Z. Wahrscheinlichkeitstheorie verw. Gebiete
43, 183–185 (1978)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete
© by Springer-Verlag 1978

An Inequality for the Weights of Two Families of Sets, Their Unions and Intersections

Rudolf Ahlswede[1] and David E. Daykin[2]

[1] Fakultät für Mathematik, Universität 4800 Bielefeld, Federal Republic of Germany
[2] Department of Mathematics, University of Reading, Whiteknights,
Reading, Berkshire, England

1. Introduction

The object of this note is to prove

Theorem 1. *Let S be the family of all subsets of the set* $\{1,2,\ldots,n\}$. *If* α, β, γ, δ *are non-negative real valued functions on S such that*

$$\alpha(a)\,\beta(b)\leqq\gamma(a\cup b)\,\delta(a\cap b)\qquad\text{for all } a,b\in S, \tag{1}$$

then

$$\alpha(A)\,\beta(B)\leqq\gamma(A\vee B)\,\delta(A\wedge B)\qquad\text{for all } A,B\subset S, \tag{2}$$

where $\alpha(A)=\sum(a\in A)\,\alpha(a)$ *and* $A\vee B=\{a\cup b;\,a\in A,b\in B\}$ *and* $A\wedge B=\{a\cap b;\,a\in A,b\in B\}$.

Since every distributive lattice can be embedded in the subsets of some set we get an immediate

Corollary. *If S is a distributive lattice and* (2) *holds whenever A, B each contain exactly one point of S then* (2) *always holds. Here S, A, B may be infinite.*

Our theorem contains as special cases results of Anderson, Daykin, Fortuin, Ginibre, Greene, Holley, Kasteleyn, Kleitman, Seymour, West and others[1]. We discovered it whilst guests at the Mathematisches Forschungsinstitut Oberwolfach and thank all concerned for their kindness to us.

2. The Proof

Case $n=1$. Write 0, 1 for ϕ, $\{1\}$ respectively. The Conditions (1) are

$$\alpha(0)\,\beta(0)\leqq\gamma(0)\,\delta(0) \tag{3}$$

[1] This is explained in detail in the forthcoming paper "Inequalities for a pair of maps $S\times S\to S$ with S a finite set" by the same authors (submitted to Math. Zeitschrift). This paper contains several new inequalities also for other binary operations

0044-3719/78/0043/0183/$01.00

$$\alpha(0)\,\beta(1) \leqq \gamma(1)\,\delta(0) = \varepsilon \quad \text{say} \tag{4}$$

$$\alpha(1)\,\beta(0) \leqq \varepsilon \tag{5}$$

$$\alpha(1)\,\beta(1) \leqq \gamma(1)\,\delta(1). \tag{6}$$

When A and B both contain two elements the result (2) becomes

$$(\alpha(0)+\alpha(1))\,(\beta(0)+\beta(1)) \leqq (\gamma(0)+\gamma(1))\,(\delta(0)+\delta(1)). \tag{7}$$

Suppose $0<\varepsilon$ for otherwise (7) is trivial. We decrease $\gamma(0)$ and $\delta(1)$ until we get equality in (3) and (6). Then (7) simplifies to

$$0 \leqq (\alpha(0)\,\beta(1)-\varepsilon)\,(\alpha(1)\,\beta(0)-\varepsilon)/\varepsilon,$$

which holds by (4) and (5). The remaining choices for A, B hold by inspection, so case $n=1$ is verified.

Induction Step. Assume the result for $n=m\geqq 1$ and consider the case $n=m+1$. Write S, T, P for the family of all subsets of $\{1,2,\ldots,m+1\}$, $\{2,3,\ldots,m+1\}$, $\{1\}$ respectively.

Given $a\in S$ put $a^*=a\backslash\{1\}\in T$ and $^*a=a\backslash\{2,3,\ldots,m+1\}\in P$. Let α, β, γ, δ, A, B be chosen and fixed. Define α_2, β_2, γ_2, δ_2 on T by

$$\alpha_2(c)=\sum(a\in A,\ a^*=c)\,\alpha(a)$$

$$\gamma_2(c)=\sum(a\in A\vee B,\ a^*=c)\,\gamma(a)$$

with similar expressions for β_2 and δ_2. Then

$$\alpha(A)=\sum_{a\in A}\alpha(a)=\sum_{c\in T}\Bigg(\sum_{\substack{a\in A\\ a^*=c}}\alpha(a)\Bigg)=\sum_{c\in T}\alpha_2(c)=\alpha_2(T)$$

and similarly

$$\beta(B)=\beta_2(T), \qquad \gamma(A\vee B)=\gamma_2(T), \qquad \delta(A\wedge B)=\delta_2(T).$$

