GENERIC ALGEBRAS: RATIONAL PARAMETRIZATION AND NORMAL FORMS

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Abstract. For every algebraically closed field $\boldsymbol{k}$ of characteristic different from 2, we prove the following:

(1) Generic finite dimensional (not necessarily associative) $\boldsymbol{k}$-algebras of a fixed dimension, considered up to isomorphism, are parametrized by the values of a tuple of algebraically independent over $\boldsymbol{k}$ rational functions in the structure constants.

(2) There exists an "algebraic normal form", to which the set of structure constants of every such algebra can be uniquely transformed by means of passing to its new basis, namely: there are two finite systems of nonconstant polynomials on the space of structure constants, $\{f_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$, such that the ideal generated by the set $\{f_i\}_{i\in I}$ is prime and, for every tuple c of structure constants satisfying the property $b_j(c)\neq 0$ for all $j\in J$, there exists a unique new basis of this algebra in which the tuple c' of its structure constants satisfies the property $f_i(c')=0$ for all $i\in I$.

1. Introduction

Hereinafter $\boldsymbol{k}$ denotes an algebraically closed field $\boldsymbol{k}$ of arbitrary characteristic different from 2. Let V be a finite dimensional vector space over the field $\boldsymbol{k}$. We put

$$n := \dim_{\boldsymbol{k}} V, \qquad \mathcal{M} := V^* \otimes V^* \otimes V.$$

Putting in correspondence to an element $\sum \ell\otimes\ell'\otimes v\in\mathcal{M}$ the bilinear map

$$V\times V\to V, \quad (a,b)\mapsto \sum \ell(a)\ell'(b)v,$$

we obtain a well-defined bijection between $\mathcal{M}$ and the set of all bilinear maps $V\times V\to V$. This bijections commutes with the natural action of the group

$$G := \mathrm{GL}(V)$$

* Supported by grants РФФИ 14-01-00160, НШ–2998.2014.1, and the granting program *Contemporary Problems of Theoretical Mathematics* of the Mathematics Branch of the Russian Academy of Sciences.

on $\mathcal{M}$ and its action on the the set of all bilinear maps $V \times V \to V$ given by the formula $(g \cdot \varphi)(a, b) = g \cdot (\varphi(g^{-1} \cdot a, g^{-1} \cdot b))$.

The problem of classification of G-orbits in $\mathcal{M}$ is linked with the applications in the theory of continuous and discrete dynamical systems; see [3], [6], [9]. In these papers, the case $n = 2$ is considered (under the restriction $\operatorname{char} \boldsymbol{k} \neq 3$ supplementary to $\operatorname{char} \boldsymbol{k} \neq 2$): in [3], [6] symmetric, and in [9] arbitrary generic maps are considered. The first main result of paper [9] is the proof of the existence of

(i) a nonempty open subset U in $\mathcal{M}$ and
(ii) four G-invariant rational functions f_1, f_2, f_3, f_4 on $\mathcal{M}$ defined everywhere on U

such that every two tensors $a, b \in U$ lie in the same G-orbit if and only if $f_i(a) = f_i(b)$ for all i. Thus generic G-orbits in $\mathcal{M}$ are bijectively parametrized by points of the image of rational map

$$U \to \mathbf{A}^4, \quad m \mapsto (f_1(m), f_2(m), f_3(m), f_4(m)). \tag{1}$$

The second main result of paper [9] is the proof of the existence of a "normal form" to which every tensor in U can be uniquely transformed by means of an element of G; the set of all normal forms is algebraic, i.e., has the appearance $U \cap \{m \in \mathcal{M} \mid h_i(m) = 0 \ \forall\ i \in I\}$ for some finite set $\{h_i\}_{i \in I}$ of polynomial functions on $\mathcal{M}$. In [9], the set U, the functions f_i, the normal forms, and the elements of G transforming tensors into normal forms are explicitly given by bulky formulas, and the proofs are largely based on cumbersome explicit calculations.

According to the classical Rosenlicht theorem [15, Thm. 2], for any action of an algebraic group on an irreducible variety, orbits of points in general position are separated by a finite tuple of invariant rational functions. Therefore, leaving aside the explicit formulas, only the claim about the number of separating invariants (4, what is equal to $\dim \mathcal{M} - \dim G$) does not follow from Rosenlicht's theorem. This claim has the following important complements (a), (b), and (c), which are not noted in [9]:

(a) The field $\boldsymbol{k}(\mathcal{M})^G$ of all G-invariant rational functions on $\mathcal{M}$ is a finite purely inseparable extension of the field $\boldsymbol{k}(f_1, \ldots, f_4)$, and for $\operatorname{char} \boldsymbol{k} = 0$ these fields coincide (this follows from [2, Sect. 18.2, Thm.] because $\boldsymbol{k}(f_1, \ldots, f_4)$ separates G-orbits of points of an open subset of $\mathcal{M}$).

(b) By [9, Prop. 4.1], in $\mathcal{M}$ there are points with finite G-stabilizers. Therefore, $\max\limits_{m \in \mathcal{M}} \dim G \cdot m = \dim G = 4$. Since the transcendence degree of $\boldsymbol{k}(\mathcal{M})^G$ over $\boldsymbol{k}$ is $\dim\,\mathcal{M} - \max\limits_{m\in\mathcal{M}} \dim G\cdot m$ (see [12, Sect. 2.3, Cor.]), this means that its is equal to 4.

(c) The functions $f_1, \dots, f_4$ are algebraically independent over $\boldsymbol{k}$, and therefore the image of map (1) contains a nonempty open subset of $\mathbf{A}^4$ (this follows from (a) and (b)).

Property (c) is specific for the considered action: given the negative solution to the Noether problem [16], for a linear action of an algebraic group, in general, it is impossible to separate orbits of points in general position by an algebraically independent system of rational invariants.

Here we show that both main results of paper [9], together with complements (a) (in a strengthened form) and (c), hold in *arbitrary* dimension n. Namely, we prove the following statements (part of them is intentionally formulated in the form close to the applications oriented papers [3], [6], [9]).

