WITT KERNELS AND BRAUER KERNELS FOR QUARTIC EXTENSIONS IN CHARACTERISTIC TWO

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ABSTRACT. Let F be a field of characteristic 2 and let E/F be a field extension of degree 4. We determine the kernel $W_q(E/F)$ of the restriction map $W_qF \to W_qE$ between the Witt groups of nondegenerate quadratic forms over F and over E, completing earlier partial results by Ahmad, Baeza, Mammone and Moresi. We also deduct the corresponding result for the Witt kernel $W(E/F)$ of the restriction map $WF \to WE$ between the Witt rings of nondegenerate symmetric bilinear forms over F and over E from earlier results by the first author. As application, we describe the 2-torsion part of the Brauer kernel for such extensions.

1. INTRODUCTION

When considering algebraic objects defined over fields such as quadratic forms or central simple algebras, an important problem is to characterize their behavior under field extensions, for example to determine which quadratic forms become hyperbolic or which central simple algebras split over a given extension, in other words to compute the Witt kernel or the Brauer kernel for that extension. The purpose of the present paper is to determine Witt kernels for quartic extensions in characteristic 2 completing earlier results by various authors, and to apply this to the determination of the 2-torsion part of the Brauer kernel. These kernels have been previously computed in characteristic not 2. Also, in all characteristics the kernels have been known for quite some time in the case of quadratic extensions, and they are trivial for odd degree extensions due to Springer's theorem, so quartic extensions in characteristic 2 are the first case where the determination of these kernels was still incomplete. We will survey the known results in §2 and §4, including the bilinear case.

There are several aspects that complicate matters when studying the Witt kernels for quartic extensions in characteristic 2. Firstly, one has to distinguish between the Witt kernel for bilinear forms and that for quadratic forms. Secondly, one has to handle carefully various cases of separability and inseparability when dealing with quartic extensions. For that reason, we provide a quick survey of quartic

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extensions in characteristic 2 in §3. Finally, when computing the Witt kernels for quadratic forms in characteristic 2, we will often have to deal with singular quadratic forms, something that can be largely ignored in characteristic not 2. Accordingly, the formulation of the main result in Theorem 5.4 concerning quadratic Witt kernels of simple quartic extensions is more complex than the corresponding result in characteristic not 2. Section 5 will be devoted to the proof of that theorem and we will also show how previously known results on generators for the quadratic Witt kernels for non purely inseparable biquadratic and purely inseparable simple quartic extensions relate to our list of generators for these kernels. In §6 we apply our knowledge of Witt kernels to determine the 2-torsion part of the Brauer kernel for quartic extensions in characteristic 2.

2. Basic definitions and facts

We refer to [9] and [15] for any undefined terminology or any basic facts about quadratic and bilinear forms especially in the case of characteristic 2 that we do not mention explicitly. All quadratic resp. bilinear forms over a field F are assumed to be finite-dimensional, and bilinear forms are always assumed to be symmetric. Let $b = (b, V)$ be a bilinear form defined on an F-vector space V. b is said to be nonsingular if for its radical one has $\mathrm{Rad}(b) = \{x \in V \mid b(x, V)\} = 0$. In the sequel, we will always assume bilinear forms to be nonsingular. We have the usual notions of isometry $\cong$, orthogonal sum $\perp$ and tensor product $\otimes$ for bilinear forms. We define the value sets $D_F(b) = \{b(x, x) \mid x \in V \setminus \{0\}\}$, $D^*_F(b) = D_F(b) \cap F^*$, $D^0_F(b) = D_F(b) \cup \{0\}$. b is said to be isotropic if $D_F(b) = D^0_F(b)$, anisotropic otherwise, i.e. if $D_F(b) = D^*_F(b)$. The 2-dimensional isotropic bilinear form is called a metabolic plane in which case there is a basis such that the Gram matrix is of shape $\mathbb{M}_a \cong \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$, $a \in F$. $\mathbb{M}_0$ is called a hyperbolic plane, and a metabolic resp. hyperbolic bilinear form is one that is isometric to an orthogonal sum of metabolic resp. hyperbolic planes. Any bilinear b with $D^*_F(b) \neq \emptyset$ can be diagonalized. This always holds in characteristic not 2, and also in characteristic 2 provided b is not hyperbolic. We write $b \cong \langle a_1, \dots, a_n\rangle_b$ for such a diagonalization. Each bilinear form b decomposes as $b \cong b_{\mathrm{an}} \perp b_m$ with b_{an} anisotropic and b_m metabolic. b_{an} is uniquely determined up to isometry. Two nonsingular bilinear forms b, b' are Witt equivalent, $b \sim b'$, if $b_{\mathrm{an}} \cong b'_{\mathrm{an}}$. The Witt equivalence classes together with addition resp. multiplication induced by orthogonal sum resp. tensor product form the Witt ring WF of bilinear forms, or the bilinear Witt ring for short.

In characteristic not 2 the theories of bilinear and quadratic forms are the same, so let us from now on assume $\mathrm{char}(F) = 2$.

Let q be a quadratic form defined on the F-vector space V. The associated bilinear form of q is defined to be $b_q(x, y) = q(x + y) - q(x) - q(y)$. Isometry, isotropy, $D_F(q)$, $D^*_F(q)$ and $D^0_F(q)$ are defined in an analogous way to the bilinear case. If (q, V), (q', V') are quadratic forms, we say that q' dominates q, denoted by $q \prec q'$, if there exists an injective linear map $t : V \to V'$ with $q(x) = q'(tx)$ for all $x \in V$.

q is said to be nonsingular if the radical $\mathrm{Rad}(q) := \mathrm{Rad}(b_q) = \{0\}$, totally singular if $\mathrm{Rad}(q) = V$. q is totally singular iff q is a diagonal form $\sum_{i=1}^n a_i x_i^2$, and we write $q \cong \langle a_1, \dots, a_n\rangle$. Note that in this case $D^0_F(q)$ is a finite-dimensional F^2-vector space inside F. Two totally singular quadratic forms q, q' are isometric iff $\dim q = \dim q'$ and $D_F^0(q) = D_F^0(q')$. q is nonsingular iff it is isometric to an orthogonal sum of forms of type $[a,b] := ax^2+xy+by^2$. A nonsingular 2-dimensional isotropic quadratic form is called a hyperbolic plane $\mathbb{H} \cong [0,0]$, and a hyperbolic quadratic form is an orthogonal sum of hyperbolic planes. Any quadratic form q can be decomposed in the following way (see, e.g., [15, Prop. 2.4]):

$$q \cong i \times \mathbb{H} \perp \widetilde{q}_r \perp \widetilde{q}_s \perp j \times \langle 0 \rangle$$

with $\widetilde{q}_r$ nonsingular, $\widetilde{q}_s$ totally singular, $\widetilde{q}_r \perp \widetilde{q}_s$ anisotropic. The form $\widetilde{q}_r \perp \widetilde{q}_s$ is uniquely determined up to isometry and is called the anisotropic part of q, denoted by q_{an}. $i = i_W(q)$ is called the Witt index of q, $j = i_d(q)$ the defect, and $i_t(q) = i_W(q) + i_d(q)$ the total index, which is the same as the dimension of a maximal totally isotropic subspace of q. Note that $\widetilde{q}_s \perp j \times \langle 0 \rangle$ is just the restriction $q|_{\mathrm{Rad}(q)}$ of q to its radical and it is therefore also uniquely determined, as is the nondefective part $i \times \mathbb{H} \perp \widetilde{q}_r \perp \widetilde{q}_s$ of q.

Two quadratic forms q, q' are called Witt equivalent, $q \sim q'$, if $q_{\mathrm{an}} \cong q'_{\mathrm{an}}$. The Witt equivalence classes of nonsingular quadratic forms over F together with addition induced by the orthogonal sum form the quadratic Witt group W_qF which becomes a WF-module in a natural way. Note that in particular $\langle a \rangle_b \otimes q \cong aq$.

