TRIALITY AND ALGEBRAIC GROUPS OF TYPE $^3\mathsf{D}_4$

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Abstract. We determine which simple algebraic groups of type $^3\mathsf{D}_4$ over arbitrary fields of characteristic different from 2 admit outer automorphisms of order 3, and classify these automorphisms up to conjugation. The criterion is formulated in terms of a representation of the group by automorphisms of a trialitarian algebra: outer automorphisms of order 3 exist if and only if the algebra is the endomorphism algebra of an induced cyclic composition; their conjugacy classes are in one-to-one correspondence with isomorphism classes of symmetric compositions from which the induced cyclic composition stems.

1. Introduction

Let G_0 be an adjoint Chevalley group of type D_4 over a field F. Since the automorphism group of the Dynkin diagram of type D_4 is isomorphic to the symmetric group $\mathfrak{S}_3$, there is a split exact sequence of algebraic groups

$$1 \longrightarrow G_0 \xrightarrow{\text{Int}} \mathbf{Aut}(G_0) \xrightarrow{\pi} \mathfrak{S}_3 \longrightarrow 1. \tag{1}$$

Thus, $\mathbf{Aut}(G_0) \cong G_0 \rtimes \mathfrak{S}_3$; in particular G_0 admits outer automorphisms of order 3, which we call *trialitarian automorphisms*. Adjoint algebraic groups of type D_4 over F are classified by the Galois cohomology set $H^1(F, G_0 \rtimes \mathfrak{S}_3)$ and the map induced by π in cohomology

$$\pi_*\colon H^1(F, G_0 \rtimes \mathfrak{S}_3) \to H^1(F, \mathfrak{S}_3)$$

associates to any group G of type D_4 the isomorphism class of a cubic étale F-algebra L. The group G is said to be of type $^1\mathsf{D}_4$ if L is split, of type $^2\mathsf{D}_4$ if $L \cong F \times \Delta$ for some quadratic separable field extension Δ/F, of type $^3\mathsf{D}_4$ if L is a cyclic field extension of F and of type $^6\mathsf{D}_4$ if L is a non-cyclic field extension. An easy argument given in Theorem 4.1 below shows that groups of type $^2\mathsf{D}_4$ and $^6\mathsf{D}_4$ do not admit trialitarian automorphisms defined over the base field. Trialitarian automorphisms of groups of type $^1\mathsf{D}_4$ were classified in [3], and by a different method in [2]: the adjoint groups of type $^1\mathsf{D}_4$ that admit trialitarian automorphisms are the groups of proper projective similitudes of 3-fold Pfister quadratic spaces; their trialitarian automorphisms are shown in [3, Th. 5.8] to be in one-to-one correspondence with the symmetric composition structures on the quadratic space. In the present paper,

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we determine the simple groups of type ${}^3\mathsf{D}_4$ that admit trialitarian automorphisms, and we classify those automorphisms up to conjugation.

Our main tool is the notion of a trialitarian algebra, as introduced in [7, Ch. X]. Since these algebras are only defined in characteristic different from 2, we assume throughout (unless specifically mentioned) that the characteristic of the base field F is different from 2. In view of [7, Th. (44.8)], every adjoint simple group G of type D_4 can be represented as the automorphism group of a trialitarian algebra $T = (E, L, \sigma, \alpha)$. In the datum defining T, L is the cubic étale F-algebra given by the map π_* above, E is a central simple L-algebra with orthogonal involution σ, known as the *Allen invariant* of G (see [1]), and α is an isomorphism relating (E, σ) with its Clifford algebra $C(E, \sigma)$ (we refer to [7, §43] for details). We show in Theorem 4.1 that if G admits an outer automorphism of order 3 modulo inner automorphisms, then L is either split (i.e., isomorphic to $F \times F \times F$), or it is a cyclic field extension of F (so G is of type ${}^1\mathsf{D}_4$ or ${}^3\mathsf{D}_4$), and the Allen invariant E of G is a split central simple L-algebra. This implies that T has the special form $T = \operatorname{End}\Gamma$ for some cyclic composition Γ. We further show in Theorem 4.2 that if G carries a trialitarian automorphism, then the cyclic composition Γ is *induced*, which means that it is built from some symmetric composition over F, and we establish a one-to-one correspondence between trialitarian automorphisms of G up to conjugation and isomorphism classes of symmetric compositions over F from which Γ is built.

The notions of symmetric and cyclic compositions are recalled in §2. Trialitarian algebras are discussed in §3, which contains the most substantial part of the argument: we determine the trialitarian algebras that have semilinear automorphisms of order 3 (Theorem 3.1) and we classify these automorphisms up to conjugation (Theorem 3.5). The group-theoretic results follow easily in §4 by using the correspondence between groups of type D_4 and trialitarian algebras.

Notation is generally as in the Book of Involutions [7], which is our main reference. For an algebraic structure S defined over a field F, we let $\operatorname{Aut}(S)$ denote the group of automorphisms of S, and write $\mathbf{Aut}(S)$ for the corresponding group scheme over F.

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2. Cyclic and symmetric compositions

Cyclic compositions were introduced by Springer in his 1963 Göttingen lecture notes ([8], [9]) to get new descriptions of Albert algebras. We recall their definition from [9][1] and [7, §36.B], restricting to the case of dimension 8.

Let F be an arbitrary field (of any characteristic). A *cyclic composition* (of dimension 8) over F is a 5-tuple $\Gamma = (V, L, Q, \rho, *)$ consisting of

- a cubic étale F-algebra L;
- a free L-module V of rank 8;
- a quadratic form $Q\colon V \to L$ with nondegenerate polar bilinear form b_Q;
- an F-automorphism ρ of L of order 3;

[1] A cyclic composition is called a *normal twisted composition* in [8] and [9].

- an F-bilinear map $*\colon V \times V \to V$ with the following properties: for all x, y, $z \in V$ and $\lambda \in L$,

$$(\lambda x) * y = \rho(\lambda)(x * y), \qquad x * (y\lambda) = (x * y)\rho^2(\lambda),$$
$$Q(x * y) = \rho\big(Q(x)\big) \cdot \rho^2\big(Q(y)\big),$$
$$b_Q(x * y, z) = \rho\big(b_Q(y * z, x)\big) = \rho^2\big(b_Q(z * x, y)\big).$$

These properties imply the following (see [7, §36.B] or [9, Lemma 4.1.3]): for all x, $y \in V$,

$$(x * y) * x = \rho^2\big(Q(x)\big)y \quad \text{and} \quad x * (y * x) = \rho\big(Q(x)\big)y. \tag{2}$$

Since the cubic étale F-algebra L has an automorphism of order 3, L is either a cyclic cubic field extension of F, and ρ is a generator of the Galois group, or we may identify L with $F \times F \times F$ and assume ρ permutes the components cyclically. We will almost exclusively restrict to the case where L is a field; see however Remark 2.3 below.

Let $\Gamma' = (V', L', Q', \rho', *')$ be also a cyclic composition over F. An *isotopy*[2] $\Gamma \to \Gamma'$ is defined to be a pair (ν, f) where $\nu\colon (L, \rho) \xrightarrow{\sim} (L', \rho')$ is an isomorphism of F-algebras with automorphisms (i.e., $\nu \circ \rho = \rho' \circ \nu$) and $f\colon V \xrightarrow{\sim} V'$ is a ν-semilinear isomorphism for which there exists $\mu \in L^\times$ such that

$$Q'\big(f(x)\big) = \nu\big(\rho(\mu)\rho^2(\mu) \cdot Q(x)\big) \quad \text{and} \quad f(x) *' f(y) = \nu(\mu) f(x * y)$$

for x, $y \in V$. The scalar μ is called the *multiplier* of the isotopy. Isotopies with multiplier 1 are *isomorphisms*. When the map ν is clear from the context, we write simply f for the pair (ν, f), and refer to f as a *ν-semilinear isotopy*.