Theorem 1 (Rationality of the field of G-invariants of $\mathcal{M}$)**.** *The field $\boldsymbol{k}(\mathcal{M})^G$ is a rational extension of the field $\boldsymbol{k}$ of transcendental degree $n^3 - n^2$.*

Theorem 1 and Rosenlicht's theorem [15, Thm. 2] imply the following statement (its part concerning the case $n = 1$ is clear):

Corollary 1 (G-equivalence of points in general position in $\mathcal{M}$)**.** *Let $n \geqslant 2$. There exist a set $\{f_i\}_{i\in I}$ of $n^3 - n^2$ algebraically independent over $\boldsymbol{k}$ rational G-invariant functions on $\mathcal{M}$, and a finite set $\{h_j\}_{j\in J}$ of nonconstant polynomial functions on $\mathcal{M}$ such that:*

(i) *for any points a and b of a dense open subset $\{m\in\mathcal{M} \mid h_j(m)\neq 0\ \forall j\in J\}$ of $\mathcal{M}$ the conditions*
 (a) $G\cdot a = G\cdot b$,
 (b) $f_i(a) = f_i(b)$ *for all* $i \in I$
are equivalent;

(ii) *the set $\{f_i\}_{i\in I}$ generates the field $\boldsymbol{k}(\mathcal{M})^G$ over $\boldsymbol{k}$.*

If $n = 1$, then $\boldsymbol{k}(\mathcal{M})^G = \boldsymbol{k}$, and $\mathcal{M} \setminus \{0\}$ is a single G-orbit.

Theorem 2 (The existence of normal forms for $\mathcal{M}$)**.** *There exist two finite sets $\{s_p\}_{p\in P}$ and $\{t_q\}_{q\in Q}$ of nonconstant polynomial functions on $\mathcal{M}$, such that:*

(i) *the closed subset $\mathcal{S} := \{m\in\mathcal{M} \mid s_p(m)=0\ \forall p\in P\}$ is irreducible, and the map $\boldsymbol{k}(\mathcal{M})^G \to \boldsymbol{k}(\mathcal{S})$, $f \mapsto f|_{\mathcal{S}}$ is a well-defined $\boldsymbol{k}$-isomorphism of fields;*

(ii) *for every point a of the open dense subset $\{m \in \mathcal{M} \mid t_q(m) \neq 0\ \forall q\in Q\}$ of $\mathcal{M}$, there exists a unique element $g \in G$, for which $g\cdot a \in \mathcal{S}$.*

Actually, we use another interpretation of elements of $\mathcal{M}$, considering them as $\boldsymbol{k}$-algebra structures (not necessarily associative) on V in the sense of Bourbaki, i.e., ring structures, for which $\boldsymbol{k}$ is the domain of external operators; see [4], [18]. Fixing such a structure on V is equivalent to fixing a bilinear map $\varphi\colon V\times V\to V$ that defines the product of elements $a, b\in V$ by formula $ab:=\varphi(a,b)$. This gives a bijection between $\mathcal{M}$ and the set of all such structures, which assigns to an element $m=\sum \ell\otimes\ell'\otimes v\in\mathcal{M}$ the $\boldsymbol{k}$-algebra structure on V such that the product of elements $a, b\in V$ is defined by the formula

$$ab:=\sum \ell(a)\ell'(b)v. \tag{2}$$

We denote this $\boldsymbol{k}$-algebra by $\{V,m\}$ and call elements of the space $\mathcal{M}$ *multiplications*. For every element $g\in G$ and every $\boldsymbol{k}$-algebra $\{V,m\}$, the map $\{V,m\}\to\{V,g\cdot m\}$, $a\mapsto g\cdot a$ is an isomorphism of $\boldsymbol{k}$-algebras and every isomorphism between $\boldsymbol{k}$-algebras is obtained in this way. Thus, $\boldsymbol{k}$-algebras $\{V,a\}$ and $\{V,b\}$ are isomorphic if and only if $G\cdot a=G\cdot b$, and the automorphism group of the $\boldsymbol{k}$-algebra $\{V,m\}$ is the G-stabilizer of the multiplication $m\in\mathcal{M}$.

Let $\theta\in\mathrm{GL}(\mathcal{M})$ be the involution induced by permuting the first two factors in $V^*\otimes V^*\otimes V$:

$$\theta\colon \mathcal{M}\to\mathcal{M},\quad \ell\otimes\ell'\otimes v\mapsto \ell'\otimes\ell\otimes v.$$

Since $\operatorname{char}\boldsymbol{k}\neq 2$, every multiplication $m\in\mathcal{M}$ can be written in the form $m=\frac{1}{2}\big(m+\theta(m)\big)+\frac{1}{2}\big(m-\theta(m)\big)$, which implies that

$$\left.\begin{aligned}\mathcal{M}\ &=\mathcal{C}\oplus\mathcal{A},\\ \mathcal{C}:\ &=\{m\in\mathcal{M}\mid \theta(m)=m\},\\ \mathcal{A}:\ &=\{m\in\mathcal{M}\mid \theta(m)=-m\}.\end{aligned}\right\} \tag{3}$$

The algebra $\{V,m\}$ is commutative, i.e., $ab=ba$ in (2) (respectively, anticommutative, i.e., $ab=-ba$ (2)) if and only if $m\in\mathcal{C}$ (respectively, $m\in\mathcal{A}$). The subspaces $\mathcal{C}$ and $\mathcal{A}$ in $\mathcal{M}$ are G-invariant.

The proofs of Theorems 1 and 2 are based on the triviality claims in the following two theorems:

Theorem 3. *For points m in general position in $\mathcal{C}$, the algebras $\{V,m\}$ are simple commutative algebras with trivial automorphism group.*

The analogous statement holds for $\mathcal{M}$:

Theorem 4. *For points m in general position in $\mathcal{M}$, the algebras $\{V,m\}$ are simple and have trivial automorphism group.*

Theorems 1, 2, and 4 imply

Corollary 2. *There exists a system of simple pairwise nonisomorphic n-dimensional $\boldsymbol{k}$-algebras rationally parametrized by n^3-n^2 parameters algebraically independent over $\boldsymbol{k}$.*

We do not know whether the field $\boldsymbol{k}(\mathcal{C})^G$ is rational over $\boldsymbol{k}$ or not, however, Theorem 3 implies the following

Corollary 3. *The field $\boldsymbol{k}(\mathcal{C})^G$ is a stably rational extension of the field $\boldsymbol{k}$ of the transcendence degree $(n-1)n^2/2$.*

Using Theorem 3, one proves the analogue of Theorem 2 for $\mathcal{C}$:

Theorem 5 (The existence of normal forms for $\mathcal{C}$)**.** *The formulation of Theorem* 2 *holds true if $\mathcal{M}$ in it is replaced by $\mathcal{C}$.*

The following analogue of Corollary 2 follows from Theorems 3, 5 and Corollary 3:

Corollary 4. *There exists a system of simple pairwise nonisomorphic commutative n-dimensional $\boldsymbol{k}$-algebras rationally parametrized by points of an $(n-1)n^2/2$-dimensional stably rational variety.*

In the proof of Theorem 1, we use the following general statement:

Lemma 1. *Let L_1, L_2 L_3 be finite dimensional vector spaces over $\boldsymbol{k}$, each of which is endowed with a linear action of an affine algebraic group H. Assume that*

(i) *the actions of H on L_1 and L_2 are locally free;*
(ii) *$\boldsymbol{k}(L_1)^H$ is rational over $\boldsymbol{k}$;*
(iii) $\dim L_3 \geqslant \dim L_1$.

