Block seminar

Arithmetic groups & Rigidity

Bielefeld, Germany

21^{st} to 23^{rd} of March 2018

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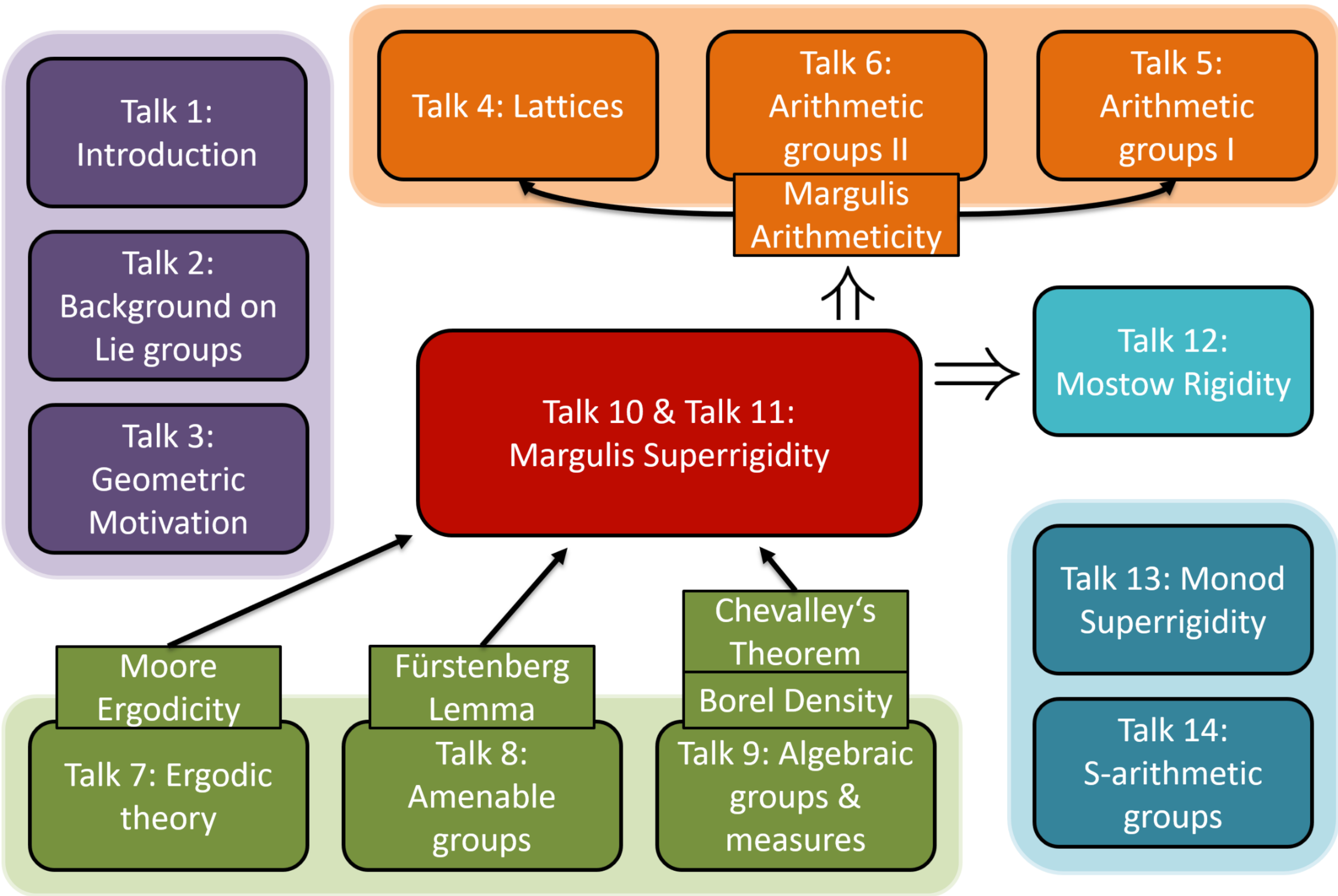
Schedule

All talks will be held in Rooms **V2-213** (Wed, Fri) and **U2-205** (Thu). The coffee breaks shall take place at the Common Room, **V3-201**.