Pfister forms will play an important role in our investigations. An n-fold bilinear Pfister form (in arbitrary characteristic) is a nonsingular bilinear form isometric to a form of type $\langle\langle a_1, \ldots, a_n \rangle\rangle_b := \langle 1, -a_1 \rangle_b \otimes \ldots \otimes \langle 1, -a_n \rangle_b$, $a_i \in F^*$. An $(n+1)$-fold quadratic Pfister form in characteristic 2 is a nonsingular quadratic form isometric to a form of type $\langle\langle a_1, \ldots, a_n, c]] := \langle\langle a_1, \ldots, a_n \rangle\rangle_b \otimes [1, c]$, $a_i \in F^*$, $c \in F$. Pfister forms are either anisotropic or metabolic (bilinear case) resp. hyperbolic (quadratic case). They are also round forms, i.e. if π is a bilinear or quadratic Pfister form and $x \in F^*$, then $\pi \cong x\pi$ iff $x \in D_F^*(\pi)$. An n-fold quasi-Pfister form $\widetilde{\pi}$ is a totally singular quadratic form such that there exists an n-fold bilinear Pfister form $\pi \cong \langle\langle a_1, \ldots, a_n \rangle\rangle_b$ with $\widetilde{\pi}(x) = \pi(x,x)$, in which case we write $\widetilde{\pi} \cong \langle\langle a_1, \ldots, a_n \rangle\rangle$. We then have $D_F^0(\widetilde{\pi}) = D_F^0(\pi) = F^2(a_1, \ldots, a_n)$. Quasi-Pfister forms are clearly also round. A quadratic form q is a Pfister neighbor of the quadratic or quasi-Pfister form π iff $\dim q > \frac{1}{2}\dim \pi$ and there exists an $x \in F^*$ with $q \prec x\pi$. In this case, q is isotropic iff π is isotropic.

We put $\wp(F) = \{\lambda^2 + \lambda \mid \lambda \in F\}$, and for $b \in F$ we write $\wp^{-1}(b)$ to denote a root of $X^2 + X + b$ in some algebraic closure of F. The Arf invariant of a nonsingular quadratic form $q \cong [c_1, d_1] \perp \ldots \perp [c_n, d_n]$ is defined to be $\Delta(q) = \sum_{i=1}^n c_i d_i \in F/\wp(F)$. A 2-dimensional nonsingular form $q = [a,b]$ is isotropic iff $\Delta(\varphi) = ab = 0 \in F/\wp(F)$.

By abuse of notation, we often use the same symbol to denote a quadratic or bilinear form and its Witt class.

If E/F is a field extension and φ is a bilinear resp. quadratic form over F, we denote by $\varphi_E = \varphi \otimes E$ the form obtained by scalar extension to E. This gives rise to the restriction homomorphisms $WF \to WE$ resp. $W_qF \to W_qE$ with kernels $W(E/F)$ resp.$W_q(E/F)$. We call these kernels the bilinear resp. quadratic Witt kernel of the extension E/F.

We will often freely use the following results.

Lemma 2.1. *Let φ and ψ be quadratic forms over F with φ nonsingular, and let $\sigma \cong \langle c_1, \ldots, c_n \rangle$ be a totally singular quadratic form over F.*

(i) *For all $d_1, \ldots, d_n \in F$ one has $[c_1, d_1] \perp \ldots \perp [c_n, d_n] \perp \sigma \sim \sigma$.*

(ii) *If $\varphi \prec \psi$ then there exists a quadratic form τ over F with $\psi \cong \varphi \perp \tau$.*

(iii) *If ψ is nonsingular, then the following are equivalent:*

(a) $\varphi \perp \sigma \prec \psi$;

(b) *there exists a nonsingular quadratic form τ over F of dimension* $\dim\psi - \dim\varphi - 2\dim\sigma$ *and $d_1,\ldots,d_n \in F$ with*

$$\psi \cong \varphi \perp [c_1,d_1] \perp \ldots \perp [c_n,d_n] \perp \tau \ ;$$

(c) *there exists a nonsingular quadratic form τ over F of dimension* $\dim\psi - \dim\varphi - 2n$ *such that $\psi \perp \varphi \perp \sigma \sim \tau \perp \sigma$.*

Proof. (i) follows from $[c,d] \perp \langle c\rangle \cong \mathbb{H} \perp \langle c\rangle$, and (ii) follows from general properties of nonsingular subspaces (see, e.g., [9, Prop. 7.22]).

(iii) The equivalence of (a) and (b) is a special case of a more general result [15, Lemma 3.1].

(b) implies (c) by adding φ on both sides noting that $\varphi \perp \varphi = (\dim\varphi)\mathbb{H} \sim 0$.

To show that (c) implies (b), note that by adding φ on both sides and comparing dimensions, we get

$$\psi \perp \sigma \cong n\mathbb{H} \perp \varphi \perp \tau \perp \sigma \ .$$

Also, $\sigma \prec n\mathbb{H} \cong [c_1,0] \perp \ldots \perp [c_n,0]$. By [15, Lemma 3.9], there exists a nonsingular form $\rho \cong [c_1,d_1] \perp \ldots \perp [c_n,d_n]$ for suitable $d_i \in F$ such that

$$\psi \perp n\mathbb{H} \cong n\mathbb{H} \perp \varphi \perp \tau \perp \rho$$

and thus $\psi \cong \varphi \perp \tau \perp \rho$ as desired. □

An essential ingredient in our studies is the behavior of quadratic forms under quadratic extensions as described in by the following well known results.

Lemma 2.2. *Let φ be an anisotropic quadratic form over F and let E/F be a quadratic extension.*

(i) (Baeza [6, 4.3], Hoffmann-Laghribi [16, Lemma 5.4].) *If $E = F(\sqrt{a})$, $a \in F \setminus F^2$, then φ_E is isotropic iff there exists $\lambda \in F^*$ such that $\lambda\langle 1,a\rangle \prec \varphi$.*

(ii) (Baeza [7, V.4.2].) *If $E = F(\wp^{-1}(a))$, $a \in F \setminus \wp(F)$, then φ_E is isotropic iff there exists $\lambda \in F^*$ and a quadratic form ψ over F with $\varphi \cong \lambda[1,a] \perp \psi$.*

As a consequence, one can readily show the following.

Proposition 2.3. *Let φ be an anisotropic nonsingular quadratic form over F and let E/F be a quadratic extension.*

(i) (Ahmad [1, Cor. 2.8], Baeza [6, 4.3].) *If $E = F(\sqrt{a})$, $a \in F \setminus F^2$, then φ_E is hyperbolic iff there exists a nonsingular quadratic form q over F with $\varphi \cong \langle 1,a\rangle_b \otimes q$. In particular, $W_q(E/F) = \langle 1,a\rangle_b \otimes W_qF$.*

(ii) (Baeza [7, V.4.11].) *If $E = F(\wp^{-1}(a))$, $a \in F \setminus \wp(F)$, then φ_E is hyperbolic iff there exists a bilinear form b over F such that $\varphi \cong b \otimes [1,a]$. In particular, $W_q(E/F) = WF \otimes [1,a]$.*

The following lemma is an essential ingredient in the determination of the Witt kernel for quartic extensions.

Lemma 2.4. *Let $\tilde{\pi}$ be an n-fold bilinear Pfister form with associated totally singular form π, q a nonsingular quadratic form and $\varphi \cong \tilde{\pi} \otimes q$. Let $x \in F^*$ and suppose that φ is anisotropic and $\pi \perp \langle x\rangle \prec \varphi$. Then $\pi \perp x\pi \prec \varphi$.*

Proof. Since $1 \in D_F(\pi)$ we have $1 \in D_F(\varphi)$ and the roundness of $\tilde{\pi}$ implies that q can be chosen to represent 1 as well, i.e. we may assume $q \cong [1,a] \perp q'$ for suitable $a \in F$ and nonsingular q'. Now $\pi \perp \langle x \rangle \prec \varphi$ implies

$$\dim(\varphi \perp \pi \perp \langle x \rangle)_{\mathrm{an}} = \dim \varphi - 2^n - 1 \ .$$

We have $\tilde{\pi} \otimes [1,a] \perp \pi \sim \pi$ and thus

$$\varphi \perp \pi \perp \langle x \rangle \sim \tilde{\pi} \otimes q' \perp \pi \perp \langle x \rangle \ .$$

The form on the right hand side has dimension $\dim \varphi - 2^n + 1 > \dim \varphi - 2^n - 1$ and is therefore isotropic. But $\tilde{\pi} \otimes q' \perp \pi$ is anisotropic as it is dominated by φ. Hence, $x \in D_F^*(\tilde{\pi} \otimes q' \perp \pi)$ and there exist $u \in D_F^0(\tilde{\pi} \otimes q')$, $v \in D_F^0(\pi)$ with $x = u + v$.

