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Giant $k-$core in Random Hypergraph.

We obtain the treshold for the number of hyperedges in $r-$uniform random hypergraph when the $k-$core of linear with number of vertices size occure.

In paper [2] we extend to the hypergraphs the switching method suggested in [1] for graphs. This allows us to obtain the asymptotics of the number of simple hypergraph with given degree sequence. The same asymptotics for graphs was obtained in [4]. By simple hypergraph we mean hypergraph without loops and such that there are no pairs of vertices which belong to more than one edge. For some fixed r consider the $r-$uniform hypergraph $H = (V, E)$ on vertices $V, |V| = n$ and hyperedges (which we call simply edges) E. We say that hypergraph $H' = (V', E') \subset (V, E)$ is $k-$core if every vertex from V' is incident to not less than k edges. It is evident procedure with hypergraph H which allow to extract the maximal (by the number of vertices V') $k-$core H': on each step we should delete all vertices which incident to less than k edges from H together with the edges which are incident to deleting vertices. The final step will be the maximal $k-$core (possibly it is empty). Next we consider the uniform distribution on the set of simple hypergraphs $\{H = (V, E)\}, |V| = n, |E| = m$ and the following deleting process is simular to the deleting process for graphs from [3] and which allows to extract from the each hypergraph the maximal $k-$core. Let's v_j $j = 0, 1, 2, \dots, k-1$ is the number of vertices in H of degree j and μ be the total number of edges, then the total number of light nonisolated vertices is $v = \sum_{i=1}^{k-1} v_i$ and the number of heavy vertices is $v_k = n - v - v_0$. Also define $\omega = (\bar{v}, \mu), \bar{v} = (v_0, v_1, \dots, v_k)$. On each step we choose randomly with equal probability the nonisolated light vertex and delete all edges incident to this vertex. Hence this vertex become isolated and possibly new light verteces are created. We stop the procedure when there will be no nonisolated light verteces and hence the set of nonisolated verteces generate the maximal $k-$core. Let $T(H)$ is the number of necessary steps and $c = rm/n$. Also we suppose that $k \geq 3$. The analysis of this procedure in main points is simular to the case of graphs [3] and here we implement the asymptotic formula for the number of labeled hypergraphs with given degree sequence from [2].

Theorem 1 *Let's $Z(\lambda)$ is a Poisson random variable with intesity $\lambda > 0$, also let's $p_k(\lambda) = P(Z(lambda) \geq k), \pi_k(\lambda) = P(Z(\lambda) \geq k-1)$ and*

$$\gamma_k = \inf_{\lambda > 0} \frac{\lambda}{\pi_k(\lambda)}$$

If $c > \gamma_k$ then suppose that $\lambda_k(c)$ is the larger root of the equation

$$c = \frac{\lambda}{\pi_k(\lambda)}.$$

If $\gamma_k > c$ then for every given $\epsilon > 0$ hypergraph $H(n, cn/r)$ has $k-$core of size $\geq [\epsilon n]$ with probability tends to zero (with n).

If $\gamma_k < c$ then hypergraph $H(n, cn/r)$ has $k-$core of size $np_k(\lambda_k(c)) + o(n)$ with probability approaching 1 as $n \to \infty$.

This theorem sounds quiet simular to the theorems 1 and 2 from [3]. Difference is in definition of c. We dont introduce here the whole proof of this theorem restricting our attention on applying the formula for the number of labeled simple hypergraphs with given degree sequence and for the last part of the proof sending to the paper [3].

Next we preliminary consider the following scheme. We choose the hypergraph from the set of simple graphs $\{H(n,m) = (V,E)\}$, $|V| = n, |E| = m$ with given parametres $\omega = (\bar{v}, m)$ under uniform distribution and start the deleting process as described before. Let's $H(t)$ is the random hypergraph which is obtained on the $t-$th step of the deletion process $t = 0, 1, \ldots$. If $P(H(t) = H) \neq 0$ and t is not final step, one can calculate the probability $P(H(t+1) = H'|H(t) = H)$. This probability depends on whether the choosed nonisolated vertex is incident to one edge, such that all verteces which are incident to this edge have multiplicity one or any other cases. In first case (I) we have

$$P(H(t+1) = H'|H(t) = H) = \frac{r}{\varphi(H)},$$

in the second (II) case

$$P(H(t+1) = H'|H(t) = H) = \frac{1}{\varphi(H)},$$

where $\varphi(H)$ is the number of light nonisolated verteces from H and H' can be obtained from H by one step deleting procedure. Next we have the following