Examples of cyclic compositions can be obtained by scalar extension from symmetric compositions over F, as we now show. Recall from [7, §34] that a *symmetric composition* (of dimension 8) over F is a triple $\Sigma = (S, n, \star)$ where (S, n) is an 8-dimensional F-quadratic space (with nondegenerate polar bilinear form b_n) and $\star\colon S \times S \to S$ is a bilinear map such that for all x, y, $z \in S$

$$n(x \star y) = n(x)n(y) \quad \text{and} \quad b_n(x \star y, z) = b_n(x, y \star z).$$

If $\Sigma' = (S', n', \star')$ is also a symmetric composition over F, an *isotopy* $\Sigma \to \Sigma'$ is a linear map $f\colon S \to S'$ for which there exists $\lambda \in F^\times$ (called the *multiplier*) such that

$$n'\big(f(x)\big) = \lambda^2 n(x) \quad \text{and} \quad f(x) \star' f(y) = \lambda f(x \star y) \quad \text{for } x,\ y \in S.$$

Note that if $f\colon \Sigma \to \Sigma'$ is an isotopy with multiplier λ, then $\lambda^{-1} f\colon \Sigma \to \Sigma'$ is an isomorphism. Thus, symmetric compositions are isotopic if and only if they are isomorphic. For an explicit example of a symmetric composition, take a Cayley (octonion) algebra $(C, \cdot)$ with norm n and conjugation map $\overline{}$. Letting $x \star y = \overline{x} \cdot \overline{y}$ for x, $y \in C$ yields a symmetric composition $\widetilde{C} = (C, n, \star)$, which is called a *para-Cayley composition* (see [7, §34.A]).

Given a symmetric composition $\Sigma = (S, n, \star)$ and a cubic étale F-algebra L with an automorphism ρ of order 3, we define a cyclic composition $\Sigma \otimes (L, \rho)$ as follows:

$$\Sigma \otimes (L, \rho) = (S \otimes_F L, L, n_L, \rho, *)$$

[2] The term used in [7, p. 490] is *similarity*.

where n_L is the scalar extension of n to L and $*$ is defined by extending $\star$ linearly to $S \otimes_F L$ and then setting

$$x * y = (\mathrm{Id}_S \otimes \rho)(x) \star (\mathrm{Id}_S \otimes \rho^2)(y) \qquad \text{for } x,\, y \in S \otimes_F L.$$

(See [7, (36.11)].) Clearly, every isotopy $f\colon \Sigma \to \Sigma'$ of symmetric compositions extends to an isotopy of cyclic compositions $(\mathrm{Id}_L, f)\colon \Sigma \otimes (L,\rho) \to \Sigma' \otimes (L,\rho)$. Observe for later use that the map $\widehat{\rho} = \mathrm{Id}_S \otimes \rho \in \mathrm{End}_F(S \otimes_F L)$ defines a ρ-semilinear automorphism

$$\widehat{\rho}\colon \Sigma \otimes (L,\rho) \xrightarrow{\sim} \Sigma \otimes (L,\rho) \tag{3}$$

such that $\widehat{\rho}^3 = \mathrm{Id}$.

We call a cyclic composition that is isotopic to $\Sigma \otimes (L,\rho)$ for some symmetric composition Σ *induced*. Cyclic compositions induced from para-Cayley symmetric compositions are called *reduced* in [9].

Remark 2.1. Induced cyclic compositions are not necessarily reduced. This can be shown by using the following cohomological argument. We assume for simplicity that the field F contains a primitive cube root of unity ω. There is a cohomological invariant $g_3(\Gamma) \in H^3(F, \mathbb{Z}/3\mathbb{Z})$ attached to any cyclic composition Γ. The cyclic composition Γ is reduced if and only if $g_3(\Gamma) = 0$ (we refer to [9, §8.3] or [7, §40] for details). We construct an induced cyclic composition Γ with $g_3(\Gamma) \neq 0$. Let $a,\, b \in F^\times$ and let $A(a,b)$ be the F-algebra with generators α, β and relations $\alpha^3 = a$, $\beta^3 = b$, $\beta\alpha = \omega\alpha\beta$. The algebra $A(a,b)$ is central simple of dimension 9 and the space A^0 of elements of $A(a,b)$ of reduced trace zero admits the structure of a symmetric composition $\Sigma(a,b) = (A^0, n, \star)$ (see [7, (34.19)]). Such symmetric compositions are called *Okubo symmetric compositions*. From the Elduque–Myung classification of symmetric compositions [4, p. 2487] (see also [7, (34.37)]), it follows that symmetric compositions are either para-Cayley or Okubo. Let $L = F(\gamma)$ with $\gamma^3 = c \in F^\times$ be a cubic cyclic field extension of F, and let ρ be the F-automorphism of L such that $\gamma \mapsto \omega\gamma$. We may then consider the induced cyclic composition $\Gamma(a,b,c) = \Sigma(a,b) \otimes (L,\rho)$. Its cohomological invariant $g_3\big(\Gamma(a,b,c)\big)$ can be computed by the construction in [9, §8.3]: Using ω, we identify the group μ_3 of cube roots of unity in F with $\mathbb{Z}/3\mathbb{Z}$, and for any $u \in F^\times$ we write $[u]$ for the cohomology class in $H^1(F, \mathbb{Z}/3\mathbb{Z})$ corresponding to the cube class $uF^{\times 3}$ under the isomorphism $F^\times/F^{\times 3} \cong H^1(F, \mu_3)$ arising from the Kummer exact sequence (see [7, p. 413]). Then $g_3\big(\Gamma(a,b,c)\big)$ is the cup-product $[a] \cup [b] \cup [c] \in H^3(F, \mathbb{Z}/3\mathbb{Z})$. Thus any cyclic composition $\Gamma(a,b,c)$ with $[a] \cup [b] \cup [c] \neq 0$ is induced but not reduced.

Another cohomological argument can be used to show that there exist cyclic compositions that are not induced. We still assume that F contains a primitive cube root of unity ω. There is a further cohomological invariant of cyclic compositions $f_3(\Gamma) \in H^3(F, \mathbb{Z}/2\mathbb{Z})$ which is zero for any cyclic composition induced by an Okubo symmetric composition[3] and is given by the class in $H^3(F, \mathbb{Z}/2\mathbb{Z})$ of the 3-fold Pfister form which is the norm of $\widetilde{C}$ if Γ is induced from the para-Cayley $\widetilde{C}$ (see for example [7, §40]). Thus a cyclic composition Γ with $f_3(\Gamma) \neq 0$ and $g_3(\Gamma) \neq 0$ is not induced. Such examples can be given with the help of the Tits process used for constructing Albert algebras (see [7, §39 and §40]). However, for example, cyclic

[3] The fact that F contains a primitive cubic root of unity is relevant for this claim.

compositions over finite fields, p-adic fields or algebraic number fields are reduced, see [9, p. 108].

Examples 2.2. (i) Let $F = \mathbb{F}_q$ be the field with q elements, where q is odd and $q \equiv 1 \bmod 3$. Thus F contains a primitive cube root of unity and we are in the situation of Remark 2.1. Let $L = \mathbb{F}_{q^3}$ be the (unique, cyclic) cubic field extension of F, and let ρ be the Frobenius automorphism of L/F. Because $H^3(F, \mathbb{Z}/3\mathbb{Z}) = 0$, every cyclic composition over F is reduced; moreover every 3-fold Pfister form is hyperbolic, hence every Cayley algebra is split. Therefore, up to isomorphism there is a unique cyclic composition over F with cubic algebra (L, ρ), namely $\Gamma = \widetilde{C} \otimes (L, \rho)$ where $\widetilde{C}$ is the split para-Cayley symmetric composition. If Σ denotes the Okubo symmetric composition on 3×3 matrices of trace zero with entries in F, we thus have $\Gamma \cong \Sigma \otimes (L, \rho)$, which means that Γ is also induced by Σ. By the Elduque–Myung classification of symmetric compositions, every symmetric composition over F is isomorphic either to the Okubo composition Σ or to the split para-Cayley composition $\widetilde{C}$. Therefore, Γ is induced by exactly two symmetric compositions over F up to isomorphism.

(ii) Assume that F contains a primitive cube root of unity and that F carries an anisotropic 3-fold Pfister form n. Let C be the non-split Cayley algebra with norm n and let $\widetilde{C}$ be the associated para-Cayley algebra. For any cubic cyclic field extension (L, ρ) the norm n_L of the cyclic composition $\widetilde{C} \otimes (L, \rho)$ is anisotropic. Thus it follows from the Elduque–Myung classification that any symmetric composition Σ such that $\Sigma \otimes (L, \rho)$ is isotopic to $\widetilde{C} \otimes (L, \rho)$ must be isomorphic to $\widetilde{C}$.

(iii) Finally, we observe that the cyclic compositions of type $\Gamma(a, b, c)$, described in Remark 2.1, have invariant g_3 equal to zero if $c = a$. Since the f_3-invariant is also zero, they are all isotopic to the cyclic composition induced by the split para-Cayley algebra. Thus we can get (over suitable fields) examples of many mutually non-isomorphic symmetric compositions $\Sigma(a, b)$ that induce isomorphic cyclic compositions $\Gamma(a, b, c)$.

Of course, besides this construction of cyclic compositions by induction from symmetric compositions, we can also extend scalars of a cyclic composition: if $\Gamma = (V, L, Q, \rho, *)$ is a cyclic composition over F and K is any field extension of F, then $\Gamma_K = (V \otimes_F K, L \otimes_F K, Q_K, \rho \otimes \mathrm{Id}_K, *_K)$ is a cyclic composition over K.