Time	Wednesday	**Time**	Thursday	Friday
9:00	**Introduction,** *B. Brück*	9:00	**Ergodic theory,** *J. Grüning*	**Mostow Rigidity,** *J. Przewocki*
9:30	**Background on Lie groups,** *M. Gruchot*	10:00	Coffee break	Coffee break
10:30	Coffee break	10:30	**Amenable groups,** *D. Luckhardt*	**Monod's Super-rigidity Theorem,** *E. Schesler*
11:00	**Geometric motivation,** *F. Bannuscher*	11:45	**Algebraic groups, actions on measures, and rationality of maps,** *M. Marcinkowski*	**Superrigidity and arithmeticity for** S**-arithmetic groups,** *S. Witzel*
12:15	**Lattices,** *M. Pfeil*	12:45	Lunch break	
13:15	Lunch break	14:30	**Margulis Superrigidity I,** *A. Litterick*	
14:45	**Arithmetic groups I,** *P. Wegener*	15:30	Coffee break	
15:45	Coffee break	16:00	**Margulis Superrigidity II,** *Y. Santos Rego*	
16:15	**Arithmetic groups II,** *S. Yahiatene*	17:15	Discussion session	
17:30	Discussion session			
		19:00	Social dinner	


Talk 1: Introduction
Talk 2: Background on Lie groups
Talk 3: Geometric Motivation
Talk 4: Lattices
Talk 6: Arithmetic groups II
Talk 5: Arithmetic groups I
Margulis Arithmeticity
Talk 10 & Talk 11: Margulis Superrigidity
Talk 12: Mostow Rigidity
Talk 13: Monod Superrigidity
Talk 14: S-arithmetic groups
Moore Ergodicity
Talk 7: Ergodic theory
Fürstenberg Lemma
Talk 8: Amenable groups
Chevalley's Theorem
Borel Density
Talk 9: Algebraic groups & measures

Abstracts

Talk 1: Introduction
Speaker: Benjamin Brück
I will shortly present an outline of the seminar's programme and summarise how the different topics are connected and will be combined for the proof of the Margulis Superrigidity Theorem.

Talk 2: Background on Lie groups
Speaker: Maike Gruchot
The speaker of this talk should introduce basic concepts, examples and possibly some well-known facts about Lie groups. Define what a (linear) real Lie group is. Give examples of both connected and non-connected as well as compact Lie groups (for geometric motivation). Give examples of abelian, soluble and unipotent Lie groups (for algebraic motivation). Present structural concepts such as simplicity, semi-simplicity, isogenies. Introduce the Haar measure and comment existence and uniqueness for Lie groups. If time permits, mention semi-simple, hyperbolic, elliptic and unipotent elements and then give some structural properties such as Jordan Decomposition, Engel's theorem, tori (in the sense of [Mor15, Chapter 8]) and parabolic subgroups. For the remainder of the talk, the speaker's favourite topic on Lie groups could be discussed.
Literature: For the necessary material for the seminar, see [Mor15, Appendix A]. For basics about Lie groups, see e.g. [Zil10]. Standard references include [Hel01; Bou02; Hal03].

Talk 3: Geometric Motivation
Speaker: Falk Bannuscher
The aim of this talk is to explain how arithmetic groups arise in the context of locally symmetric spaces. Explain (locally) symmetric spaces and give an idea of how to produce a symmetric space from a connected Lie group and vice versa. Motivate the definition of a lattice. Finally, state the Mostow Rigidity Theorem as in [Mor15, Theorem 1.3.10].

Literature: [Mor15, Chapter 1]. For basics about Lie groups, see [Zil10] or [Hel01]. A richer source on symmetric spaces is given by Loos' classic books [Loo69a; Loo69b].

Talk 4: Lattices

Speaker: Mareike Pfeil

Give the definition of lattices in Lie groups following [Mor15, p. 4.1] and work out some example of a lattice. Introduce the notion of irreducibility of a lattice and give an example of an irreducible lattice in a product of non compact simple Lie groups. The speaker could also discuss properties of lattices and, if the time allows, give an idea of how to show it for the case of $SL_n(\mathbb{Z}) \subset SL_n(\mathbb{R})$, e.g. Selberg's Lemma as in [Mor15, p. 4.8].
Literature: [Mor15] and [Zim84].