If $u = 0$ then $x = v \in D_F^*(\pi)$ and $\pi \perp \langle x \rangle$ would be isotropic, a contradiction. Thus, $u \neq 0$ and we may assume, again by the roundness of Pfister forms, $q' \cong [u,w] \perp q''$ for some $w \in F$ and nonsingular q'' and we see that

$$\pi \perp u\pi \prec \tilde{\pi} \otimes ([1,a] \perp [u,w] \perp q'') \cong \varphi \ .$$

Now suppose $\pi \cong \langle\langle r_1, \ldots, r_n \rangle\rangle$, $r_i \in F$. Then $\pi \perp x\pi \cong \langle\langle r_1, \ldots, r_n, x \rangle\rangle$ and $\pi \perp u\pi \cong \langle\langle r_1, \ldots, r_n, u \rangle\rangle$. Since $v \in D_F^0(\pi) = F^2(r_1, \ldots, r_n)$ and with $u = x + v$, we get

$$D_F^0(\pi \perp u\pi) = F^2(r_1, \ldots, r_n, u) = F^2(r_1, \ldots, r_n, x) = D_F^0(\pi \perp ux\pi)$$

and thus $\pi \perp x\pi \cong \pi \perp u\pi$, hence $\pi \perp x\pi \prec \varphi$. □

3. Quartic field extensions in characteristic 2

In this section we recall well known and some perhaps lesser known facts about quartic extensions in characteristic 2 which we will need later on. We omit proofs as these results belong to basic field and Galois theory.

Let F be a field of characteristic 2 and let E/F be a field extension with $[E : F] = 4$. Such extensions can be classified as follows.

3.1. **Simple extensions of degree** 4. In this case, let $E = F(\alpha)$ and let $f(X) = X^4 + aX^3 + bX^2 + cX + d \in F[X]$ be the minimal polynomial of α over F. We distinguish 4 (sub)cases.

Case 1. E/F is separable. This is the case iff $a \neq 0$ or $c \neq 0$. We may assume (possibly after replacing α by α^{-1} and/or a linear change of variables) that the minimal polynomial of α is of shape $X^4 + aX^3 + cX + d \in F[X]$ with $a \neq 0$.

Case 2. E/F is inseparable but not purely inseparable. This is the case iff the minimal polynomial is of shape $f(X) = X^4 + bX^2 + d$ with $b \neq 0$. Note that the separable closure of F inside E is then the (uniquely determined) separable quadratic subextension $F(\alpha^2)$.

We now consider the extension $F(\sqrt{b}, \sqrt{d})/F$. Since f is irreducible, it is not possible that both $b, d \in F^2$. Hence $[F(\sqrt{b}, \sqrt{d}) : F] = 2$ or 4 and we distinguish the respective subcases.

2a. $[F(\sqrt{b}, \sqrt{d}) : F] = 2$. This is the case iff the inseparable closure of F in E is a (uniquely determined) inseparable quadratic extension $F(\sqrt{c})$ for a certain $c \in F \setminus F^2$. In this case, one therefore has that $E = F(\sqrt{c}, \alpha^2)$ is "mixed" biquadratic. In field theory, such an algebraic extension is often called balanced as it is the compositum of its maximal inseparable and maximal separable subextensions.

One can furthermore show that this subcase holds iff the equation $x^2 + by^2 + dz^2 = 0$ has a nontrivial solution $(x, y, z) \neq (0, 0, 0)$ with $x, y, z \in F$, i.e. iff the totally singular quadratic form $\langle 1, b, d\rangle$ is isotropic over F. We will refer to this case also as the mixed biquadratic case.

2b. $[F(\sqrt{b}, \sqrt{d}) : F] = 4$. This is the case iff the inseparable closure of F in E is just F itself, i.e. E/F does *not* contain an inseparable quadratic subextension. So in particular, this extension is unbalanced. This subcase holds iff $\langle 1, b, d\rangle$ is anisotropic over F.

Case 3. E/F is purely inseparable. This is the case iff the minimal polynomial is of shape $f(X) = X^4 + d$, so $E = F(\sqrt[4]{d})$ for some $d \in F \setminus F^2$.

3.2. **Nonsimple extensions of degree** 4. E/F is nonsimple iff E/F is biquadratic purely inseparable: $E = F(\sqrt{a}, \sqrt{b})$ for suitable $a, b \in F^*$.

3.3. **The cubic resolvent of a polynomial of degree** 4. Let $f(X) = X^4 + aX^3 + bX^2 + cX + d \in F[X]$ and let α_i, $1 \leq i \leq 4$ be the roots of f in an algebraic closure of F. The α_i need not be distinct. Let now

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4\ , \quad \beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4\ , \quad \beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3\ .$$

The cubic resolvent of f (in any characteristic) is then given by

$$f_C(X) = \prod_{i=1}^{3}(X - \beta_i) = X^3 - bX^2 + (ac - 4d)X - (a^2d + c^2 - 4bd) \in F[X]\ .$$

To avoid confusion, let us mention that in the literature, one often finds an alternative version of the cubic resolvent where the β_i are replaced by $\gamma_1 = (\alpha_1+\alpha_2)(\alpha_3+\alpha_4)$, $\gamma_2 = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)$, $\gamma_3 = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3)$. Note that $\beta_i + \gamma_i = b$, so one readily gets that the cubic resolvent $\widetilde{f}_C$ in this version is given by

$$\widetilde{f}_C(X) = \prod_{i=1}^{3}(X-\gamma_i) = X^3-2bX^2+(b^2+ac-4d)X+(c^2+a^2d-abc) = -f_C(-X+b)$$

For our purposes, we will need f_C.

Since we are in characteristic 2, the formula simplifies:

$$f_C(X) = X^3 + bX^2 + acX + (a^2d + c^2)\ .$$

Now for a simple degree 4 extension $E = F(\alpha)$ in characteristic 2 where α has minimal polynomial $f(X)$ as described in the above cases 1–3, we get the following corresponding cubic resolvents:

1. Minimal polynomial $f(X) = X^4 + aX^3 + cX + d$: $f_C(X) = X^3 + acX + (a^2d + c^2)$.
2. Minimal polynomial $f(X) = X^4 + bX^2 + d$: $f_C(X) = X^3 + bX^2$.
3. Minimal polynomial $f(X) = X^4 + d$: $f_C(X) = X^3$.

4. Results on Witt kernels for finite field extensions

One has $W(E/F) = 0$ resp. $W_q(E/F) = 0$ for odd degree extensions E/F since anisotropic forms stay anisotropic over odd degree extensions, a result often referred to as Springer's theorem [23] but that has apparently been proved earlier by E. Artin in a communication to E. Witt (1937).

Let us now assume that $\mathrm{char}(F) \neq 2$. It is well known that if $E = F(\sqrt{d})$ is a quadratic extension, then $W(E/F)$ is generated by the norm form $\langle 1, -d\rangle$ of that extension. The case $[E : F] = 4$ is considerably more difficult to treat. For biquadratic extensions $E = F(\sqrt{a}, \sqrt{b})$ it was shown by Elman-Lam-Wadsworth [11] that $W(E/F)$ is generated by $\langle 1, -a\rangle$ and $\langle 1, -b\rangle$, and the case of degree 4 extensions containing a quadratic subextension can be found in Lam-Leep-Tignol [19].

The complete determination of Witt kernels for arbitrary degree 4 extensions is due to Sivatski [22]. To formulate his result, first note that in characteristic not 2 the degree 4 extension E/F will be separable and hence simple, say, $E = F(\alpha)$. Let $f(X) \in K[X]$ be the minimal polynomial of α. We may assume (after a linear change of variables) that $f(X) = X^4 + bX^2 + cX + d$. Recall from subsection 3.3 that then $\widetilde{f}_C(X) = X^3 - 2bX^2 + (b^2 - 4d)X + c^2$

Theorem 4.1. (Sivatski [22, Cor. 2, Cor. 4].) *Let F be a field of characteristic not 2 and E/F a field extension of degree 4. Let $E = F(\alpha)$ be such that $f(X) = X^4 + bX^2 + cX + d \in K[X]$ is the minimal polynomial of α. Then $W(E/F)$ is generated by 1-fold Pfister forms $\langle\langle D\rangle\rangle$ for all $D \in F^* \setminus F^{*2}$ with $F(\sqrt{D}) \subseteq E$, and 2-fold Pfister forms $\langle\langle \widetilde{f}_C(r), -r\rangle\rangle$ with $r \in F^*$ such that $\widetilde{f}_C(r) \neq 0$.*

Little is known for Witt kernels of higher even degree extensions. For triquadratic extensions $E = F(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$ it is shown by Elman-Lam-Tignol-Wadsworth [10] that generally $\sum_{i=1}^{3} \langle 1, -a_i\rangle W(F) \subsetneq W(E/F)$, but an explicit description of generators of $W(E/F)$ seems not to be known. Under strong assumptions on the base field, more can be said. For example, for local or global fields, it is known that $W(F(\sqrt{a_1}, \dots, \sqrt{a_n})/F) = \sum_{i=1}^{n} \langle 1, -a_i\rangle W(F)$, see Elman-Lam-Wadsworth [12].

