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NOTE

KATONA'S INTERSECTION THEOREM: FOUR PROOFS

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It is known from a previous paper [3] that Katona's Intersection Theorem follows from the Complete Intersection Theorem by Ahlswede and Khachatrian via a Comparison Lemma. It also has been proved directly in [3] by the pushing–pulling method of that paper. Here we add a third proof via a new $(k,k+1)$-shifting technique, whose impact will be exploared elsewhere. The fourth and last of our proofs is a gift from heaven for Gyula's birthday.

1. Introduction

We begin right away with notation and basic concepts in the study of intersection properties. In standard notation in combinatorics $\mathbb{N}$ is the set of positive integers, $[i,j]=\{i,i+1,\dots,j\}$ for $i,j\in\mathbb{N}$, $[n]=[1,n]$ and for $k<n$ $\quad 2^{[n]}=\{A:A\subset[n]\}$ are the unrestricted subsets of $[n]$ and $\binom{[n]}{k}=\{A\subset 2^{[n]}:|A|=k\}$ stands for the subsets restricted to cardinality k.

A system of sets $\mathcal{A}\subset 2^{[n]}$ is called t-intersecting, if $|A_1\cap A_2|\geq t$ for all $A_1,A_2\in\mathcal{A}$. Basic in our presentation are the following sets and quantities:

$$I(n,t)=\text{the set of all such systems, } I(n,k,t)=\left\{\mathcal{B}\in I(n,t):\mathcal{B}\subset\binom{[n]}{k}\right\},$$

$$M(n,t)=\max_{\mathcal{A}\in I(n,t)}|\mathcal{A}|,\ \text{and}\ M(n,k,t)=\max_{\mathcal{A}\in I(n,k,t)}|\mathcal{A}|.$$

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The Katona sets

$$\mathcal{K}(n,t)=\begin{cases}\left\{A\subset 2^{[n]}:|A|\geq \frac{n+t}{2}\right\}, & 2\mid(n+t);\\ \left\{A\subset 2^{[n]}:|A|\geq \frac{n+t+1}{2}\right\}\cup\left\{A\in\binom{[n]}{\frac{n+t-1}{2}}:1\in A\right\}, & 2\nmid(n+t).\end{cases}$$

The Frankl sets

$$\mathcal{F}_i(n,k,t)=\left\{F\in\binom{[n]}{k}:|F\cap[1,t+2i]|\geq t+i\right\},\quad 0\leq i\leq\frac{n-t}{2};$$

and finally d-diametric ($d\in\mathbb{N}$) systems of sets $\mathcal{B}\subset 2^{[n]}$, for which

$$|B_1\triangle B_2|\leq d \text{ for all } B_1,B_2\in\mathcal{B},$$

the set of all such systems $D(n,d)$ and their maximal cardinality $N(n,d)=\max\limits_{\mathcal{B}\in D(n,d)}|\mathcal{B}|$.

Let us next recall:

Theorem (Katona 1964).

$$M(n,t)=\begin{cases}\sum\limits_{i=\frac{n+t}{2}}^{n}\binom{n}{i}, & \textit{if } 2\mid(n+t);\\ 2\sum\limits_{i=\frac{n+t-1}{2}}^{n-1}\binom{n-1}{i}, & \textit{if } 2\nmid(n+t).\end{cases}$$

Well-known is also the isodiametric

Theorem (Kleitman 1966).

$$N(n,n-t)=M(n,t).$$

Ahlswede and Katona [1] observed that this theorem easily implies Katona's Theorem and vice versa. Katona's Theorem settles the intersection problem raised by Erdős, Ko, and Rado [4] in the unrestricted case.

In the restricted case these authors gave the answer for $t=1$ and for $t>1$, if n is sufficiently large. A complete solution was established later.

Theorem (Ahlswede and Khachatrian 1997). *For $1\leq t\leq k\leq n$ with*

(i) $(k-t+1)(2+\frac{t-1}{r+1})<n<(k-t+1)(2+\frac{t-1}{r})$ for some $r\in\mathbb{N}\cup\{0\}$, we have

$$M(n,k,t)=|\mathcal{F}_r(n,k,t)|$$

and $\mathcal{F}_r(n,k,t)$ is – up to permutations – the unique optimum (by convention $\frac{t-1}{r}=\infty$ for $r=0$).