Talk 5: Arithmetic groups I

Speaker: Patrick Wegener

The first goal of this talk is to introduce arithmetic groups as in [Mor15, Definition 5.1.19]. Introduce the necessary definitions such as $\mathbb{Q}-$subgroups. Give some examples of arithmetic groups and, if possible, discuss results which describe the structure of arithmetic subgroups in a fixed ambient Lie group.
Literature: [Mor15, Chapters 4 to 6] and [Zim84].

Talk 6: Arithmetic groups II

Speaker: Sophiane Yahiatene

In this talk the notions of lattices and arithmetic groups should be connected by Margulis' Arithmeticity Theorem. Different versions of this theorem can be found for example in [Mor15, p. 5.2.1] and [Zim84, p. 6.1.2]. (One should be aware of the slightly different notions of arithmeticity in those books.) Give an idea of the proof of the arithmeticity theorem. Here one can derive arithmeticity by using superrigidity as in [Mor15, p. 16.3] or give an idea of the more self-contained proof in [Zim84, p. 6.1].
Literature: [Mor15] and [Zim84].

Talk 7: Ergodic theory

Speaker: Julius Grüning

In this talk the basic notions of ergodic theory should be introduced. In order to prove Margulis Superrigidity it is necessary to speak about Moore's ergodicity theorem [Zim84, Thm. 2.2.6]. Give a sketch of the proof of Moore's result, which involves the vanishing theorem of matrix coefficients [Zim84,

p. 2.2.20].
Literature: [Zim84, Chapter 2]. See also [Mor15, Sections 14.1 and 14.2].

Talk 8: Amenable groups
Speaker: Daniel Luckhardt
This talk is meant to be a brief introduction into the theory on amenability. Define what an amenable group is. Mention or give an idea of why soluble groups are amenable and maybe discuss some consequences of amenability. Present Fürstenberg's Lemma as stated in [Mor15, p. 12.6.1], which will also be needed in the proof of superrigidity. The remaining time can be used either to give more examples, to present equivalent definitions of amenability or to talk about the historical motivation to introduce amenability.
Literature: Various definitions of amenability, Fürstenberg's Lemma and basic examples can be found in [Mor15, Chapter 12] and [Zim84, Chapter 4]. A lot more details on amenability are contained in [Pat88]. For the historical background, see [Mor15, Remarks 12.4.3] and [TW16].

Talk 9: Algebraic groups, actions on measures, and rationality of maps
Speaker: Michał Marcinkowski
The goal of this talk is to introduce the third main theory involved in the proof of Margulis' theorem. Since it is not as self-contained as the others, it might be presented e.g. in a "crash course" style, focusing on elucidating the concepts to be introduced rather than on proofs. Briefly define algebraic varieties and the Zariski topology and rationality of morphisms between varieties, then recall the definition of a linear algebraic group. Give examples of varieties, including projective spaces, and examples of actions of algebraic groups on varieties, possibly listing some well-known facts. Then, present just enough material to state (and, if time permits, sketch the proof of) Chevalley's stabiliser lemma (Prop. 3.1.4 of [Zim84]), Borel's density theorem (e.g. as in [Zim84, p. 3.2.5]) and possibly some corollaries [Zim84, Section 3.2].
Literature: [Zim84, Chapter 3]. For the measure-theoretic part, see [Zim84, Appendix] and [Mor15, Appendix B.6]. A good short reference for the relevant concepts from algebraic geometry is [Spr98, Chapter 1]. A more complete source on linear algebraic groups is [Bor91].

Talk 10: Margulis Superrigidity I
Speaker: Alastair Litterick
The goal of this talk and the next is to present the proof of the Margulis Superrigidity Theorem for (real) connected semi-simple Lie groups of real

rank ≥ 2, as done in Chapter 5 of [Zim84], using the tools presented so far in the seminar. The proof makes use of ergodic theory, results on amenability, and measure-theoretic aspects of actions of algebraic groups. The key steps in the proof involve the construction of some equivariant measurable maps, for which the contents of Talks 7 and 8 (especially Moore's theorem and Fürstenberg's lemma) are needed, and checking that such maps have properties discussed in Talk 9. This first talk recollects some of the needed results and restates the problem in terms of measurable maps between certain coset spaces.
Literature: [Zim84, Chapter 5], [Mar91, Chapter VII].