Statment 1 *For nonfinal step t when $P(\omega) \neq 0$, then the probability $p(\omega'|\omega)$ to receive the hypergraph with parametres $\omega' = (\bar{v}', \mu')$ when previous hypergraph has parametres $\omega = (\bar{v}, \mu)$ is as follows*

$$p(\omega'|\omega) \triangleq$$
$$= \frac{h(\omega')}{\varphi h(\omega)} \frac{\binom{i(r-1)}{r-1, r-1, \ldots, r-1}}{i!} v'_0 \prod_{j=0}^{k} \binom{v'_j - \delta_{j,0}}{u_{j+1}}, \qquad (1)$$

where $\{u_j\}_1^{k+1}$ *is the solution of the linear equations*

$$\begin{aligned}
v_j &= v'_j - u_{j+1} + u_j + \delta_{ij},\ j = 1, 2, \dots, k-1, u_0 \triangleq -1, \\
v_k &= v'_k + u_k, \\
i &\triangleq \tfrac{1}{r-1} \sum_{j=1}^{k+1} u_j = \mu - \mu'
\end{aligned} \tag{2}$$

and $\varphi = \sum_{j=1}^{k-1} v_j$.

The distribution on the set of random hypergraphs $H(t)$, *conditioned on the event* $U = \{\omega(\tau) = \omega^0(\tau), \tau = 1, 2, \dots, t\}$, *such that* $P(U) > 0$ *is uniform:*

$$P(G(t) = G|U) = \frac{1}{h(\omega^0(t))}, G \in K(\omega). \tag{3}$$

The proofs in the case I (when $i = 1$ and $v_0 - v'_0 = k$) and II are slightly different. We suppose that (3) is valid for t and will prove it for $t+1$. In the case II we have

$$\begin{aligned}
& P(H(t+1) = H'|\omega(\tau) = \omega^0(\tau),\ \tau = 1, 2, \dots, t) = \\
= & \sum_{H \in \mathcal{K}(\omega^0(t))} P(H(t+1) = H', H(t) = H|\omega(\tau) = \omega^0(\tau),\ \tau = 1, 2, \dots, t) = \\
= & \sum_{H \in \mathcal{K}(\omega^0(t))} P(H(t+1) = H', H(t) = H) P(G(t) = G|\omega(\tau) = \omega^0(\tau),\ \tau = 1, 2, \dots, t) = \\
= & \frac{1}{h(\omega^0(t))\varphi} N(H', \omega^0(t)).
\end{aligned} \tag{4}$$

Here $N(H', \omega^0(t))$ is the number of hypergraphs from $\mathcal{K}(\omega^0(t))$ such that H' can be obtained from H by deleting edges incident to one given light vertex.

Next we have the formula for $N(H', \omega^0(t))$:

$$N(H', \omega^0(t)) = \xi(\omega^0, \omega'^0) \triangleq v'_o \frac{\binom{\sum_{j=0}^{k} u_j}{r-1, \dots, r-1}}{\left(\frac{\sum_{j=0}^{k} u_j}{r-1}\right)!} \prod_{j=0}^{k} \binom{v'_j - \delta_{0j}}{u_{j+1}}. \tag{5}$$

Note, that for given $\omega(t)$ it does not depends on $\omega(\tau), \tau < t$ and on choosing $G' \in \mathcal{K}(\omega'^0(\tau))$.

Hence

$$\begin{aligned}
& P(\omega(t+1) = \omega'|\omega(\tau) = \omega_o(\tau),\ \tau = 1, 2, \dots, t) = \\
= & P(\omega(t+1) = \omega'|\omega(t) = \omega_o(t)) = \\
= & \frac{h(\omega')\xi(\omega_o, \omega')}{\varphi h(\omega)}.
\end{aligned} \tag{6}$$

In the first case we have

$$N(H', \omega^0(t)) = \binom{v'_0}{r}$$

and from (4) we have

$$P(H(t) = H' | \omega(\tau) = \omega^0(\tau),\ \tau = 1, 2, \ldots, t) = \frac{r\binom{v'_0}{r}}{h(\omega^0(t))\varphi}.$$

From here and from (4), (5) the statement follows in this case.

The simple proof of (3) for $t+1$ is from [3]:

$$\begin{aligned}
& P(H(t+1) = H' | \omega(\tau) = \omega^0(\tau, \tau = 0, 1, \ldots, t+1) = \\
= \ & \frac{P(H(t+1) = H' | \omega(\tau) = \omega^0(\tau), \tau = 1, 2, \ldots, t)}{P(\omega(\tau+1) = \omega' | \omega(\tau) = \omega^0(\tau), \tau = 0, 1, \ldots, t)} = \\
= \ & \frac{p(\omega'|\omega)}{h(\omega')p(\omega'|\omega)} = \frac{1}{h(\omega')}.
\end{aligned}$$