Let us now consider Witt kernels $W_q(E/F)$ for quadratic forms in characteristic 2. For quadratic extensions E/F, see Proposition 2.3. The following summarizes the known results for separable biquadratic extensions and for multiquadratic extensions of separability degree at most 2.

Theorem 4.2. *Let F be a field of characteristic 2, let $\alpha_1, \dots, \alpha_n, \beta_1, \beta_n$ be nonzero elements in an algebraic closure of F such that there exist $a_1, \dots, a_n$, $b_1, b_2 \in F^*$ with $\alpha_i^2 = a_i$ $(1 \leq i \leq n)$ and $\beta_i^2 + \beta_i = b_i$ $(i = 1, 2)$. Let $E = F(\alpha_1, \dots, \alpha_n)$, $E' = F(\alpha_1, \dots, \alpha_n, \beta_1)$, $E'' = F(\beta_1, \beta_2)$.*

(i) (Aravire-Laghribi [5]; Mammone-Moresi [20] for $n = 2$.)

$$W_q(E/F) = \sum_{i=1}^{n} \langle 1, a_i\rangle_b \otimes W_q(F) \ .$$

(ii) (Aravire-Laghribi [5]; Ahmad [2] for $n = 1$)

$$W_q(E'/F) = \sum_{i=1}^{n} \langle 1, a_i\rangle_b \otimes W_q(F) + W(F) \otimes [1, b_1] \ .$$

(iii) (Baeza [7, 4.16].)

$$W_q(E''/F) = W(F)[1, b_1] + W(F)[1, b_2] \ .$$

For a different proof of parts (i) and (ii), see [14].

For simple totally inseparable degree 4 extensions, i.e. Case 3 in section 3.1, Ahmad has shown the following.

Theorem 4.3. (Ahmad [3].) *Let F be a field of characteristic 2, let α be a nonzero element in an algebraic closure of F such that there exists $a \in F \setminus F^2$ with $\alpha^4 = a$. Let $E = F(\alpha)$. Then $W_q(E/F)$ is generated by 2-fold quadratic Pfister forms of type $\langle\langle a, x]]$ and $\langle\langle x, ax^2y^2]]$ with $x \in F^*$ and $y \in F^2(a)^*$.*

Again, not much else is known for other types of finite algebraic extensions in characteristic 2.

Finally, consider Witt kernels $W(E/F)$ for Witt rings of bilinear forms in characteristic 2. It is not difficult to show that if E/F is separable, then $W(E/F) = 0$ (Knebusch [18]). In [14], Witt kernels for a large class of purely inseparable algebraic extensions have been determined. In particular, the following was shown.

Theorem 4.4. (Hoffmann [13].) *Let E be a purely inseparable extension of exponent 1 over a field F of characteristic 2 (i.e. $E^2 \subset F \subset E$). Then $W(E/F)$ is generated by bilinear forms $\langle 1, t\rangle_b$ where $t \in E^{*2}$.*

Now the nonsimple degree 4 extensions E of a field F of characteristic 2 are exactly the biquadratic purely inseparable extensions $E = F(\sqrt{a_1}, \sqrt{a_2})$ and thus they are exactly the purely inseparable exponent 1 extensions of degree 4, so this case is covered by the previous theorem.

The case of simple degree 4 extensions follows from a much more general result shown in [8], where (in characteristic 2) $W(E/F)$ was determined for *arbitrary* function fields of hypersurfaces, i.e. for extensions E/F where E is the quotient field of $F[X]/(f(X))$ for an irreducible polynomial $f(X) = f(X_1, \ldots, X_n) \in F[X_1, \ldots, X_n]$. In the case of a simple extension of even degree, this result boils down to the following.

Theorem 4.5. *Let $E = F(\alpha)$ be a simple extension of even degree n of a field F of characteristic 2, and let $X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 \in F[X]$ be the minimal polynomial of α.*

(i) *If E/F is separable (i.e. $a_i \neq 0$ for some odd i), then $W(E/F) = 0$.*

(ii) *If E/F is inseparable (i.e. $a_i = 0$ for all odd i), let $K = F(\sqrt{a_0}, \ldots, \sqrt{a_{n-2}})$. Then $[K : F] = 2^s$ for some $1 \leq s \leq \frac{n}{2}$, and $W(E/F)$ is generated by the s-fold bilinear Pfister forms $\langle\langle b_1, \ldots, b_s\rangle\rangle_b$ with $b_i \in F^*$ such that $F(\sqrt{b_1}, \ldots, \sqrt{b_s}) = K$.*

More precisely, any anisotropic bilinear form φ over F with $\varphi \in W(E/F)$ can be written as $\varphi \cong \lambda_1\pi_1 \perp \ldots \perp \lambda_n\pi_n$ for suitable $n \in \mathbb{N}$, $\lambda_i \in F^*$ and where the π_i are s-fold bilinear Pfister forms of the above type.

Proof. Part (i) of this theorem follows from Knebusch's result [18] mentioned above. Part (ii) follows from [8, Cor. 11.4]. □

5. Witt kernels of simple quartic extensions

We start with a proposition that is essentially due to Sivatski [22, Prop. 1]. He assumed the characteristic to be different from 2 but the proof goes through basically without any changes also in characteristic 2 and also for singular quadratic forms. Our formulation is slightly different.

Proposition 5.1. *Let m be a positive integer. Let $f(X) \in F[X]$ be an irreducible polynomial of even degree $2m$, α be a root of F in some algebraic closure of F and $E = F(\alpha)$. Let φ be an anisotropic quadratic form over F and assume that over*

E, one has $i_t(\varphi_E) > \left(\frac{1}{2} - \frac{1}{2m}\right)\dim\varphi$. Then there exists a form $\varphi_0 \prec \varphi$ such that $2 \leq \dim\varphi_0 \leq m+1$ and φ_0 represents $\lambda f(X)$ for some $\lambda \in F^*$ over $F[X]$. In particular, $(\varphi_0)_E$ is isotropic.

Corollary 5.2. *Let E/F be a simple field extension with $[E : F] = 4$ and let φ be an anisotropic form in $W_q(E/F)$. Then there exists a form $\varphi_0 \prec \varphi$ with $2 \leq \dim\varphi_0 \leq 3$ and such that $(\varphi_0)_E$ is isotropic.*

¿From now on, let F be a field of characteristic 2 and $E = F(\alpha)$ be a simple degree 4 extension where α has minimal polynomial $f(X) = X^4+aX^3+bX^2+cX+d$ with cubic resolvent $f_C(X) = X^3 + bX^2 + acX + a^2d + c^2$.

Lemma 5.3. *Let $e \in F^*$ be such that $f_C(e) \neq 0$. Then $\langle\langle f_C(e), \frac{d}{e^2}]] \in W(E/F)$.*

Proof. Note that $\varphi := [1, \frac{d}{e^2}] \perp \langle f_C(e)\rangle$ is a Pfister neighbor of $\langle\langle f_C(e), \frac{d}{e^2}]]$. Thus, $\langle\langle f_C(e), \frac{d}{e^2}]]_E$ is hyperbolic iff φ_E is isotropic iff the polynomial equation

$$\varphi(X, Y, Z) = X^2 + XY + \frac{d}{e^2}Y^2 + (e^3 + be^2 + ace + a^2d + c^2)Z^2 = 0$$

has a nontrivial solution over E. Now put

$$(X, Y, Z) = (e\alpha^2 + c\alpha, e(a\alpha + e), \alpha) \neq (0, 0, 0)$$

By substituting we get

$$\begin{aligned}
\varphi(e\alpha^2 + c\alpha, e(a\alpha + e), \alpha) &= e^2\alpha^4 + c^2\alpha^2 + (e\alpha^2 + c\alpha)e(a\alpha + e) \\
&\quad + da^2\alpha^2 + de^2 + (e^3 + be^2 + ace + a^2d + c^2)\alpha^2 \\
&= e^2(\alpha^4 + a\alpha^3 + b\alpha^2 + c\alpha + d) \\
&= e^2 f(\alpha) = 0
\end{aligned}$$

as desired. □

By subsection 3.1, we may from now on assume the following regarding the minimal polynomial $f(X) = X^4 + aX^3 + bX^2 + cX + d$.

