UNITARY GRASSMANNIANS OF DIVISION ALGEBRAS

NIKITA A. KARPENKO

Abstract. We consider a central division algebra over a separable quadratic extension of a base field endowed with a unitary involution and prove 2-incompressibility of certain varieties of isotropic right ideals of the algebra. The remaining related projective homogeneous varieties are shown to be 2-compressible in general. Together with [13], where a similar issue for orthogonal and symplectic involutions has been treated, the present paper completes the study of grassmannians of isotropic ideals of division algebras.

Let F be a field, K/F a separable quadratic field extension, n an integer ≥ 1, and D a central division K-algebra of degree 2^n endowed with a K/F-unitary involution σ. For definitions as well as for basic facts about involutions on central simple algebras, we refer to [16].

For any integer i, we write X_i for the F-variety of *isotropic* (with respect to σ) right ideals in D of reduced dimension i. (The reduced dimension of an ideal in D is its dimension over K divided by $\deg D := \sqrt{\dim_K D}$.) For any i, the variety X_i is smooth and projective. It is nonempty if and only if $0 \leq i \leq 2^{n-1}$ (X_0 is simply $\operatorname{Spec} F$) in which case it is geometrically connected and has dimension

$$\dim X_i = i(2 \deg D - 3i).$$

For any i, the variety X_i is a closed subvariety of the Weil transfer $\mathcal{R}_{K/F} \mathrm{SB}_i D$, where $\mathrm{SB}_i D$ is the ith generalized Severi-Brauer variety of D – the K-variety of all right ideals in D of reduced dimension i. We recall that according to [10] (see [15] for a more recent and simple proof), for any $r = 0, 1, \ldots, n-1$, the variety $\mathcal{R}_{K/F} \mathrm{SB}_{2^r} D$ is 2-*incompressible*. This means, roughly speaking, that any self-correspondence

$$\mathcal{R}_{K/F} \mathrm{SB}_{2^r} D \rightsquigarrow \mathcal{R}_{K/F} \mathrm{SB}_{2^r} D$$

of odd multiplicity is dominant. In particular, any rational self-map

$$\mathcal{R}_{K/F} \mathrm{SB}_{2^r} D \dashrightarrow \mathcal{R}_{K/F} \mathrm{SB}_{2^r} D$$

is dominant.

The following theorem is the main result of this note. It extends to the unitary setting the results on orthogonal and symplectic involutions obtained in [13].

Theorem 1. *For any $r = 0, 1, \ldots, n-1$, the variety X_{2^r} is 2-incompressible.*

Date: 22 August 2014.

Key words and phrases. Algebraic groups, quadratic forms, projective homogeneous varieties, Chow groups and motives. *Mathematical Subject Classification (2010):* 14L17; 14C25.

The author acknowledges a partial support of the French Agence Nationale de la Recherche (ANR) under reference ANR-12-BL01-0005; his work has been also supported by the start-up grant of the University of Alberta and a Discovery Grant from the National Science and Engineering Board of Canada.

The proof will be given right after some preparation work. It extensively uses the notion of *upper motives* introduced in [14] and [8]. In our exposition, we go along the lines of [13] undertaking the necessary modifications.

Examples 7 and 9 show that Theorem 1 precisely detects the types of those projective homogeneous varieties under the projective unitary group $\mathrm{Aut}(D,\sigma)$ of a *division* algebra of given degree, which are 2-incompressible *in general*, i.e., for any F, K, D and σ. Note that $\mathrm{Aut}(D,\sigma)$ is an absolutely simple adjoint affine algebraic group of outer type $\mathcal{A}_{2^n-1}$. The varieties X_i for $i=1,\dots,2^{n-1}-1$ correspond to the pairs of vertices of the Dynkin diagram exchanged by the action of $\mathrm{Gal}(K/F)$; the variety $X_{2^{n-1}}$ corresponds to the unique $\mathrm{Gal}(K/F)$-stable vertex.

We start the preparation for the proof of Theorem 1. For any $i=0,1,\dots,2^{n-1}$, the tensor product $K\otimes_F F(X_i)$ is a field (namely, the field $K(X_i)$) and the tensor product $D\otimes_F F(X_i)$ is a central simple $K(X_i)$-algebra.