Talk 11: Margulis Superrigidity II
Speaker: Yuri Santos Rego
A further point in the proof of Margulis' theorem is the verification of some lemmata (see [Zim84, Chapter 5]) involving tori and unipotent subgroups, as well as showing the existence of the desired maps. The remaining steps, to be proved in this talk, will also make use of ergodic theory as well as the structure theory of semisimple linear algebraic groups.
Literature: [Zim84, Chapter 5], [Mar91, Chapter VII] and [Ste16].

Talk 12: Mostow Rigidity
Speaker: Janusz Przewocki
Prove a variant of the Mostow Rigidity Theorem (in real rank ≥ 2) as an application of Margulis' theorem, either as in [Zim84, Chapter 5] or working out the outline given in [Mor15, Chapter 15]. The remainder of the talk should be devoted to the geometric translation of the above mentioned version, that is, obtaining the classical geometric formulation of rigidity [Spa04] from the version of Mostow's theorem proved above (with the appropriate modifications, if needed).
Literature: [Zim84, Chapter 5], [Mor15, Chapter 15], [Spa04] and [Loo69a; Loo69b].

Talk 13: Monod's superrigidity
Speaker: Eduard Schesler
This talk is about more recent results on rigidity which were proven by Monod [Mon06]. The goals of this talk are to understand the differences and similarities of the rigidity results of Margulis and Monod. Introduce the basic notions of CAT(0)-geometry (see e.g. [Mon06, p. 3]) which are necessary to formulate the rigidity result of Monod ([Mon06, Theorem 6]). Explain why Monod's results generalise Margulis' results [Mar91] in the case

where the given Lie group decomposes into at least 2 factors and the lattice is uniform. To do so, explain where CAT(0) spaces appear in the context of Margulis' rigidity theorem. If time allows, elucidate why our version of superrigidity proved in the seminar does not necessarily fit the framework of Monod's theorem (so that both results might be understood as mutually complementary).
Literature: [Mon06], [BH99] and [Mar91, Chapter VII].

Talk 14: Superrigidity and arithmeticity for S-arithmetic groups
Speaker: Stefan Witzel
I will talk about what S-arithmetic groups are and how rigidity statements for arithmetic groups extend to them. I will also state the Arithmeticity Theorem in the S-arithmetic case which Margulis proved using his superrigidity result. Time permitting I will say a bit about how to get from one to the other.

Recommended literature

[Bor91] A. Borel. *Linear algebraic groups.* 2nd ed. Vol. 126. Graduate Texts in Mathematics. Springer-Verlag, New York, 1991, pp. xii+288. DOI: 10.1007/978-1-4612-0941-6.

[Bou02] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics (Berlin). Translated from the 1968 French original by Andrew Pressley. Springer-Verlag, Berlin, 2002, pp. xii+300. DOI: 10.1007/978-3-540-89394-3.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature.* Vol. 319. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 1999, pp. xxii+643. DOI: 10.1007/978-3-662-12494-9.

[Hal03] Brian C. Hall. *Lie groups, Lie algebras, and representations.* Vol. 222. Graduate Texts in Mathematics. An elementary introduction. Springer-Verlag, New York, 2003, pp. xiv+351. DOI: 10.1007/978-0-387-21554-9.

[Hel01] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces.* Vol. 34. Graduate Studies in Mathematics. Corrected reprint of the 1978 original. American Mathematical Society, Providence, RI, 2001, pp. xxvi+641. DOI: 10.1090/gsm/034.

[Loo69a] Ottmar Loos. *Symmetric spaces. I: General Theory.* Vol. 36. Mathematics lecture notes series. W. A. Benjamin, Inc., New York, 1969, pp. vii+198. ISBN: 0805366210.

[Loo69b] Ottmar Loos. *Symmetric spaces. II: Compact spaces and classification.* Vol. 37. Mathematics lecture notes series. W. A. Benjamin, Inc., New York, 1969, pp. vii+183. ISBN: 0805366229.

[Mar91] Gregory A. Margulis. *Discrete subgroups of semisimple Lie groups.* Vol. 17. Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer-Verlag, Berlin, 1991, pp. x+388. ISBN: 9783540121794.

[Mon06] Nicolas Monod. “Superrigidity for irreducible lattices and geometric splitting”. In: *J. Amer. Math. Soc.* 19.4 (2006), pp. 781–814. DOI: `10.1090/S0894-0347-06-00525-X`.