Remark 2.3. Let $\Gamma = (V, L, Q, \rho, *)$ be an arbitrary cyclic composition over F with L a field. Write θ for ρ^2. We have an isomorphism of L-algebras

$$\nu\colon L \otimes_F L \xrightarrow{\sim} L \times L \times L \quad \text{given by} \quad \ell_1 \otimes \ell_2 \mapsto (\ell_1\ell_2, \rho(\ell_1)\ell_2, \theta(\ell_1)\ell_2).$$

Therefore, the extended cyclic composition Γ_L over L has a split cubic étale algebra. To give an explicit description of Γ_L, note first that under the isomorphism ν the automorphism $\rho \otimes \mathrm{Id}_L$ is identified with the map $\widetilde{\rho}$ defined by $\widetilde{\rho}(\ell_1, \ell_2, \ell_3) = (\ell_2, \ell_3, \ell_1)$. Consider the twisted L-vector spaces ${}^\rho V$, ${}^\theta V$ defined by

$${}^\rho V = \{{}^\rho x \mid x \in V\}, \qquad {}^\theta V = \{{}^\theta x \mid x \in V\}$$

with the operations

$${}^\rho(x+y) = {}^\rho x + {}^\rho y,\ {}^\theta(x+y) = {}^\theta x + {}^\theta y, \text{ and } {}^\rho(x\lambda) = ({}^\rho x)\rho(\lambda),\ {}^\theta(x\lambda) = ({}^\theta x)\theta(\lambda)$$

for x, $y \in V$ and $\lambda \in L$. Define quadratic forms ${}^\rho Q\colon {}^\rho V \to L$ and ${}^\theta Q\colon {}^\theta V \to L$ by

$${}^\rho Q({}^\rho x) = \rho\big(Q(x)\big) \quad \text{and} \quad {}^\theta Q({}^\theta x) = \theta\big(Q(x)\big) \quad \text{for } x \in V,$$

and L-bilinear maps

$$*_{\mathrm{Id}}\colon {}^\rho V\times{}^\theta V\to V,\quad *_\rho\colon {}^\theta V\times V\to{}^\rho V,\quad *_\theta\colon V\times{}^\rho V\to{}^\theta V$$

by

$${}^\rho x *_{\mathrm{Id}} {}^\theta y = x*y,\quad {}^\theta x *_\rho y = {}^\rho(x*y),\quad x *_\theta {}^\rho y = {}^\theta(x*y)\quad \text{for } x,\, y\in V.$$

We may then consider the quadratic form

$$Q\times{}^\rho Q\times{}^\theta Q\colon V\times{}^\rho V\times{}^\theta V\to L\times L\times L$$

and the product $\diamond\colon (V\times{}^\rho V\times{}^\theta V)\times(V\times{}^\rho V\times{}^\theta V)\to(V\times{}^\rho V\times{}^\theta V)$ defined by

$$(x,{}^\rho x,{}^\theta x)\diamond(y,{}^\rho y,{}^\theta y) = ({}^\rho x *_{\mathrm{Id}} {}^\theta y,\, {}^\theta x *_\rho y,\ x *_\theta {}^\rho y).$$

Straightforward calculations show that the F-vector space isomorphism $f\colon V\otimes_F L\to V\times{}^\rho V\times{}^\theta V$ given by

$$f(x\otimes\ell) = (x\ell, ({}^\rho x)\ell, ({}^\theta x)\ell)\qquad\text{for } x\in V \text{ and } \ell\in L$$

defines with ν an isomorphism of cyclic compositions

$$\Gamma_L \xrightarrow{\sim} (V\times{}^\rho V\times{}^\theta V,\ L\times L\times L,\ Q\times{}^\rho Q\times{}^\theta Q,\ \widetilde{\rho},\ \diamond).$$

3. Trialitarian algebras

In this section, we assume that the characteristic of the base field F is different from 2. Trialitarian algebras are defined in [7, §43] as 4-tuples $T=(E,L,\sigma,\alpha)$ where L is a cubic étale F-algebra, (E,σ) is a central simple L-algebra of degree 8 with an orthogonal involution, and α is an isomorphism from the Clifford algebra $C(E,\sigma)$ to a certain twisted scalar extension of E. We just recall in detail the special case of trialitarian algebras of the form $\operatorname{End}\Gamma$ for Γ a cyclic composition, because this is the main case for the purposes of this paper.

Let $\Gamma=(V,L,Q,\rho,*)$ be a cyclic composition (of dimension 8) over F, with L a field, and let $\theta=\rho^2$. Let also σ_Q denote the orthogonal involution on $\operatorname{End}_L V$ adjoint to Q. We will use the product $*$ to see that the Clifford algebra $C(V,Q)$ is split and the even Clifford algebra $C_0(V,Q)$ decomposes into a direct product of two split central simple L-algebras of degree 8. Using the notation of Remark 2.3, to any $x\in V$ we associate L-linear maps

$$\ell_x\colon {}^\rho V\to{}^\theta V\quad\text{and}\quad r_x\colon {}^\theta V\to{}^\rho V$$

defined by

$$\ell_x({}^\rho y) = x *_\theta {}^\rho y = {}^\theta(x*y)\quad\text{and}\quad r_x({}^\theta z) = {}^\theta z *_\rho x = {}^\rho(z*x)$$

for $y,\, z\in V$. From (2) it follows that for $x\in V$ the L-linear map

$$\alpha_*(x) = \begin{pmatrix} 0 & r_x\\ \ell_x & 0\end{pmatrix}\colon {}^\rho V\oplus{}^\theta V\to{}^\rho V\oplus{}^\theta V\quad\text{given by}\quad ({}^\rho y,{}^\theta z)\mapsto\big(r_x({}^\theta z),\ell_x({}^\rho y)\big)$$

satisfies $\alpha_*(x)^2 = Q(x)\,\mathrm{Id}$. Therefore, there is an induced L-algebra homomorphism

$$\alpha_*\colon C(V,Q)\to\operatorname{End}_L({}^\rho V\oplus{}^\theta V). \tag{4}$$

This homomorphism is injective because $C(V,Q)$ is a simple algebra, hence it is an isomorphism by dimension count. It restricts to an L-algebra isomorphism

$$\alpha_{*0}\colon C_0(V,Q)\xrightarrow{\sim}\operatorname{End}_L({}^\rho V)\times\operatorname{End}_L({}^\theta V),$$

see [7, (36.16)]. Note that we may identify $\operatorname{End}_L({}^\rho V)$ with the twisted algebra ${}^\rho(\operatorname{End}_L V)$ (where multiplication is defined by ${}^\rho f_1\cdot{}^\rho f_2 = {}^\rho(f_1\circ f_2)$) as follows: for

$f \in \operatorname{End}_L V$, we identify ${}^\rho f$ with the map ${}^\rho V \to {}^\rho V$ such that ${}^\rho f({}^\rho x) = {}^\rho(f(x))$ for $x \in V$. On the other hand, let σ_Q be the orthogonal involution on $\operatorname{End}_L V$ adjoint to Q. The algebra $C_0(V,Q)$ is canonically isomorphic to the Clifford algebra $C(\operatorname{End}_L V, \sigma_Q)$ (see [7, (8.8)]), hence it depends only on $\operatorname{End}_L V$ and σ_Q. We may regard α_{*0} as an isomorphism of L-algebras

$$\alpha_{*0}\colon C(\operatorname{End}_L V, \sigma_Q) \xrightarrow{\sim} {}^\rho(\operatorname{End}_L V) \times {}^\theta(\operatorname{End}_L V).$$

Thus, α_{*0} depends only on $\operatorname{End}_L V$ and σ_Q. The trialitarian algebra $\operatorname{End}\Gamma$ is the 4-tuple

$$\operatorname{End}\Gamma = (\operatorname{End}_L V, L, \sigma_Q, \alpha_{*0}).$$

An *isomorphism* of trialitarian algebras $\operatorname{End}\Gamma \xrightarrow{\sim} \operatorname{End}\Gamma'$, for $\Gamma' = (V', L', Q', \rho', *')$ a cyclic composition, is defined to be an isomorphism of F-algebras with involution $\varphi\colon (\operatorname{End}_L V, \sigma_Q) \xrightarrow{\sim} (\operatorname{End}_{L'} V', \sigma_{Q'})$ subject to the following conditions:

(i) the restriction of φ to the center of $\operatorname{End}_L V$ is an isomorphism $\varphi|_L\colon (L,\rho) \xrightarrow{\sim} (L', \rho')$, and
(ii) the following diagram (where $\theta' = {\rho'}^2$) commutes:

$$\begin{tikzcd}
C(\operatorname{End}_L V, \sigma_Q) \arrow[r, "\alpha_{*0}"] \arrow[d, "C(\varphi)"'] & {}^\rho(\operatorname{End}_L V) \times {}^\theta(\operatorname{End}_L V) \arrow[d, "{}^\rho\varphi \times {}^\theta\varphi"] \\
C(\operatorname{End}_{L'} V', \sigma_{Q'}) \arrow[r, "\alpha_{*'0}"] & {}^{\rho'}(\operatorname{End}_{L'} V') \times {}^{\theta'}(\operatorname{End}_{L'} V')
\end{tikzcd}$$

For example, it is straightforward to check that every isotopy $(\nu, f)\colon \Gamma \to \Gamma'$ induces an isomorphism $\operatorname{End}\Gamma \to \operatorname{End}\Gamma'$ mapping $g \in \operatorname{End}_L V$ to $f \circ g \circ f^{-1} \in \operatorname{End}_{L'} V'$. As part of the proof of the main theorem below, we show that every isomorphism $\operatorname{End}\Gamma \xrightarrow{\sim} \operatorname{End}\Gamma'$ is induced by an isotopy; see Lemma 3.4. (A cohomological proof that the trialitarian algebras $\operatorname{End}\Gamma$, $\operatorname{End}\Gamma'$ are isomorphic if and only if the cyclic compositions Γ, Γ' are isotopic is given in [7, (44.16)].)

We show that the trialitarian algebra $\operatorname{End}\Gamma$ admits a ρ-semilinear automorphism of order 3 if and only if Γ is reduced. More precisely:

Theorem 3.1. *Let $\Gamma = (V, L, Q, \rho, *)$ be a cyclic composition over F, with L a field.*

(i) *If Σ is a symmetric composition over F and $f\colon \Sigma \otimes (L,\rho) \to \Gamma$ is an L-linear isotopy, then the automorphism $\tau_{(\Sigma,f)} = \operatorname{Int}(f \circ \widehat{\rho} \circ f^{-1})|_{\operatorname{End}_L V}$ of $\operatorname{End}\Gamma$, where $\widehat{\rho}$ is defined in (3), is such that $\tau^3_{(\Sigma,f)} = \mathrm{Id}$ and $\tau_{(\Sigma,f)}|_L = \rho$. The automorphism $\tau_{(\Sigma,f)}$ only depends, up to conjugation in $\operatorname{Aut}_F(\operatorname{End}\Gamma)$, on the isomorphism class of Σ.*
(ii) *If $\operatorname{End}\Gamma$ carries an F-automorphism τ such that $\tau|_L = \rho$ and $\tau^3 = \mathrm{Id}$, then Γ is reduced. More precisely, there exists a symmetric composition Σ over F and an L-linear isotopy $f\colon \Sigma \otimes (L,\rho) \to \Gamma$ such that $\tau = \tau_{(\Sigma,f)}$.*

Proof. (i) It is clear that $\tau^3_{(\Sigma,f)} = \mathrm{Id}$ and $\tau_{(\Sigma,f)}|_L = \rho$. For the last claim, note that if $g\colon \Sigma \otimes (L,\rho) \to \Gamma$ is another L-linear isotopy, then $f \circ g^{-1}$ is an isotopy of Γ, hence $\operatorname{Int}(f \circ g^{-1})$ is an automorphism of $\operatorname{End}\Gamma$, and

$$\tau_{(\Sigma,f)} = \operatorname{Int}(f \circ g^{-1}) \circ \tau_{(\Sigma,g)} \circ \operatorname{Int}(f \circ g^{-1})^{-1}.$$

The proof of claim (ii) relies on three lemmas. Until the end of this section, we fix a cyclic composition $\Gamma = (V, L, Q, \rho, *)$, with L a field. We start with some general observations on ρ-semilinear automorphisms of $\operatorname{End}_L V$. For this, we consider the inclusions

$$L \hookrightarrow \operatorname{End}_L V \hookrightarrow \operatorname{End}_F V.$$

The field L is the center of $\operatorname{End}_L V$, hence every automorphism of $\operatorname{End}_L V$ restricts to an automorphism of L.

Lemma 3.2. *Let $\nu \in \{\mathrm{Id}_L, \rho, \theta\}$ be an arbitrary element in the Galois group $\mathrm{Gal}(L/F)$. For every F-linear automorphism φ of $\operatorname{End}_L V$ such that $\varphi|_L = \nu$, there exists an invertible transformation $u \in \operatorname{End}_F V$ such that $\varphi(f) = u \circ f \circ u^{-1}$ for all $f \in \operatorname{End}_L V$. The map u is uniquely determined up to a factor in $L^\times$; it is ν-semilinear, i.e., $u(x\lambda) = u(x)\nu(\lambda)$ for all $x \in V$ and $\lambda \in L$. Moreover, if $\varphi \circ \sigma_Q = \sigma_Q \circ \varphi$, then there exists $\mu \in L^\times$ such that*

$$Q\bigl(u(x)\bigr) = \nu\bigl(\mu \cdot Q(x)\bigr) \qquad \text{for all } x \in V.$$

Proof. The existence of u is a consequence of the Skolem–Noether theorem, since $\operatorname{End}_L V$ is a simple subalgebra of the simple algebra $\operatorname{End}_F V$: the automorphism φ extends to an inner automorphism $\mathrm{Int}(u)$ of $\operatorname{End}_F V$ for some invertible $u \in \operatorname{End}_F V$. Uniqueness of u up to a factor in $L^\times$ is clear because L is the centralizer of $\operatorname{End}_L V$ in $\operatorname{End}_F V$, and the ν-semilinearity of u follows from the equation $\varphi(f) = u \circ f \circ u^{-1}$ applied with f the scalar multiplication by an element in L.

Now, suppose φ commutes with σ_Q, hence for all $f \in \operatorname{End}_L V$

$$u \circ \sigma_Q(f) \circ u^{-1} = \sigma_Q(u \circ f \circ u^{-1}). \tag{5}$$

Let $\mathrm{Tr}_*(Q)$ denote the transfer of Q along the trace map $\mathrm{Tr}_{L/F}$, so $\mathrm{Tr}_*(Q)\colon V \to F$ is the quadratic form defined by $\mathrm{Tr}_*(Q)(x) = \mathrm{Tr}_{L/F}\bigl(Q(x)\bigr)$. The adjoint involution $\sigma_{\mathrm{Tr}_*(Q)}$ coincides on $\operatorname{End}_L V$ with σ_Q, hence from (5) it follows that $\sigma_{\mathrm{Tr}_*(Q)}(u)u$ centralizes $\operatorname{End}_L V$. Therefore, $\sigma_{\mathrm{Tr}_*(Q)}(u)u = \mu$ for some $\mu \in L^\times$. We then have $b_{\mathrm{Tr}_*(Q)}\bigl(u(x), u(y)\bigr) = b_{\mathrm{Tr}_*(Q)}(x, y\mu)$ for all x, $y \in V$, which means that

$$\mathrm{Tr}_{L/F}\bigl(b_Q(u(x), u(y))\bigr) = \mathrm{Tr}_{L/F}\bigl(\mu b_Q(x, y)\bigr). \tag{6}$$

Now, observe that since u is ν-semilinear, the map $c\colon V \times V \to L$ defined by $c(x, y) = \nu^{-1}\bigl(b_Q(u(x), u(y))\bigr)$ is L-bilinear. From (6), it follows that $c - \mu b_Q$ is a bilinear map on V that takes its values in the kernel of the trace map. But the value domain of an L-bilinear form is either L or $\{0\}$, and the trace map is not the zero map. Therefore, $c - \mu b_Q = 0$, which means that

$$\nu^{-1}\bigl(b_Q(u(x), u(y))\bigr) = \mu b_Q(x, y) \qquad \text{for all } x,\ y \in V,$$

hence $Q\bigl(u(x)\bigr) = \nu\bigl(\mu \cdot Q(x)\bigr)$ for all $x \in V$. □

Note that the arguments in the preceding proof apply to any quadratic space (V, Q) over L. By contrast, the next lemma uses the full cyclic composition structure: Let again $\nu \in \{\mathrm{Id}_L, \rho, \theta\}$. Given an invertible element $u \in \operatorname{End}_F V$ and $\mu \in L^\times$ such that for all $x \in V$ and $\lambda \in L$

$$u(x\lambda) = u(x)\nu(\lambda) \quad \text{and} \quad Q\bigl(u(x)\bigr) = \nu\bigl(\mu \cdot Q(x)\bigr),$$

we define an L-linear map $\beta_u\colon {}^\nu V \to \operatorname{End}_L({}^\rho V \oplus {}^\theta V)$ by

$$\beta_u({}^\nu x) = \begin{pmatrix} 0 & \nu(\mu)^{-1} r_{u(x)} \\ \ell_{u(x)} & 0 \end{pmatrix} \in \operatorname{End}_L({}^\rho V \oplus {}^\theta V) \quad \text{for } x \in V.$$