Lemma 2. *For any $r=0,1,\dots,n-1$, the Schur index of the central simple algebra $D\otimes_F F(X_{2^r})$ is equal to 2^r.*

Proof. First of all, although $X:=X_{2^r}$ is an F-variety, the center of D is K, not F. Therefore we do not need the index reduction formula [19, (9.29)] for the F-variety X here, we rather need an index reduction formula for the K-variety X_K. The variety X_K is isomorphic to the variety of flags of right ideals in D of reduced dimensions 2^r and 2^n-2^r. This flag variety is equivalent (in the sense of existence of rational maps in both directions) to $\mathrm{SB}_{2^r}D$ so that the desired result on

$$\mathrm{ind}(D\otimes_F F(X))=\mathrm{ind}(D\otimes_K K(X))$$

is contained in [23]. It is also a consequence of the index reduction formula for the generalized Severi-Brauer varieties [3] (see also [18, (5.11)]). □

Lemma 3. *Theorem 1 holds for $r=n-1$.*

Proof. This is a particular case of Proposition A4. □

We are working with Chow groups modulo 2. In particular, *multiplicities of correspondences*, [6, §75], take values in $\mathbb{F}_2:=\mathbb{Z}/2\mathbb{Z}$.

Lemma 4. *Assume that for some $r=0,1,\dots,n-2$ there is no multiplicity 1 correspondence $X_{2^r}\rightsquigarrow X_{2^{r+1}}$. Then the variety X_{2^r} is 2-incompressible.*

Proof. This is a particular case of Proposition A4. □

Before proving the general case of Theorem 1, as a warm up, we prove the case of maximal r among yet unproved ones:

Proposition 5. *Theorem 1 holds for $r=n-2$.*

Proof. By Lemma 4, we may assume that there exists a multiplicity 1 correspondence $X_{2^{n-2}}\rightsquigarrow X_{2^{n-1}}$. We set $T:=X_{2^{n-1}}$. The involution $\sigma_{F(T)}$ is hyperbolic. Besides, by Lemma 2, the Schur index of the algebra $D\otimes_F F(T)$ is 2^{n-1}.

For $X:=X_{2^{n-2}}$, we have $\mathrm{ind}\,D\otimes_F F(X)=2^{n-2}$ by Lemma 2. By Lemma A3 (applied twice), the complete motivic decomposition of $X_{F(X)}$ contains four Tate summands: $\mathbb{F}_2$,

$\mathbb{F}_2(2^{2n-3})$, $\mathbb{F}_2(\dim X - 2^{2n-3})$, $\mathbb{F}_2(\dim X)$. Note that $\dim X = 2^{2n-4} \cdot 5$, so that $2^{2n-3} < \dim X - 2^{2n-3}$, showing that the four Tate summands have pairwise different shifts.

Each of the remaining summands of the complete motivic decomposition of $X_{F(X)}$ is of even rank because by Lemma A3 it is a summand of the motive of an anisotropic variety. (The definition of anisotropic variety is given right before Lemma Lemma A3; the rank of any summand of the motive of an anisotropic variety is even by [14, Lemma 2.21]). For the upper motive $U(X)$, we are going to show that $U(X)_{F(X)}$ contains all the 4 Tate summands; this will imply that X is 2-incompressible, cf. [7, Theorem 5.1].

By definition of $U(X)$, $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2$.

By Corollary A6, $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2(\dim X - 2^{2n-3})$.

Let C be a central division $K(T)$-algebra (of degree 2^{n-1}) Brauer-equivalent to

$$D \otimes_F F(T) = D \otimes_K K(T).$$

Since there exist multiplicity 1 correspondences

$$X_{F(T)} \leftrightsquigarrow \mathcal{R}_{K(T)/F(T)} \mathrm{SB}_{2^{n-2}} C,$$

the upper motive of the variety $X_{F(T)}$ is isomorphic to the upper motive of the variety $\mathcal{R}_{K(T)/F(T)} \mathrm{SB}_{2^{n-2}} C$, [14, Corollary 2.15]. Since the latter variety is 2-incompressible and has dimension

$$[K(T) : F(T)] \cdot \dim \mathrm{SB}_{2^{n-2}} C = 2^{2n-3},$$

$U(X_{F(T)})_{F(T)(X)}$ contains the Tate summand $\mathbb{F}_2(2^{2n-3})$. In particular, $U(X)_{F(T)(X)}$ contains this Tate summand. Since the field $F(T)(X) = F(X)(T)$ is purely transcendental over $F(X)$ (because of the assumption that there exists a multiplicity 1 correspondence $X \rightsquigarrow T$), $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2(2^{2n-3})$.