[Mor15] Dave Witte Morris. *Introduction to arithmetic groups*. Deductive Press, Canada, 2015, pp. xii+475. ISBN: 9780986571602. URL: `https://arxiv.org/abs/math/0106063v6`.

[Pat88] Alan L. T. Paterson. *Amenability*. Vol. 29. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1988, pp. xx+452. DOI: `10.1090/surv/029`.

[Spa04] Ralf J. Spatzier. “An invitation to rigidity theory”. In: *Modern dynamical systems and applications*. Cambridge Univ. Press, Cambridge, 2004, pp. 211–231. ISBN: 0521840732.

[Spr98] T. A. Springer. *Linear algebraic groups*. 2nd ed. Vol. 9. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1998, pp. xiv+334. DOI: `10.1007/978-0-8176-4840-4`.

[Ste16] R. Steinberg. *Lectures on Chevalley groups*. Vol. 66. University Lecture Series. Notes prepared by John Faulkner and Robert Wilson, revised and corrected edition of the 1968 original, with a foreword by Robert R. Snapp. American Mathematical Society, Providence, RI, 2016, pp. xi+160. DOI: `10.1090/ulect/066`.

[TW16] Grzegorz Tomkowicz and Stan Wagon. *The Banach-Tarski paradox*. Second. Vol. 163. Encyclopedia of Mathematics and its Applications. With a foreword by Jan Mycielski. Cambridge University Press, New York, 2016, pp. xviii+348. ISBN: 9781107042599.

[Zil10] Wolfgang Ziller. *Lie Groups. Representation Theory and Symmetric Spaces, Lecture Notes*. 2010. URL: `https://www.math.upenn.edu/~wziller/math650/LieGroupsReps.pdf`.

[Zim84] Robert J. Zimmer. *Ergodic theory and semisimple groups*. Vol. 81. Monographs in Mathematics. Birkhäuser Verlag, Basel, 1984, pp. x+209. DOI: `10.1007/978-1-4684-9488-4`.

Glossary of basic concepts

Definition.

1. A *real Lie group* is an abstract group that is also a smooth finite dimensional manifold such that the product map $G \times G \to G$, $(a, b) \mapsto a \cdot b$ and the inversion map $G \to G$, $a \mapsto a^{-1}$ are smooth.

2. If the underlying manifold is a complex manifold and both the product and inversion maps are holomorphic, then the Lie group is a *complex Lie group.*

3. A *linear Lie group* is a any closed subgroup of $\mathrm{SL}(l, \mathbb{R})$ or $\mathrm{SL}(l, \mathbb{C})$ for some l.

4. A Lie group G is *simple* if it is not abelian and has no non-trivial, closed, connected, proper normal subgroup.

5. A Lie group G_1 is *isogenous* to a Lie group G_2 if, for $i = 1, 2$, there exist finite normal subgroups N_i of finite-index subgroups $G'_i \leq G_i$ such that $G'_1/N_1 \cong G'_2/N_2$.

6. A Lie group is semisimple if it isogenous to a direct product of simple Lie groups.

Definition.

1. Let H be a Lie group. A σ-finite Borel measure μ of H is called a *left Haar measure* if $\mu(C) < \infty$ for all compact subgroups C of H and $\mu(hA) = \mu(A)$ for all $h \in H$ and all Borel sets $A \subseteq H$. The *right Haar measure* is defined analogously.

2. A Lie group H is *unimodular* if its left Haar measure equals its right Haar measure.

Definition.

1. An element $g \in \mathrm{GL}(n, \mathbb{R})$ is *semisimple* if it is diagonalisable over $\mathbb{C}$.
2. An element $g \in \mathrm{GL}(n, \mathbb{R})$ is *hyperbolic* if it is semisimple and its eigenvalues are real and positive.
3. An element $g \in \mathrm{GL}(n, \mathbb{R})$ is *elliptic* if it is semisimple and its eigenvalues lie on the unit circle in $\mathbb{C}$.
4. An element $g \in \mathrm{GL}(n, \mathbb{R})$ is *unipotent* if 1 is the only eigenvalue of g.

Definition.

1. A closed connected subgroup T of G is a *torus* if T is diagonalisable over $\mathbb{C}$.
2. A torus is $\mathbb{R}$-split if it is diagonalisable over $\mathbb{R}$.
3. $\mathrm{rank}_{\mathbb{R}}(G)$ is the dimension of any maximal $\mathbb{R}$-split torus of G.