Then $\boldsymbol{k}(L_2 \oplus L_3)^H$ is also rational over $\boldsymbol{k}$.

Theorems 3 and 4 are proved in Section 2. Lemma 1, Theorems 1, 2, 5, and Corollary 3 are proved in Section 3.

Terminology and notation

In what follows, variety means algebraic variety over $\boldsymbol{k}$ in the sense of Serre.

The topological terms are related to the Zariski topology.

If K/L is a field extension, then K is called *rational*[1] over L if either K is a finitely generated purely transcendental extension of L or $K=L$. If there exists a field extension S/K such that S is rational over K and over L, then K is called *stably rational* over L.

For every nonempty set T of transformations of a set M, we denote by M^T the set of all joint fixed points of all transformation from T.

[1] *pure* in the terminology of [5].

We freely use the standard notation and conventions of the theory of algebraic groups and invariant theory [2], [12], [19]. In particular, we say that a certain property holds for *points in general position* in variety X if it holds for all points of an open dense subset of X (depending on the property under consideration). Action of an algebraic group G on a variety X means regular action, i.e., such that the map $G\times X\to X$, $(g,x)\mapsto g\cdot x$ defining this action is a morphism. If X is irreducible, then $\pi_{G,X}\colon X\dashrightarrow X/\!\!/G$ denotes a rational quotient for this action, i.e., $X/\!\!/G$ is a variety (defined up to birational isomorphism) and $\pi_{G,X}$ is a dominant rational map such that $\pi_{G,X}^*(\boldsymbol{k}(X/\!\!/G)=\boldsymbol{k}(X)$. An action of G on X is called *locally free* if the G-stabilizers of points in general position in X are trivial.

2. Proofs of Theorems 3 and 4

It will be convenient to fix a basis $e_1,\ldots,e_n$ in V and assume that $G=\mathrm{GL}_n(\boldsymbol{k})$ (indentifying elements of G with their matrices in this basis).

Let $\ell^1,\ldots,\ell^n$ be the dual basis in V^*. Then the elements

$$c_r^{pq}:=(\ell^p\otimes\ell^q+\ell^q\otimes\ell^p)\otimes e_r\in\mathcal{C},\quad \text{where } 1\leqslant p\leqslant q\leqslant n,\ 1\leqslant r\leqslant n, \tag{4}$$

(somehow ordered) constitute a basis in $\mathcal{C}$, and so

$$\dim\mathcal{C}=n^2(n+1)/2. \tag{5}$$

¿From (5), (3) and $\dim\mathcal{M}=n^3$ we get

$$\dim\mathcal{A}=(n-1)n^2/2. \tag{6}$$

Below we use the following known (see, e.g., [1, Lemma 3]) statement:

Lemma 2. *Let an algebraic group H act on a variety X and let Y be a closed subset of X. Then $N_H(Y):=\{h\in H\mid h\cdot Y\subseteq Y\}$ is a closed subgroup of H and dimension of the closure in X of the set* $\dim H\cdot Y$ *is at most* $\dim H+\dim Y-\dim N_H(Y)$.

Proof. Regarding the closedness see [2, Prop. I, 1,7]. The inequality is proved by applying the fiber dimension theorem to the morphism $H\times Y\to X$, $(h,y)\mapsto h\cdot y$. □

Proof of the claim of simplicity in Theorem 3.

Here, mutatis mutandis, the same arguments are applicable as in the proof of simplicity of $\{V,m\}$ for m in general position in $\mathcal{M}$, given in [10, p. 129]. For $n=1$ the claim is clear, therefore let $n\geqslant 2$.

For every integer $d\in\mathbf{Z}$, $0<d\leqslant n$, put

$$\mathcal{I}_d:=\{m\in\mathcal{C}\mid\{V,m\}\ \text{has a } d\text{-dimensional two-sided ideal}\}.$$

If a commutative $\boldsymbol{k}$-algebra $\{V, m\}$ has a two-sided d-dimensional ideal I, then for every element $g \in G$ the subspace $g \cdot I$ is a two-sided d-dimensional ideal of $\{V, g \cdot m\}$. Since G acts transitively on the set of all d-dimensional linear subspaces of V, there exists g such that $g \cdot I$ is the $\boldsymbol{k}$-linear span V_d of the vectors $e_1, \ldots, e_d$. Hence

$$\mathcal{I}_d = G \cdot \mathcal{L}_d, \text{ where} \tag{7}$$

$$\mathcal{L}_d := \{m \in \mathcal{C} \mid V_d \text{ is the two-sided ideal of } \{V, m\}\}. \tag{8}$$

Consider in G the subgroup

$$P_d := \{g \in G \mid g \cdot V_d = V_d\}. \tag{9}$$

It follows from (8), (9) that $P_d \cdot \mathcal{L}_d = \mathcal{L}_d$, which, in view of (7) and Lemma 2, implies that

$$\dim \mathcal{I}_d \leqslant \dim G + \dim \mathcal{L}_d - \dim P_d \tag{10}$$

and, since P_d is a parabolic subgroup of G, that the set $\mathcal{I}_d$ is closed in $\mathcal{C}$ (see, e.g., [19, Sect. 2.13, Lemma 2]).

It follows from (4) that V_d is the two-sided ideal of a commutative $\boldsymbol{k}$-algebra $\{V, m\}$, where

$$m = \sum_{\substack{1 \leqslant p \leqslant q \leqslant n \\ 1 \leqslant r \leqslant n}} \alpha_{pq}^r c_r^{pq}, \quad \alpha_{pq}^r \in \boldsymbol{k}, \tag{11}$$

if and only if $\alpha_{pq}^r = 0$ for all triples (p, q, r) from the range of summation in (11) satisfying the conditions:

(i) p or q is not greater than d;

(ii) $r > d$.