- $b = 0$ and $a \neq 0$ if E/F is separable (case 1 in subsection 3.1);
- $a = c = 0$, $b \neq 0$ if E/F inseparable but not purely inseparable (case 2 in subsection 3.1);
- $a = b = c = 0$ if E/F is purely inseparable (case 3 in subsection 3.1).

Theorem 5.4. *$W_q(E/F)$ is generated as WF-module by*

(a) *$[1, g]$ for those $g \in F$ such that $F(\wp^{-1}(g)) \subset E$;*
(b) *$\langle\langle g, h]]$ for $h \in F$ and those $g \in F^*$ with $F(\sqrt{g}) \subset E$;*
(c) *$\langle\langle f_C(e), \frac{d}{e^2}]]$ for $e \in F^*$ with $f_C(e) \neq 0$;*
(d) *(only in case 2) $\langle\langle b, d, h]]$ with $h \in F$.*

Remark 5.5. (i) Note that $[1, g] \neq 0 \in W_q(F)$ iff $g \notin \wp(F)$ iff $[F(\wp^{-1}(g)) : F] = 2$, in which case $F(\wp^{-1}(g))/F$ is a separable quadratic extension. Thus, such nonhyperbolic binary forms cannot show up in the list of generators in $W_q(E/F)$ in case 3 (purely inseparable extensions).

(ii) Simlarly, if $\langle\langle g, h]]$ is such that $F(\sqrt{g}) \subset E$, then in case 1 (separable extensions) this would imply $g \in F^{*2}$ and thus $\langle\langle g, h]] = 0 \in W_q(F)$, so nonhyperbolic forms of this type cannot show up in the list of generators in $W_q(E/F)$ in case 1.

(iii) For forms of type $\langle\langle b, d, h]]$ with $h \in F$ in case 2, note that in the mixed biquadratic case 2a we have $[F(\sqrt{b}, \sqrt{d}) : F] = 2$ which is equivalent to $\langle 1, b, d\rangle_b$

being isotropic by subsection 3.1, hence $\langle\langle b,d,h]]$ is isotropic and hence $\langle\langle b,d,h]] = 0 \in W_q(F)$ in that case. So nonhyperbolic forms of this type cannot show up in the list of generators in $W_q(E/F)$ in case 2a.

Proof of Theorem 5.4. We first show that the above generators are indeed in $W_q(E/F)$.

Generators of type (a): If $F(\wp^{-1}(g)) \subset E$ for $g \in F$, then $g \in \wp(E)$, hence $\Delta([1,g]) = g = 0 \in E/\wp(E)$ and thus $[1,g]_E = 0 \in W_q(E)$.

Generators of type (b): If $F(\sqrt{g}) \subset E$ for $g \in F^*$ then $g \in E^{*2}$, thus $(\langle 1,g\rangle_b)_E$ is isotropic and hence metabolic, so $(\langle\langle g\rangle\rangle_b)_E = 0 \in WE$ and thus $\langle\langle g,h]]_E = 0 \in W_q(E)$.

Generators of type (c): This is Lemma 5.3

Generators of type (d): Here, by the previous remark, we may assume that we are in case 2b with minimal polynomial $f(X) = X^4 + bX^2 + d$ with $[F(\sqrt{b},\sqrt{d}) : F] = 4$. By Theorem 4.5 we have $\langle\langle b,d\rangle\rangle_b \in W(E/F)$ and thus $\langle\langle b,d,h]] \in W_q(E/F)$.

We now prove that any form in $W_q(E/F)$ can be written as a sum of scalar multiples of these generators. Let q be a form with $q \in W_q(E/F)$. We use induction on $\dim q = n$. We may assume without loss of generality that q is anisotropic

If $n = 2$, then we may assume after scaling that $q \cong [1,g]$ for some $g \in F \setminus \wp(F)$. Since $q_E = 0 \in W_q(E)$, we have $g \in \wp(E)$ and it follows that $F(\wp^{-1}(g)) \subset E$ and therefore q is of type (a).

Now assume $n \geq 4$. By Corollary 5.2, there exists a form $q_0 \prec q$ with $\dim q_0 \in \{2,3\}$ and $(q_0)_E$ isotropic. If $\dim q_0 = 2$, we may have either (after scaling) $q_0 \cong [1,g]$ for some $g \in F \setminus \wp(F)$, or $q_0 \cong \langle 1,g\rangle$ for some $g \in F^* \setminus F^{*2}$.

If $q_0 \cong [1,g]$, then q_0 is a generator of type (a) (as in case $n = 2$), and by Lemma 2.1 there exists a nonsingular form q' over F with $q \cong q_0 \perp q'$. But $q, q' \in W_q(E)$, hence also $q' \in W_q(E/F)$, and since $\dim q' = \dim q - 2$, we are done by induction.

If $q_0 \cong \langle 1,g\rangle$, then $(q_0)_E$ being isotropic is equivalent to $g \in E^{*2}$, hence $F(\sqrt{g}) \subset E$. Furthermore, by Lemma 2.1, there exist $u,v \in F$ and a nonsingular form q' over F with $q \cong [1,u] \perp g[1,v] \perp q'$. Consider $\pi := \langle\langle g,v]]$. This is a generator of type (b), and in $W_q(F)$ we have

$$\pi + q = [1,v] \perp g[1,v] \perp [1,u] \perp g[1,v] \perp q' = [1,u+v] \perp q' \ .$$

Now $[1,u+v] \perp q' \in W_q(E/F)$ since $q,\pi \in W_q(E/F)$, and $\dim([1,u+v] \perp q') = \dim(q) - 2$. Again, we are done by induction.

¿From now on we may therefore in addition assume that there is no 2-dimensional form $\varphi \prec q$ with φ_E isotropic. In particular, by Lemma 2.2, q will not become isotropic over any quadratic intermediate extension $F \subset K \subset E$.

So let $q_0 \prec q$ with $(q_0)_E$ isotropic and $\dim q_0 = 3$. By Proposition 5.1, q_0 represents a scalar multiple of $f(X)$ over $F[X]$. After scaling q, we may assume that q_0 represents $f(X)$. Let U be the underlying F-vector space of q_0, and let B be the bilinear form associated with q_0. The anisotropy of q_0 and a simple degree argument (using $\deg(f) = 4$) show that there exist $u,v,w \in U$ such that $q_0(uX^2 + vX + w) = f(X)$, and therefore

$$\begin{aligned} &q_0(u)X^4 + B(u,v)X^3 + (q_0(v) + B(u,w))X^2 + B(v,w)X + q_0(w) \\ &= X^4 + aX^3 + bX^2 + cX + d \ . \end{aligned}$$

Comparing coefficients yields

$$q_0(u) = 1,\ B(u,v) = a,\ q_0(v) + B(u,w) = b,\ B(v,w) = c,\ q_0(w) = d\ .$$

Note also that $q_0(u\alpha^2 + v\alpha + w) = f(\alpha) = 0$, thus, if $U' = \mathrm{span}(u,v,w)$, we see that the form $q_0|_{U'}$ becomes isotropic over E. Since we assumed that no 2-dimensional form over F dominated by q becomes isotropic over E, we thus necessarily have that $U = U'$ and u,v,w are linearly independent. In particular, $e := q_0(v) \neq 0$ because q_0 is anisotropic. Writing the form q_0 as a homogeneous degree 2 polynomial in three variables, we get

$$q_0(X,Y,Z) = X^2 + aXY + eY^2 + (e+b)XZ + cYZ + dZ^2\ .$$

We now make a further case distinction according to the types of field extension. We first treat the cases 1 and 3 before treating the more difficult cases 2a and 2b.