(ii) $(k-t+1)(2+\frac{t-1}{r+1})=n$ *for* $r\in\mathbb{N}\cup\{0\}$ *we have*

$$M(n,k,t)=|\mathcal{F}_r(n,k,t)|=|\mathcal{F}_{r+1}(n,k,t)|$$

and an optimal system equals up to permutations – either $\mathcal{F}_r(n,k,t)$ *or* $\mathcal{F}_{r+1}(n,k,t)$.

For the proof we introduced a concept of "generating sets". However we found no direct way to prove (the easier) Theorem of Katona by this approach. Instead we derived it from Theorem AK via a simple analytical approach, which we called

Comparison Lemma. *Let* $\alpha_t\ge\alpha_{t+1}\ge\cdots\ge\alpha_{t+2r}\ge 0$, $\alpha_t\ne 0$ *be a non-increasing sequence of real numbers such that* $\max\limits_{\mathcal{A}\in I(t+2r,t)}\sum_{i=t}^{t+2r}|\mathcal{A}_i|\cdot\alpha_i$ *is assumed at* $\mathcal{A}=\mathcal{K}(t+2r,t)$, *where* $\mathcal{A}_i=\{A\in\mathcal{A}:|A|=i\}$.

Then the same holds if

$$\alpha_t=\cdots=\alpha_{t+2r}=1,$$

which is equivalent to Katona's Theorem.

Using Theorem AK we showed that such a sequence $\alpha_t,\ldots,\alpha_{t+2r}$ exists, which implies (in view of the Comparison Lemma) *Katona's Theorem*.

Later we found a new proof of Theorem AK based on a new shifting technique, which we called "pushing–pulling".

By the same method we proved also Katona's Theorem – *our second proof* [3].

In the next Section we *present our third proof* and then finally in the last Section *our fourth proof*, simpler than anyone we have seen. It is remarkably simple and makes the Theorem appear to be a triviality. But we are convinced that our most important message is *the new shifting technique* in the third proof. Whereas the standard shifting, which is originally due to Erdős, Ko and Rado involves exchanges of two positions we operate on more positions! Is shifting an art?

We have already other problems where "2 by 2" switches are adequate and a whole theory of shifting is ready to be born! It may dramatically change the field of extremal set theory.

2. Third proof: A new shifting technique

For a family $\mathcal{A}\subset 2^{[n]}$ and disjoint sets $J,K\in 2^{[n]}$ define

$$\mathcal{B}=\{A\in\mathcal{A}: A\cap J=J,\quad A\cap K=\phi\quad\text{and}\quad (A\smallsetminus J)\cup K\in\mathcal{A}\};$$
$$\mathcal{C}=\{A\in\mathcal{A}: A\cap J=J,\quad A\cap K=\phi\quad\text{and}\quad (A\smallsetminus J)\cup K\notin\mathcal{A}\};$$

and $\mathcal{D}=\mathcal{A}\smallsetminus(\mathcal{B}\cup\mathcal{E})$.

The $(|J|,(K))$-shift $S_{J,K}$ applied to $\mathcal{A}$ gives $\mathcal{E}=S_{I,J}(\mathcal{A})=\mathcal{B}\cup\bar{\mathcal{C}}\cup\mathcal{D}$, where $\bar{\mathcal{C}}=\{\bar{C}:\bar{C}=(C\smallsetminus J)\cup K,\ C\in\mathcal{C}\}$.

Clearly, $|\mathcal{E}|=|\mathcal{A}|$ and it is easy to show that for $\mathcal{A}\in I(n,t)$ also $\mathcal{E}\in I(n,t)$ for $(1,2)$-shifts. After finitely many applications of $(1,2)$-shifts we get a family $\mathcal{A}^*$ which is $(1,2)$-stable that is stable with respect to $(1,2)$-shifts.

We iterate than with $(2,3)$-shifts, but before we apply one of them we first guarantee that the family is $(1,2)$-stable.[1] This procedure ends with a family which is $(1,2)$-stable and $(2,3)$-stable. Then we go on to $(3,4)$-shifting and come to a $(1,2)$-, $(2,3)$-, and $(3,4)$-stable family. Finally we continue this until we end with a $(1,2)$-$,\dots,(k,k+1)$-stable $\mathcal{A}^*$ family.