Then from (2) we get $\beta_u(x)^2 = \nu\big(Q(x)\big) = {}^\nu Q({}^\nu x)$. Therefore, the map β_u extends to an L-algebra homomorphism

$$\beta_u \colon C({}^\nu V, {}^\nu Q) \to \operatorname{End}_L({}^\rho V \oplus {}^\theta V).$$

Just like α_* in (4), the homomorphism β_u is an isomorphism. We also have an isomorphism of F-algebras $C({}^\nu\cdot)\colon C(V,Q) \to C({}^\nu V, {}^\nu Q)$ induced by the F-linear map $x \mapsto {}^\nu x$ for $x \in V$, so we may consider the F-automorphism ψ_u of $\operatorname{End}_L({}^\rho V \oplus {}^\theta V)$ that makes the following diagram commute:

$$\begin{tikzcd} C(V,Q) \arrow[r, "\alpha_*"] \arrow[d, "C({}^\nu\cdot)"'] & \operatorname{End}_L({}^\rho V \oplus {}^\theta V) \arrow[d, "\psi_u"] \\ C({}^\nu V, {}^\nu Q) \arrow[r, "\beta_u"] & \operatorname{End}_L({}^\rho V \oplus {}^\theta V) \end{tikzcd} \tag{7}$$

Lemma 3.3. *The F-algebra automorphism ψ_u restricts to an F-algebra automorphism ψ_{u0} of $\operatorname{End}_L({}^\rho V) \times \operatorname{End}_L({}^\theta V)$. The restriction of ψ_{u0} to the center $L \times L$ is either $\nu \times \nu$ or $(\nu \times \nu) \circ \varepsilon$ where ε is the switch map $(\ell_1, \ell_2) \mapsto (\ell_2, \ell_1)$. Moreover, if $\psi_{u0}|_{L\times L} = \nu \times \nu$, then there exist invertible ν-semilinear transformations u_1, $u_2 \in \operatorname{End}_F V$ such that*

$$\psi_u(f) = \begin{pmatrix} {}^\rho u_1 & 0 \\ 0 & {}^\theta u_2 \end{pmatrix} \circ f \circ \begin{pmatrix} {}^\rho u_1^{-1} & 0 \\ 0 & {}^\theta u_2^{-1} \end{pmatrix} \quad \textit{for all } f \in \operatorname{End}_L({}^\rho V \oplus {}^\theta V).$$

For any pair (u_1, u_2) satisfying this condition, we have

$$u_2(x * y) = u(x) * u_1(y) \ \textit{and}\ u_1(x * y) = \theta\nu(\mu)^{-1}\big(u_2(x) * u(y)\big) \ \textit{for all } x,\ y \in V.$$

Proof. The maps α_* and β_u are isomorphisms of graded L-algebras for the usual $(\mathbb{Z}/2\mathbb{Z})$-gradings of $C(V,Q)$ and $C({}^\nu V, {}^\nu Q)$, and for the "checker-board" grading of $\operatorname{End}_L({}^\rho V \oplus {}^\theta V)$ defined by

$$\operatorname{End}_L({}^\rho V \oplus {}^\theta V)_0 = \operatorname{End}_L({}^\rho V) \times \operatorname{End}_L({}^\theta V)$$

and

$$\operatorname{End}_L({}^\rho V \oplus {}^\theta V)_1 = \begin{pmatrix} 0 & \operatorname{Hom}_L({}^\theta V, {}^\rho V) \\ \operatorname{Hom}_L({}^\rho V, {}^\theta V) & 0 \end{pmatrix}.$$

Therefore, ψ_u also preserves the grading, and it restricts to an automorphism ψ_{u0} of the degree 0 component. Because the map $C({}^\nu\cdot)$ is ν-semilinear, the map ψ_u also is ν-semilinear, hence its restriction to the center of the degree 0 component is either $\nu \times \nu$ or $(\nu \times \nu) \circ \varepsilon$.

Suppose $\psi_{u0}|_{L\times L} = \nu \times \nu$. By Lemma 3.2 (applied with ${}^\rho V \oplus {}^\theta V$ instead of V), there exists an invertible ν-semilinear transformation $v \in \operatorname{End}_F({}^\rho V \oplus {}^\theta V)$ such that $\psi_u(f) = v \circ f \circ v^{-1}$ for all $f \in \operatorname{End}_F({}^\rho V \oplus {}^\theta V)$. Since ψ_{u0} fixes $\begin{pmatrix} \mathrm{Id}_{{}^\rho V} & 0 \\ 0 & 0 \end{pmatrix}$, the element v centralizes $\begin{pmatrix} \mathrm{Id}_{{}^\rho V} & 0 \\ 0 & 0 \end{pmatrix}$, hence $v = \begin{pmatrix} {}^\rho u_1 & 0 \\ 0 & {}^\theta u_2 \end{pmatrix}$ for some invertible u_1, $u_2 \in \operatorname{End}_F V$. The transformations u_1 and u_2 are ν-semilinear because v is ν-semilinear. From the commutativity of (7) we have $v \circ \alpha_*(x) = \beta_u({}^\nu x) \circ v = \alpha_*\big(u(x)\big) \circ v$ for all $x \in V$. By the definition of α_*, it follows that

$$u_1(z * x) = \theta\nu^{-1}(\mu)\big(u_2(z) * u(x)\big) \text{ and } u_2(x * y) = u(x) * u_1(y) \text{ for all } y,\ z \in V.$$

$\square$

Lemma 3.4. *Let $\nu \in \{\mathrm{Id}_L, \rho, \theta\}$. For every F-linear automorphism φ of $\operatorname{End}\Gamma$ such that $\varphi|_L = \nu$, there exists an invertible transformation $u \in \operatorname{End}_F V$, uniquely determined up to a factor in $L^\times$, such that $\varphi(f) = u \circ f \circ u^{-1}$ for all $f \in \operatorname{End}_L V$. Every such u is a ν-semilinear isotopy $\Gamma \to \Gamma$.*

Proof. The existence of u, its uniqueness up to a factor in $L^\times$, and its ν-semilinearity, were established in Lemma 3.2. It only remains to show that u is an isotopy.

Since φ is an automorphism of $\operatorname{End}\Gamma$, it commutes with σ_Q, hence Lemma 3.2 yields $\mu \in L^\times$ such that $Q\big(u(x)\big) = \nu\big(\mu \cdot Q(x)\big)$ for all $x \in V$. We may therefore consider the maps β_u and ψ_u of Lemma 3.3. Now, recall from [7, (8.8)] that $C_0(V,Q) = C(\operatorname{End}_L V, \sigma_Q)$ by identifying $x \cdot y$ for x, $y \in V$ with the image in $C(\operatorname{End}_L V, \sigma_Q)$ of the linear transformation $x \otimes y$ defined by $z \mapsto x \cdot b_Q(y,z)$ for $z \in V$. We have

$$\varphi(x \otimes y) = u \circ (x \otimes y) \circ u^{-1} \colon z \mapsto u\big(x \cdot b_Q(y, u^{-1}(z))\big) \qquad \text{for } x,\ y,\ z \in V.$$

Since u is ν-semilinear and $Q\big(u(x)\big) = \nu\big(\mu \cdot Q(x)\big)$ for all $x \in V$, it follows that

$$u\big(x \cdot b_Q(y, u^{-1}(z))\big) = u(x) \cdot \nu\big(b_Q(y, u^{-1}(z))\big) = u(x) \cdot \nu(\mu)^{-1} b_Q(u(y), z).$$

Therefore, $\varphi(x \otimes y) = \nu(\mu)^{-1} u(x) \otimes u(y)$ for x, $y \in V$, hence the following diagram (where β_u and $C({}^\nu\cdot)$ are as in (7)) is commutative:

$$\begin{tikzcd}
C_0(V,Q) \arrow[r, "C({}^\nu\cdot)|_{C_0(V,Q)}"] \arrow[d, "C(\varphi)"'] & C_0({}^\nu V, {}^\nu Q) \arrow[d, "\beta_u|_{C_0({}^\nu V, {}^\nu Q)}"] \\
C_0(V,Q) \arrow[r, "\alpha_{*0}"] & \operatorname{End}_L({}^\rho V) \times \operatorname{End}_L({}^\theta V)
\end{tikzcd}$$