Cases 1 and 3 (separable and purely inseparable extension): Here, we have $b = 0$ (in the separable case we assumed this without loss of generality), and we get

$$q_0(X,Y,Z) = X^2 + aXY + eY^2 + eXZ + cYZ + dZ^2\ .$$

We perform an invertible linear change of variables and obtain

$$q_0(X + cY, eY, \frac{1}{e}Z + aY) = X^2 + (e^3 + ace + da^2 + c^2)Y^2 + XZ + \frac{d}{e^2}Z^2$$

and therefore

$$q_0 \cong [1, \tfrac{d}{e^2}] \perp \langle f_C(e)\rangle \prec \langle\langle f_C(e), \tfrac{d}{e^2}]]\ .$$

But then $\pi = \langle\langle f_C(e), \frac{d}{e^2}]]$ is a generator of type (c). Since $q_0 \prec q$, again by Lemma 2.1, we can write $q \cong [1, \frac{d}{e^2}] \perp f_C(e)[1,t] \perp q'$ for some $t \in F$ and a nonsingular form q'. But then, in $W_q(F)$,

$$\begin{aligned} \pi + q &= [1, \tfrac{d}{e^2}] \perp f_C(e)[1, \tfrac{d}{e^2}] \perp [1, \tfrac{d}{e^2}] \perp f_C(e)[1,t] \perp q' \\ &= \underbrace{f_C(e)[1, t + \tfrac{d}{e^2}] \perp q'}_{q''}\ . \end{aligned}$$

Since $\pi, q \in W_q(E/F)$ we have $q'' \in W_q(E/F)$, and also $\dim(q'') = \dim q - 2$ and we are done by induction.

Case 2 (inseparable but not purely inseparable extension): Here, we have $a = c = 0$ and $b \neq 0$ and we get

$$q_0(X,Y,Z) = X^2 + eY^2 + (e+b)XZ + dZ^2\ .$$

First, assume $e' = e + b \neq 0$. Then

$$q_0(X, e'Y, \frac{1}{e'}Z) = X^2 + (e'^3 + be'^2)Y^2 + XZ + \frac{d}{e'^2}Z^2$$

and thus $q_0 \cong [1, \frac{d}{e'^2}] \perp \langle f_C(e')\rangle$ and we can conclude as before.

Now suppose $e + b = 0$, i.e. $e = b$. Then $q_0 \cong \langle 1, b, d\rangle$ is totally singular. The anisotropy of q_0 then implies that we must be in case 2b. After scaling q by b, we may assume $bq_0 \cong \langle 1, b, bd\rangle \prec q$.

Recall that $E = F(\alpha)$ with $\alpha^4 + b\alpha^2 + d = 0$. Put $\beta = \alpha^2$. Then $L = F(\beta)$ is the unique quadratic intermediate extension $F \subset L \subset E$, and we have $L = F(\wp^{-1}(\frac{d}{b^2}))$ and $E = L(\sqrt{\beta})$.

Recall also, that by an earlier assumption, q will be anisotropic over L but hyperbolic over $E = L(\sqrt{\beta})$. By Proposition 2.3, there exists a nonsingular form ψ over L such that $q_L \cong \langle\langle\beta\rangle\rangle_b \otimes \psi$. Note also that $d = \beta b + \beta^2$ and thus

$$\langle 1, b, bd\rangle_L \cong \langle 1, b, b(\beta b + \beta^2)\rangle \cong \langle 1, \beta, b\rangle \prec q_L$$

where the last isometry holds since $\{1, b, b(\beta b + \beta^2)\}$ and $\{1, b, \beta\}$ generate the same L^2-vector space inside L. By Lemma 2.4, we then have $\langle\langle\beta, b\rangle\rangle \prec q_L$. Also, $L^2(\beta, b) = L^2(\beta b + \beta^2, b) = L^2(d, b)$ and therefore $\langle\langle\beta, b\rangle\rangle \cong \langle\langle b, d\rangle\rangle_L$ and $\langle\langle b, d\rangle\rangle_L \prec q_L$.

Suppose first that $\langle\langle b, d\rangle\rangle \prec q$. Then there exist $x, y, z, t \in F$ and a nonsingular form q' over F such that

$$q \cong [1, x] \perp b[1, y] \perp d[1, z] \perp bd[1, t] \perp q'$$

and thus, in WF,

$$q + \langle\langle b, d, x]] = \underbrace{b[1, x + y] \perp d[1, x + z] \perp bd[1, x + t] \perp q'}_{q''}$$

Since $\langle\langle b, d, x]]$ is a generator of type (d), we have $\langle\langle b, d, x]], q \in W_q(E/F)$ and thus also $q'' \in W_q(E/F)$. But $\dim(q'') = \dim(q) - 2$ and we are done by induction.

Now suppose that that $\langle\langle b, d\rangle\rangle \not\prec q$. Together with $\langle 1, b, bd\rangle \prec q$, $\langle\langle b, d\rangle\rangle_L \prec q_L$ and q_L anisotropic, we get that $i_W(q \perp \langle\langle b, d\rangle\rangle) = 3$ and $i_W(q_L \perp \langle\langle b, d\rangle\rangle_L) = 4$. Hence, there exists a nonsingular form q' over F with

- $q \perp \langle\langle b, d\rangle\rangle \cong 3\mathbb{H} \perp q' \perp \langle\langle b, d\rangle\rangle$,
- $q' \perp \langle\langle b, d\rangle\rangle$ anisotropic,
- $(q' \perp \langle\langle b, d\rangle\rangle)_L$ isotropic, and

By Lemma 2.2 applied to $L = F(\wp^{-1}(\frac{d}{b^2}))$, there exists $\lambda \in F^*$ such that $\lambda[1, \frac{d}{b^2}] \prec q' \perp \langle\langle b, d\rangle\rangle$, and hence there exists a nonsingular form q'' over F of dimension $\dim q' - 2 = \dim q - 8$ with $q' \perp \langle\langle b, d\rangle\rangle \cong q'' \perp \lambda[1, \frac{d}{b^2}] \perp \langle\langle b, d\rangle\rangle$. We get by comparing dimensions

$$q \perp \lambda[1, \tfrac{d}{b^2}] \perp \langle\langle b, d\rangle\rangle \cong 5\mathbb{H} \perp q'' \perp \langle\langle b, d\rangle\rangle\ ,$$

so $q \perp \lambda[1, \frac{d}{b^2}]$ cannot be anisotropic because then $i_W(q \perp \lambda[1, \frac{d}{b^2}] \perp \langle\langle b, d\rangle\rangle) \leq \dim \langle\langle b, d\rangle\rangle = 4$, a contradiction. On the other hand $\lambda[1, \frac{d}{b^2}] \not\prec q$ because otherwise q_L would be isotropic, again a contradiction to an earlier assumption. As a consequence, $i_W(q \perp \lambda[1, \frac{d}{b^2}]) = 1$.

Put $\widehat{q} \cong (q \perp \lambda[1, \frac{d}{b^2}])_{\text{an}}$. We then have $\dim \widehat{q} = \dim q$. Since $q \in W_q(E/F)$ and since $[1, \frac{d}{b^2}]$ is a generator of type (a), it follows that $\widehat{q} \in W_q(E/F)$. Furthermore, the above shows that $i_W(\widehat{q} \perp \langle\langle b, d\rangle\rangle) = 4$ and thus $\langle\langle b, d\rangle\rangle \prec \widehat{q}$. By the same reasoning as before, we are done by induction if we replace q by $\widehat{q}$. But since q and $\widehat{q}$ only differ by a scalar multiple of a generator of type (a), we are done. □

The next corollary will be crucial for determining Brauer kernels in the next section.

Corollary 5.6. *Let φ be an anisotropic nonsingular quadratic form over F with $\varphi \in W_q(E/F)$. Suppose $\dim \varphi = 4$ or $\dim \varphi > 4$ and $\dim \varphi = 2 \bmod 4$. Then there exists a 2-fold quadratic Pfister form $\pi \in W_q(E/F)$ and a 3-dimensional Pfister neighbor ψ of π with $\psi \prec \varphi$. In particular, the 2-fold Pfister forms in $W_q(E/F)$ are exactly the ones of the following types:*

(a') $\langle\langle h, g]]$ *for* $h \in F^*$ *and those* $g \in F$ *such that* $F(\wp^{-1}(g)) \subset E$;
(b) $\langle\langle g, h]]$ *for* $h \in F$ *and those* $g \in F^*$ *with* $F(\sqrt{g}) \subset E$;
(c) $\langle\langle f_C(e), \frac{d}{e^2}]]$ *for* $e \in F^*$ *with* $f_C(e) \neq 0$.

Proof. If φ becomes isotropic already over a quadratic subextension in E, then by Lemma 2.2 we may assume that after scaling $\langle 1, g\rangle \prec \varphi$ with $F(\sqrt{g}) \subset E$, or $[1, g] \prec \varphi$ with $F(\wp^{-1}(g)) \subset E$. It is then clear that one can find an $h \in F$ such that in the first case $\psi \cong [1, h] \perp \langle g\rangle \prec \varphi$, and then ψ is a Pfister neighbor of $\langle\langle g, h]]$, and such that in the second case $\psi \cong [1, g] \perp \langle h\rangle \prec \varphi$, and then ψ is a Pfister neighbor of $\langle\langle h, g]]$.