This and (8) imply that $\mathcal{L}_d$ is a linear subspace in $\mathcal{C}$ and, since in total there are $(n-d)\left(\binom{n+1}{2} - \binom{n-d+1}{2}\right) = (n-d)d(2n-d+1)/2$ these triples, we obtain

$$\dim \mathcal{L}_d = \dim \mathcal{C} - (n-d)d(2n-d+1)/2. \tag{12}$$

¿From $\dim G = n^2$, $\dim P_d = n^2 - d(n-d)$, and (12) we deduce that the right-hand side of inequality (10) is equal to $\dim \mathcal{C} - d(n-d)(2n - d + 1)/2 + d(n-d)$, which, for $d < n$, is smaller than $\dim \mathcal{C}$. Hence $\mathcal{C} \setminus \mathcal{I}_d$ is, for every $d < n$, a nonempty open subset of $\mathcal{C}$. Therefore, the intersection of all these open subsets is a nonempty open subset of $\mathcal{C}$ such that for every its point m the $\boldsymbol{k}$-algebra $\{V, m\}$ is simple. This proves the claim of simplicity of algebra in Theorem 3. □

Proof of the claim of triviality in Theorem 3.

1. As above, since for $n = 1$ the claim is clear, we assume further that $n \geqslant 2$.

Since the G-stabilizer G_x of every point $x \in \mathcal{C}$ is an algebraic subgroup of G, the claim will be proved if shall prove the existence in $\mathcal{C}$ the nonempty open subsets $\mathcal{C}_{(s)}$ and $\mathcal{C}_{(u)}$ such that G_x for every $x \in C_{(s)}$ (respectively, for every $x \in C_{(u)}$) does not contain nonidentity semisimple (respectively, unipotent) elements.

2. First we prove the existence of $\mathcal{C}_{(s)}$.

The plan is the following. We shall use that every semisimple element of the group G is conjugate to an element of its fixed maximal torus, see [2, Thms. 11.10, 10.6, Prop. 11.19]. Namely, let $\mathrm{X}(T)$ be the group (algebraic) characters $T \to \boldsymbol{k}^*$, and let Δ be the weight system of the T-module $\mathcal{C}$, i.e., the set of all characters $\mu \in \mathrm{X}(T)$ such that the dimension of its weight subspace

$$\mathcal{C}_\mu := \{m \in \mathcal{C} \mid t \cdot m = \mu(t)m \ \text{ for every element } t \in T\} \tag{13}$$

is positive. We have the decomposition

$$\mathcal{C} = \bigoplus\nolimits_{\mu \in \Delta} \mathcal{C}_\mu. \tag{14}$$

In view of (13), (14), for every element $t \in T$, the equality

$$\mathcal{C}^t := \{m \in \mathcal{C} \mid t \cdot m = m\} = \bigoplus\nolimits_{\{\mu \in \Delta \,|\, \mu(t)=1\}} \mathcal{C}_\mu. \tag{15}$$

holds. Since the set Δ is finite, it follows from (15) that, when t runs through T, the space $\mathcal{C}^t$, being the sum of some of the weight subspaces $\mathcal{C}_\mu$, runs through only finitely many T-invariant linear subspaces in $\mathcal{C}$. We shall find an integer h such that for every nonidentity $t \in T$ the inequalities

$$\dim \mathcal{C}^t \leqslant h, \tag{16}$$

$$h + \dim G - \dim T = h + n^2 - n < \dim \mathcal{C} = n^2(n+1)/2. \tag{17}$$

hold. In view of Lemma 2, it then follows from (16), (17) that the closure of the set $G \cdot \mathcal{C}^t$ in $\mathcal{C}$ is a proper subvariety of $\mathcal{C}$. Since, as is explained above, the set of these closures is finite, the complement to their union is a nonempty open subset of $\mathcal{C}$. ¿From the above it follows that it can be taken as $\mathcal{C}_{(s)}$.

3. We now proceed to implement this plan. As T we take the maximal torus consisting of diagonal matrices. Put

$$\varepsilon_i \colon T \to \boldsymbol{k}^*, \quad \operatorname{diag}(t_1, \ldots, t_n) \mapsto t_i. \tag{18}$$

Then $\varepsilon_1, \ldots, \varepsilon_n$ is a basis of the free abelain group $\mathrm{X}(T)$ and

$$t \cdot e_i = \varepsilon_i(t) e_i, \quad t \cdot \ell_i = \varepsilon_i^{-1}(t) \ell_i \quad \text{for every element } t \in T. \tag{19}$$

Formulas (19) and (4) imply the inclusion $c_r^{pq} \in \mathcal{C}_{\varepsilon_p^{-1}\varepsilon_q^{-1}\varepsilon_r}$, from which we obtain

$$\left.\begin{aligned} &\Delta = \Delta' \cup \Delta'', \ \text{ where}\\ &\Delta' := \{\varepsilon_p^{-1}\varepsilon_q^{-1}\varepsilon_r \mid 1 \leqslant p \leqslant q \leqslant n,\ 1 \leqslant r \leqslant n,\ r \neq p,\ r \neq q\},\\ &\Delta'' := \{\varepsilon_s^{-1} \mid 1 \leqslant s \leqslant n\}, \ \text{ and}\\ &\dim \mathcal{C}_\mu = \begin{cases} 1 & \text{if } \mu \in \Delta',\\ n & \text{if } \mu \in \Delta''. \end{cases} \end{aligned}\right\} \qquad (20)$$

Every character $\alpha \in \mathrm{X}(T)$ defines a partition of the set Δ as the union of mutually disjoint sequences of weights $\mu_1, \ldots, \mu_d \in \Delta$ having the properties

(i) $\mu_{i+1} = \alpha\mu_i$ for every $i = 1, \ldots, d-1$,
(ii) $\alpha^{-1}\mu_1 \notin \Delta$ and $\alpha\mu_d \notin \Delta$.

We call such sequences α-*series*.

The kernel of action of G on $\mathcal{C}$ is trivial: being a normal subgroup, it lies in the center of G, hence also in T, and it follows from (18), (19), and the inclusion $\Delta'' \subset \Delta$ that the kernel of action of T on $\mathcal{C}$ is trivial.

Fix in T a nonidentity element $t = \operatorname{diag}(t_1, \ldots, t_n)$.

If $t_1 = \ldots = t_n \neq 1$, then it follows from (20) that $\mathcal{C}^t = \{0\}$, so $\mathcal{C} \setminus \mathcal{C}^t$ is a nonempty open subset of $\mathcal{C}$.