On the other hand, the following diagram is commutative because φ is an automorphism of $\operatorname{End}\Gamma$:

$$\begin{tikzcd}
C_0(V,Q) \arrow[r, "\alpha_{*0}"] \arrow[d, "C(\varphi)"'] & \operatorname{End}_L({}^\rho V) \times \operatorname{End}_L({}^\theta V) \arrow[d, "{}^\rho\varphi \times {}^\theta\varphi"] \\
C_0(V,Q) \arrow[r, "\alpha_{*0}"] & \operatorname{End}_L({}^\rho V) \times \operatorname{End}_L({}^\theta V)
\end{tikzcd}$$

Therefore, $\beta_u|_{C_0({}^\nu V, {}^\nu Q)} \circ C({}^\nu\cdot)|_{C_0(V,Q)} = ({}^\rho\varphi \times {}^\theta\varphi) \circ \alpha_{*0}$. By comparing with (7), we see that $\psi_{u0} = {}^\rho\varphi \times {}^\theta\varphi$, hence $\psi_{u0}|_{L \times L} = \nu \times \nu$. Lemma 3.3 then yields ν-semilinear transformations u_1, $u_2 \in \operatorname{End}_F V$ such that

$$\psi_u(f) = \begin{pmatrix} {}^\rho u_1 & 0 \\ 0 & {}^\theta u_2 \end{pmatrix} \circ f \circ \begin{pmatrix} {}^\rho u_1^{-1} & 0 \\ 0 & {}^\theta u_2^{-1} \end{pmatrix} \qquad \text{for all } f \in \operatorname{End}_L({}^\rho V \oplus {}^\theta V),$$

hence $\psi_{u0} = \operatorname{Int}({}^\rho u_1) \times \operatorname{Int}({}^\theta u_2)$. But we have $\psi_{u0} = {}^\rho\varphi \times {}^\theta\varphi = \operatorname{Int}({}^\rho u) \times \operatorname{Int}({}^\theta u)$. Therefore, multiplying (u_1, u_2) by a scalar in $L^\times$, we may assume $u = u_1$ and $u_2 = \zeta u$ for some $\zeta \in L^\times$. Lemma 3.3 then gives

$$\zeta u(x * y) = u(x) * u(y) \text{ and } u(x * y) = \theta\nu(\mu)^{-1}\big((\zeta u(x)) * u(y)\big) \text{ for all } x,\ y \in V.$$

The second equation implies that $u(x * y) = \rho(\zeta)\theta\nu(\mu)^{-1}\big(u(x) * u(y)\big)$. By comparing with the first equation, we get $\rho(\zeta)\theta\nu(\mu)^{-1} = \zeta^{-1}$, hence $\nu(\mu) = \rho(\zeta)\theta(\zeta)$. Therefore, (ν, u) is an isotopy $\Gamma \to \Gamma$ with multiplier $\nu^{-1}(\zeta)$. □

We start with the proof of claim (ii) of Theorem 3.1. Suppose τ is an F-automorphism of $\operatorname{End}\Gamma$ such that $\tau|_L = \rho$ and $\tau^3 = \mathrm{Id}$. By Lemma 3.4, we may find an invertible ρ-semilinear transformation $t \in \operatorname{End}_F V$ such that $\tau(f) = t \circ f \circ t^{-1}$ for all $f \in \operatorname{End}_L V$, and every such t is an isotopy of Γ. Since $\tau^3 = \mathrm{Id}$, it follows that t^3 lies in the centralizer of $\operatorname{End}_L V$ in $\operatorname{End}_F V$, which is L. Let $t^3 = \xi \in L^\times$. We have $\rho(\xi) = t\xi t^{-1} = \nu$, hence $\xi \in F^\times$. The F-subalgebra of $\operatorname{End}_F V$ generated by L and t is a crossed product (L, ρ, ξ); its centralizer is the F-subalgebra $(\operatorname{End}_L V)^\tau$ fixed under τ, and we have

$$\operatorname{End}_F V \cong (L, \rho, \xi) \otimes_F (\operatorname{End}_L V)^\tau.$$

Now, $\deg(L, \rho, \xi) = 3$ and $\deg(\operatorname{End}_L V)^\tau = 8$, hence (L, ρ, ξ) is split. Therefore $\xi = N_{L/F}(\eta)$ for some $\eta \in L^\times$. Substituting $\eta^{-1}t$ for t, we get $t^3 = \mathrm{Id}_V$, and t is still a ρ-linear isotopy of Γ. Let $\mu \in L^\times$ be the corresponding multiplier, so that for all x, $y \in V$

$$Q\big(t(x)\big) = \rho\big(\rho(\mu)\theta(\mu)Q(x)\big) \quad \text{and} \quad t(x) * t(y) = \rho(\mu)t(x * y). \tag{8}$$

From the second equation we deduce that $t^3(x) * t^3(y) = N_{L/F}(\mu)t^3(x * y)$ for all x, $y \in V$, hence $N_{L/F}(\mu) = 1$ because $t^3 = \mathrm{Id}_V$. By Hilbert's Theorem 90, we may find $\zeta \in L^\times$ such that $\mu = \zeta\theta(\zeta)^{-1}$. Define $Q' = \rho(\zeta)\theta(\zeta)Q$ and let $x *' y = \zeta(x * y)$ for x, $y \in V$. Then Id_V is an isotopy $\Gamma \to \Gamma' = (V, L, Q', \rho, *')$ with multiplier ζ, and (8) implies that

$$Q'\big(t(x)\big) = \rho\big(Q'(x)\big) \quad \text{and} \quad t(x) *' t(y) = t(x *' y) \quad \text{for all } x,\, y \in V.$$

Now, observe that because t is ρ-semilinear and $t^3 = \mathrm{Id}_V$, the Galois group of L/F acts by semilinear automorphisms on V, hence we have a Galois descent (see [7, (18.1)]): the fixed point set $S = \{x \in V \mid t(x) = x\}$ is an F-vector space such that $V = S \otimes_F L$. Moreover, since $Q'\big(t(x)\big) = \rho\big(Q'(x)\big)$ for all $x \in V$, the restriction of Q' to S is a quadratic form $n\colon S \to F$, and we have $Q' = n_L$. Also, because $t(x *' y) = t(x) *' t(y)$ for all x, $y \in V$, the product $*'$ restricts to a product $\star$ on S, and $\Sigma = (S, n, \star)$ is a symmetric composition because Γ' is a cyclic composition. The canonical map $f\colon S \otimes_F L \to V$ yields an isomorphism of cyclic compositions $f\colon \Sigma \otimes (L, \rho) \xrightarrow{\sim} \Gamma'$, hence also an isotopy $f\colon \Sigma \otimes (L, \rho) \to \Gamma$. We have $t = f \circ \widehat{\rho} \circ f^{-1}$, hence τ is conjugation by $f \circ \widehat{\rho} \circ f^{-1}$. □

Theorem 3.5. *The assignment $\Sigma \mapsto \tau_{(\Sigma, f)}$ induces a bijection between the isomorphism classes of symmetric compositions Σ for which there exists an L-linear isotopy $f\colon \Sigma \otimes (L, \rho) \to \Gamma$ and conjugacy classes in $\operatorname{Aut}_F(\operatorname{End}\Gamma)$ of automorphisms τ of $\operatorname{End}\Gamma$ such that $\tau^3 = \mathrm{Id}$ and $\tau|_L = \rho$.*

Proof. We already know by Theorem 3.1 that the map induced by $\Sigma \mapsto \tau_{(\Sigma, f)}$ is onto. Therefore, it suffices to show that if the automorphisms $\tau_{(\Sigma, f)}$ and $\tau_{(\Sigma', f')}$ associated to symmetric compositions Σ and Σ' are conjugate, then Σ and Σ' are isomorphic. Assume $\tau_{(\Sigma', f')} = \varphi \circ \tau_{(\Sigma, f)} \circ \varphi^{-1}$ for some $\varphi \in \operatorname{Aut}_F(\operatorname{End}\Gamma)$, and let $t = f \circ \widehat{\rho} \circ f^{-1}$, $t' = f' \circ \widehat{\rho} \circ f'^{-1} \in \operatorname{End}\Gamma$ be the ρ-semilinear transformations such that $\tau_{(\Sigma, f)} = \operatorname{Int}(t)|_{\operatorname{End}_L V}$ and $\tau_{(\Sigma', f')} = \operatorname{Int}(t')|_{\operatorname{End}_L V}$. By Lemma 3.4 we may find an isotopy $(\nu, u)\colon \Gamma \to \Gamma$ such that $\varphi = \operatorname{Int}(u)|_{\operatorname{End}_L V}$. The equation $\tau_{(\Sigma', f')} = \varphi \circ \tau_{(\Sigma, f)} \circ \varphi^{-1}$ then yields $\operatorname{Int}(t')|_{\operatorname{End}_L V} = \operatorname{Int}(u \circ t \circ u^{-1})|_{\operatorname{End}_L V}$, hence there exists $\xi \in L^\times$ such that $u \circ t \circ u^{-1} = \xi t'$. Because $t^3 = t'^3 = \mathrm{Id}_V$, we have $N_{L/F}(\xi) = 1$, hence Hilbert's Theorem 90 yields $\eta \in L^\times$ such that $\xi = \rho(\eta)\eta^{-1}$.