Finally, since $U(X)$ has even rank, $U(X)_{F(X)}$ contains the remaining (fourth) Tate summand $\mathbb{F}_2(\dim X)$. □

For the general case of Theorem 1 we need one more observation:

Lemma 6. *For some $r = 0, 1, \ldots, n-1$, let us consider the biggest i such that there exists a multiplicity 1 correspondence $X_{2^r} \rightsquigarrow X_i$. (In particular, $X_i \neq \emptyset$, so that $i \leq 2^{n-1}$.) Then $i = 2^s$ for some $s \in \{r, r+1, \ldots, n-1\}$.*

Proof. Assuming that $i > 2^s$ for some $s = r, r+1, \ldots, n-2$, we show that $i \geq 2^{s+1}$. Since $\operatorname{ind} D \otimes_F F(X_{2^r}) = 2^r$ by Lemma 2, it is a priori clear that $i \geq 2^s + 2^r$.

Note that $\operatorname{ind} D \otimes_F F(T) = 2^s$ for $T := X_{2^s}$ by Lemma 2. Let I be an isotropic right ideal of reduced dimension 2^s in $D \otimes_F F(T)$. Let

$$C := \operatorname{End}_{D \otimes_F F(T)} I$$

so that C is a central division $K(T)$-algebra of degree 2^s Brauer-equivalent to $D \otimes_F F(T) = D \otimes_K K(T)$. Let A be a central simple $K(T)$-algebra with a $K(T)/F(T)$-unitary involution obtained out of I by Construction A2. Let X be the variety of isotropic right ideals in A of reduced dimension 2^r. The upper motives of X and of $\mathcal{R}_{K(T)/F(T)} \mathrm{SB}_{2^r} C$ are isomorphic. Since $\mathcal{R}_{K(T)/F(T)} \mathrm{SB}_{2^r} C$ is 2-incompressible and has dimension

$$d := \dim \mathcal{R}_{K(T)/F(T)} \mathrm{SB}_{2^r} C = [K(T) : F(T)] \cdot \dim \mathrm{SB}_{2^r} C = 2^{r+1}(2^s - 2^r),$$

the motive of $X_{F(T)(X)}$ contains the Tate motive $\mathbb{F}_2(d)$ as a summand. It follows by Lemma A3 that the maximum of the Witt index of the unitary involution on $A \otimes_{F(T)} E$ for E running over finite odd-degree field extensions of $F(T)(X)$ is at least 2^s. Therefore the maximum of the Witt index of σ_E is at least $2^s + 2^s = 2^{s+1}$ and it follows that $i \geq 2^{s+1}$. $\square$

Proof of Theorem 1. We set $X := X_{2^r}$. Let i be the maximal integer such that there exists a multiplicity 1 correspondence $X \rightsquigarrow X_i$. By Lemma 6, $i = 2^s$ for some $s \in \{r, r+1, \dots, n-1\}$.

By Lemma 2, $\operatorname{ind} D \otimes_F F(X) = 2^r$. By Lemma A3 (applied 2^{s-r} times), the complete motivic decomposition of the variety $X_{F(X)}$ contains the Tate summands with the shifts $j2^{2r+1}$ and $\dim X - j2^{2r+1}$ for $j = 0, 1, \dots, 2^{s-r} - 1$ (precisely one Tate summand for each shifting number). Note that $(2^{s-r} - 1)2^{2r+1} < \dim X - (2^{s-r} - 1)2^{2r+1}$ so that the shifting numbers are pairwise different. Each of the remaining summands in the complete motivic decomposition of $X_{F(X)}$ is of even rank. For the upper motive $U(X)$ it suffices to show that $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2(\dim X)$.