We show first that for $\mathcal{A}\in I(n,t)$ and all $S_{J,K}(\mathcal{A})\in I(n,t)$ for $|K|=|J|+1<k+1$, this is also the case for $|K|=|J|+1=k+1$. Clearly, by our assumptions $\mathcal{B},\mathcal{D}$ are t-intersecting and also $|B\cap D|\geq t$ for $B\in\mathcal{B}$, $D\in\mathcal{D}$. Furthermore, since $\mathcal{C}$ is t-intersecting $\bar{\mathcal{C}}$ is even $(t+1)$-intersecting, and also

$$|\bar{C}\cap B|=|C\cap((B\smallsetminus J)\cup K|\geq t.$$

So the only non-obvious case is

$$|\bar{C}\cap D|\geq t \text{ for } \bar{C}\in\mathcal{C}, D\in\mathcal{D}. \tag{1}$$

To see this, define

$$\delta=\min(|D\cap J),|K|-|D\cap K|-1). \tag{2}$$

There are $J'\subset D\cap J$, $K'\subset D\cap K$ with $|J'|=\delta$ and $|K'|=\delta+1$. Since $\delta<k$ and $\mathcal{A}$ is δ-stable necessarily $D'=(D\smallsetminus J')\cup K'\in\mathcal{A}$. Furthermore

$$(C\cap D)=|C\cap D'|+\delta\geq t+\delta \tag{3}$$

and since $|\bar{C}\cap D|=|C\cap D|-|D\cap J|+|D\cap K|$ by (3) and (2) $|\bar{C}\cap D|\geq t+\min(|D\cap K|,|K|-1-|D\cap J|)\geq t$, because $|K|-1=k=|J|\geq|D\cap J|$.

We establish now the bound on $M(n,t)$.

For $A\in\mathcal{A}^*$ and $k=\min(|A|,n-|A|-1)$ there is a $B\in\mathcal{A}^*$ with $|A\cap B|=|A|-k$ and $|([n]\smallsetminus A)\cap B|=k+1$. Necessarily $|A|-k\geq t$ and therefore $|A|\geq t+\min(|A|,n-|A|+1)$.

Hence $|A|\geq t+n-|A|+1$ and $|A|\geq\frac{n+t-1}{2}$. In the case $2\mid(n+t)$ this implies $|A|\geq\frac{n+t}{2}$ and thus $M(n,t)=\sum\limits_{i=\frac{n+t}{2}}^{n}\binom{n}{2}$.

Finally by Lemma 1 of [5] $M(n,t)=2(M(n–1,t)$, if $n+t$ is odd, completing the proof of the Theorem. Alternatively, the maximum number of sets of size $\frac{n+t-1}{2}$ can be determined by using complementation and the classical EKR-theorem for $t=1$ on level $\ell=\frac{n-t+1}{2}$.

[1] A referee kindly asked whether this is not automatically the case. Indeed it is and we prove this in the Appendix.

3. Fourth: The simplest (possible?) proof (for mathematicians above 60)

Case $\boldsymbol{2\,|\,(n+t)}$ (the case $2\nmid(n+t)$ is similar).

We can assume that the optimal family $\mathcal{A}\in I(n,t)$, $t>1$ is left-compressed (in the sense of EKR).

Let $\mathcal{A}_1=\{A\in\mathcal{A}:1\in A\},\mathcal{A}_0=\{A\in\mathcal{A}:1\notin A\}$, and

$$\mathcal{A}_j^*=\{A\cap[2,n]:A\in\mathcal{A}_j\},\ j=0,1.$$

Simple observations:

$\mathcal{A}_1^*\in I(n-1,t-1)$ (trivial) and $\mathcal{A}_0^*\in I(n-1,t+1)$ (since $\mathcal{A}$ is left-compressed).

Induction: For $t=1$, $t=n$ the statement is true and by Pascal's identity

$$|\mathcal{A}|=|\mathcal{A}_1^*|+|\mathcal{A}_0^*|\le\sum_{i=\frac{n+t}{2}-1}^{n-1}\binom{n-1}{i}+\sum_{\frac{n+t}{2}}^{n-1}\binom{n-1}{i}=\sum_{i=\frac{n+t}{2}}^{n}\binom{n}{i}. \qquad \blacksquare$$

Remark: The uniqueness of the optimal family also follows.