Then $\eta^{-1}u\colon \Gamma \to \Gamma$ is a ν-semilinear isotopy such that $(\eta^{-1}u) \circ t \circ (\eta^{-1}u)^{-1} = \xi t'$, and we have a commutative diagram

$$\begin{CD}
\Sigma \otimes (L,\rho) @>{f'^{-1}\circ(\eta^{-1}u)\circ f}>> \Sigma' \otimes (L,\rho)\\
@V{\widehat{\rho}}VV @VV{\widehat{\rho}}V\\
\Sigma \otimes (L,\rho) @>{f'^{-1}\circ(\eta^{-1}u)\circ f}>> \Sigma' \otimes (L,\rho)
\end{CD}$$

The restriction of $f'^{-1} \circ (\eta^{-1}u) \circ f$ to Σ is an isotopy of symmetric compositions $\Sigma \to \Sigma'$; a scalar multiple of this map is an isomorphism $\Sigma \xrightarrow{\sim} \Sigma'$. □

4. Trialitarian automorphisms of groups of type D_4

Let F be a field of characteristic different from 2. By [7, (44.8)], for every adjoint simple group G of type D_4 over F there is a trialitarian algebra $T = (E, L, \sigma, \alpha)$ such that G is isomorphic to $\mathbf{Aut}_L(T)$. Since the correspondence between trialitarian algebras and adjoint simple groups of type D_4 is actually shown in [7, (44.8)] to be an equivalence of groupoids, we have $\mathbf{Aut}(G) \cong \mathbf{Aut}_F(T)$ if $G = \mathbf{Aut}_L(T)$. We then have a commutative diagram with exact rows:

$$\begin{CD}
1 @>>> \mathbf{Aut}_L(T) @>>> \mathbf{Aut}_F(T) @>>> \mathbf{Aut}_F(L) @>>> 1\\
@. @| @VV{\Phi}V @| @.\\
1 @>>> G @>>> \mathbf{Aut}(G) @>{\pi}>> (\mathfrak{S}_3)_L @>>> 1
\end{CD} \tag{9}$$

where Φ maps every F-automorphism τ of T to conjugation by τ, and $(\mathfrak{S}_3)_L$ is a (non-constant) twisted form of the symmetric group $\mathfrak{S}_3$. Here $\mathbf{Aut}_F(L)$ is the group scheme given by $\mathbf{Aut}_F(L)(R) = \mathrm{Aut}_{R\text{-alg}}(L \otimes_F R)$ for any commutative F-algebra R. Thus, the type of the group G is related as follows to the type of L and to $\mathbf{Aut}_F(L)$:

(i) type ${}^1\mathsf{D}_4$: $L \cong F \times F \times F$ and $\mathbf{Aut}_F(L)(F) \cong \mathfrak{S}_3$;
(ii) type ${}^2\mathsf{D}_4$: $L \cong F \times \Delta$ (with Δ a quadratic field extension of F) and $\mathbf{Aut}_F(L)(F) \cong \mathfrak{S}_2$;
(iii) type ${}^3\mathsf{D}_4$: L a cyclic cubic field extension of F and $\mathbf{Aut}_F(L)(F) \cong \mathbb{Z}/3\mathbb{Z}$;
(iv) type ${}^6\mathsf{D}_4$: L a non-cyclic cubic field extension of F and $\mathbf{Aut}_F(L)(F) = 1$.

Theorem 4.1. *Let G be an adjoint simple group of type D_4 over F. If $\mathbf{Aut}(G)(F)$ contains an outer automorphism φ such that φ^3 is inner, then G is of type ${}^1\mathsf{D}_4$ or ${}^3\mathsf{D}_4$, and in the trialitarian algebra $T = (E, L, \sigma, \alpha)$ such that $G \cong \mathbf{Aut}_L(T)$, the central simple L-algebra E is split.*

Proof. Since the image $\pi(\varphi) \in (\mathfrak{S}_3)_L(F)$ has order 3, it is clear from the characterization of the various types above that G cannot be of type ${}^2\mathsf{D}_4$. If G is of type ${}^6\mathsf{D}_4$, then after extending scalars from F to L we get as new cubic algebra $L \otimes_F L \cong L \times (\Delta \otimes_F L)$, where Δ, the discriminant of L, is a quadratic field extension. Thus, the group G_L has type ${}^2\mathsf{D}_4$; but the outer automorphism φ extends to an outer automorphism of G_L such that φ^3 is inner, in contradiction to the preceding case. Therefore, the type of G is ${}^1\mathsf{D}_4$ or ${}^3\mathsf{D}_4$. If G is of type ${}^1\mathsf{D}_4$, then the algebra E is split by [5, Example 15] or by [2, Theorem 13.1]. If G is of type ${}^3\mathsf{D}_4$, then after scalar extension to L the group G_L has type ${}^1\mathsf{D}_4$, so $E \otimes_F L$ is split. Therefore, the Brauer class of E has 3-torsion since it is split by a cubic extension. But it also has 2-torsion since E carries an orthogonal involution, hence E is split. $\square$

For the rest of this section, we focus on trialitarian automorphisms (i.e., outer automorphisms of order 3) of groups of type $^3\mathsf{D}_4$. Let G be an adjoint simple group of type $^3\mathsf{D}_4$ over F, and let L be its associated cyclic cubic field extension of F. Thus,

$$(\mathfrak{S}_3)_L(F) = \mathrm{Gal}(L/F) \cong \mathbb{Z}/3\mathbb{Z}.$$

If G carries a trialitarian automorphism φ defined over F, then $\pi\colon \mathbf{Aut}(G)(F) \to \mathrm{Gal}(L/F)$ is a split surjection, hence $\mathbf{Aut}(G)(F) \cong G(F) \rtimes (\mathbb{Z}/3\mathbb{Z})$. Therefore, it is easy to see that for any other trialitarian automorphism φ' of G defined over F, the elements φ and φ' are conjugate in $\mathbf{Aut}(G)(F)$ if and only if there exists $g \in G(F)$ such that $\varphi' = \mathrm{Int}(g) \circ \varphi \circ \mathrm{Int}(g)^{-1}$. When this occurs, we have $\pi(\varphi) = \pi(\varphi')$.

Theorem 4.2. (i) *Let G be an adjoint simple group of type $^3\mathsf{D}_4$ over F. The group G carries a trialitarian automorphism defined over F if and only if the trialitarian algebra $T = (E, L, \sigma, \alpha)$ (unique up to isomorphism) such that $G \cong \mathbf{Aut}_L(T)$ has the form $T \cong \mathrm{End}\,\Gamma$ for some reduced cyclic composition Γ.*

(ii) *Let $G = \mathbf{Aut}_L(\mathrm{End}\,\Gamma)$ for some reduced cyclic composition Γ. Every trialitarian automorphism φ of G has the form $\varphi = \mathrm{Int}(\tau)$ for some uniquely determined F-automorphism τ of $\mathrm{End}\,\Gamma$ such that $\tau^3 = \mathrm{Id}$ and $\tau|_L = \pi(\varphi)$. For a given nontrivial $\rho \in \mathrm{Gal}(L/F)$, the assignment $\Sigma \mapsto \mathrm{Int}(\tau_{(\Sigma,f)})$ defines a bijection between the isomorphism classes of symmetric compositions for which there exists an L-linear isotopy $f\colon \Sigma \otimes (L,\rho) \to \Gamma$ and conjugacy classes in $\mathbf{Aut}(G)(F)$ of trialitarian automorphisms φ of G such that $\pi(\varphi) = \rho$.*

Proof. Suppose first that φ is a trialitarian automorphism of G, and let $G = \mathbf{Aut}_L(T)$ for some trialitarian algebra $T = (E, L, \sigma, \alpha)$. Theorem 4.1 shows that the central simple L-algebra E is split, hence $T = \mathrm{End}\,\Gamma$ for some cyclic composition $\Gamma = (V, L, Q, \rho, *)$ over F. Substituting φ^2 for φ if necessary, we may assume $\pi(\varphi) = \rho$. The preimage of φ under the isomorphism $\Phi_F\colon \mathbf{Aut}_F(T)(F) \xrightarrow{\sim} \mathbf{Aut}(G)(F)$ (from (9)) is an F-automorphism τ of T such that $\varphi = \mathrm{Int}(\tau)$, $\tau^3 = \mathrm{Id}$, and $\tau|_L = \rho$. Since Φ_F is a bijection, τ is uniquely determined by φ. By Theorem 3.1(ii), the existence of τ implies that the cyclic composition Γ is reduced.

