quad (\sigma,\sigma') \mapsto A_{\Pi/\Xi}(\sigma\sigma')$$

*is non-degenerate, and $\{A_\Xi(B_{I_u^{\mathrm{rev}}}(x_\Pi f_e))\}_{u\in W^\Xi}$, $\{X_{I_u}^*\}_{u\in W^\Xi}$ are dual $S$-bases of $(\mathbf{D}_F^\star)^{W_\Xi}$.*

*Proof.* By Corollary 11.6, the pairing is well-defined (i.e. it does map into $S$). By Lemma 6.7, Lemma 6.3 and Theorem 10.7, we obtain

$$\begin{aligned} A_{\Pi/\Xi}(X_{I_u}^* A_\Xi(B_{I_w^{\mathrm{rev}}}(x_\Pi f_e))) &= A_{\Pi/\Xi}(A_\Xi(X_{I_u}^* B_{I_w^{\mathrm{rev}}}(x_\Pi f_e))) \\ &= A_\Pi(X_{I_u}^* B_{I_w^{\mathrm{rev}}}(x_\Pi f_e)) = \delta_{w,u}\mathbf{1}. \end{aligned}$$ □

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