Now let $t_s \neq t_{s+1}$ for some s, or, in other words,

$$\alpha_s(t) \neq 1, \quad \text{where} \quad \alpha_s := \varepsilon_s\varepsilon_{s+1}^{-1}. \qquad (21)$$

It follows from (20) and the definition of α-series that all sequences of forms (22)–(27) listed below are the α_s-series for every p, q, and r satisfying the specified restrictions:

$$\varepsilon_s^2\varepsilon_{s+1},\ \varepsilon_s^{-1},\ \varepsilon_{s+1}^{-1},\ \varepsilon_{s+1}^{-2}\varepsilon_s; \qquad (22)$$

$$\varepsilon_p^{-1}\varepsilon_s^{-1}\varepsilon_{s+1},\ \varepsilon_p^{-1},\ \varepsilon_j^{-1}\varepsilon_{s+1}^{-1}\varepsilon_s, \quad p \neq s, s+1; \qquad (23)$$

$$\varepsilon_p^{-1}\varepsilon_q^{-1}\varepsilon_r, \quad r \neq p, q \text{ and among } p, q, r \text{ there is no } s \text{ and } s+1; \qquad (24)$$

$$\varepsilon_p^{-1}\varepsilon_q^{-1}\varepsilon_{s+1},\ \varepsilon_p^{-1}\varepsilon_q^{-1}\varepsilon_s, \quad p \neq s, s+1,\ q \neq s, s+1; \qquad (25)$$

$$\varepsilon_s^{-1}\varepsilon_q^{-1}\varepsilon_r,\ \varepsilon_{s+1}^{-1}\varepsilon_q^{-1}\varepsilon_r, \quad r \neq s, s+1,\ q \neq s, s+1,\ r \neq q; \qquad (26)$$

$$\varepsilon_s^{-2}\varepsilon_r,\ \varepsilon_s^{-1}\varepsilon_{s+1}^{-1}\varepsilon_r,\ \varepsilon_{s+1}^{-2}\varepsilon_r, \quad r \neq s, s+1. \qquad (27)$$

Since for every weight from Δ there exists only one of the α_s-series of the forms (22)–(27) in which it lies, there are no other α_s-series. The number of different α_s-series of every form (22)–(27) is given in the following table where in the top raw are listed the forms of α_s-series and in the bottom the specified numbers:

(22)	(23)	(24)	(25)	(26)	(27)
1	$n-2$	$(n-2)^2(n-3)/2$	$(n-1)(n-2)/2$	$(n-2)(n-3)$	$n-2$

It follows from (21) and property (i) in the definition of α-series that t can not lie in the intersection of kernels of any two neighboring weights of every α_s-series. Therefore, the set of weights of any α_s-series of form (22) that contains t in its kernel is either empty, or contains only one weight, or exactly two non-neighboring weights. In view of (20) this shows that the sum of dimensions of the weight spaces of this set is not greater than $n+1$.

Similarly, we obtain that if μ runs through all the weights of some α_s-series of form (23), (24), (25), (26), or (27) that contains t in its kernel, then the number $\sum \dim \mathcal{C}_\mu$ is not greater than, respectively, n, 1, 1, 1, or 2 (for form (23), we use here that $n \geqslant 2$).

It follows from this and the above table that if μ runs through *all* α_s-series of form (22), (23), (24), (25), (26), or (27) that contains t in its kernel, then the number $\sum \dim \mathcal{C}_\mu$ is not bigger than, respectively, $n+1$, $n(n-2)$, $(n-2)^2(n-3)/2$, $(n-1)(n-2)/2$, $(n-2)(n-3)$, or $2(n-2)$.

Since Δ is the union of all α_s-series, this shows that the inequality (16) holds for

$$\begin{aligned} h = n+1+n(n-2)+(n-2)^2(n-3)/2 \\ +(n-1)(n-2)/2+(n-2)(n-3)+2(n-2). \end{aligned} \tag{28}$$

A simple direct calculation then shows that inequality (17) follows from (28). This completes the proof of the existence of $\mathcal{C}_{(s)}$.

4. We now turn to the proof of the existence of $\mathcal{C}_{(u)}$.

First, we prove that the G-stabilizers of points m in general position in $\mathcal{C}$ are finite, or, equivalently, that the automorphism groups of the $\boldsymbol{k}$-algebras $\{V, m\}$ are finite. Since the set of points of $\mathcal{C}$, whose G-stabilizer has minimal dimension, is open in $\mathcal{C}$, in order to prove the finiteness it suffices to find a single multiplication $m_0 \in \mathcal{C}$ such that the automorphism group of the $\boldsymbol{k}$-algebra $\{V, m_0\}$ is finite. Note, however, that from the triviality of the automorphism group of the $\boldsymbol{k}$-algebra $\{V, m_0\}$ we can *not* conclude that the automorphism group of the $\boldsymbol{k}$-algebra $\{V, m\}$ is trivial for points m in general position in $\mathcal{C}$, see [12, Sect. 6.1, Example 1].

Define $m_0 \in \mathcal{C}$ by the following multiplication table:

$$e_ie_j := \begin{cases} e_i & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases} \quad \text{for all } 1 \leqslant i, j, \leqslant n. \tag{29}$$

Then $A := \{V, m_0\}$ is the direct sum of one-dimensional subalgebras: $A = A_1 \oplus \cdots \oplus A_n$, where e_i is the basis of A_i, and the multiplication is given by the formula $e_i^2 = e_i$ showing that e_i is the identity in the $\boldsymbol{k}$-algebra A_i.

All one-dimensional two-sided ideals I of A are exhausted by the ideals $A_1, \ldots, A_n$. Indeed, let a vector $v = \alpha_1 e_1 + \cdots + \alpha_n e_n$, where $\alpha_1, \ldots, \alpha_n \in \boldsymbol{k}$, be a basis of I, and let $\alpha_p \neq 0$. Then $e_p v = \alpha_p e_p$ in view of (29). On the other hand, since I is a one-dimensional two-sided ideal, $e_p v = \lambda v$ for some $\lambda \in \boldsymbol{k}$. It then follows from $\alpha_p \neq 0$ that $\lambda \neq 0$, hence $v = \alpha_p \lambda^{-1} e_p$. Therefore, $I = A_p$.

Now let σ be an automorphism of the $\boldsymbol{k}$-algebra A. Then $\sigma(A_i)$ for every i is a one-dimensional two-sided ideal of A, hence, as is proved, it coincides with A_j for some j and the restriction of σ to A_i is an isomorphism of $\boldsymbol{k}$-algebras $A_i \to A_j$. Since isomorphism maps identity to identity, $\sigma(e_i) = e_j$. Thus the automorphism group of the $\boldsymbol{k}$-algebra A leaves invariant the finite set $\{e_1, \ldots, e_n\}$. Since $e_1, \ldots, e_n$ is a basis in V, its action on this set has no kernel. Therefore, this group is finite[2].