References

[1] R. Ahlswede and G. Katona: Contributions to the geometry of Hamming spaces, *Discrete Mathematics* **17** (1977), 1–22.

[2] R. Ahlswede and L. H. Khachatrian: The complete intersection theorem for systems of finite sets, *Europ. J. Combinatorics* **18** (1997), 125–136.

[3] R. Ahlswede and L. H. Khachatrian: A pushing–pulling method: new proofs of intersection theorems, *Combinatorica* **19(1)** (1999), 1–15.

[4] P. Erdős, C. Ko and R. Rado: Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser.* **12** (1961), 313–320.

[5] G. O. H. Katona: Intersection theorems for systems of finite sets, *Acta Math. Acad. Sci. Hungar.* **15** (1964), 329–337.

[6] D. J. Kleitman: On a combinatorial conjecture of Erdős, *J. Combin. Theory* **1** (1966), 209–214.

Appendix: Improvement of the shifting technique

Proposition. *Given $\mathcal{A}\subset 2^{[n]}$, $1<k,n$ let $\mathcal{E}\triangleq S_{J,K}(\mathcal{A})$ for some $J,K\subset[n]$ with $|J|=|K|-1=k$. Then if $\mathcal{A}$ is $(i,i+1)$-stable for all $1<i\le k-1$ so is $\mathcal{E}$.*

Proof. We have $\mathcal{A}=(\mathcal{B}\cup\mathcal{D})\cup\mathcal{C}$ and $\mathcal{E}=(\mathcal{B}\cup\mathcal{D})\cup\bar{\mathcal{C}}$ (keeping the notation above).

The proof is based on the following two observations.

Observation 1. $\mathcal{B}\cup\mathcal{D}$ is $(i,i+1)$-stable for all $1<i\leq k-1$, i.e. for every disjoint $J,K\subset[n]$ with $|J|=|K|-1\leq k-1$ holds $S_{J,K}(\mathcal{B}\cup\mathcal{D})=\mathcal{B}\cup\mathcal{D}$.

Proof. Suppose for contradiction $C\triangleq S_{J',K'}(E)\in\mathcal{C}$ for some $E\in(\mathcal{B}\cup\mathcal{D})$ with $|J'|=i\leq k-1$, $|K'|=i+1$ and $C\cap(J\cup K)=J$. Denote $F=E\cap J$. Clearly $|F|=|J|-|J'|-1=k-i-1$. Let also $j\in J\setminus F$, $K_1\subset K$ with $|K_1|=|J'|=i$ and $K_2\triangleq K\setminus K_1$. It is not hard to observe now that

$$S_{F\cup\{j\},K_2}(S_{J',K_1\cup\{j\}}(E))=(C\setminus J)\cup K.$$

This is a contradiction with $C\in\mathcal{C}$ since $|F\cup\{j\}|=|K_2|-1=k-i\leq k-1$ and $|J'|=|K_1\cup\{j\}|-1=i\leq k-1$. ∎

Observation 2. For every $(i,i+1)$-shift $S_{J',K'}$ with $1<i\leq k-1$ we have

$$S_{J',K'}(\mathcal{C})\subset(\mathcal{B}\cup\mathcal{D}).$$

Proof. Let $E\triangleq S_{J',K'}(\bar{C})$ $(\bar{C}\triangleq S_{J,K}(C))$ for some $\bar{C}\in\bar{\mathcal{C}}$ and $|J'|=|K'|-1=i\leq k-1$.

Define $K_1=K'\setminus J$, $K_2=E\cap K$ and $F=K'\cap J$. Note that $|K_2|=k+i-1$. Consider now some $J_1\subset J\setminus F$ with $|J_1|=|K_1|-1$ and $J_2\triangleq J\setminus(F\cup J_1)$.

Clearly $|J_2|=k-i$. Observe now that

$$S_{J_2,K_2}(S_{J_1,K_1}(C))=E.$$

This completes the proof since $|J_1|=|K_1|-1$, $|J_2|=|K_2|-1$ and $|J_1|,|J_2|\leq k-1$. ∎

Evidently Observations 1 and 2 imply the statement. ∎

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