Assume for the moment that

$$\alpha_2(c)\,\beta_2(d) \leqq \gamma_2(c\cup d)\,\delta_2(c\cap d) \qquad \text{for all } c,d\in T. \tag{8}$$

Using $T\vee T=T$ and our induction hypothesis we get

$$\alpha(A)\,\beta(B)=\alpha_2(T)\,\beta_2(T) \leqq \gamma_2(T\vee T)\,\delta_2(T\wedge T)=\gamma(A\vee B)\,\delta(A\wedge B)$$

which is (2) as required. Hence it remains to prove (8).

Let $c,d\in T$ be fixed arbitrarily. Write $e=c\cup d$ and $f=c\cap d$. Define $\alpha_1, \beta_1, \gamma_1, \delta_1$ on P by

$$\alpha_1(p)=\begin{cases}\alpha(p\cup c) & \text{if } p\cup c\in A\\ 0 & \text{otherwise}\end{cases}$$

$$\beta_1(p)=\begin{cases}\beta(p\cup d) & \text{if } p\cup d\in B\\ 0 & \text{otherwise}\end{cases}$$

$$\gamma_1(p)=\begin{cases}\gamma(p\cup e) & \text{if } p\cup e\in A\vee B\\ 0 & \text{otherwise}\end{cases}$$

$$\delta_1(p)=\begin{cases}\delta(p\cup f) & \text{if } p\cup f\in A\wedge B\\ 0 & \text{otherwise}\end{cases}$$

Then

$$\alpha_2(c)=\sum_{\substack{a\in A\\ a^*=c}}\alpha(a)=\sum_{p\in P}\left(\sum_{\substack{a\in A\\ a^*=c\\ {}^*a=p}}\alpha(a)\right)=\sum_{p\in P}\alpha_1(p)=\alpha_1(P)$$

and similarly

$$\beta_2(d)=\beta_1(P),\qquad \gamma_2(e)=\gamma_1(P),\qquad \delta_2(f)=\delta_1(P).$$

Assume for the moment that

$$\alpha_1(p)\,\beta_1(q)\leqq\gamma_1(p\cup q)\,\delta_1(p\cap q)\qquad\text{for all } p,q\in P.$$

Then by the case $n=1$ we have

$$\alpha_1(P)\,\beta_1(P)\leqq\gamma_1(P\vee P)\,\delta_1(P\wedge P),$$

or in other words (8) holds. So it now remains for (9) to be proved.

The left hand side of (9) is zero unless $p\cup c\in A$ and $q\cup d\in B$ in which case it is $\alpha(p\cup c)\,\beta(a\cup d)$. We then have

$$(p\cup c)\cup(q\cup d)=(p\cup q)\cup e\in A\vee B\quad\text{and}$$
$$(p\cup c)\cap(q\cup d)=(p\cap q)\cup f\in A\wedge B.$$

Hence the right hand side of (9) is $\gamma((p\cup q)\cup e)\,\delta((p\cap q)\cup f)$ so (9) holds by hypothesis (1) and Theorem 1 follows inductively.

References

1. Anderson, I.: Intersection theorems and a lemma of Kleitman. Discrete Math. (To appear)
2. Daykin, D.E.: A lattice is distributive iff $|A||B|\leqq|A\vee B||A\wedge B|$. Nanta Math. (To appear)
3. Daykin, D.E.: Poset functions commuting with the product and yielding Čebyčev type inequalities. C.N.R.S. Colloque, Paris (1976) (To appear)
4. Daykin, D.E., Kleitman, D.J., West. D.B.: The number of meets between two subsets of a lattice. J. Combinatorial Theory [submitted]
5. Greene, C., Kleitman, D.J.: Proof techniques in the theory of finite sets. M.A.A. Studies in Combinatorics. Editor G.C. Rota (To appear)
6. Fortuin, C.M., Kasteleyn, P.W., Ginibre, J.: Correlation inequalities on some partially ordered sets. Comm. Math. Phys. **22** 89–103 (1971)
7. Holley, R.: Remarks on the FKG inequalities. Comm. Math. Phys. **36**, 227–231 (1974)
8. Kleitman, D.J.: Families of non-disjoint subsets. J. Combinatorial Theory **1**, 153–155 (1966)
9. Seymour, P.D.: On incomparable collections of sets. Mathematika Period. Sb. Pererodov Inostran Statei. **20**, 208–209 (1973)

Received August 25, 1977