5. If $\operatorname{char} k = 0$, then the order of every nonidentity unipotent element is infinite, therefore, the existence of $\mathcal{C}_{(u)}$ follows from the proved finiteness of the G-stabilizers of points in general position in $\mathcal{C}$. In the case of $\operatorname{char} \boldsymbol{k} > 0$, the order of every unipotent element is finite, therefore, this argument is not applicable. Consider this case.

Let $u \in G$ be a nonidentity unipotent element, let C be the finite cyclic group that it generates, and let $\boldsymbol{k}C$ be its group algebra over $\boldsymbol{k}$. The order of C is a positive power $\operatorname{char} \boldsymbol{k}$. It follows from the Jordan normal form theory (see also [8]), that for every positive integer d not bigger than C, there is a unique up to isomorphism indecomposable d-dimensional $\boldsymbol{k}C$-module M_d, and there are no other indecomposable $\boldsymbol{k}C$-modules. In particular, $I := M_1$ is a one-dimensional trivial $\boldsymbol{k}C$-module. The said implies that M_d is isomorphic to its dual module M_d^*. In a certain basis, the matrix of the linear operator $M_d \to M_d$, $m \mapsto u \cdot m$ is the Jordan block with eigenvalue 1, and therefore,

$$\dim_{\boldsymbol{k}} M_d^C = 1. \tag{30}$$

If the $\boldsymbol{k}C$-modules P and Q are isomorphic, we write $P \simeq Q$. If N is a finite dimensional $\boldsymbol{k}C$-module, we denote by $N^{\oplus r}$ and $N^{\otimes r}$ respectively the direct sum and the tensor product of $r > 0$ copies of the $\boldsymbol{k}C$-module N; we put $N^{\otimes 0} := I^{\oplus \dim_{\boldsymbol{k}} N}$. By $|N|$ we denote the number of indecomposable $\boldsymbol{k}C$-submodules, whose direct sum is N (by

[2]Its order is $n!$ since the element of G inducing a permutation of $e_1, \ldots, e_n$ is an automorphism of the $\boldsymbol{k}$-algebra A.

the Krull–Schmidt theorem this number is well-defined). According to [13, Lemma 2.1],

$$|M_p \otimes M_q| = \min\{p, q\} \text{ for every } p \text{ and } q. \tag{31}$$

Lemma 3. *For every finite dimensional* $\boldsymbol{k}C$*-modules* P *and* Q *and every positive integer* d*, the following inequality holds:*

$$|P \otimes Q| \leqslant |P \otimes I^{\oplus \dim_{\boldsymbol{k}} Q}|. \tag{32}$$

Proof of Lemma 3. Decompose P and Q as the direct sums of indecomposable $\boldsymbol{k}C$-submodules:

$$P \simeq \bigoplus\nolimits_p M_{s_p}, \quad Q \simeq \bigoplus\nolimits_q M_{r_q}. \tag{33}$$

¿From these decompositions we obtain

$$\begin{aligned}|P \otimes Q| &\overset{(33)}{=} |\bigoplus\nolimits_{p,q} \big(M_{s_p} \otimes M_{r_q}\big)| = \sum\nolimits_{p,q} |M_{s_p} \otimes M_{r_q}| \\ &\overset{(31)}{\leqslant} \sum\nolimits_{p,q} |M_{s_p} \otimes I^{\oplus r_q}| = |\bigoplus\nolimits_{p,q} (M_{s_p} \otimes I^{\oplus r_q})| \overset{(33)}{=} |P \otimes I^{\oplus \dim_{\boldsymbol{k}} Q}|.\end{aligned}$$

This proves Lemma 3. □

Since $u \in G$, and V and $\mathcal{C}$ are the $\boldsymbol{k}G$-modules, then V and $\mathcal{C}$ are also $\boldsymbol{k}C$-modules. Since u is a nonidentity and the kernel of the $\boldsymbol{k}G$-module V is trivial, the decomposition of the $\boldsymbol{k}C$-module V as a direct sum of the indecomposable submodules contains at least one summand of dimension $s \geqslant 2$. Therefore, for some $\boldsymbol{k}C$-module N we have

$$V \simeq M_s \oplus N, \quad s \geqslant 2. \tag{34}$$

Since the $\boldsymbol{k}G$-modules $\mathcal{C}$ and $(\mathrm{Sym}^2 V^*) \otimes V$ are isomorphic, from (34) and the self-duality of finite dimensional $\boldsymbol{k}C$-modules we obtain

$$\mathcal{C} \simeq (\mathrm{Sym}^2(M_s \oplus N)) \otimes (M_s \oplus N). \tag{35}$$

Using that $\mathrm{Sym}^2(A \oplus B)$ and $(\mathrm{Sym}^2 A) \oplus (A \otimes B) \oplus (\mathrm{Sym}^2 B)$ are isomorphic for every modules A and B (see, e.g., [7, (B.2), p. 473]), from (35) we obtain

$$\begin{aligned}\mathcal{C} \simeq ((\mathrm{Sym}^2 M_s) \otimes M_s) \oplus (M_s^{\otimes 2} \otimes N) \oplus ((\mathrm{Sym}^2 N) \otimes M_s) \\ \oplus ((\mathrm{Sym}^2 M_s) \otimes N) \oplus (N^{\otimes 2} \otimes M_s) \oplus ((\mathrm{Sym}^2 N) \otimes N).\end{aligned} \tag{36}$$

Note also that $\mathrm{Sym}^2 M_s \subsetneq M_s^{\otimes 2}$ and (31) imply the inequality

$$|\mathrm{Sym}^2 M_s| \leqslant s - 1. \tag{37}$$

¿From Lemma 3, taking into account (37), (31) and the definition of $|\cdot|$, we obtain

$$
\left.\begin{aligned}
|(\mathrm{Sym}^2 M_s)\otimes M_s| &\overset{(32)}{\leqslant} |(\mathrm{Sym}^2 M_s)\otimes I^{\oplus s}| \overset{(37)}{\leqslant} s(s-1),\\
|M_s^{\otimes 2}\otimes N| &\overset{(32)}{\leqslant} |M_s^{\otimes 2}\otimes I^{\oplus(n-s)}| \overset{(31)}{=} s(n-s),\\
|(\mathrm{Sym}^2 N)\otimes M_s| &\overset{(32)}{\leqslant} \big|M_s\otimes I^{\oplus\frac{(n-s+1)(n-s)}{2}}\big| = \frac{(n-s+1)(n-s)}{2},\\
|(\mathrm{Sym}^2 M_s)\otimes N| &\overset{(32)}{\leqslant} |(\mathrm{Sym}^2 M_s)\otimes I^{\oplus(n-s)}| \overset{(37)}{\leqslant} (n-s)(s-1),\\
|N^{\otimes 2}\otimes M_s| &\overset{(32)}{\leqslant} |M_s\otimes I^{\oplus(n-s)^2}| = (n-s)^2,\\
|(\mathrm{Sym}^2 N)\otimes N| &\leqslant \dim((\mathrm{Sym}^2 N)\otimes N) = \frac{(n-s+1)(n-s)^2}{2}.
\end{aligned}\right\}\quad (38)
$$