By Corollary A6, $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2(\dim X - (2^{s-r} - 1)2^{2r+1})$.

By Lemma 2, $\operatorname{ind} D \otimes_F F(T) = 2^s$, where $T := X_{2^s}$. Let C be a central division $K(T)$-algebra of degree 2^s Brauer-equivalent to $D \otimes_F F(T)$. The upper motives of the varieties $X_{F(T)}$ and $S := \mathcal{R}_{K(T)/F(T)} \mathrm{SB}_{2^r} C$ are isomorphic. Passing to the dual motives and shifting, we get that

$$U(X_{F(T)})^*(\dim X) \simeq U(S)^*(\dim X).$$

Since the variety S is 2-incompressible, the motive $U(S)_{F(T)(X)}$ contains the Tate summands $\mathbb{F}_2$ and $\mathbb{F}_2(\dim S)$. Consequently, $U(S)^*_{F(T)(X)}(\dim X)$ contains the Tate summands $\mathbb{F}_2(\dim X)$ and $\mathbb{F}_2(\dim X - \dim S)$. In particular, $U(X)^*_{F(T)(X)}(\dim X)$ contains both of these Tate summands. Since the field extension $F(T)(X)/F(X)$ is purely transcendental, $U(X)^*_{F(X)}(\dim X)$ contains both of these Tate summands. Note that

$$\dim S = (2^{s-r} - 1)2^{2r+1}$$

and $U(X)^*(\dim X)$ is an indecomposable summand of $M(X)$. Since $U(X)_{F(X)}$ also contains the Tate summand $\mathbb{F}_2(\dim X - \dim S)$, the Krull-Schmidt principle of [5] (see also [8]) tells us that $U(X) \simeq U(X)^*(\dim X)$ and therefore $U(X)_{F(X)}$ contains $\mathbb{F}_2(\dim X)$ as desired. $\square$

The following example shows that for $G = \operatorname{Aut}(D, \sigma)$, the varieties listed in Theorem 1 are the only projective G-homogeneous varieties which are 2-incompressible in general (i.e., for any field F, any separable quadratic field extension K/F, and any central division K-algebra D of degree 2^n endowed with a K/F-unitary involution σ). We recall that an arbitrary (different from $\operatorname{Spec} F$) projective G-homogeneous variety is isomorphic to the variety $X_{l_1 \dots l_k}$ of flags of isotropic right ideals in D of some fixed reduced dimensions $1 \leq l_1 < \dots < l_k \leq 2^{n-1}$ with some $k \geq 1$.

Example 7. Let F, K, D, and σ be as in Lemma 8 and assume that D is a division algebra. An arbitrary projective G-homogeneous variety is isomorphic to the variety $X_{l_1 \dots l_k}$ with some $k \geq 1$ and some $1 \leq l_1 < \dots < l_k \leq 2^{n-1}$. By Lemma 8, this variety is equivalent (in the sense of existing of rational maps in both directions) to the variety X_{2^r}, where 2^r is the largest 2-power dividing $l_1, \dots, l_k$. In particular, canonical 2-dimensions of these two varieties coincide (see e.g. [15, Lemma 3.3]). Hence $\{l_1, \dots, l_k\} = \{2^r\}$ (i.e., $k = 1$ and $l_1 = 2^r$) if the variety $X_{l_1 \dots l_k}$ is 2-incompressible (because $\dim X_{l_1 \dots l_k} > \dim X_{2^r}$ otherwise).

Lemma 8. *For any given $n \geq 2$, let us consider over an appropriate field F of characteristic $\neq 2$, a central division F-algebra D' of degree 2^n endowed with an orthogonal or a symplectic involution σ' such that (D', σ') is the tensor product of n quaternion algebras with involutions. Let K/F be a separable quadratic field extension. We define a K/F-unitary involution σ on $D := D' \otimes_F K$ as the tensor product of σ' by the nontrivial automorphism of K/F. Then for any field extension L/F, the unitary involution on $D \otimes_F L$ given by σ is either anisotropic or hyperbolic.*

Proof. We only need to consider the case where $K \otimes_F L$ is a field (because the involution is hyperbolic otherwise). We assume that the involution is isotropic for such an L and we want to show that it is hyperbolic. Replacing F by L, we simply assume that σ (over F) is isotropic and we want to show that σ is hyperbolic.