To find the asymptotic formula for $h(\omega)$ we use the formula for the number $f(\bar{k})$ of simple labeled hypergraphs on n vertices with given degree sequence $\bar{k} = (k_1, k_2, \ldots, k_n)$ from [2], which is as follows:

$$\begin{aligned}
f(\bar{k}) = & \frac{\kappa!}{\left(\frac{\kappa}{r}\right)!(r!)^{\kappa/r}\prod_{j=1}^{n} k_j!} \times \\
\times \ \exp & \left\{ -\frac{k(2)}{2\kappa} - \frac{k^2(2)}{4\kappa^2} - \frac{k^2(2)k(3)}{2\kappa^4} + \frac{k^4(2)}{4\kappa^5} + \frac{k^2(3)}{6\kappa^3} + O\left(\frac{k_{\max}^3}{\kappa}\right) \right\}, \qquad (7)
\end{aligned}$$

where

$$\begin{aligned}
k(j) &= \sum_{j=1}^{n} (k_i)_{(j)}, \\
\ell_{(j)} &= \ell(\ell-1)\ldots(\ell-j+1), \\
k_{\max} &= \max_i k_i, \\
\kappa &\triangleq k_{(1)}, \\
k_{\max} &= o\ (\kappa^{1/3}).
\end{aligned}$$

Then

$$h(\omega) = \frac{n!}{\prod_{j=1}^{k} v_j!} \sum_{\bar{k} \in K} f(\bar{k}), \qquad (8)$$

where K is the set of degree sequences $\bar{k}$ such that

$$
\begin{aligned}
& k_1 = k_2 = \ldots = k_{v_0} = 0, \\
& k_{v_0+1} = k_{v_0+2} = \ldots = k_{v_0+v_1} = 1, \\
& \vdots \\
& k_{\sum_{j=0}^{k-2} v_j + 1} = \ldots = k_{\sum_{j=0}^{k-1} v_j} = k-1, \\
& k_{\sum_{j=0}^{k-1} v_j + 1}, \ldots, k_n \geq k, \\
& \sum_{j=1}^{n} k_j = r\mu. \qquad (9)
\end{aligned}
$$

In the simular way as for the case of graphs it can be shown (see [3]) that in our case only two first terms of exponent from (7) have the unvanishing evaluation into $h(\omega)$:

$$
\begin{aligned}
h(\omega) &= (1 + O(n^{3b-1})) \frac{n!}{\prod_{j=0}^{k} v_j!} \sum_{\bar{k} \in K_1} f_1(\bar{k}), \\
f_1(\bar{k} &= \frac{\kappa!}{\left(\frac{\kappa}{r}\right)! (r!)^{\kappa/r} \prod_{j=1}^{n} k_j!} \exp(-\lambda/2 - \lambda^2/4), \\
\lambda &= \frac{k(2)}{\kappa}, \; b \in (0, 1/3). \qquad (10)
\end{aligned}
$$

The only change in the proof of (10) is in the asymptotics of the number of hypergraphs $H(n,m)$ on n vertices and with m edges such that no two vertices are incident to two edges. If A_n is the such event among all possibilities of choosing of edges for hypergraph, then it is not difficult to estimate the probability $P(A_n)$:

$$
\begin{aligned}
& \frac{\binom{n}{r}\left(\binom{n}{r} - \binom{r}{2}\binom{n-2}{r-2} + 2\binom{r}{3}\binom{n-3}{r-3}\right)\ldots}{n^{rm}} \times \\
\times \;& \ldots \left(\binom{n}{r} - i\binom{r}{2}\binom{n-2}{r-2} + 2i\binom{r}{3}\binom{n-3}{r-3}\right) \ldots \times \\
\times \;& \left(\binom{n}{r} - (m-1)\binom{r}{2}\binom{n-2}{r-2} - 2(m-1)\binom{r}{3}\binom{n-3}{r-3}\right) r!^m \geq \\
\geq \;& P(A_n) \geq \\
\geq \;& \frac{\binom{n}{r}\left(\binom{n}{r} - \binom{r}{2}\binom{n-2}{r-2}\right)\ldots\left(\binom{n}{r} - i\binom{r}{2}\binom{n-2}{r-2}\right)\ldots\left(\binom{n}{r} - (m-1)\binom{r}{2}\binom{n-2}{r-2}\right) r!^m}{n^{rm}}.
\end{aligned}
$$

Hence

$$
\lim_{n\to\infty} P(A_n) = exp\left(-\frac{c(r-1)}{2} - \frac{(c(r-1))^2}{4}\right) > 0.
$$

Actually all what is needed for the proof of (10) is the proof of the relation (11) which is quite simple. Next statement is the same as proposition 2 from [3] and has the simular simple proof.