It follows from (36) and (38) that

$$
\begin{aligned}
|\mathcal{C}| \leqslant s(s-1) + s(n-s) + \frac{(n-s+1)(n-s)}{2} + (n-s)(s-1)&\\
+ (n-s)^2 + \frac{(n-s+1)(n-s)^2}{2}.&
\end{aligned}\qquad (39)
$$

But (30) implies that for every finite dimensional $\boldsymbol{k}C$-module N the equality $N^C = |N|$ holds. Therefore, (39) means that $\dim_{\boldsymbol{k}} \mathcal{C}^u$ is not bigger than the right-hand side of inequality (39). As is known, dimension of the centralizer of the element u in G is not smaller than the rank of G that is equal to n (see, e.g., [19, Prop. 1, p. 94]). Since this centralizer lies in $N_G(\mathcal{C}^u)$, it now follows from Lemma 2 that dimension of the closure $\overline{G\cdot\mathcal{C}^u}$ of the set $G\cdot\mathcal{C}^u$ in $\mathcal{C}$ is not bigger than the number

$$
\begin{aligned}
s(s-1) + s(n-s) + \frac{(n-s+1)(n-s)}{2} + (n-s)(s-1)&\\
+ (n-s)^2 + \frac{(n-s+1)(n-s)^2}{2} + n^2 - n.&
\end{aligned}\qquad (40)
$$

Twice the difference between $\dim\mathcal{C} = (n^3+n^2)/2$ and number (40) is equal to

$$
n^2(3s-5) + n(-3s^2+4s+3) + (s^3-2s^2+s). \qquad (41)
$$

Consider (41) is the polynomial in n. Its leading coefficient is positive for every $s \geqslant 2$, and the discriminant is equal to $-3s^3(s-20/3)-54s(s-22/27)+9$, which shows that the latter is negative for every $s \geqslant 7$. Direct calculation shows that it is negative also for $s = 2, 3, 4, 5, 6$. Therefore, for every $s \geqslant 2$ the specified difference is positive.

Hence $\mathcal{C}\setminus\overline{G\cdot\mathcal{C}^u}$ is a nonempty open subset of $\mathcal{C}$ such that the G-stabilizers of its points do not contain elements conjugate to u. But there are only finitely many conjugacy classes of unipotent elements in G in view of the finiteness of the set of possible Jordan normal forms of such elements (see also [19]). Therefore, if u runs through the set of representatives of nonidentity conjugacy classes of unipotent elements in G, then the intersection of the open subsets $\mathcal{C} \setminus \overline{G \cdot \mathcal{C}^u}$ is a nonempty open subsubset of $\mathcal{C}$. The said implies that this subset may be taken as $\mathcal{C}_{(u)}$.

This completes the proof of Theorem 3. $\square$

Remark. Actually, Parts 2 and 3 of the proof give more than we used. Namely, the arguments used there show that for every constant $\alpha \in \boldsymbol{k}$ and every element $t \in T$ which does not lie in the center of the group G (i.e., which is not scalar), inequalitites (16) and (17), where h is given by formula (28), still hold if $\mathcal{C}^t$ in (16) is replaced by $\{m \in \mathcal{C} \mid t \cdot m = \alpha m\}$. This means that we have proved the following

Theorem 6. *For the natural action of the group G on the projectivization $P\mathcal{C}$ of the vector space $\mathcal{C}$, the stabilizers of points in general position in $P\mathcal{C}$ coincide with the center of the group G.*

Proof of Theorem 4.

The claim of simplicity is proved in [10, Thm. 4], and that of triviality follows from Theorem 3 in view of the first equality in (3). $\square$

3. Proofs of Lemma 1, Theorems 1, 2, 5, and Corollary 3

The proof of Lemma 1 is based on one frequently used statement known as the No-name Lemma. We recall its formulation.

Let $\pi\colon E \to X$ be an algebraic vector bundle over a variety X. Assume that E and X are endowed with the actions of an affine algebraic group G such that π is the G-equivariant morphism and for every elements $g \in G$, $x \in X$ the map of fibers $\pi^{-1}(x) \to \pi^{-1}(g \cdot x)$ defined by the transformation g is $\boldsymbol{k}$-linear.

Lemma 4 (No-name Lemma)**.** *Using the above notation, assume that the action of G on X is locally free and put $d = \dim E - \dim X$. Consider the action of G on $X \times \mathbf{A}^d$ via the first factor and let $\pi_1\colon X \times \mathbf{A}^d \to X$ be the natural projection. Then there exists a G-equivariant birational isomorphism $\varphi\colon E \dashrightarrow X \times \mathbf{A}^d$ such that the following diagram is commutative*

$$\begin{tikzcd} E \arrow[rr, dashed, "\varphi"] \arrow[dr, "\pi"'] & & X \times \mathbf{A}^d \arrow[dl, "\pi_1"] \\ & X & \end{tikzcd}.$$

Proof. If the group G is finite, it is the classical "Speiser Lemma" [20]; in the general case, the proof, valid in arbitrary characteristic, see, e.g., in [14]. $\square$

¿From now on we use the following notation. If X and Y are the irreducible varieties, then $X \approx Y$ means that X and Y are birationally isomorphic; if an algebraic group H acts on X and Y, then $X \overset{H}{\approx} Y$ means that there is an H-equivariant birational isomorphism between X and Y.

Proof of Lemma 1.

Put $d_i := \dim L_i$. The natural projection $L_2 \oplus L_3 \to L_2$ is a vector bundle over L_2, to which, in view of (i), is applicable Lemma 4. It implies that

$$L_2 \oplus L_3 \overset{H}{\approx} L_2 \times \mathbf{A}^{d_3}, \tag{42}$$

where H acts on $L_2 \times \mathbf{A}^{d_3}$ via the first factor. Similarly, applying Lemma 4 to the natural projections $L_1 \oplus L_2 \to L_1$ and $L_1 \oplus L_2 \to L_2$ and considering the actions of H on $L_1 \times \mathbf{A}^{d_2}$ and $L_2 \times \mathbf{A}^{d_1}$ via the first factors, we obtain

$$L_1 \times \mathbf{A}^{d_2} \overset{H}{\approx} L_1 \oplus L_2 \overset{H}{\approx} L_2 \times \mathbf{A}^{d_1}. \tag{43}$$

It follows from (42) and (iii) that

$$(L_2 \oplus L_3)/H \approx (L_2/H) \times \mathbf{A}^{d_3} = (L_2/H) \times \mathbf{A}^{d_1} \times \mathbf{A}^{d_3-d_1}, \tag{44}$$

and from (43) that

$$(L_1/H) \times \mathbf{A}^{d_2} \approx (L_1 \oplus L_2)/H \approx (L_2/H) \times \mathbf{A}^{d_1}. \tag{45}$$

In view of (ii), we have $(L_1/H) \approx \mathbf{A}^{\dim L_1/H}$, from where, in view of (44) and (45), we obtain

$$(L_2 \oplus L_3)/H \approx \mathbf{A}^{d_2+d_3-d_1+\dim L_1/H}.$$

This completes the proof. $\square$

Proof of Theorem 1.