Conversely, if Γ is reduced, then by Theorem 3.1(i), the trialitarian algebra $\mathrm{End}\,\Gamma$ carries automorphisms τ such that $\tau^3 = \mathrm{Id}$ and $\tau|_L \neq \mathrm{Id}_L$. For any such τ, conjugation by τ is a trialitarian automorphism of G.

The last statement in (ii) readily follows from Theorem 3.5 because trialitarian automorphisms $\mathrm{Int}(\tau)$, $\mathrm{Int}(\tau')$ are conjugate in $\mathbf{Aut}(G)(F)$ if and only if τ, τ' are conjugate in $\mathrm{Aut}_F(\mathrm{End}\,\Gamma)$. $\square$

The following proposition shows that the algebraic subgroup of fixed points under a trialitarian automorphism of the form $\mathrm{Int}(\tau_{(\Sigma,f)})$ is isomorphic to $\mathbf{Aut}(\Sigma)$, hence in characteristic different from 2 and 3 it is a simple adjoint group of type G_2 or A_2, in view of the classification of symmetric compositions (see [3, §9]).

Proposition 4.3. *Let $G = \mathbf{Aut}_L\big(\mathrm{End}(\Sigma \otimes (L,\rho))\big)$ for some symmetric composition $\Sigma = (S, n, \star)$ over F and some cyclic cubic field extension L/F with nontrivial* *automorphism ρ. The subgroup of G fixed under the trialitarian automorphism* $\mathrm{Int}(\widehat{\rho})$ *is canonically isomorphic to* $\mathbf{Aut}(\Sigma)$.

Proof (Sketch). Mimicking the construction of the map α_* in (4), we may use the product $\star$ to construct an F-algebra isomorphism

$$\alpha_\star\colon C(S,n) \xrightarrow{\sim} \mathrm{End}_F(S \oplus S)$$

such that $\alpha_\star(x)(y,z) = (z \star x, x \star y)$ for x, y, $z \in S$. This isomorphism restricts to an isomorphism

$$\alpha_{\star 0}\colon C_0(S,n) \xrightarrow{\sim} (\mathrm{End}_F S) \times (\mathrm{End}_F S).$$

Let $\mathbf{Aut}(\mathrm{End}\,\Sigma)$ be the group scheme whose rational points are the F-algebra automorphisms φ of $(\mathrm{End}_F S, \sigma_n)$ that make the following diagram commute:

$$\begin{tikzcd}
C(\mathrm{End}_F S, \sigma_n) \arrow[r, "\alpha_{\star 0}"] \arrow[d, "C(\varphi)"'] & (\mathrm{End}_F S) \times (\mathrm{End}_F S) \arrow[d, "\varphi \times \varphi"] \\
C(\mathrm{End}_F S, \sigma_n) \arrow[r, "\alpha_{\star 0}"] & (\mathrm{End}_F S) \times (\mathrm{End}_F S)
\end{tikzcd}$$

Arguing as in Lemma 3.4, one proves that every such automorphism has the form $\mathrm{Int}(u)$ for some isotopy u of Σ. But if u is an isotopy of Σ with multiplier μ, then $\mu^{-1}u$ is an automorphism of Σ. Therefore, mapping every automorphism u of Σ to $\mathrm{Int}(u)$ yields an isomorphism $\mathbf{Aut}(\Sigma) \xrightarrow{\sim} \mathbf{Aut}(\mathrm{End}\,\Sigma)$. The extension of scalars from F to L yields an isomorphism

$$\mathbf{PGL}(S) \xrightarrow{\sim} R_{L/F}\big(\mathbf{PGL}(S \otimes_F L)\big)^{\mathrm{Int}(\widehat{\rho})},$$

which carries the subgroup $\mathbf{Aut}(\mathrm{End}\,\Sigma)$ to $G^{\mathrm{Int}(\widehat{\rho})}$. □

To conclude, we briefly mention without proof the analogue of Theorem 4.2 for simply connected groups, which we could have considered instead of adjoint groups. (Among simple algebraic groups of type D_4, only adjoint and simply connected groups may admit trialitarian automorphisms.)

Theorem 4.4. (i) *For any cyclic composition $\Gamma = (V, L, Q, \rho, *)$ over F, the group $\mathbf{Aut}_L(\Gamma)$ is simple simply connected of type ${}^3\mathsf{D}_4$, and there is an exact sequence of algebraic groups*

$$1 \longrightarrow \boldsymbol{\mu}_2^2 \longrightarrow \mathbf{Aut}_L(\Gamma) \xrightarrow{\mathrm{Int}} \mathbf{Aut}_L(\mathrm{End}\,\Gamma) \longrightarrow 1.$$

(ii) *A simple simply connected group of type ${}^3\mathsf{D}_4$ admits trialitarian automorphisms defined over F if and only if it is isomorphic to the automorphism group of a reduced symmetric composition $\Gamma = (V, L, Q, \rho, *)$. Conjugacy classes of trialitarian automorphisms of $\mathbf{Aut}_L(\Gamma)$ defined over F are in bijection with isomorphism classes of symmetric compositions Σ for which there is an isotopy $\Sigma \otimes (L, \rho) \to \Gamma$.*

Corollary 4.5. *Every simple adjoint or simply connected group of type ${}^3\mathsf{D}_4$ over a finite field admits trialitarian automorphisms.*

Proof. The Allen invariant is trivial, and cyclic compositions are reduced, see [9, §4.8]. □

Examples 4.6. (i) Let $F = \mathbb{F}_q$ be the field with q elements, where q is odd and $q \equiv 1 \bmod 3$. As observed in Example 2.2(i), every symmetric composition over F is isomorphic either to the Okubo composition Σ or to the split para-Cayley composition $\widetilde{C}$, and (up to isomorphism) there is a unique cyclic composition $\Gamma \cong \widetilde{C} \otimes (L, \rho) \cong \Sigma \otimes (L, \rho)$ with cubic algebra (L, ρ). Therefore, the simply connected group $\mathbf{Aut}_L(\Gamma)$ and the adjoint group $\mathbf{Aut}_L(\operatorname{End}\Gamma)$ have exactly two conjugacy classes of trialitarian automorphisms defined over F. See also [6, (9.1)].

(ii) Example 2.2(ii) describes a cyclic composition induced by a unique (up to isomorphism) symmetric composition. Its automorphism group is a group of type $^3\mathsf{D}_4$ admitting a unique conjugacy class of trialitarian automorphisms.

(iii) In contrast to (i) and (ii) we get from Example 2.2(iii) examples of groups of type $^3\mathsf{D}_4$ with many conjugacy classes of trialitarian automorphisms.

References

[1] Bruce N. Allison. Lie algebras of type D_4 over number fields. *Pacific J. Math.*, 156(2):209–250, 1992.

[2] Vladimir Chernousov, Alberto Elduque, Max-Albert Knus, and Jean-Pierre Tignol. Algebraic groups of type D_4, triality, and composition algebras. *Documenta Math.*, 18:413–468, 2013.

[3] Vladimir Chernousov, Max-Albert Knus, and Jean-Pierre Tignol. Conjugate classes of trialitarian automorphisms and symmetric compositions. *J. Ramanujan Math. Soc.*, 27:479–508, 2012.

[4] Alberto Elduque and Hyo Chul Myung. On flexible composition algebras. *Comm. Algebra*, 21(7):2481–2505, 1993.

[5] Skip Garibaldi. Outer automorphisms of algebraic groups and determining groups by their maximal tori. *Michigan Math. J.*, 61(2):227–237, 2012.

[6] Daniel Gorenstein and Richard Lyons. The local structure of finite groups of characteristic 2 type. *Mem. Amer. Math. Soc.*, 42(276):vii+731, 1983.

[7] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The Book of Involutions*. Number 44 in American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, R.I., 1998. With a preface in French by J. Tits.

[8] Tonny A. Springer. Oktaven, Jordan-Algebren und Ausnahmegruppen. Mathematisches Institut der Universität Göttingen, 1963. Vorlesungsausarbeitung von P. Eysenbach, 101 S.

[9] Tonny A. Springer and Ferdinand D. Veldkamp. *Octonions, Jordan algebras and exceptional groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

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