By [12, Theorem A.2] (see also [9]), we may replace F by the function field of the F-variety $\mathcal{R}_{K/F} \mathrm{SB}(D)$. Since now the F-algebra D' splits over the quadratic extension K/F, it is equivalent to a quaternion F-algebra Q. It follows by [2] that the algebra D' together with the (orthogonal or symplectic) involution σ' is isomorphic to a tensor product of n quaternions algebras with involutions, where the first quaternion algebra is Q and the remaining $n-1$ quaternion algebras are split. Since the quadratic extension K/F splits Q, our algebra D together with the unitary involution σ is the tensor product (over K) of n split quaternion K-algebras with K/F-unitary involutions.

Let h be a K/F-hermitian form on an n-dimensional K-vector space V such that σ is adjoint to h. Because of the tensor decomposition we have for (D, σ), the quadratic form $q : v \mapsto h(v, v)$ on V over F is similar to an $(n+1)$-fold Pfister form. Isotropy of σ implies isotropy of h, which implies isotropy of q, which implies hyperbolicity of q, which in its turn implies hyperbolicity of h and of σ (cf. [11, Lemma 9.1]). □

The following example shows that in the case when $\deg D$ is not a power of 2, i.e., when $\deg D = 2^n \cdot m$ with $n \geq 0$ and with odd $m \geq 3$, none (but $\operatorname{Spec} F$) of the projective homogeneous varieties under $G = \operatorname{Aut}(D, \sigma)$ is 2-incompressible in general.

Example 9. Let n and m be integers with $n \geq 0$ and with odd $m \geq 3$. Let F, K, D, and σ be as in Lemma 10 with the above m. Changing notation, we write D_m and σ_m for these D and σ. Replacing F by a purely transcendental extension field of sufficiently large transcendence degree, we may find a central division K-algebra D' of degree 2^n with a K/F-unitary involution σ'. (For instance, one may take the degree $2n$ purely transcendental field extension $F(x_1, y_1, \dots, x_n, y_n)$, consider the tensor product of quaternion algebras $(x_1, y_1) \otimes \cdots \otimes (x_n, y_n)$ with the tensor product of their canonical involutions, and then take the induced unitary involution over $K(x_1, y_1, \dots, x_n, y_n)$.) We define the central division K-algebra D as the tensor product $D' \otimes_K D_m$ and we define a K/F-unitary involution σ on D as the tensor product of σ' and σ_m. We claim that none of the projective homogeneous varieties given by (D, σ) is 2-incompressible. Indeed, canonical 2-dimension of any such variety X remains the same over any finite field extension of F

of odd degree, [17, Proposition 1.5]. Replacing F by an odd degree extension L/F given by Lemma 10, we come to the situation where σ_m is isotropic. Therefore σ is isotropic and it follows that X is not 2-incompressible.

Lemma 10. *For any odd integer $m \geq 3$, there exist a field F, a separable quadratic field extension K/F, a degree m central division K-algebra D, a K/F-unitary involution σ on D, and a finite field extension L/F of odd degree such that σ is isotropic over L.*

Proof. For prime $m \geq 5$, such examples can be derived from [21] and [20]. For general m, one may proceed by a generic construction as follows.

Let $m \geq 3$ be an odd integer and let D be a central division algebra of degree m (over an appropriate field K) possessing a unitary involution σ. Let $F \subset K$ be the subfield of the σ-invariant elements. There exists a finite field extension L/F of odd degree such that the algebra $D \otimes_F L$ is split (cf. [1, 3.3.1] or [9, Page 938]). Let h be a hermitian form such that the involution σ_L on the split algebra $D \otimes_F L$ is adjoint to h. Let q be the quadratic form over L (of dimension $2m$) given by h. Let q' be a nondegenerate subform of q of dimension $2m-1$. Let X be the projective quadric of q'. We are going to replace the base field F by the function field of the Weil transfer $\mathcal{R}_{L/F}X$. The quadratic form q' (and therefore q, h and σ_L as well) will then become isotropic. We only need to check that D will remain a division algebra. We check this with the help of the index reduction formula for Weil transfers of quadrics. This formula is simpler for odd-dimensional q' then for even-dimensional q and this is the reason to use q', not q itself.