So we may assume that φ stays anisotropic over any quadratic subextension in E. The proof of Theorem 5.4 then shows that there exists a 3-dimensional form $\psi \prec \varphi$ such that either ψ is a Pfister neighbor of $\langle\langle f_C(e), \frac{d}{e^2}]]$ for some $e \in F^*$ with $f_C(e) \neq 0$, or ψ is totally singular. But the proof also showed that if ψ is totally singular then this can only hold in the case 2b. In that situation and with $E = F(\alpha)$ as in the proof and $L = F(\beta)$ with $\beta = \alpha^2$, it was shown that $\varphi_L \cong \langle\langle \beta\rangle\rangle \otimes \tau$ for some nonsingular form τ over L. In particular, $\dim \varphi \equiv 0 \bmod 4$. By assumption, this requires $\dim \varphi = 4$, but this is impossible as φ is nonsingular and can therefore not dominate a 3-dimensional totally singular form.

Now if $\varphi \in W_q(E/F)$ is a 2-fold quadratic Pfister form, then by the above there exists a 2-fold quadratic Pfister form π of type (a'), (b) or (c) and a 3-dimensional Pfister neighbor ψ of π with $\psi \prec \varphi$. After scaling φ (and thus also ψ), we may assume $\psi \prec \pi$, hence $\dim(\varphi \perp \pi)_{\mathrm{an}} \leq 2$. But $\Delta(\varphi \perp \pi) = 0 \in F/\wp(F)$, thus $\dim(\varphi \perp \pi)_{\mathrm{an}} = 0$, therefore $\varphi \cong \pi$. □

A 6-dimensional nonsingular quadratic form with trivial Arf invariant is called an Albert form. We will need them in the determination of the 2-torsion part of the Brauer kernel $\mathrm{Br}_2(E/F)$ in section 6

Corollary 5.7. *Let* $\varphi \in W_q(E/F)$ *be anisotropic with* $\dim \varphi = 6$ *and* $\Delta(\varphi) = 0 \in F/\wp(F)$. *Then there exist* 2*-fold quadratic Pfister forms* π_1, π_2 *as in Corollary 5.6 (a'), (b), (c) and* $\lambda \in F^*$ *such that* $\varphi = \lambda(\pi_1 + \pi_2) \in W_qF$.

Proof. After scaling and by Corollary 5.6, there exists such a 2-fold quadratic Pfister form π_1 and a 3-dimensional $\psi \prec \pi_1$ with $\psi \prec \varphi$. Hence, there exists a nonsingular ρ with $\dim \rho = 4$ and $\varphi \perp \pi_1 = \rho \in W_qF$. But then $\Delta(\rho) = 0 \in F/\wp(F)$, so there exists $\lambda \in F$ such that $\rho \cong \lambda\pi_2$ for a 2-fold quadratic Pfister form π_2. Since $0 = (\varphi \perp \pi_1)_E = (\lambda\pi_2)_E \in W_qE$, it follows that π_2 is as in Corollary 5.6 (a'), (b) or (c). Also $\varphi = \pi_1 \perp \lambda\pi_2 \in W_qF$. Comparing dimensions yields that $\pi_1 \perp \lambda\pi_2$ is isotropic, so $D_F(\pi_1)^* \cap D_F(\lambda\pi_2)^* \neq \emptyset$. The roundness of π_i then shows that one may assume without loss of generality that λ is represented by π_1 and thus $\varphi = \lambda(\pi_1 \perp \pi_2) \in W_qF$. □

We now would like to align the result in Theorem 5.4 with the descriptions of generators in Theorem 4.2 for separable and mixed biquadratic extensions (cases 1 and 2a) and in Theorem 4.3 for simple purely inseparable degree 4 extensions (case 3). To do so, we use the following relations in the Witt group $W_q(F)$ that are obvious or can be readily verified:

- $[r, s] + [u, v] = [r + u, s] + [u, s + v]$. In particular $[1, u] + [1, v] = [1, u + v]$ and thus $\langle\langle w, u]] + \langle\langle w, v]] = \langle\langle w, u + v]]$;
- $[1, u + v^2] = [1, u + v]$, in particular $[1, v + v^2] = 0$;

- $\langle\langle u,u]] = 0$ and thus also $\langle\langle u,u+v]] = \langle\langle u,v]]$ and $\langle\langle u+v,v]] = \langle\langle u+v,u]]$;
- $\langle\langle uv,w]] = \langle\langle u,w]] + u\langle\langle v,w]]$ and thus $\langle\langle uv,v]] = \langle\langle u,v]]$;
- $\langle\langle uv^2,w]] = \langle\langle u,w]]$, in particular $\langle\langle v^2,w]] = 0$.

5.1. **Generators in the separable biquadratic case.** Let us assume that the degree 4 extension E/F is separable biquadratic, so $E = F(\mu,\nu)$ with $\mu^2+\mu+u = 0$ and $\nu^2+\nu+v = 0$ for some $u,v \in F^*$. By Theorem 4.2(iii), $W_q(E/F)$ is generated (as WF-module) by $[1,u]$ and $[1,v]$. In our list of generators, there are forms $[1,w]$ with $F(\wp^{-1}(w)) \subset E$. But the only proper quadratic subextensions are $F(\mu) = F(\wp^{-1}(u))$, $F(\nu) = F(\wp^{-1}(v))$, and $F(\mu+\nu) = F(\wp^{-1}(u+v))$, thus in the list of generators in Theorem 5.4(a), the only anisotropic forms up to isometry are $[1,u]$, $[1,v]$, $[1,u+v] = [1,u]+[1,v]$.

By Remark 5.5(ii), there are no nonhyperbolic generators of type (b). So the last type to consider are generators $\langle\langle f_C(e), \frac{d}{e^2}]]$ of type (c). To do so, let us first write E as a simple extension. Put $\gamma := \mu\nu + u + v$. One checks that $E = F(\gamma)$ and that the minimal polynomial of γ is

$$f(X) = X^4 + X^3 + (u^2+v^2+uv)X + u^2v + uv^2 + u^2v^2 + u^4 + v^4$$

with cubic resolvent

$$f_C(X) = X^3 + (u^2+v^2+uv)X + u^2v + uv^2 = (X+u)(X+v)(X+u+v)$$

The X^0 term of $f(X)$ is $d = u^2v + uv^2 + u^2v^2 + u^4 + v^4$ and we get in $W_q(F)$:

$$\begin{aligned} \left[1, \tfrac{d}{e^2}\right] &= \left[1, \tfrac{u^2v+v^2u}{e^2} + \left(\tfrac{uv+u^2+v^2}{e}\right)^2\right] \\ &= \left[1, \tfrac{u^2v+v^2u}{e^2} + \tfrac{uv+u^2+v^2}{e}\right] \\ &= \left[1, \tfrac{f_C(e)}{e^2} + e\right] \end{aligned}$$

and hence, with $r = e+u$, $s = e+v$, $t = e+u+v$ and the above relations:

$$\begin{aligned} \langle\langle f_C(e), \tfrac{d}{e^2}]] &= \langle\langle \tfrac{f_C(e)}{e^2}, \tfrac{f_C(e)}{e^2} + e]] = \langle\langle \tfrac{f_C(e)}{e^2}, e]] \\ &= \langle\langle f_C(e), e]] = \langle\langle (e+u)(e+v)(e+u+v), e]] \\ &= \langle\langle e+u, e]] + (e+u)\langle\langle e+v, e]] + (e+u)(e+v)\langle\langle e+u+v, e]] \\ &= \langle\langle e+u, u]] + r\langle\langle e+v, v]] + rs\langle\langle e+u+v, u+v]] \\ &= \langle 1, r, rs, rst\rangle_b \otimes [1,u] + \langle r, rst\rangle_b \otimes [1,v] \end{aligned}$$

This shows that indeed $W_q(E/F)$ is already generated as WF-module by $[1,u]$ and $[1,v]$, providing a new proof for Theorem 4.2(iii).