Definition. A **lattice** Λ in a Lie group G is a discrete subgroup $\Lambda \leq G$ such that the space G/Λ has finite volume.

Let as now make some **standing assumptions.** In the following, we let

- G be a linear semisimple Lie group with finitely many connected components and
- Γ be a lattice in G.

Definition. Two subgroups Λ_1 and Λ_2 of a group H are **commensurable** if $\Lambda_1 \cap \Lambda_2$ is a finite-index subgroup of both Λ_1 and Λ_2.

Definition. A lattice Γ is **irreducible** if for every noncompact closed normal subgroup N of G°, ΓN is dense in G (where G° denotes the identity component of G).

Definition.

1. $\mathbb{R}[x_{1,1}, \ldots, x_{n,n}]$ denotes the polynomial ring over $\mathbb{R}$ in the n^2 variables $\{x_{i,j} \mid 1 \leq i, j \leq n\}$.
2. For $\mathcal{Q} \subseteq \mathbb{R}[x_{1,1}, \ldots, x_{n,n}]$ put

$$\mathrm{Var}(\mathcal{Q}) = \{g \in \mathrm{SL}_n(\mathbb{R}) \mid f(g) = 0 \ \forall f \in \mathcal{Q}\}$$

and call it the **variety** associated to $\mathcal{Q}$.

3. $H \subseteq \mathrm{SL}_n(\mathbb{R})$ is called **Zariski closed** if $H = \mathrm{Var}(\mathcal{Q})$ for some $\mathcal{Q} \subseteq \mathbb{R}[x_{1,1}, \ldots, x_{n,n}]$.

Definition. Let $H \leq \mathrm{SL}_n(\mathbb{R})$ be a closed subgroup. We say that H is **defined over** $\mathbb{Q}$ (or is a $\mathbb{Q}$**-subgroup**) if there exists a subset $\mathcal{Q} \subseteq \mathbb{Q}[x_{1,1}, \ldots, x_{n,n}]$ such that

(i) $\mathrm{Var}(\mathcal{Q}) = \{g \in \mathrm{SL}_n(\mathbb{R}) \mid f(g) = 0 \ \forall f \in \mathcal{Q}\}$ is a subgroup of $\mathrm{SL}_n(\mathbb{R})$;

(ii) $H^\circ = \mathrm{Var}(\mathcal{Q})^\circ$;

(iii) H has only finitely many connected components.

In other words, H is commensurable to $\mathrm{Var}(\mathcal{Q})$ for some $\mathcal{Q} \subseteq \mathbb{Q}[x_{1,1}, \ldots, x_{n,n}]$.

Definition. For a subring $\mathcal{O}$ of $\mathbb{R}$ (containing 1) put $G_{\mathcal{O}} = G \cap \mathrm{SL}_n(\mathcal{O})$.

Essentially, a lattice of the form $G_{\mathbb{Z}}$ should be called arithmetic (if G is defined over $\mathbb{Q}$, then $G_{\mathbb{Z}}$ is a lattice). But we also want to include the following properties for arithmetic subgroups:

- $\phi : G_1 \to G_2$ an isomorphism, $\Gamma_1 \leq G$ arithmetic $\Rightarrow \phi(\Gamma_1) \leq G_2$ arithmetic.
- $K \leq G$ compact normal subgroup, $\Gamma \leq G$ a lattice: Γ is arithmetic if and only if $\Gamma K/K \leq G/K$ is arithmetic.
- Arithmeticity should be independent of commensurability.

Therefore we end up with the following definition:

Definition. Γ is called an **arithmetic** subgroup of G if and only if there exist

(i) a closed, connected, semisimple subgroup G' of some $\mathrm{SL}_n(\mathbb{R})$ such that G' is defined over $\mathbb{Q}$,

(ii) compact normal subgroups K and K' of G° and G', repectively,

(iii) an isomorphism $\phi : G^\circ/K \to G'/K'$,

such that $\phi(\overline{\Gamma})$ is commensurable to $\overline{G'_{\mathbb{Z}}}$, where $\overline{\Gamma}$ is the image of $\Gamma \cap G^\circ$ in G°/K and $\overline{G'_{\mathbb{Z}}}$ is the image of $G'_{\mathbb{Z}}$ in G'/K'.

List of participants

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