The index reduction formula we need is given in [19, (7.25)]. We want to check that the index of $D \otimes_K K(\mathcal{R}_{L/F}X)$ is m. By [19, (7.25)], this index (note that $(\mathcal{R}_{L/F}X)_K \simeq \mathcal{R}_{K\otimes_F L/K}(X_{K\otimes_F L})$) is the greatest common divisor of certain products

$$n_E \cdot [E : K] \cdot \mathrm{ind}(D \otimes_K A_E),$$

where n_E is an integer, E/K is a finite field extension, and A_E is a central simple E-algebra whose exponent divides 2. To see that the exponent of A_E divides 2, it suffices to note that A_E is obtained out of the even Clifford algebra of q' using the operations of restriction and corestriction of scalars; the even Clifford algebra has exponent dividing 2 and the exponent of a restriction/corestriction of a central simple algebra divides the exponent of the algebra.

It follows that for any E, the index of $D \otimes_K A_E$ is divisible by the index of $D \otimes_K E$. On the other hand, $m = \mathrm{ind}\, D$ always divides the product $[E : K] \cdot \mathrm{ind}(D \otimes_K E)$. Therefore m divides $\mathrm{ind}(D \otimes_K K(\mathcal{R}_{L/F}X)$. □

Appendix. Quadric-like behavior

In this Appendix we establish some results on grassmannians of isotropic ideals which are very close (in the statement as well as in the proof) to results on projective quadrics in the spirit of [24].

Let F be a field, K/F a separable quadratic field extension, A a central simple K-algebra endowed with a K/F-unitary involution σ.

For a right ideal $J \subset A$, its orthogonal complement $J^\perp$ is defined as the (right) annihilator of the left ideal $\sigma(J)$. This is a right ideal of reduced dimension $\mathrm{rdim}\, J^\perp = \deg A - \mathrm{rdim}\, J$, [16, Proposition 6.2].

A right ideal J is *nondegenerate* if $J \cap J^\perp = 0$.

Construction A1. Given a nondegenerate right ideal $J \subset A$, the right A-module A is a direct sum of the submodules J and $J^\perp$. The image $e \in J$ of $1 \in A$ with respect to the projection $A \to J$ is a symmetric (with respect to the involution σ) idempotent generating J: $\sigma(e) = e$, $e^2 = e$, and $J = eA$. The K-algebra $\operatorname{End}_A J$ is identified with the subalgebra eAe of A (see [16, Corollary 1.13]) stable under the involution σ. (Note that the unit of the algebra eAe is the element e which may differ from the unit 1 of A so that the unital algebra eAe is, in general, not a unital subalgebra of A.) The restriction of σ to eAe is a K/F-unitary involution. Note that the degree of the algebra eAe is equal to the reduced dimension of the ideal J.

In contrast to [16], we define the (Witt) index $\operatorname{ind}\sigma$ of σ as the maximum of reduced dimension of an isotropic right ideal in A. The information given by the Witt index of σ in the sense of [16], or equivalently by the Tits index of the algebraic group $\operatorname{Aut}(A, \sigma)$, is equivalent to the information given by $\operatorname{ind}\sigma$ and $\operatorname{ind} A$.

Construction A2. Given an isotropic right ideal I in A, we have $I \subset I^\perp$. Let us choose an ideal $J \subset I^\perp$ such that $I^\perp = I \oplus J$. The ideal J is nondegenerate so that, using Construction A1, we get the algebra eAe with restriction of σ. Note that $\deg(eAe) = \operatorname{rdim} J = \deg A - 2 \operatorname{rdim} I$. The (Witt) index of this restriction is equal to $\operatorname{ind}\sigma - \operatorname{rdim} I$. (Note that Construction A1 applied to the ideal $J^\perp$ produces an algebra with *hyperbolic* unitary involution.)