Statment 2 *If* $h(\omega) > 0$ *and*

$$P\left(k_{\max}(H) \geq n^b \ or \sum_{heavy j} k_j^4 \geq 2nE(Y^4(c))\right) < \frac{1}{n} \tag{11}$$

then uniformly on choosing ω*, such that*

$$\begin{aligned} t &\triangleq r\mu - \sum_{j=1}^{k-1} jv_j \geq (k+a)v_k, \\ v_k &\geq an, \ a > 0, \end{aligned} \tag{12}$$

the following relations are valid

$$\begin{aligned} h(\omega) &= (1 + O(n^{3b-1} + n^{-1/2}\ln n))\frac{n!\kappa!}{\left(\frac{\kappa}{r}\right)!v_k!(r!)^{\kappa/r}\prod_{j=1}^{k-1}(j!)^{v_j}v_j!} \times \\ &\times \frac{(e_k(z))^{v_k}}{z^t}\exp(-\bar{\lambda}/2 - \bar{\lambda}^2/4)\frac{1}{\sqrt{2\pi v_k Var(Y(z))}}, \end{aligned} \tag{13}$$

where $Y(z)$ *is the random variable with distribution*

$$P(Y(z) = \ell) = \frac{P(Z(z) = \ell)}{p_k(z)},$$

$v_k E(Y(z)) = t$ *and*

$$\begin{aligned} e_k(z) &= \sum_{j \geq k} \frac{z^j}{j!}, \\ \bar{\lambda} &= \frac{\sum_{j=1}^{k-1} j(j-1)v_j + v_k E(Y(z)(Y(z)-1))}{\sum_{j=1}^{k-1} jv_j + v_k E(Y(z))} \end{aligned}$$

Next we have

$$\frac{v_k!\prod_{j=0}^{k-1}(j!)^{v_j}v_j!}{v_k'!\prod_{j=0}^{k-1}(j!)^{v_j'}v_j'!} = i!\prod_{j=0}^{k}\frac{v_j!}{v_j'!}\prod_{j=1}^{k-1}j^{u_j}\left(\frac{1}{(k-1)!}\right)^{u_k} \tag{14}$$

$$\frac{\binom{i(r-1)}{r-1, r-1, \ldots, r-1}(\kappa'-1)!\left(\frac{\kappa}{r}\right)!(r!)^{\kappa/r}}{i!(\kappa-1)!\left(\frac{\kappa'}{r}\right)!(r!)^{\kappa'/r}} = \frac{(i(r-1))!}{i!\mu^{(r-1)i}}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{15}$$

$$v_0'\prod_{j=0}^{k}\binom{v_j' - \delta_{j0}}{u_{j+1}} = v_i\prod_{j=0}^{k}\frac{v_j'!}{v_j!}\prod_{j=0}^{k-1}\binom{v_j - \delta_{ij}}{u_j}\frac{v_k!}{h_k!u_{k+1}!(v_k - u_k - u_{k+1})!}. \tag{16}$$

Because

$$|z - z'| = O(n^{-1})),$$

it follows that

$$\begin{aligned} Var(Y(z')) &= (1 + O(n^{-1}))Var(Y(z)), \\ \bar{\lambda}' &= (1 + O(n^{-1})\bar{\lambda}, \\ \frac{(e_k(z'))^{v'_k} z^t}{(e_k(z))^{v_k} (z')^{t'}} &= (1 + O(n^{-1})) \left(\frac{z^k}{e_k(z)} \right)^{u_k} z^{u_{k+1}}. \end{aligned}$$

From here and from (4), (5), (13), (14), (15) and (16) it follows that if ω is nonterminal, $h(\omega) > 0$ and the relations (11), (12) are valid and there exists nonegative solution of (2), then

$$P(\omega(t+1) = \omega' | w(t) = \omega^0(t)) = (1+O(n^{-1+3b} + n^{-1/2} \ln n)) \left(1 + O \left(\sum_{j=1}^{k-1} \frac{(u_j - 1)^+}{v_j + 1)} \right) \right) \bar{p}(\omega'|\omega), \tag{17}$$

where

$$\bar{p}(\omega'|\omega) = \frac{v_i}{v} P(multin(i(r-1), \bar{p}) = \bar{u}),$$

where $\bar{u} = (u_1, u_2, \ldots, u_{k+1})$, $multin$ is multinomial distribution with $i(r-1)$ choices and probabilities

$$\begin{aligned} p_j &= \frac{j(v_j - \delta_{ij})}{r\mu - i}, \ j = 1, 2, \ldots, k-1, \\ p_k &= \frac{z^k v_k}{(r\mu - i)(k-1)! e_k(z)}, \\ p_{k+1} &= \frac{z v_k}{r\mu - i}. \end{aligned}$$

Formula for the dynamic of the random sequence $w(t)$ (17) is differ from the case of graphs only by factor $r - 1$ in the number of choices and by r in the denominator for probabilities. This minor difference leads to the minor difference in all other analysis which is made in [3]; we omit the futher proof of the theorem sending the reader for the proof from [3].

References

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