5.2. **Generators in the mixed biquadratic case.** Let us assume that the degree 4 extension E/F is mixed biquadratic, so $E = F(\mu,\nu)$ with $\mu^2 = u$ and $\nu^2+\nu+v = 0$ for some $u,v \in F^*$. In this case, there are exactly two proper quadratic subextensions, the separable quadratic extension $F(\wp^{-1}(v))$ and the inseparable quadratic extension $F(\sqrt{u})$. The only nonhyperbolic forms of type (a) or type (b) in Theorem 5.4 are thus, up to isometry, $[1,v] \in WF \otimes [1,v]$ and $\langle\langle u,w]] \in \langle 1,u\rangle_b \otimes W_q(F)$ for suitable $w \in F^*$. So the last type to consider are generators $\langle\langle f_C(e), \frac{d}{e^2}]]$ of type (c). We proceed as before and write E as a simple extension. Put $\gamma = \mu\nu$. Then $E = F(\gamma)$ and the minimal polynomial of γ is $f(X) = X^4 + uX^2 + u^2v^2$ with cubic resolvent $f_C(X) = X^3 + uX^2$. In $W_q(F)$, we then get with $d = u^2v^2$ and $r = \frac{uv}{e}$:

$$\begin{aligned}\langle\langle f_C(e), \tfrac{d}{e^2}]] &= \langle\langle e^3 + e^2u, (\tfrac{uv}{e})^2]] = \langle\langle e+u, \tfrac{uv}{e}]]\\ &= \langle\langle \tfrac{uv}{e}(e+u), \tfrac{uv}{e}]] = \langle\langle u(v+\tfrac{uv}{e}), \tfrac{uv}{e}]]\\ &= \langle\langle u, \tfrac{uv}{e}]] + u\langle\langle v+\tfrac{uv}{e}, \tfrac{uv}{e}]] = \langle\langle u, r]] + u\langle\langle v+r, v]]\\ &= \langle 1, u\rangle_b \otimes [1, r] + \langle u, u(v+r)\rangle_b \otimes [1, v]\end{aligned}$$

and we recover Theorem 4.2(ii) in the case $n = 1$ there, a result that is originally due to Ahmad [2, Theorem 2.1].

5.3. **Generators in the simple purely inseparable case.** Here, $E = F(\alpha)$ with $\alpha = \sqrt[4]{d}$ a root of $X^4 + d$ where $d \in F \setminus F^2$. By Ahmad's result Theorem 4.3, $W_q(E/F)$ is generated by 2-fold quadratic Pfister forms of type $\langle\langle d, x]]$ and $\langle\langle x, dx^2y^2]]$ with $x \in F^*$ and $y \in F^2(d)^*$. Forms of type $\langle\langle d, x]]$ are generators of type (b) in our list. We now show how to express $\pi = \langle\langle x, dx^2y^2]]$ in terms of generators of type (c) in our list.

Let us write $y \in F^2(d)^*$ as $y = u^2 + dv^2$ for some $u, v \in F$ with $u \neq 0$ or $v \neq 0$. Hence, in $W_q(F)$:

$$\langle\langle x, dx^2y^2]] = \langle\langle x, dx^2(u^4 + d^2v^4)]] = \langle\langle x, dx^2u^4]] + \langle\langle x, d^3x^2v^4]] \ .$$

If $u \neq 0$ put $s = \frac{1}{xu^2}$ and if $v \neq 0$ put $t = \frac{1}{dxv^2}$. If $u = 0$ (and thus $v \neq 0$), we get

$$\begin{aligned}\pi &= \langle\langle x, dx^2u^4]] = \langle\langle x(\tfrac{1}{xu})^2, \tfrac{d}{s^2}]]\\ &= \langle\langle s, \tfrac{d}{s^2}]] = \langle\langle s^3, \tfrac{d}{s^2}]]\\ &= \langle\langle f_C(s), \tfrac{d}{s^2}]] \ .\end{aligned}$$

If $v = 0$ (and thus $u \neq 0$), we get

$$\begin{aligned}\pi &= \langle\langle x, d^3x^2v^4]] = \langle\langle x(d^3x^2v^4), \tfrac{d}{t^2}]]\\ &= \langle\langle x^3d^3v^4(vt^3)^2, \tfrac{d}{t^2}]] = \langle\langle t, \tfrac{d}{t^2}]]\\ &= \langle\langle f_C(t), \tfrac{d}{t^2}]] \ .\end{aligned}$$

Finally, if both $u, v \neq 0$, we have $\pi = \langle\langle f_C(s), \frac{d}{s^2}]] + \langle\langle f_C(t), \frac{d}{t^2}]]$.

6. An application to Brauer kernels

Let F be a field of characteristic 2. Recall that the central simple F-algebras of degree 2 are exactly the quaternion algebras $(a, b]$, $a, b \in F$, $b \neq 0$ generated by elements e, f satisfying the relations $e^2 = a$, $f^2 + f = b$, $ef = (f+1)e$. Such an algebra $(a, b]$ is a division algebra iff its norm form $\langle\langle a, b]]$ is anisotropic, and $(a, b] \cong (a', b']$ iff $\langle\langle a, b]] \cong \langle\langle a', b']]$ (see, e.g., [7, Prop. I.1.19]).

An Albert form q is a 6-dimensional nonsingular quadratic form with $\Delta(q) = 0 \in F/\wp(F)$. In particular, there exist $\lambda, x, y \in F^*$, $u, v \in F$ such that $\lambda q \cong [1, u+v] \perp x[1, u] \perp y[1, v]$. In W_qF, we have $\lambda q = \langle\langle x, u]] + \langle\langle y, v]]$, and the Clifford algebra $C(q)$ of q is Brauer equivalent to the biquaternion algebra $A = (x, u] \otimes (y, v]$ and it only depends on the similarity class of q. Conversely, given such a biquaternion algebra A, any nonsingular form q of dimension 6 with trivial Arf-invariant that satisfies $C(q) = A \in \mathrm{Br}(F)$ will be called an Albert form for A. Note that the index $\mathrm{ind}(A)$ will be 1, 2 or 4. The following well known theorem is due to Jacobson [17] (see also [21]).

Theorem 6.1. *Let q and q' be Albert forms for the biquaternion algebras A and A', respectively.*

(i) *q is similar to q' iff $A \cong A'$.*
(ii) *q is anisotropic iff A is a division algebra, i.e.* $\mathrm{ind}(A) = 4$.
(iii) *$i_W(q) = 1$ iff $A = Q \in \mathrm{Br}(F)$ for a quaternion division algebra Q, i.e.* $\mathrm{ind}(A) = 2$.
(iv) *$i_W(q) = 3$ iff A is split, i.e.* $\mathrm{ind}(A) = 1$.

Theorem 6.2. *Let E/F be a quartic extension.*

(i) *Let Q be a quaternion algebra over F Then $Q \in \mathrm{Br}_2(E/F)$ iff Q is of one of the following types:*

(a) *$(h, g]$ for $h \in F^*$ and $g \in F$ such that $F(\wp^{-1}(g)) \subset E$;*
(b) *$(g, h]$ for $h \in F$ and $g \in F^*$ such that $F(\sqrt{g}) \subset E$;*
(c) *$(f_C(e), \frac{d}{e^2}]$ for $e \in F^*$ with $f_C(e) \neq 0$.*

(ii) *If D is a nontrivial division algebra with $D \in \mathrm{Br}_2(E/F)$, then either $D \cong Q$ or $D \cong Q_1 \otimes Q_2$ where Q, Q_1, Q_2 are quaternion algebras of type (a), (b), or (c). In particular, $\mathrm{Br}_2(E/F)$ is generated by such quaternion algebras.*

Proof. (i) Let $Q = (x, y]$. By the above remarks, $Q \in \mathrm{Br}(E/F)$ iff $\langle\langle x, y]] \in W_q(E/F)$. The result then follows readily from Corollary 5.6.

(ii) Let D be a nontrivial division algebra with $D \in \mathrm{Br}_2(E/F)$. Since $[E : F] = 4$, one necessarily has $\mathrm{ind}(D) = 2$ or $\mathrm{ind}(D) = 4$. If $\mathrm{ind}(D) = 2$, then D is a quaternion algebra and the result follows from (i). If $\mathrm{ind}(D) = 4$, then D is a biquaternion algebra by Albert's theorem [4, p. 174]. Let φ be an Albert form associated with D. By Theorem 6.1, φ is anisotropic and $\varphi \in W_q(E/F)$. By Corollary 5.7, there exist quaternion algebras Q_1, Q_2 as in (i) with norm forms π_1 and π_2, respectively, and $\lambda \in F^*$ such that $\varphi = \lambda(\pi_1 + \pi_2) \in W_q F$. But then φ is also an Albert form for $Q_1 \otimes Q_2$, and again by Theorem 6.1, we have $D \cong Q_1 \otimes Q_2$. □

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