A variety is called *anisotropic* here if every its closed point has even degree. The following statement is an analogue of the motivic decomposition [6, Proposition 70.1] of smooth projective quadrics, observed originally by M. Rost. It is also the unitary analogue of [13, Lemma A.3] which contains a mistake in the statement: the motive of Y occurs in the decomposition of $M(X)$ with the shift $(\operatorname{ind} A)^2$, not with the shift $2 \operatorname{ind} A$ as claimed there. The shift we have in the unitary setting is $2(\operatorname{ind} A)^2$:

Lemma A3. *Assume that* $\operatorname{ind} A$ *is a power of* 2*. Let* I *be an isotropic ideal of reduced dimension* $\operatorname{ind} A$ *in* A*. Let* X *be the variety of isotropic right ideals of reduced dimension* $\operatorname{ind} A$ *in* A*. Let* B *be an algebra* eAe *given by Construction A2. Let* Y *be the variety of isotropic right ideals of reduced dimension* $\operatorname{ind} A = \operatorname{ind} B$ *in* B *(*Y *is nonempty iff* $\deg A \geq 4 \operatorname{ind} A$*). Then there exists a motivic decomposition of* X *with summands* $\mathbb{F}_2$*,* $\mathbb{F}_2(\dim X)$*, and – in the case of nonempty* Y *–*

$$M(Y)(2(\operatorname{ind} A)^2) = M(Y)((\dim X - \dim Y)/2)$$

such that each of the remaining summands of the decomposition is the motive of an anisotropic variety.

Proof. See [9, Lemma 2.3]. In order to determine the shift of $M(Y)$, one may use [4]. □

For any integer i, we write X_i for the variety of isotropic right ideals in A of reduced dimension i. The variety X_i is nonempty if and only if $0 \leq i \leq (\deg A)/2$.

Proposition A4 and Corollary A6 below are analogues of computation of canonical 2-dimension of smooth projective quadrics [6, Theorem 90.2]. We refer to [7] for definition and basic properties of canonical dimension.

Proposition A4. *For some $r \geq 0$ with $2^{r+1} \leq \deg A$ and 2^r dividing $\operatorname{ind} A$, assume that the variety $X := X_{2^r}$ is anisotropic and has no multiplicity 1 correspondence to $X_{2^{r+1}}$. Then the variety X is 2-incompressible.*

Proof. By the index reduction formula as in Lemma 2, we see that the index of $A_{F(X)}$ is 2^r. The $F(X)$-variety Y as in Lemma A3 is anisotropic (because of absence of a multiplicity 1 correspondence $X \rightsquigarrow X_{2^{r+1}}$). It follows that all summands of the complete motivic decomposition of the variety $X_{F(X)}$ but $\mathbb{F}_2$ and $\mathbb{F}_2(\dim X)$ have even ranks. On the other hand, since X is anisotropic, the motive $U(X)$ is also of even rank. It follows that $U(X)_{F(X)}$ contains $\mathbb{F}_2(\dim X)$. Therefore X is 2-incompressible. □

Lemma A5. *For any multiple m of $\operatorname{ind} A$ satisfying $0 \leq m \leq \deg A$, there exists a nondegenerate right ideal in A of reduced dimension m.*

Proof. We write $A = \operatorname{End}_D V$ for some central division K-algebra D with a fixed K/F-unitary involution and a right D-module V with a hermitian form h such that the involution σ on A is adjoint to h. By [22, Theorem 6.3 of Chapter 7], h can be diagonalized. □

Corollary A6. *Assume that $\operatorname{ind} A = 2^r$ for some $r \geq 0$. Assume that the variety X_{2^r} is anisotropic. Let i be the maximal integer such that there exists a multiplicity 1 correspondence $X \rightsquigarrow X_{(i+1)2^r}$. Then the canonical 2-dimension $\operatorname{cdim}_2 X$ of X is equal to $\dim X - i2^{2r+1}$. In particular, the Tate motive $\mathbb{F}_2(\dim X - i2^{2r+1})$ is a summand of $U(X)_{F(X)}$.*

Proof. Note that the case of $i = 0$ follows by Proposition A4.