The claim is clear for $n = 1$, so further we assume that $n \geqslant 2$.

Let $V^{\oplus n}$ be the direct sum of n copies of the space V. One of the orbits of the diagonal action of G on $V^{\oplus n}$ is open in $V^{\oplus n}$, therefore, $\boldsymbol{k}(V^{\oplus n})^G = \boldsymbol{k}$. Besides, this action is locally free. It follows from this, (6), and Theorem 3 that for $H = G$, $L_1 = V^{\oplus n}$, $L_2 = \mathcal{C}$, $L_3 = \mathcal{A}$, and $n \geqslant 3$ the conditions of Lemma 1 hold (for $n = 2$ condition (iii) of this lemma does not hold). Hence for $n \geqslant 3$ the claim that we are proving follows from (3) and Lemma 1.

All multiplications m from the G-module $\mathcal{C}$ (respectively, $\mathcal{A}$) such that, for every element $v \in \{V, m\}$, the linear operator $V \to V$, $a \mapsto va$ has zero trace, constitute a submodule $\mathcal{C}_0$ (respectively, $\mathcal{A}_0$). Besides, every element $\ell \in V^*$ defines the multiplications $m_{\ell+} \in \mathcal{C}$, $m_{\ell-} \in \mathcal{A}$, for which the products of elements $a, b \in V$ are given, respectively, by the formulas

$$ab := \ell(a)b + \ell(b)a, \tag{46}$$

$$ab := \ell(a)b - \ell(b)a. \tag{47}$$

The subsets $\mathcal{C}' := \{m_{\ell+} \mid \ell \in V^*\}$ and $\mathcal{A}' := \{m_{\ell-} \mid \ell \in V^*\}$ also are the submodules of, respectively, the G-modules $\mathcal{C}$ and $\mathcal{A}$. Both of these submodules are isomorphic to the G-module V^*. For every of the $\boldsymbol{k}$-algebras $\{V, m_{\ell+}\}$ and $\{V, m_{\ell+}\}$, the trace of the operator of left multiplication by an element $v \in V$ is equal to $n\ell(v)$. This implies that

$$\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}' \ \text{ and } \ \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}' \ \text{ if } \operatorname{char} \boldsymbol{k} \text{ does not divide } n. \tag{48}$$

Now let $n = 2$. Then $\mathcal{A} = \mathcal{A}'$ in view of (6), and it follows from (3), (48), and the condition $\operatorname{char} \boldsymbol{k} \neq 2$ that

$$\mathcal{M} = \mathcal{C}_0 \oplus \mathcal{C}' \oplus \mathcal{A}'. \tag{49}$$

Since one of the G-orbits is open in $V^* \oplus V^*$, we have $\boldsymbol{k}(V^* \oplus V^*)^G = \boldsymbol{k}$. Besides, the action of G on $V^* \oplus V^*$ is locally free. This implies that for $H = G$, $L_1 = L_2 = \mathcal{C}' \oplus \mathcal{A}'$, and $L_3 = \mathcal{C}_0$ the conditions of Lemma 1 hold. Therefore, for $n = 2$, the claim under the proof follows from this lemma and (49). $\square$

In view of Theorems 3 and 4, Theorems 2 and 5 result from the following general statement (Theorem 7 below) about *special* (in the sense of Serre) algebraic groups. Recall from [17, Sect. 4.1] that these are algebraic groups S such that every S-torsor is trivial in the Zariski topology (or, equivalently, for every field extension $K/\boldsymbol{k}$ the Galois cohomology $H^1(K, S)$ is trivial).

Theorem 7. *Let X be an irreducible variety endowed with a locally free action of a special algebraic group S. Then there exists an irreducible closed subset Z of X such that:*

(i) *the map $\boldsymbol{k}(\mathcal{M})^G \to \boldsymbol{k}(\mathcal{S})$, $f \mapsto f|_{\mathcal{S}}$ is well-defined and is a $\boldsymbol{k}$-isomorphism of fields;*
(ii) *for every point x of an open dense subset of X, there exists a unique element $s \in S$ such that $s \cdot x \in \mathcal{S}$.*

Proof. Consider a rational quotient $\pi_{S,X} \colon X \dashrightarrow X/S$. By Rosenlicht's theorem [15, Thm. 2] there exists an S-invariant open subset U of X

lying in the domain of definition of $\pi_{S,X}$ such that $W := \pi_{S,X}(U)$ is open in $\mathcal{X}/S$ and every fiber of the morphism

$$\pi_{S,X}|_U \colon U \to W \tag{50}$$

is an S-orbit. Since the action is locally free, replacing U by a suitable invariant open subset, we may assume that the S-stabilizer of every point of U is trivial. In turn, this implies (see [14, Remark 4]) that making another such a replacement, we may assume that (50) is a torsor. It then follows from the specialness of the group G that morphism (50), and hence the rational map $\pi_{S,X}$, admit a rational section $\sigma\colon X/S \dashrightarrow X$ (i.e., $\pi \circ \sigma = \mathrm{id}$). This implies that one can take the closure of the set $\sigma(X/S)$ in X as Z from the formulation of Theorem 2. $\square$

Proof of Theorems 2 *and* 5.

According to [17, §4], the group G is special. Hence, in view of Theorem 7, Theorems 2 and 5 follows, respectively, from Theorems 4 and 3. $\square$

Proof of Corollary 3.

This follows from the known fact (see, e.g., [11, Cor. 2(i)]) that if X in Theorem 7 is an affine space and the action is linear, then the field $\boldsymbol{k}(X)^S$ is stably rational over $\boldsymbol{k}$ (indeed, X is birationally isomorphic to $G{\times}X/G$ by Theorem 7, therefore, the claim follows from the rationality of the underlying variety of G). $\square$

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