For arbitrary i, let $J \subset A$ be a nondegenerate right ideal of reduced dimension $\deg A - i2^r$ (existing by Lemma A5). Let B be the corresponding nonunital subalgebra of A (obtained by Construction A1) and let Y be the variety of isotropic right ideals of reduced dimension 2^r in B. Since there is a multiplicity 1 correspondence $X \rightsquigarrow X_{(i+1)2^r}$, there is a multiplicity 1 correspondence from X to Y. Note that there also is a multiplicity 1 correspondence $Y \rightsquigarrow X$ so that $U(X) \simeq U(Y)$. It follows by [7, Theorem 5.1] that $\operatorname{cdim}_2 X = \operatorname{cdim}_2 Y$.

The variety Y satisfies the conditions of Proposition A4: it is anisotropic and has no multiplicity 1 correspondence to $Y_{2^{r+1}}$. Therefore Y is 2-incompressible. It follows that $U(Y)_{F(Y)}$ as well as $U(X)_{F(X)}$ contain $\mathbb{F}_2(\dim Y) = \mathbb{F}_2(\dim X - i2^{2r+1})$ as a summand. □

Acknowledgements. I am grateful to Jean-Pierre Tignol for useful comments. This work has been accomplished during my stay at the Universität Duisburg-Essen (research group of Marc Levine), under ideal work conditions.

References

[1] Bayer-Fluckiger, E., and Parimala, R. Galois cohomology of the classical groups over fields of cohomological dimension ≤ 2. *Invent. Math. 122*, 2 (1995), 195–229.
[2] Becher, K. J. A proof of the Pfister factor conjecture. *Invent. Math. 173*, 1 (2008), 1–6.
[3] Blanchet, A. Function fields of generalized Brauer-Severi varieties. *Comm. Algebra 19*, 1 (1991), 97–118.
[4] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.

[5] Chernousov, V., and Merkurjev, A. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups 11*, 3 (2006), 371–386.

[6] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[7] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.

[8] Karpenko, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.

[9] Karpenko, N. A. Hyperbolicity of unitary involutions. *Sci. China Math. 55*, 5 (2012), 937–945.

[10] Karpenko, N. A. Incompressibility of quadratic Weil transfer of generalized Severi-Brauer varieties. *J. Inst. Math. Jussieu 11*, 1 (2012), 119–131.

[11] Karpenko, N. A. Unitary Grassmannians. *J. Pure Appl. Algebra 216*, 12 (2012), 2586–2600.

[12] Karpenko, N. A. Isotropy of orthogonal involutions. *Amer. J. Math. 135*, 1 (2013), 1–15. With an appendix by Jean-Pierre Tignol.

[13] Karpenko, N. A. Orthogonal and symplectic Grassmannians of division algebras. *J. Ramanujan Math. Soc. 28*, 2 (2013), 213–222.

[14] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.

[15] Karpenko, N. A., and Reichstein, Z. A numerical invariant for linear representations of finite groups. Linear Algebraic Groups and Related Structures (preprint server) 534 (2014, May 15, revised: 2014, June 18), 24 pages.

[16] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[17] Merkurjev, A. S. Essential dimension. In *Quadratic Forms – Algebra, Arithmetic, and Geometry*, vol. 493 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2009, pp. 299–326.

[18] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. I. *K-Theory 10*, 6 (1996), 517–596.

[19] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. II. *K-Theory 14*, 2 (1998), 101–196.

[20] Parimala, R. Homogeneous varieties—zero-cycles of degree one versus rational points. *Asian J. Math. 9*, 2 (2005), 251–256.

[21] Parimala, R., Sridharan, R., and Suresh, V. Hermitian analogue of a theorem of Springer. *J. Algebra 243*, 2 (2001), 780–789.

[22] Scharlau, W. *Quadratic and Hermitian forms*, vol. 270 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[23] Schofield, A., and Van den Bergh, M. The index of a Brauer class on a Brauer-Severi variety. *Trans. Amer. Math. Soc. 333*, 2 (1992), 729–739.

[24] Vishik, A. Motives of quadrics with applications to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*, vol. 1835 of *Lecture Notes in Math.* Springer, Berlin, 2004, pp. 25–101.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
E-mail address: karpenko *at* ualberta.ca, *web page*: www.ualberta.ca/~karpenko